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Stochastic Dominance of Any Type and Any Integer Degree, and Expected Utility: A Unifying Approach

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Abstract

This paper provides a general framework for a unifying treatment of stochastic dominance of any integer degree and of any type (direct or inverse for each final or intermediate level). It gives the conditions for the congruence between stochastic dominance and classes of utility functions in this general framework and shows, as particular cases, the properties of some varieties of stochastic dominance usually neglected in standard literature.

Keywords

Stochastic Dominance, Expected Utility, Risk, Decision

1. Introduction

Stochastic dominance as a way for partially ordering probability distributions was introduced in economics and decision theory through some pioneering articles published from the 1960's onwards[1] (Quirk & Saposnik, 1962; Hadar & Russell, 1969; Hanoch & Levy, 1969; Rothschild & Stiglitz, 1970; Whitmore, 1970) and spawned later an important body of literature. From the very beginning, emphasis was laid on the equivalence between the attitude toward risk of a decision maker expressed by various types of stochastic dominance, and corresponding classes of utility functions, within an expected utility framework. Called "congruence" by Fishburn (1976), this equivalence between a partial order, generated by stochastic

[1]The first demonstrations on stochastic dominance appeared long before its introduction in economics. Le Breton (1987) pointed out an alternative demonstration concerning second degree stochastic dominance when both distributions have the same mean, published by Karamata as early as 1932 (Karamata, 1932).

dominance, and the intersection between complete orders, generated by a class of utility functions, led to the powerful conclusion that preference for a stochastically dominating distribution of such type might be represented by any utility function belonging to a specific class; and, conversely, that a decision maker endowed with any utility function from this class always prefers stochastically dominating distributions of this type. General theories of integral stochastic orders and of *s*-convex orders already provide broad characterizations of stochastic orders through classes of test functions (Müller, 1997; Denuit, Lefèvre, & Shaked, 1998; Denuit, Lefèvre, & Scarsini, 2001; Shaked & Shanthikumar, 2007). The purpose of the present framework is not to define a class of abstract integral orders broader than those theories. Its distinctive feature is instead constructive: the primitive object is a word of length n which records, at every step, whether the preceding index is integrated from the lower or the upper endpoint. The intermediate primitives and their endpoint terms are therefore retained, rather than only the final test-function cone. This gives 2^n explicit dominance criteria and, through the transformation defined below, a one-to-one correspondence with all 2^n sign patterns of the first n derivatives of utility. Complete direct dominance, the usual inverse cases and the relevant *s*-convex specifications are recovered as particular patterns; what is added is the possibility of switching tail orientation independently at successive degrees, together with the associated boundary conditions and utility-sign class.

The first results relative to this topic (Quirk & Saposnik, 1962; Hadar & Russell, 1969; Hanoch & Levy, 1969; Rothschild & Stiglitz, 1970) connected first degree stochastic dominance (FSD) to the class of non-decreasing utility functions—that is, to functions whose first derivative was non-negative—so that both risk-averse and risk-seeking decision makers preferred first degree stochastically dominating distributions[2]. In the last three papers (Hadar & Russell, 1969; Hanoch & Levy, 1969; Rothschild & Stiglitz, 1970), it was also shown that second degree stochastic dominance (SSD) was linked to the class of non-decreasing concave utility functions—whose first and second derivatives are respectively non-negative and non-positive—which characterize risk-aversion for expected utility decision makers. This result was extended to a more specific type of risk-aversion, taking skewness into account, by Whitmore (1970), who linked third degree stochastic dominance (TSD) to the class of non-decreasing concave utility functions with non-negative third derivative, provided the mean of the dominating distribution was at least equal to that of the dominated one. Later on, such attitude toward risk has been named "prudence" (Kimball, 1990), and unified accounts of the three first degrees of stochastic dominance could be provided (see, for instance, Thorlund-Petersen, 2001). Again, the result on third degree stochastic dominance was generalized to

[2]An intuitive knowledge of first degree stochastic dominance in relation to decision seems to have been widely spread long before the contributions of 1969 and 1970. As Pradier (2006) pointed out, Jacob Bernoulli, for instance, argued as early as the very beginning of 18th century, in the fourth part of his *Ars Conjectandi*, that "what may be advantageous in one case, and can never harm, should be preferred [*praeferendum est*] to what is in no case beneficial or harmful", and related this to popular wisdom expressed in a German saying, "*Hilffdt es nicht/ so schadt es nicht*" (Bernoulli, [1713] 2006).

n-th degree stochastic dominance, which was related to utility functions whose odd and even derivatives are respectively non-negative and non-positive, till the *n*-th degree (Fishburn, 1976), therefore allowing a broader exploration of these refinements of the attitude toward risk involved similarly in higher degree stochastic dominance and in the signing of higher degree derivatives of the utility function, like what was called "temperance" (4th degree, after Kimball, 1992) or "edginess" (5th degree, Lajeri-Chaherli, 2004). This line of analysis is also connected to increasing *n*-th degree risk (Ekern, 1980) and to the apportionment of risks through higher-order stochastic dominance (Eeckhoudt, Schlesinger, & Tsetlin, 2009).

Symmetrically, characterizations of risk-seeking appeared some years later, giving rise to what was to be called (in spite of an imperfectly fixed vocabulary) "inverse" stochastic dominance[3]. Goovaerts, De Vylder, & Haezendonck (1984) introduced second degree inverse stochastic dominance (SISD), which amounts to what Levy (2016: pp. 110-115) called "risk-seeking stochastic dominance" (RSSD), as related to the class of non-decreasing convex utility functions (with non-negative first and second derivatives). Like in the case of direct dominance, taking skewness into account led to third degree inverse stochastic dominance, which was divided by Zaras (1989) between two types: third degree type 1 inverse stochastic dominance (TISD1), linked to the class of convex utility functions with non-positive third derivative (Zaras, 1989) therefore denoting imprudence; and third degree type 2 inverse stochastic dominance (TISD2), which also corresponds to convex utility functions, but with non-negative third derivative (Goovaerts, De Vylder, & Haezendonck, 1984) which might be interpreted as prudence, like for TSD[4].

Since the pioneering works of the 1960's and 70's, the concept of stochastic dominance has been subjected to several refinements and extensions: quantile approach, introduction of riskless assets, extension to the measure of inequalities, multicriteria decision, almost dominance or fuzzy measures, consistency with non-expected utility theories like rank-dependent utility or cumulative prospect theory, etc. A review of this literature can be found in Shaked & Shanthikumar (2007), Sriboonchitta et al. (2010), Guo (2013), and Levy (2016). But what seems to have been the most salient evolution concerns the way the orders generated by stochastic dominance were considered. The original approach leads to view stochastic dominance of various types as generating possible partial preference preorders over a set of lotteries. This justifies raising the question of the congruence

[3]For instance, in a pioneering paper where they applied stochastic dominance to the question of inequality measures, Muliere & Scarsini (1989) named "inverse" stochastic dominance the integration of differences between the inverses of cumulative distribution functions, F^{-1} and G^{-1}, and not between decumulative functions.

[4]For systematic accounts of risk attitudes involved in the signs of the various derivatives of the utility function, see Eeckhoudt & Schlesinger (2006), Eeckhoudt, Schlesinger, & Tsetlin (2009), and Deck & Schlesinger (2014). Risk-seeking alone does not determine the sign of the third derivative: the prudence result for "mixed risk lovers" uses a more restrictive notion (Crainich, Eeckhoudt, & Trannoy, 2013; Ebert, 2013).

with a class of utility functions within a context of expected utility. However, once the idea of congruence has been accepted in principle, attention seems to have shifted toward the nature of the concept of stochastic dominance which allows the efficient selection of a lottery among a subset of lotteries, according to some utility functions belonging to the congruent class. Typical of this approach is the paper by Post & Kopa (2013), which allows for comparing a given lottery with a discrete set of alternative lotteries, or with a set of linear combinations of these lotteries, on the basis of a generalization of the concept of convex stochastic dominance, which Fishburn introduced as early as 1974 (Fishburn, 1974). Subsequent work developed tractable tests and efficiency criteria for richer higher-order classes (Post, Fang, & Kopa, 2015; Fang & Post, 2017).

The present analysis concerns positive integer degrees $n \geq 1$. Fractional-degree stochastic dominance addresses a different question—interpolation between adjacent integer orders—and is therefore complementary to the direct/inverse classification developed here (Müller et al., 2017; Bi & Zhu, 2019; Huang, Tzeng, & Zhao, 2020; Mao, Wang, & Zhao, 2026).

This paper goes a step back to the original approach, acknowledging that the question of congruence has been solved only for the few cases noted above. It follows on from Fishburn's solution, which concerned *n*-th degree direct stochastic dominance (Fishburn, 1976) but gives up this limitation to a specific type of stochastic dominance. Therefore, it first provides a unifying treatment, now permitting any type of stochastic dominance, direct or inverse, and of any integer degree, in relation to classes of utility functions defined by the signing of their successive derivatives (section 2). This allows giving general conditions of congruence, that is conditions for the representation of each kind of stochastic dominance by a class of utility functions and reciprocally (section 3).

2. A Formal Framework

Let X and Y be two random variables with distinct distributions that are concentrated on a common compact bounding interval $[a,b] \subset \mathbb{R}$. The endpoints need not be the exact endpoints of both supports: either density is understood to be zero on any part of $[a,b]$ outside its actual support. The reference interval $[a,b]$ is held fixed throughout. This restriction is substantive for higher-order dominance: above the third order, dominance may depend on the chosen reference interval (Wang & Wu, 2025). Assume that their respective distribution functions, F and G, are absolutely continuous, with non-negative densities $f, g \in L^1([a,b])$ of unit integral. The recursively defined indices $H^i_{A,n}$ are then absolutely continuous for $i \geq 1$. Utility functions are taken in $C^n([a,b])$; these assumptions are sufficient for every application of integration by parts in the Appendix[5]. Throughout, $u_i := u^{(i)}$ for $i \in [n]$, and $E_u(f) := \int_a^b u(x) f(x) \mathrm{d}x$ and

[5]The compact-support and absolute-continuity assumptions are made for simplicity, in order to avoid the additional qualifications required by Riemann-Stieltjes or Lebesgue-Stieltjes integration. See the clarification and restatement of the first results of Hanoch and Levy (1969) and Hadar and Russell (1971) by Tesfatsion (1976).

$E_u(g) := \int_a^b u(x)g(x)\mathrm{d}x$. Stochastic dominance of degree i ($i = 1, \cdots, n$) of f over g (X over Y) amounts to the non-positivity of an index of dominance for each value of x on $[a,b]$. This index, which is said *final* when $i = n$ and *intermediate* otherwise, is the i-th integral of $f(x) - g(x)$, each step of integration being either from a to x — *direct* dominance—or from x to b — *inverse* dominance. The situation usually favored in the literature, where each step of integration from 1 to n is direct, will be termed hereafter "complete direct dominance" of degree n. Symmetrically, in an expected utility framework, preference given to f over g amounts to the non-negativity of the difference between expected utilities, $E_u(f) - E_u(g)$, which depends on the properties of a utility function $u \in C^n([a,b])$, expressed in the signs of its successive derivatives, $u_1(x), \cdots, u_n(x)$.

The formal framework within which the relation between stochastic dominance and expected utility is investigated results from the definitions of 1) the index and condition of stochastic dominance; 2) the classes of utility functions and of resulting preferences according to expected utility; and 3) a transformation procedure from the degrees of integration of the index of stochastic dominance to the degrees of derivation of the corresponding utility function.

Definition 1. Stochastic dominance

Let $A \subseteq [n]$, *where* $[n]$ *is the set of integers* $\{1, \cdots, n\}$. *Then*:

1) *The* stochastic dominance index *of degree* $i \in [n]$, *denoted* $H_{A,n}^i(x)$ *is defined by*[6]:

$$H_{A,n}^i(x) = \mathbb{1}_A(i)\int_a^x H_{A,n}^{i-1}(y)\mathrm{d}y + (1 - \mathbb{1}_A(i))\int_x^b H_{A,n}^{i-1}(y)\mathrm{d}y$$
$$\left(\text{with } H_{A,n}^0(x) = f(x) - g(x)\right).$$

This index is final if $i = n$ *and intermediate if* $i < n$.

2) *Let* $fD_{A,n}g$ *denote the stochastic dominance of* f *over* g *at final degree* n, direct *for all degrees in A and* inverse *for all degrees not in A. The condition of stochastic dominance of* f *over* g *is defined as*:

$$fD_{A,n}g \Leftrightarrow \forall x \in [a,b], H_{A,n}^n(x) \leq 0$$

(*since the distributions* F *and* G *are distinct, strict inequality holds for at least one value of* x).

Note. Throughout this paper, subscripts n *or A in* $H_{A,n}^i$ *are omitted when the context avoids any misunderstanding.*

Example 1. Assume $1 < l < m < n$, with $1, l, n \in A$ and $m \in \complement_{[n]}A$ (that is, in the complement of A with respect to $[n]$). The dominance index of final degree n and of intermediate degrees $1, l, m$ results from repeated integrations of $f(x) - g(x)$, where the first integration is from a to x_1, the l-th from a to x_l, the m-th from x_m to b, and the n-th from a to x:

[6]Recall that the indicator function $\mathbb{1}_A$ in Definition 1 is an application from $[n]$ to $\{0,1\}$ which, for $i \in [n]$, yields 1 if i belongs to A, and 0 if it does not.

$$H_{A,n}^{n}(x)=\int_{a}^{x}\cdots\left[\int_{x_m}^{b}\cdots\left[\int_{a}^{x_l}\cdots\left[\int_{a}^{x_1}\left(f(y)-g(y)\right)\mathrm{d}y\right]\cdots\mathrm{d}x_{l-1}\right]\cdots\mathrm{d}x_{m-1}\right]\cdots\mathrm{d}x_{n-1}.$$

If $H_{A,n}^{n}(x)\le 0$ for each value of x on $[a,b]$, f is said to dominate stochastically g, directly at final degree n and at intermediate degrees 1 and l, and inversely at intermediate degree m.

Remark 1. Let $A_{(i)}$ and $A_{\setminus(i)}$ be identical subsets of $[n]$, except that i belongs to $A_{(i)}$ and not to $A_{\setminus(i)}$. By construction, the following properties always hold:

$$H_{A_{(i)}}^{i}(a)=H_{A_{\setminus(i)}}^{i}(b)=0 \quad \text{(a)}$$

$$H_{A_{(i)}}^{i}(b)=H_{A_{\setminus(i)}}^{i}(a) \quad \text{(b)}$$

$$H_{A_{(i)}}^{i}(x)=H_{A_{\setminus(i)}}^{i}(a)-H_{A_{(i)}}^{i}(x) \quad \text{(c)}$$

$$H_{A_{\setminus(i)}}^{i}(x)=H_{A_{(i)}}^{i}(b)-H_{A_{(i)}}^{i}(x) \quad \text{(d)}$$

$$H_{A_{(1)}}^{i}(x)=-H_{A_{\setminus(1)}}^{i}(x) \quad \text{(e)}$$

Remark 2. Standard literature often favors situations of *complete* direct stochastic dominance, where the complementary of A, $\complement_{[n]}A$, is empty. For instance: first (FSD), second (SSD), third (TSD) and *n*-th (NSD) degree stochastic dominance respectively correspond to configurations where the bipartitions of $[n]$ are respectively such that $\left(A,\complement_{[1]}A\right)=\left(\{1\},\varnothing\right)$, $\left(A,\complement_{[2]}A\right)=\left(\{1,2\},\varnothing\right)$, $\left(A,\complement_{[3]}A\right)=\left(\{1,2,3\},\varnothing\right)$ and $\left(A,\complement_{[n]}A\right)=\left(\{1,\cdots,n\},\varnothing\right)$. The possibility that $\complement_{[n]}A$ be not empty (that is, that f stochastically dominates g *inversely* at some intermediate or final degree) is typically dealt with in order to explore risk-seeking, through either second degree inverse stochastic dominance (SISD or RSSD—risk-seeking stochastic dominance), where $\left(A,\complement_{[2]}A\right)=\left(\{1\},\{2\}\right)$, or third degree inverse stochastic dominance, either of type 1 (TISD1) when the third integration is from a to x so that $\left(A,\complement_{[3]}A\right)=\left(\{1,3\},\{2\}\right)$, or of type 2 (TISD2) when the third integration is from x to b so that $\left(A,\complement_{[3]}A\right)=\left(\{1\},\{2,3\}\right)$.

The construction of relevant classes of n-differentiable utility functions u defined on $[a,b]$ obeys the same principles as those which have governed the construction of an index of stochastic dominance. The starting point is the identification of the sets of degrees of derivation of a utility function, corresponding to either positive or negative derivatives.

Definition 2. Utility

Let $B\subseteq[n]$. *Then*:

1) *Define a set* $U_{B,n}$ *of n-times continuously differentiable utility functions as*:

$$\begin{aligned}U_{B,n}=\Big\{&u\in C^{n}\left([a,b]\right):\forall i\in B,\forall x\in[a,b],u_i(x)\ge 0,\\&\forall i\in\complement_{[n]}B,\forall x\in[a,b],u_i(x)\le 0,\\&\forall i\in[n],\exists x_i\in[a,b]\text{ such that }u_i(x_i)\ne 0\Big\}.\end{aligned}$$

2) *Following the expected utility approach, all decision makers whose utility*

function belongs to $U_{B,n}$ *are said to prefer* f *to* g *when for each of them, the expected value of the utility of* f *is not smaller than that of* g. *Their common preference is denoted* $fR_{B,n}g$:

$$fR_{B,n}g \Leftrightarrow \forall u \in U_{B,n}, E_u(f) - E_u(g) \geq 0.$$

Remark 3. The signing of the derivatives of the utility function is usually linked to some typical attitudes toward risk through the ranking of random variables according to their expected utility. For instance, it is well-known that increasing utility functions ($1 \in B$) characterize decision-makers with monotonous increasing preferences, whatever their attitude toward risk. When $2 \in \complement_{[n]}B$, the decision-maker is risk-averse and, conversely, he or she is risk-seeking when $2 \in B$. The intuitive meaning of $3 \in B$ is this of "prudence" (Kimball, 1990), so that $3 \in \complement_{[n]}B$ corresponds to "imprudence". Higher degrees of derivation are not that easy to interpret. However, $4 \in \complement_{[n]}B$ and $4 \in B$ are currently viewed as, respectively, "temperance" (Kimball, 1992) and "intemperance". Similarly, $5 \in B$ and $5 \in \complement_{[n]}B$ correspond to "edginess" (Lajeri-Chaherli, 2004) and to what might be called "calmness". Typical utility functions like the logarithmic ones, which give rise to monotone increasing preferences, risk-aversion, prudence, temperance, edginess and so on, are characterized by alternatively positive and negative derivatives, and were sometimes called "mixed risk averse" utility functions (Caballé & Pomansky, 1996): $B = \{i \in [n] : i \text{ is odd}\}$ and $\complement_{[n]}B = \{i \in [n] : i \text{ is even}\}$. Eeckhoudt & Schlesinger (2006) have shown the equivalence between such higher order risk attitudes and preferences over particular classes of lottery pairs, involving zero-mean independent noise random variables. Later work has examined the consistency and experimental content of higher-order risk attitudes (Deck & Schlesinger, 2014; Trautmann & van de Kuilen, 2018), as well as nonmonotonic sign patterns that are neither uniformly prudent nor uniformly imprudent (Bi & Zhu, 2022).

The relation between stochastic dominance and expected utility is taken up through a relation between two elements A and B of the powerset of $[n]$. Consider the Procedure AB hereafter, which allocates i ($i = 1, \cdots, n$) between B and $\complement_{[n]}B$ as a degree of derivation, according to its belonging, as a degree of integration, to A or $\complement_{[n]}A$, and to the belonging of $i-1$ to B or $\complement_{[n]}B$:

Procedure AB

- *For all* $i \in A$

$$\begin{cases} \text{if } i-1 \in B, \text{ then } i \in \complement_{[n]}B \\ \text{if } i-1 \in \complement_{[n]}B \text{ or } i = 1, \text{ then } i \in B \end{cases}$$

- *For all* $i \in \complement_{[n]}A$

$$\begin{cases} \text{if } i-1 \in \complement_{[n]}B \text{ or } i = 1, \text{ then } i \in \complement_{[n]}B \\ \text{if } i-1 \in B, \text{ then } i \in B \end{cases}$$

Definition 3. Transformation

The transformation of $i = 1, \cdots, n$, *as degrees of integration belonging to either*

A or $\complement_{[n]}A$, into degrees of derivation belonging to either B or $\complement_{[n]}B$, is performed by

$$\phi: 2^{[n]} \to 2^{[n]}, A \mapsto \phi(A) = B \tag{1}$$

3. Congruence between Stochastic Dominance and Expected Utility

The issue is now that of the conditions of consistency between two partial orders: according to stochastic dominance, $D_{A,n}$, and to expected utility, $R_{B,n}$. Though presented differently, such consistency was called "congruence" by P. Fishburn (1976: p. 303):

Definition 4. Congruence

A stochastic dominance order $D_{A,n}$ and an expected utility order $R_{B,n}$ are congruent *when, for each f and g,*

$$fD_{A,n}g \Leftrightarrow fR_{B,n}g.$$

It will be shown (Proposition 1) that, provided additional conditions on the bounds of the distributions are satisfied, congruence is achieved when B is the image of A through the transformation ϕ defined in (1). The structure of the relation between the orders generated by stochastic dominance and expected utility is further investigated in Proposition 2.

The following lemma shows that each possible allocation of the degrees of integration corresponds through ϕ to an allocation of the degrees of derivation, and reciprocally:

Lemma 1. The transformation $\phi: 2^{[n]} \to 2^{[n]}$ is a bijection.

Proof. Write $a_i = \mathbb{1}_A(i)$ and $b_i = \mathbb{1}_B(i)$. Procedure AB gives $b_1 = a_1$ and, for $i \geq 2$,

$$b_i = \begin{cases} b_{i-1}, & a_i = 0, \\ 1 - b_{i-1}, & a_i = 1. \end{cases}$$

Thus $b_i \equiv \sum_{j=1}^{i} a_j \pmod 2$, which determines $B = \phi(A)$ uniquely. Conversely, given B, define

$$a_1 = b_1, \quad a_i = \mathbb{1}_{\{b_i \neq b_{i-1}\}} \quad (i = 2, \cdots, n).$$

This construction yields a unique A and reverses the preceding recursion. It is therefore an explicit inverse of ϕ, which proves that ϕ is a bijection of $2^{[n]}$ onto itself. □

A general condition for congruence between stochastic dominance and expected utility orders is presented in the following proposition.

Proposition 1. *For all $A, B \in 2^{[n]}$ such that $B = \phi(A)$, the two following propositions are equivalent for all f and g:*

1) $fR_{B,n}g$ (*expected utility*).

2) $fD_{A,n}g$ (*stochastic dominance*)

and, if $n \geq 3$, for all $i = 2, \cdots, n-1$:

$H^i_{A,n}(\gamma_i) \leq 0$ (*where* $\gamma_i = \mathbb{1}_A(i)b + (1 - \mathbb{1}_A(i))a$, for every i in $[n]$)

(*conditions on upper or lower bounds*).

Proof. See Appendix B. □

An intuitive interpretation of Proposition 1 is that provided conditions on bounds are satisfied, the partial order on random variables generated by any type of stochastic dominance, say $D_{A,n}$, is identical to the intersection between all the complete preference preorders on random variables underlying all the possible utility functions belonging to $U_{B,n}$, when $B=\phi(A)$. Proposition 1 may also be viewed as establishing a link between subjective assessments and objective properties. It means that on the one hand, what all possible decision makers, characterized by a utility function from the same class, have in common evidently rests on their respective subjective preferences; but, on the other hand, this common ranking of random variables may alternatively be viewed as model-free, that is, as depending only on some objective properties of the distribution functions expressed in stochastic dominance.

We know that in the case of complete direct n-th degree stochastic dominance, the conditions on upper bounds can be viewed as algebraic combinations of the differences between the successive moments around zero of the distributions (Jean & Helms, 1988). A typical illustration, anticipated by Whitmore (1970), is that in order to have congruence between complete direct third degree stochastic dominance ($A=\{1,2,3\}$) and the class of increasing concave utility functions with a non-negative third derivative ($B=\{1,3\}$), the difference between means (first moments), $E(f)-E(g)$, had to be non-negative[7]. The conclusion of Jean & Helms (1988) still holds in the case of Proposition 1 where dominance is not necessarily direct at each step since, by construction, the value of any index of dominance at the relevant bound is itself an algebraic sum of some complete direct dominance indices of equal and smaller degrees at the upper bound: any index of degree i at the relevant bound is therefore a linear combination of the differences between the moments around zero ${}^{j}M_{f}(0)-{}^{j}M_{g}(0)$ of all degrees j from 1 to $i-1$. Related refinements connect higher-degree stochastic dominance to decreasing higher-order absolute risk aversion (Denuit & Liu, 2014).

Example 2. A worked mixed pattern: $A=\{1,2\}$ at degree three

Let $[a,b]=[0,1]$, $f(x)=2x$ and $g(x)=2(1-x)$. Both are densities on the common reference interval. For $A=\{1,2\}$, the first two integrations are from 0 to x, whereas the third is from x to 1. Direct calculation gives

$$H_{A,3}^{1}(x)=2x^{2}-2x,$$

$$H_{A,3}^{2}(x)=\frac{2}{3}x^{3}-x^{2},$$

$$H_{A,3}^{3}(x)=\int_{x}^{1}H_{A,3}^{2}(t)\,\mathrm{d}t=-\frac{1+x^{4}-2x^{3}}{6}.$$

Since $1+x^{4}-2x^{3}$ decreases from 1 to 0 on $[0,1]$, $H_{A,3}^{3}(x)\le 0$ throughout the

[7]Recall that $E(f)-E(g)\ge 0$ is already a consequence of complete direct stochastic dominance at degrees 1 or 2.

interval: the final dominance condition is satisfied. The only additional boundary condition in Proposition 1 is

$$H_{A,3}^{2}(\gamma_2) = H_{A,3}^{2}(1) = -\frac{1}{3} \leq 0,$$

because $2 \in A$. Procedure AB gives $B = \phi(A) = \{1\}$, so the associated utility class is

$$U_{\{1\},3} = \left\{u \in C^3([0,1]) : \forall x \in [0,1], u_1 \geq 0, u_2 \leq 0, u_3 \leq 0\right\},$$
$$\forall x \in [1,2,3], \exists x_i \in [0.1] : u_i(x_i) \neq 0$$

Here $\gamma_3 = 0$, and equation (2) becomes

$$E_u(f) - E_u(g) = -u_1(1)H_{A,3}^{2}(1) + u_2(0)H_{A,3}^{3}(0) + \int_0^1 u_3(x)H_{A,3}^{3}(x)\mathrm{d}x.$$

Every term on the right-hand side is non-negative for $u \in U_{\{1\},3}$. This example displays explicitly the lower-, lower-, upper-tail sequence and the boundary term that accompanies it.

Example 3. Usual and neglected types of stochastic dominance in relation to expected utility

Such representation of congruence allows finding again the usual formulations of stochastic dominance from the end of the 1960's to the 1980's, and their correspondence with certain classes of utility functions. In return, it also allows finding the neglected or missing categories which can now be put to the fore.

This bringing together is obvious in the case of complete direct dominance in the following usual cases, A_n and B_n denoting elements of the power set $2^{[n]}$:

FSD (first degree stochastic dominance: Quirk & Saposnik (1962) (discrete case); Hadar & Russell, 1969; Hanoch & Levy, 1969) corresponds to $A_1 = \{1\}$. It was acknowledged as congruent with

$$B_1 = \phi(\{1\}) = \{1\}$$

that is with monotonous non-decreasing preferences, entailing universal preference for the dominating random variable, whatever the decision maker's attitude toward risk.

SSD (second degree stochastic dominance: Hadar & Russell, 1969; Hanoch & Levy, 1969; Rothschild & Stiglitz, 1970 ("mean preserving spread" subcase-MPS)) corresponds to $A_2 = \{1,2\}$, and was acknowledged congruent with

$$B_2 = \phi(\{1,2\}) = \{1\}$$

that is with monotonous non-decreasing preferences and risk-aversion.

TSD (third degree stochastic dominance: Whitmore, 1970) corresponds to $A_3 = \{1,2,3\}$, and was acknowledged congruent with

$$B_3 = \phi(\{1,2,3\}) = \{1,3\}$$

that is with monotonous non-decreasing preferences, risk-aversion, positive skewness-seeking and negative skewness-aversion (typically, a decision maker interested in insuring his or her house, and in buying public lottery tickets). This

corresponds to what is usually called "prudence" since Kimball (1990)'s paper, in relation to the positive sign of the third derivative of the utility function.

NSD (n -th degree stochastic dominance: Fishburn, 1976) corresponds to complete direct stochastic dominance with $A_n = [n]$ and was acknowledged congruent with

$$B_n = \phi([n]) = \{i \in [n] : i \text{ is odd}\}$$

that is, with a class of utility functions whose odd and even derivatives are respectively non-negative and non-positive.

The same bringing together is a bit less immediate in the cases of stochastic dominance of degrees 2 and 3, related to monotonous non-decreasing preferences and risk-seeking, from the point of view of expected utility. These different cases were usually called "inverse" stochastic dominance (see, for example, Zaras, 1989), and have in common that $2 \notin A_n$. A technical but simple problem of presentation arises from the fact that the corresponding indices of stochastic dominance generally relied on decumulative distribution functions, of type $\bar{F}(x) = \int_x^b f(y)\mathrm{d}y$, instead of cumulative functions $F(x) = \int_a^x f(y)\mathrm{d}y$. The resulting successive integrals were therefore computed on the basis of $H^1_{\varnothing,1}(x)$, instead of $H^1_{\{1\},1}(x)$, although the concerned decision makers were endowed with non-decreasing preferences. Nonetheless, since $H^1_{\varnothing,1}(x) = -H^1_{\{1\},1}(x)$, any higher degree of the intermediate or final index is such that, assuming that 1 is an element of A_n , $H^i_{A_n\setminus\{1\},n}(x) = -H^i_{A_n,n}(x)$ (see Remark 1 on Definition 1). So that the non-positivity requirement for the index of stochastic dominance was usually replaced by a non-negativity condition insofar as it was applied to inverse dominance at intermediate or final degree 2. Taking this into account allows establishing the following three correspondences:

SISD (second degree inverse stochastic dominance: Goovaerts, De Vylder, & Haezendonck, 1984) or **RSSD** (risk-seeking stochastic dominance: Levy, 2016) corresponds to $A_2 = \{1\}$, and was acknowledged congruent with

$$B_2 = \phi(\{1\}) = \{1,2\}$$

that is, with monotonous non-decreasing preferences and risk-seeking.

TISD1 (third degree type 1 inverse stochastic dominance: Zaras, 1989) corresponds to $A_3 = \{1,3\}$ and was acknowledged congruent with

$$B_3 = \phi(\{1,3\}) = \{1,2\}$$

that is, with monotonous non-decreasing preferences, risk-seeking, positive skewness-aversion and negative skewness-seeking (imprudence).

TISD2 (third degree type 2 inverse stochastic dominance: Goovaerts, De Vylder, & Haezendonck, 1984) corresponds to $A_3 = \{1\}$ and was acknowledged congruent with

$$B_3 = \phi(\{1\}) = \{1,2,3\}$$

that is, with monotonous non-decreasing preferences, risk-seeking, positive skew-

ness-seeking and negative skewness-aversion (prudence).

However, the three following instances do not appear to have received a systematic treatment within the present 2^n lower-/upper-tail integration framework, and are worth being noted. Related work on nonmonotonic higher-order risk preferences is discussed by Bi & Zhu (2022).

1) The first instance is that of situations in which a decision maker would prefer the distribution whose probability of getting at least this income is the *smaller* that is, the stochastically dominating random variable at degree 1 when $A_1 = \varnothing$:

FISD (stochastic dominance *inverse* for the degree 1). $A_1 = \varnothing$. This amounts to congruence with

$$B_1 = \phi(\varnothing) = \varnothing$$

that is with non-increasing utility functions. Of course, such situations might seem of little practical interest—except in case the purpose were to consider the behavior toward risk of a decision maker whose objective would be to ruin himself or herself. However, it makes clear that, contrary to a hasty conclusion, though preference for distributions which are stochastically directly dominated at degree 1 by other distributions might seem a bit strange, it is by no means constitutively irrational.

2) But it is also obvious that, by contrast to TSD which has been extensively studied after Whitmore's (1970) paper, in relation to the more familiar idea of DARA (decreasing absolute risk aversion), a second type of third stochastic dominance TSD2, homologous to TISD2, doesn't seem to have aroused special interest:

TSD2 (third degree stochastic dominance, type 2) corresponds to $A_3 = \{1,2\}$. According to Proposition 1, granted that third degree conditions on bounds are satisfied, TSD2 is congruent with

$$B_3 = \phi(\{1,2\}) = \{1\}$$

that is, with monotonous non-decreasing preferences, risk-aversion, positive skewness-aversion and negative skewness-seeking—which amounts to imprudence. After all, we all know people who dislike risk, never buy public lottery tickets, and nonetheless would reject the idea of insuring their house if they were not legally obliged to subscribe to an insurance contract.

3) Finally, whereas *complete* direct stochastic dominance of any degree n (NSD, where $A_n = \{1,\cdots,n\}$) has been regularly taken up after Fishburn's (1976) paper (see, for instance, Levy, 2016: pp. 115-116), such was not the case for stochastic dominance direct or indirect at any intermediate degree i or final degree n, that is, when A_n is any element of $2^{[n]}$. Filling this gap was the main purpose of this article.

Remark 4. Stochastic dominance orders and classes of utility functions

According to a well-known "sufficient rule" of stochastic dominance orders, if f stochastically dominates g at degree k, it also dominates g at degrees $k+1, k+2, \cdots$ (see, for instance, Levy, 2016: pp. 104-105, for the relations between complete direct dominances of degrees 1, 2 and 3). Typically, this means that risk-

averse and risk-seeking decision makers with positive monotone preferences might disagree on the order between random variables not ordered by first degree direct stochastic dominance, but that they agree on the order of random variables generated by the latter, which is consistent with both direct and inverse stochastic dominance of degree 2. This result can be easily generalized and related to preferences according to expected utility in the following proposition. For each positive integer r, let $\phi_r : 2^{[r]} \to 2^{[r]}$ denote the transformation defined by Procedure AB on $[r]$.

Proposition 2. *Let* m *and* n *be two positive integers such that* $1 \le m \le n$. *Let* $A_m \in 2^{[m]}$ *and* $A_n \in 2^{[n]}$, *and set* $B_m = \phi_m(A_m)$ *and* $B_n = \phi_n(A_n)$. *If*

$$A_m \subseteq A_n, \complement_{[m]} A_m \subseteq \complement_{[n]} A_n \quad or \quad B_m \subseteq B_n, \complement_{[m]} B_m \subseteq \complement_{[n]} B_n$$

then:

1) $D_{A_m,m} \subseteq D_{A_n,n}$

2) $U_{B_n,n} \subseteq U_{B_m,m}$

(By abuse of notation, $D_{A_m,m}$ and $D_{A_n,n}$ stand for the set-equivalents of the corresponding binary relations).

Proof. By the definitions of ϕ_m and ϕ_n, the two alternative conditions are equivalent: $A_m \subseteq A_n, \complement_{[m]} A_m \subseteq \complement_{[n]} A_n \Leftrightarrow B_m \subseteq B_n, \complement_{[m]} B_m \subseteq \complement_{[n]} B_n$.

1) The first condition gives $A_n \cap [m] = A_m$. Hence, if $fD_{A_m,m}g$, then $H^m_{A_n,n}(x) = H^m_{A_m,m}(x) \le 0$ for every $x \in [a,b]$. Each subsequent lower- or upper-tail integration preserves non-positivity, through $H^n_{A_n,n}(x) \le 0$. Therefore $fD_{A_m,m}g \Rightarrow fD_{A_n,n}g$.

2) The second equivalent condition gives $B_n \cap [m] = B_m$. Hence $u \in U_{B_n,n} \Rightarrow u \in U_{B_m,m}$. □

Bringing together the two parts of Proposition 2 shows the structure of the relation between the two partial orders generated either by stochastic dominance, or by expected utility. At the final stage, Proposition 1 makes clear that when conditions on bounds are satisfied, they lead to the same result: $D_{A_m,m}$ and $R_{B_m,m}$ order the same random variables (that is, $D_{A_m,m} = R_{B_m,m}$), just like $D_{A_n,n}$ and $R_{B_n,n}$ (that is, $D_{A_n,n} = R_{B_n,n}$) and it can be concluded from Proposition 2(1) that their set is expanding from m to n. However, Proposition 2(2) shows that the same conclusion is obtained differently from the expected utility point of view. Moving from m to n first generates a contraction: for instance, such utility function, which was included in $U_{B_m,m}$ does not belong any more to $U_{B_n,n}$, because of the signs of its derivatives from $m+1$ to n. But since the resulting partial order R is the intersection between the complete orders underlying each utility function belonging to U, R depends on a (weakly) smaller class of utility functions when moving from m to n. So that it is expanding from $R_{B_m,m}$ to $R_{B_n,n}$, whereas U is contracting from $U_{B_m,m}$ to $U_{B_n,n}$.

4. Concluding Remarks

The aim of this paper was to set out a framework for analyzing the relations be-

tween subjective preferences between random variables founded on specific classes of utility functions, and the model-free, objective properties of the probability distributions of these random variables expressed in corresponding types of stochastic dominance. This framework first consists in the construction of an index of stochastic dominance, which allows either direct or inverse dominance at any intermediate or final degree. A proposition is demonstrated which establishes the conditions of congruence (in the sense of Fishburn, 1976) between the orderings generated by stochastic dominance and by classes of utility functions, extending the results of the pioneering works of Hadar & Russell (1969), Hanoch & Levy (1969), Whitmore (1970), Fishburn (1976), Goovaerts, De Vylder, & Haezendonck (1984) and Zaras (1989). Comparison with previous contributions also leads to the identification of neglected kinds of stochastic dominance, like a third degree type 2 stochastic dominance (TSD2) and the congruent class of utility functions. The contribution concerns integer degrees and a fixed common reference interval; fractional-degree orders and the sensitivity of higher-order dominance to the reference interval are complementary issues (Huang, Tzeng, & Zhao, 2020; Wang & Wu, 2025).

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix: Congruence between Stochastic Dominance and Expected Utility (Proposition 1)

A. Preliminaries Concerning the Construction of the Expression of $E_u(f)-E_u(g)$

[*At least till Equation* (2), *this sub-section can be skipped by the readers familiar with the technique of demonstration already used in most pioneering papers since Hadar & Russell* (1969) *or Hanoch & Levy* (1969)].

Under the assumptions stated in Section 2, $u\in C^n\left(\left[a,b\right]\right)$ and each $H_{A,n}^i$, $i\geq 1$, is absolutely continuous. All the boundary values below are therefore defined, and integration by parts may be applied successively at every degree. The proof of Proposition 1 starts from $E_u(f)-E_u(g)=\int_a^b u(x)\left(f(x)-g(x)\right)\mathrm{d}x=\int_a^b u(x)H_{A,n}^0(x)\mathrm{d}x$, the non-negativity of $E_u(f)-E_u(g)$ for all $u\in U_{B,n}$ being equivalent to $fR_{B,n}g$. The result is quite classical, except that at each step of integration, it has to be given according to the belonging of i to either A or $\complement_{[n]}A$, which draws on Remark 1 (a)-(e). Starting from the first degree of integration gives:

- Degree 1:

$$E_u(f)-E_u(g)=\int_a^b u(x)H_A^0(x)\mathrm{d}x\begin{cases}1\in A:\\ \quad=\left[u(x)H_A^1(x)\right]_a^b-\int_a^b u_1(x)H_A^1(x)\mathrm{d}x\\ \quad=-\int_a^b u_1(x)H_A^1(x)\mathrm{d}x\\ 1\in\complement_{[n]}A:\\ \quad=-\left[u(x)H_A^1(x)\right]_a^b+\int_a^b u_1(x)H_A^1(x)\mathrm{d}x\\ \quad=\int_a^b u_1(x)H_A^1(x)\mathrm{d}x.\end{cases}$$

Note that the sign before the integrals in the right-hand side of the equations is negative when 1 belongs to A, and positive when it does not.

- Degree 2:

$$E_u(f)-E_u(g)\begin{cases}1\in A:\\ \quad=-\int_a^b u_1(x)H_A^1(x)\mathrm{d}x\begin{cases}2\in A:\\ \quad=-u_1(b)H_A^2(b)+\int_a^b u_2(x)H_A^2(x)\mathrm{d}x\\ 2\in\complement_{[n]}A:\\ \quad=-u_1(a)H_A^2(a)-\int_a^b u_2(x)H_A^2(x)\mathrm{d}x\end{cases}\\ 1\in\complement_{[n]}A:\\ \quad=\int_a^b u_1(x)H_A^1(x)\mathrm{d}x\begin{cases}2\in A:\\ \quad=u_1(b)H_A^2(b)-\int_a^b u_2(x)H_A^2(x)\mathrm{d}x\\ 2\in\complement_{[n]}A:\\ \quad=u_1(a)H_A^2(a)+\int_a^b u_2(x)H_A^2(x)\mathrm{d}x.\end{cases}\end{cases}$$

To sum up, the sign before the integral of $u_2(x)H_A^2(x)$ in the right-hand side of each equation is positive when $2 \in A$ and when, at degree 1, the sign before the integral of $u_1(x)H_A^1(x)$ in the right-hand side is negative; or when $2 \notin A$ and when, at degree 1, the sign before the integral of $u_1(x)H_A^1(x)$ in the right-hand side is positive. Conversely, it is negative when $2 \notin A$ and when, at degree 1, the sign before the integral of $u_1(x)H_A^1(x)$ in the right-hand side is negative; or when $2 \in A$ and when, at degree 1, the sign before the integral of $u_1(x)H_A^1(x)$ in the right-hand side is positive. On the other hand, the sign before $u_1(\gamma_2)H_A^2(\gamma_2)$, where γ_2 equals a or b, still in the right-hand side of each equation, is positive when at degree 1, the sign before the integral of $u_1(x)H_A^1(x)$ in the right-hand side is itself positive, and it is negative otherwise.

Carrying on till degree n yields:

- Degree n:

$$E_u(f)-E_u(g)=\sum_{i=2}^{n}\sigma_{i-1}u_{i-1}(\gamma_i)H_{A,n}^i(\gamma_i)+\sigma_n\int_a^b u_n(x)H_{A,n}^n(x)\mathrm{d}x \qquad (2)$$

where for all $i \in [n]$, $\sigma_i = 1-2\mathbb{1}_C(i)$.

The set $C \in 2^{[n]}$ is the outcome of Procedure AC hereafter, which allocates each i to either C or $\complement_{[n]}C$ and, therefore, determines the values of σ_n and σ_{i-1} in (2).

Procedure AC

- *For all* $i \in A$

$$\begin{cases} \text{if } i-1 \in C, \text{ then } i \in \complement_{[n]}C \\ \text{if } i-1 \in \complement_{[n]}C \text{ or } i=1, \text{ then } i \in C \end{cases}$$

- *For all* $i \in \complement_{[n]}A$

$$\begin{cases} \text{if } i-1 \in \complement_{[n]}C \text{ or } i=1, \text{ then } i \in \complement_{[n]}C \\ \text{if } i-1 \in C, \text{ then } i \in C \end{cases}$$

B. Proof of Proposition 1

In an expected utility framework, $fR_{B,n}g$ is equivalent to the non-negativity of $E_u(f)-E_u(g)$ for all $u \in U_{B,n}$, as given in (2).

Note that Procedure AB and Procedure AC above are identical, C in AC standing for B in AB, so that $B=C$.

As a prerequisite, the following lemma will be used to prove necessity. Let

$$\varepsilon_j = 2\mathbb{1}_B(j)-1,$$

so that the sign restrictions defining $U_{B,n}$ are $\varepsilon_j u_j \geq 0$ and $\sigma_j = -\varepsilon_j$. Consequently, $\sigma_j u_j(x) \leq 0$ for every $j \in [n]$, every $x \in [a,b]$ and every $u \in U_{B,n}$.

Lemma 2. Suppose $B=\phi(A)$. Then the following holds:

1) Let J be a non-empty open subinterval of (a,b) and let $h \in C([a,b])$ be non-negative, non-zero and supported in J. There is a utility function $u_h \in U_{B,n}$ such that

$$\left(u_h\right)_n = \varepsilon_n h,\ \ \left(u_h\right)_j\left(\gamma_{j+1}\right) = 0\ \ (j = 1,\cdots,n-1).$$

2) For every $k \in \{2,\cdots,n-1\}$ and $c > 0$, there is a weakly admissible utility function $v_k \in C^n\left([a,b]\right)$, satisfying all the weak sign restrictions of $U_{B,n}$, for which

$$\left(v_k\right)_{k-1}\left(\gamma_k\right) = \varepsilon_{k-1} c,$$

while every other boundary value in the sum in (2) and $\left(v_k\right)_n$ is zero. Moreover, $v_k + \eta u_h \in U_{B,n}$ for every $\eta > 0$.

Proof. Procedure AB implies that, for $i \geq 2$, if $i \in A$, then $\gamma_i = b$ and $\varepsilon_i = -\varepsilon_{i-1}$; if $i \notin A$, then $\gamma_i = a$ and $\varepsilon_i = \varepsilon_{i-1}$.

For 1), set $\left(u_h\right)_n = \varepsilon_n h$ and define recursively for $j = n-1,\cdots,1$,

$$\left(u_h\right)_j(x) = \int_{\gamma_{j+1}}^{x} \left(u_h\right)_{j+1}(t)\,\mathrm{d}t.$$

If $\gamma_{j+1} = b$, this integration reverses the sign; if $\gamma_{j+1} = a$, it preserves it. The relations just noted therefore give $\varepsilon_j\left(u_h\right)_j \geq 0$ at every order and also give $\left(u_h\right)_j\left(\gamma_{j+1}\right) = 0$. Integrating $\left(u_h\right)_1$ once more defines u_h up to an irrelevant additive constant. Since h is non-zero, every derivative has the required strict sign somewhere, so $u_h \in U_{B,n}$.

For 2), take $\left(v_k\right)_{k-1} = \varepsilon_{k-1} c$, put $\left(v_k\right)_j = 0$ for $j \geq k$, and use the same recursion for $j = k-2,\cdots,1$. It gives the required signs and makes $\left(v_k\right)_j\left(\gamma_{j+1}\right) = 0$ for $j < k-1$. Thus only the boundary term indexed by k can be non-zero. Adding ηu_h supplies strict inequalities at every order without changing any of these boundary values, which proves the last assertion. □

The proof of Proposition 1 is then as follows:

Sufficiency:

$$\left.\begin{array}{l} f D_{A,n} g \\ \text{and, if } n \geq 3, \\ \text{for all } i = 2,\cdots,n-1: \\ H^i_{A,n}\left(\gamma_i\right) \leq 0 \end{array}\right\} \Rightarrow f R_{B,n} g.$$

By hypothesis, when $n \geq 3$, for all i between 2 and $n-1$, $H^i_{A,n}\left(\gamma_i\right) \leq 0$ on the right-hand-side of (2). Still by hypothesis, $H^n_{A,n}(x) \leq 0$ for all x on $[a,b]$, which also means that at the particular value γ_n, $H^n_{A,n}\left(\gamma_n\right) \leq 0$.

Since $\sigma_i u_i(x) \leq 0$ for all $i \in [n]$, all $x \in [a,b]$ and all $u \in U_{B,n}$, $\sum_{i=2}^{n} \sigma_{i-1} u_{i-1}\left(\gamma_i\right) H^i_{A,n}\left(\gamma_i\right) \geq 0$. In the same way, $\sigma_n \int_a^b u_n(x) H^n_{A,n}(x)\mathrm{d}x \geq 0$, on the right-hand-side of (2). And since the whole right-hand-side of (2) is non-negative, $E_u(f) - E_u(g) \geq 0$ for all $u \in U_{B,n}$, which amounts to $f R_{B,n} g$.

Necessity:

$$f R_{B,n} g \Rightarrow \left\{\begin{array}{l} f D_{A,n} g \\ \text{and, if } n \geq 3, \\ \text{for all } i = 2,\cdots,n-1: \\ H^i_{A,n}\left(\gamma_i\right) \leq 0. \end{array}\right.$$

Assume there exists y such that $H_{A,n}^{n}(y)>0$. By continuity, there is a non-empty open interval $J\subset(a,b)$ on which $H_{A,n}^{n}>0$. Choose a non-negative, non-zero continuous function h supported in J and the corresponding utility u_h from Lemma 2(1). Every boundary term in (2) then vanishes, while

$$E_{u_h}(f)-E_{u_h}(g)=\sigma_n\varepsilon_n\int_a^b h(x)H_{A,n}^{n}(x)\mathrm{d}x=-\int_a^b h(x)H_{A,n}^{n}(x)\mathrm{d}x<0.$$

This contradicts $fR_{B,n}g$. Hence $H_{A,n}^{n}(x)\le 0$ for all $x\in[a,b]$, which is precisely $fD_{A,n}g$.

Since this non-positivity of $H_{A,n}^{n}(x)$ also holds for the upper and lower bounds of x, and consequently for γ_n, $H_{A,n}^{n}(\gamma_n)\le 0$ in (2).

Let us turn now to the sum of boundary terms on the right-hand side of (2), *i.e.*,

$$\sum_{i=2}^{n}\sigma_{i-1}u_{i-1}(\gamma_i)H_{A,n}^{i}(\gamma_i).$$

This sum is empty (has value zero) when n=1, and we have just shown that it is non-negative for $n=2$, since $H_{A,n}^{n}(\gamma_n)\le 0$.

Suppose now that $n\ge 3$ and, for some $k\in\{2,\cdots,n-1\}$, $H_{A,n}^{k}(\gamma_k)>0$. Take v_k from Lemma 2(2). Its only non-zero contribution to (2) is

$$\sigma_{k-1}(v_k)_{k-1}(\gamma_k)H_{A,n}^{k}(\gamma_k)=-cH_{A,n}^{k}(\gamma_k)<0.$$

Although v_k may have zero derivatives at some orders, the strictly admissible utility functions $v_k+\eta u_h$ belong to $U_{B,n}$ for every $\eta>0$. Their expected-utility differences converge, as $\eta\downarrow 0$, to the negative quantity displayed above. They are therefore negative for all sufficiently small η, contradicting $fR_{B,n}g$. Since k was arbitrary, every intermediate boundary condition follows:

$$H_{A,n}^{i}(\gamma_i)\le 0,\ i=2,\cdots,n-1.$$

This completes the proof. □