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A Modified Arrow Learning-by-Doing Growth Model in a Ramsey-Cass-Koopmans Optimal Control Framework

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Abstract

The Solow-Swan or S-S growth model has been, and still is, the workhorse of standard neoclassical growth theory. The S-S growth model has two distinguishing features: the saving rate is an exogenously fixed constant, and all technical change is exogenous. The present paper's growth model relaxes both features by modifying Arrow's learning-by-doing model with endogenous technical change and embedding it in a Ramsey-Cass-Koopmans growth setup that derives an endogenously determined optimal saving rate. This optimizing framework produces empirically plausible and testable predictions about the per capita output growth effects of changes in consumer preferences, population growth, and public policies affecting the degree of learning by doing associated with the economy's stock of capital per efficient worker. Numerical simulations indicate the model's faster adjustment to equilibrium from an initial disequilibrium position.

Keywords

Learning-by-Doing, Ramsey-Cass-Koopmans

The Solow (1956)-Swan (1956), henceforth S-S, model has been, and still is, the workhorse of basic neoclassical growth theory. The S-S growth model solved the Harrod (1939)-Domar (1946), or H-D, *knife-edge* problem by incorporating a fully flexible warranted rate (saving-investment balance) and a well-behaved neoclassical production function with the usual properties, assuring full employment and balanced growth, i.e., equality between warranted rate (capital growth) and natural rate (labor growth).[1] The natural rate is the growth of effective labor[2] adjusted for labor-augmenting technical change. The S-S growth model has two distinguishing features: (1) the saving rate is an exogenously fixed constant; and (2) all technical change is exogenous. Feature (1) is relaxed with a control procedure (Ramsey, 1928; Cass, 1965; Koopmans, 1965) that maximizes intertemporal consumption and derives an endogenously determined optimal saving rate. Feature (2) is broadened to include endogenous technical change in a modified learning-by-doing feature of the Arrow (1962) growth model, as was done in Villanueva (1994). Although the latter study is not micro-founded, public policy, such as a higher saving rate, is effective in raising both temporary and permanent (equilibrium) growth rates of per capita output or income. The following eight papers motivate the current paper: Ramsey (1928), Solow (1956)-Swan (1956), Arrow (1962), Cass (1965), Koopmans (1965), Phelps (1966), and Conlisk (1967). Arrow (1962) proposed the learning-by-doing feature in explaining labor-augmenting technical change, modified in the present paper. Ramsey (1928), Cass (1965), and Koopmans (1965), henceforth, RCK, pioneered the use of optimal control to determine saving decisions (*Golden Utility Rule*). Phelps employed the *Golden Rule* in choosing the optimal saving rate. Finally, Conlisk (1967) was the first to modify

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[1]The S-S model consists of the following 6 equations: a unit-homogeneous (constant returns) neoclassical production function: 1) $Y = F(K,L) = Lf(k)$; 2) $\dot{K} = I - \delta K$; 3) $S = sY$; 4) $I = S$; 5) $\frac{\dot{L}}{L} = \lambda + n$; 6) $k = \frac{K}{L}$. K = capital, L = effective labor, satisfying the Inada (1963) conditions: $\lim \frac{\partial F}{\partial K} = \infty$ as $K \to 0$; $\lim \frac{\partial F}{\partial K} = 0$ as $K \to \infty$; $f(0) \ge 0$; $f'(k) > 0$, $f''(k) < 0$ for all $k > 0$; $\frac{\dot{K}}{K} = sF\left(1, \frac{1}{k}\right) - \delta$; $\frac{\dot{L}}{L} = \lambda + n$; $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = sF\left(1, \frac{1}{k}\right) - (\lambda + n + \delta)$. The formal constraints on the S-S model are: (a) $sF(1,\infty) - (\lambda + n + \delta) > 0$, (b) $sF(1,0) - (\lambda + n + \delta) < 0$, and (c) $s, \lambda, n, \delta > 0$; s = exogenously fixed constant saving rate, λ = constant rate of exogenous labor-augmenting technical change, n = constant rate of exogenous population growth, and δ = constant rate of depreciation of K. The production function $F(.)$ and the first two constraints (a) and (b) relax the H-D fixed factor proportions model in favor of full factor substitution. For a small (large) enough k, capital's marginal product is larger (smaller) than labor's, implying a higher substitution from labor (capital) to capital (labor), resulting in positive (negative) growth of k. The S-S model economy is characterized by *perfect* markets with full wage-price flexibility. These features sharply contrast with new endogenous growth models that feature *increasing* returns to capital in *imperfect* markets (see, among others, Romer (1986, 1990), Lucas (1988), Grossman and Helpman (1990, 1991), Rivera-Batiz and Romer (1991), Rebelo (1991), Aghion and Howitt (1992), and Barro and Sala-i-Martin (1997)). Even with *increasing* returns to scale, Conlisk (1971) shows that balanced growth can occur if the elasticity of substitution between capital and labor is unitary, showcasing a more nuanced interaction between capital and labor growth.

[2]In *efficiency* units, i.e., if a 2024 man-hour is equivalent as an input in the production function to two man-hours in the base period, say, 2000, then k is the amount of capital per half-hour 2024 or per man-hour 2000.

the S-S model by making labor-augmenting technical change partly endogenous (see Section 1.1 on Conlisk's pioneering contribution). The paper's optimizing framework produces empirically plausible and testable predictions about the per capita output growth effects of changes in preferences, population growth, and public policies affecting the degree of learning by doing associated with the economy's stock of capital per efficient worker. Numerical simulations indicate the model's faster adjustment to equilibrium from an initial disequilibrium position. Holding the learning-by-doing coefficient constant, adjustment to the steady state is faster as the elasticity of intertemporal substitution increases. Section 1 develops and analyzes the growth model, its reduced model, and comparative dynamics. Section 2 derives the model's speed of adjustment to its steady state (equilibrium). Section 3 summarizes and discusses implications for growth policies.

1. Development of the Model

1.1. Background: Conlisk Model

Conlisk (1967) was the first to introduce endogenous technical change in a closed-economy neoclassical growth model. Employing a constant-returns, well-behaved neoclassical production function $Y = F(K, L) = Lf(k)$, $Y = GDP$, K = capital, and L = effective labor, the Conlisk model consists of the following relationships:

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta = s\frac{f(k)}{k} - \delta\,;\quad \frac{\dot{L}}{L} = h\frac{Y}{L} + \mu + n = hf(k) + \mu + n\,, \tag{1}$$

Y = output or income, K = capital stock, $L = AN$ = effective labor, A = technology or productivity index, N = exogenous working population, $k = \frac{K}{L}$, s = exogenously fixed constant rate of saving, δ = rate of depreciation, h = fraction of income spent on labor-augmenting technical change, μ = rate of exogenous productivity change, n = constant rate of exogenous population growth rate, and a dot over a variable = time derivative, $\dot{K} = \frac{\mathrm{d}K}{\mathrm{d}t}$. A constant fraction s of Y is invested in K and another fraction h is used to increase A.[3] The equilibrium growth rate of GDP is

$$\left(\frac{\dot{Y}}{Y}\right)^* = \left(\frac{\dot{K}}{K}\right)^* = \left(\frac{\dot{L}}{L}\right)^* = h\left(\frac{Y}{L}\right)^* + \mu + n = h\left(f\left(k^*\right)\right) + \mu + n, \tag{2}$$

a positive function of the equilibrium capital-labor ratio k^*, which is, in turn, a function of *all* the model's structural parameters s, h, μ, n and δ, and of the form of the intensive production function $f\left(k^*\right)$.

Villanueva (1994) develops and discusses a variant of the Conlisk (1967) model, combining it with a modified Arrow (1962) learning-by-doing model wherein experience on the job plays a critical role in raising labor productivity, that is,

[3] Conlisk (1967) provides no specific interpretation of the parameter, h, other than it is a composite parameter that translates expenditures on labor-augmenting technical change in dollars into units of L in man-hours (or index number). He references to Phelps (1966) research expenditures on labor-augmenting technical change in the R&D sector.

$$\frac{\dot{A}}{A} = \phi k + \mu \quad 0 < \phi < 1, \tag{3}$$

ϕ = learning coefficient. The idea is that as the *per capita* stock of capital with embodied advanced technology gets larger, the learning experience makes workers more productive.[4]

Together with $\frac{\dot{N}}{N} = n$, the equilibrium growth rate of *GDP* is,

$$\left(\frac{\dot{Y}}{Y}\right)^* = \left(\frac{\dot{K}}{K}\right)^* = \left(\frac{\dot{L}}{L}\right)^* = \phi k^* + \mu + n, \tag{4}$$

which is similar to the Conlisk growth Equation (2), with $hf\left(k^*\right)$ replaced by ϕk^*.

The major innovation of the current paper's model is a modification of the Arrow (1962) growth model's learning-by-doing feature. This key modification requires elaboration. More than six decades ago, Arrow proposed the following learning-by-doing relationship,

$$\frac{\dot{A}}{A} = \gamma\left(\frac{\dot{K}}{K}\right) + \mu, \quad 0 < \gamma < 1 \tag{5}$$

Capital-augmenting technological improvements are embedded in the capital stock K. Laborers working with modern capital equipment that incorporates advanced technology constantly learn and become more productive as time passes. Equation (5) states that the proportionate growth in labor productivity A is the sum of the learning coefficient γ multiplied by the proportionate growth in the capital stock K plus a constant rate of exogenous productivity change μ. The faster is the growth of the capital stock, the more intensive is the learning experience on the job, and the higher is the growth in labor productivity.

Given Equation (5), definition $L = AN$, assumptions $\frac{\dot{N}}{N} = n$, $\frac{\dot{A}}{A} = \mu$, a constant equilibrium capital-labor ratio $k^* = \left(\frac{K}{L}\right)^*$ implies the following equilibrium growth rate of output Y:

$$\left(\frac{\dot{K}}{K}\right)^* = \left(\frac{\dot{L}}{L}\right)^* = \left(\frac{\dot{A}}{A}\right)^* + n = \left(\frac{\dot{Y}}{Y}\right)^* = g^{Y*} = (\mu + n)/(1-\gamma). \tag{6}$$

Although a multiple of ($\mu + n$), equilibrium output growth $\left(\frac{\dot{Y}}{Y}\right)^* = g^{Y*}$ remains a constant involving only three parameters: μ, γ, and n. That is, g^{Y*} is independent of preference, risk parameters, and form of the intensive production function $f\left(k^*\right)$.[5] Besides, the Arrow model has the property that

[4]Using $L = AN$ and $k = K/L$, rewrite Equation (3) as $\dot{A} = \phi\left(\frac{K}{N}\right) + \mu A$.

[5]Assuming a CRRA instantaneous utility function, such as Equation (11) in Section 1.2.

$\frac{\partial\left(g^{Y*}-n\right)}{\partial n}=\frac{\gamma}{1-\gamma}>0$, i.e., an *increase* in the population growth rate n *raises* the long-run growth rate of per capita output, $g^{Y*}-n$.[6] This prediction is counter-intuitive and rejected by empirical evidence.[7] If Arrow (1962) is interpreted as a learning-by-doing model where $\frac{\dot{A}}{A}$ = a positive function of the capital-labor k ratio, not of capital growth $\frac{\dot{K}}{K}$, then Equation (3) is operative and equilibrium output growth is given by Equation (4), not Equation (6).

As an alternative to endogenous labor-augmenting technical change, such as Equation (3), Villanueva (2020) suggested another way for the natural rate to adjust is an endogenous labor participation rate P.[8] From the definition $L = APN$, $0 < P \leq 1$,

$\frac{\dot{L}}{L}=\frac{\dot{A}}{A}+\frac{\dot{P}}{P}+\frac{\dot{N}}{N}$. In practically all models, $\frac{\dot{N}}{N}=n$. In Villanueva's (1994) learning-by-doing model, $\frac{\dot{A}}{A}=\phi k+\mu$. In Villanueva's labor participation model (2020), $\frac{\dot{P}}{P}=\theta k$, where $\theta>0$ is a composite parameter reflecting expenditures on secondary and tertiary education and real wage effects. Villanueva (2023) combines a modified version of Arrow's learning by doing (Villanueva, 1994) and endogenous labor participation (Villanueva, 2020), so that $\frac{\dot{L}}{L}$ is a positive function of k.

1.2. The Micro-Founded Model

Assume the following institutional arrangements of a closed, perfectly competitive economy populated with rational combined producer-households that use a unit-homogeneous production function with the usual properties. One good is produced that can be consumed or invested. Enterprises rent capital K from households and hire workers L to produce output in each period. Producer-households own the physical capital stock and receive income from working, renting capital, and managing the enterprises. Profits π from managing enterprises are

$$\pi=\left(K,L\right)-rK-wL\,, \tag{7}$$

π = profits, r = rental rate, and w = the real wage rate. The budget constraint of a representative producer-household is

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[6]Subtracting n from both sides of Equation (6) yields $\left(g^{Y*}-n\right)=\frac{\mu}{1-\varphi}+\left[\frac{\varphi}{1-\varphi}\right]n$.

[7]See, among others, Conlisk (1967), Otani and Villanueva (1990), Knight *et al.* (1993), and Villanueva (1994).

[8]The labor participation rate and unemployment rate are metrics used to gauge the health of the labor market. The key difference between the two indicators is that the participation rate measures the percentage of people who are in the labor force, while the unemployment rate is the percentage within the labor force that is currently unemployed.

$$C + \dot{K} + \delta K = rK + wL + \pi \,, \tag{8}$$

C = consumption and δ = depreciation. Dividing both sides by L, the household budget constraint is,

$$c + \dot{k} + \left(\delta + g^L\right)k = rk + w + \pi \,. \tag{9}$$

Lower case letters are expressed as ratios to effective labor L, and g^L is given by

$$\frac{\dot{L}}{L} = g^L = \frac{\dot{A}}{A} + n \,. \tag{10}$$

A = labor-augmenting technology multiplier, and n = constant rate of exogenous population growth.

As discussed in Section 1.1, the labor-augmenting productivity variable A changes according to Villanueva's (1994) modified specification of Arrow's (1962) learning by doing, Equation (3), repeated here,

$$\frac{\dot{A}}{A} = \phi k + \mu \,, \tag{11}$$

driven by the *per capita* stock of capital with embodied advanced technology ($L = AN$),

$$\dot{A} = \phi\left(\frac{K}{N}\right) + \mu A \,. \tag{12}$$

The representative producer-household maximizes a discounted stream of lifetime consumption C, subject to the budget constraint (9), in which instantaneous utility is of the CRRA form (for brevity, time t is suppressed for all variables):

$$N(0)^{1-\theta} \int_0^\infty \frac{\left(\frac{C}{L}\right)^{1-\theta}}{1-\theta} A^{1-\theta} \mathrm{e}^{-\rho^* t} \mathrm{d}t \,. \tag{13}$$

For the integral to converge, the following standard assumption is adopted:

$$\rho^* = \rho - (1-\theta) > 0 \,. \tag{14}$$

In maximizing Equation (13) subject to Equations (9)-(10), each household takes as parametrically given the time paths of r, w, π, and A. When making decisions about consumption and capital accumulation, each household is small enough to affect r, w, π, and A.

The household's Hamiltonian is

$$H = \mathrm{e}^{-\rho^* t}\left[\frac{c^{1-\theta}}{1-\theta}\right] A^{1-\theta} + \varphi\left[rk + w + \pi - c - \left(\delta + g^L\right)k\right] \tag{15}$$

After substituting Equations (9) and (10), the first-order conditions are:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[r - \delta - \rho - \theta n - \theta\left(\frac{\dot{A}}{A}\right)\right] \tag{16}$$

$$\dot{k} = rk + w + \pi - c - \left[\delta + n + \frac{\dot{A}}{A}\right] \tag{17}$$

$\frac{\dot{A}}{A} = \phi k + \mu$, Equation (11).

The economy-wide resource constraint is

$$C + \dot{K} + \delta K = F(K, L). \tag{18}$$

Dividing both sides by L,

$$c + \dot{k} + (\delta + g^L) = f(k), \tag{19}$$

which is identical to Equation (9) since $f(k) = rk + w + \pi$. In competitive equilibrium,

$r = f'(k)$ and $w = f(k) - kf'(k)$, implying $\pi = 0$. Substituting these expressions for r, w, and π and Equation (11) into Equations (16) and (17), the optimal paths for c and k are as follows:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[f'(k) - \delta - \rho - \theta(n + \mu) - \theta\phi k\right] \tag{20}$$

$$\dot{k} = f(k) - c - (\delta + n + \mu) - \phi k^2 \tag{21}$$

The transversality condition is[9]

$$\lim_{t\to\infty} \mathrm{e}^{-\rho^* t}\varphi k = 0. \tag{22}$$

If there is no learning by doing, $\phi = 0$, Equations (20)-(21) reduce to the modified RCK model that allows for population growth n and exogenous technical change μ, with the key property that the equilibrium growth rate of per capita output is fixed entirely by the rate of exogenous labor-augmenting technical change μ and is independent of consumer preferences (discount rate ρ and coefficient of risk aversion $1/\theta$) and of public policy that can raise the equilibrium capital intensity k^* and the learning coefficient ϕ.[10]

1.3. Reduced Model

Equations (20) and (21) represent the reduced model in c, k, and time t (suppressed). The equilibrium (asymptotic) values c^* and k^* are the roots of Equations (20) and (21) equated to zero.

$$f'(k^*) - \delta - \rho - \theta(n + \mu) - \theta\phi k^* = 0 \tag{23}$$

$$f(k^*) - c - (\delta + n + \mu)k^* - \phi k^{*2} = 0 \tag{24}$$

Let J = Jacobian associated with the system (23) and (24):

$$J = \begin{vmatrix} 0 & \frac{c}{\theta}\left(f''(k) - \theta\phi\right) \\ -1 & f'(k) - \delta - \mu - n - 2\phi k \end{vmatrix}$$

[9]The no-Ponzi game is imposed, i.e., non-negative present value of the household holding of k.

[10]In open-economy models with a fixed saving rate, Otani and Villanueva (1989, 1990) and Villanueva (1994) found that in developing countries, the learning coefficient is positively influenced by public expenditures on education and health, and the openness of the economy, measured by imports of advanced capital goods and through larger export receipts that relieve the foreign exchange constraint (all in % of GDP).

Evaluated in the neighborhood of the steady state k^*, c^*:[11]

$$J = \begin{vmatrix} 0 & -0.0249 \\ -1 & 0.0400 \end{vmatrix}$$

The associated eigenvalues are $\lambda_1 = -0.13904$ and $\lambda_2 = 0.17907$.

Figure 1 shows the phase diagram in *c*, *k* space with equilibrium values k^*, c^*.

The pair k^*, c^* is saddle path stable and is the *Golden Utility* solution, while the pair k^*, c^* is the *Golden Rule* solution.[12] While the latter maximizes *c* at c^{**}, the latter maximizes intertemporal utility at $c = c^*$. The equilibrium capital intensity k^* is a function of all parameters of the model, including those of the preference or utility function, namely the discount rate ρ and the coefficient of relative risk aversion ($1/\theta$) and the other parameters, notably the learning coefficient ϕ and the parameters and form of the intensive production function $f\left(k^*\right)$. Since the equilibrium growth rate of per capita output $g^{Y*} - n = \phi k^* + \mu$, any public policy that enhances the equilibrium capital-labor ratio k^* and the learning coefficient ϕ raises the long-run growth rate of per capita output.

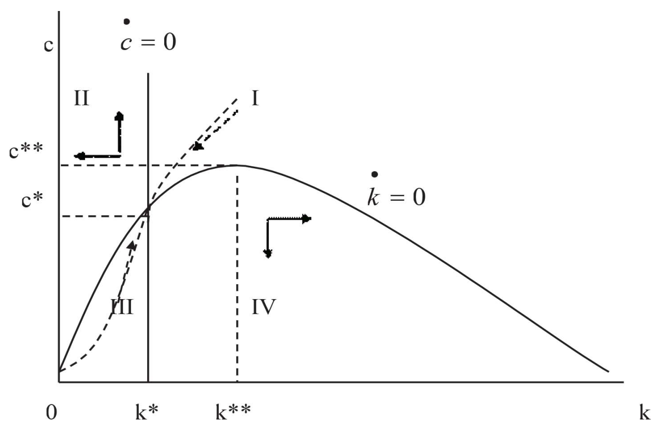

Figure 1. Phase Diagram of the reduced model.

1.4. Comparative Dynamics

Figure 2 illustrates the growth effects of an increase in the learning coefficient ϕ through for example, increased allocations of budgetary expenditures on education, training, and skills upgrade, and/or private sector's similar expenditures.

In the upper panel, the intersection at point $A\left(k_0^*, c_0^*\right)$ shows the initial equilibrium corresponding to a given level of the learning coefficient ϕ_0. In the lower panel, the vertical axis measures the equilibrium growth rate of per capita output, and the horizontal axis measures the capital-labor ratio. The $g^{Y*} - n$ curve, $\phi k^* + \mu$, has a positive slope equal to ϕ. Assume that public policy subsidizes

[11]Parameter values used are: $\alpha = 0.3$, $\delta = 0.04$, $\mu = 0.01$, $\theta = 1.5$, $\phi = 0.003$, $\rho = 0.04$, and $n = 0.02$. The solutions are: $k^* = 3.31$, $c^* = 1.17$. The optimal saving rate is 0.1846 and the steady state growth rate of per capita output is 0.02, of which the endogenous component is half at 0.01 and the other half is the exogenous component μ.

[12]The transversality condition (22) rules out quadrants II and IV in **Figure 1**. On the *Golden Rule* solution, see Phelps (1966).

on-the-job training at enterprises, resulting in an increase in the learning coefficient from ϕ_0 to ϕ_1. In the upper panel, the new equilibrium shifts to point $B\left(k_1^*, c_1^*\right)$, with both equilibrium consumption per worker and equilibrium capital per worker lower than at point *A*. In the lower panel, the $g^{Y*} - n$ curve shifts upward to $g^{Y*} - n = \phi_1 k_1^* + \mu$, with a higher equilibrium growth rate of per capita output.

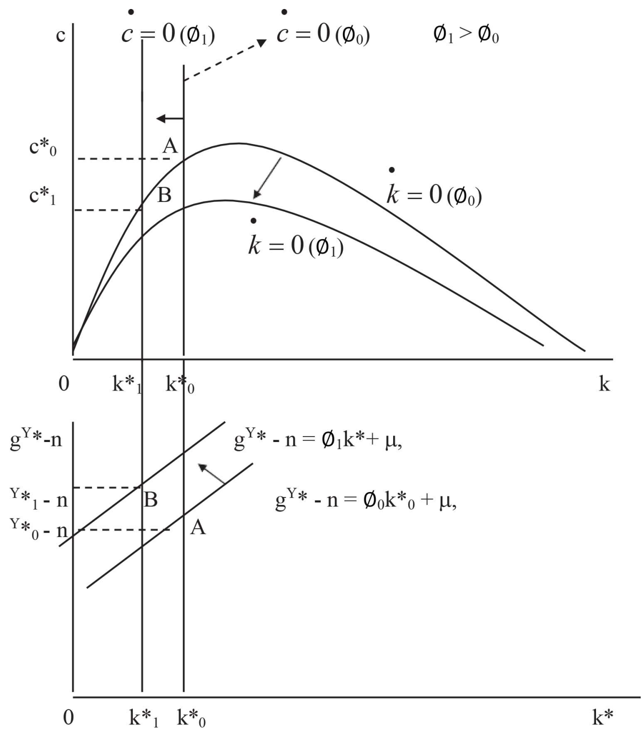

Figure 2. Growth effects of an increase in learning by doing.

Figure 3 illustrates the effects of a higher discounting of future consumption or a higher degree of relative risk aversion (lower intertemporal elasticity of substitution). The initial equilibrium is at point $A\left(k_0^*, c_0^*\right)$, shown in the upper panel. The increase in ρ shifts the $\dot{c} = 0$ curve to the left, intersecting the stationary $\dot{k} = 0$ curve at point $B\left(k_1^*, c_1^*\right)$. Both equilibrium consumption per effective worker and equilibrium capital intensity are lower than before. In the lower panel, as k^* falls from k^* to k^*, learning by doing drops and so does the equilibrium growth rate of per capita output, from g^{Y*} - n to g^{Y*} - n (downward movement along the g^{Y*} - n curve). For similar reasons, a higher degree of relative risk aversion θ or a lower intertemporal elasticity of substitution, $1/\theta$, would have similar negative effects on the steady state growth rate of per capita output.

Finally, the model yields a more empirically plausible prediction about the effect of population growth n on the steady-state growth rate of per capita output, as illustrated in **Figure 4**, a result that is particularly relevant to developing countries. In the upper panel, an increase in n from n_0 to n_1 shifts the $\dot{c} = 0$ curve to the left and the $\dot{k} = 0$ curve downward.

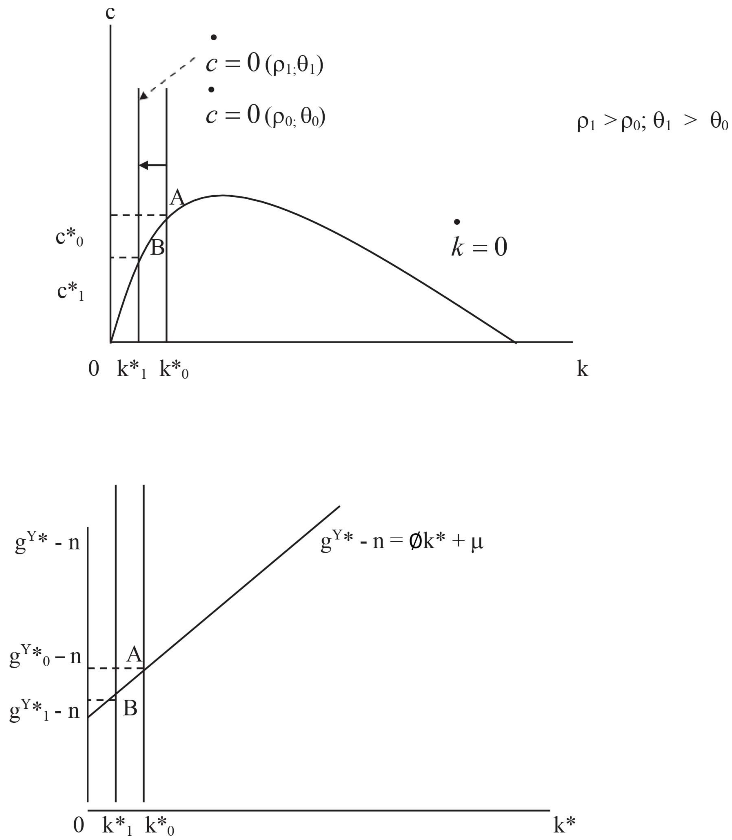

Figure 3. Growth effects of higher discounting and lower inter-temporal elasticity of substitution.

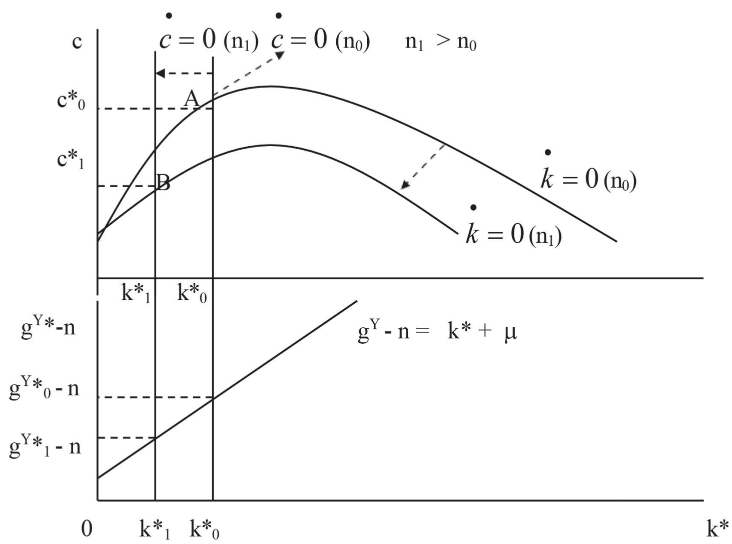

Figure 4. Growth effects of an increase in population growth.

The steady-state equilibrium moves from point *A* to point *B*, with lower equilibrium consumption per efficient labor and a lower equilibrium level of capital intensity. In the lower panel, the decline in the equilibrium stock of capital per efficient worker cuts learning by doing and leads to lower steady-state growth rates of productivity and per capita output.[13]

1.5. Optimal Saving Rate

From Equations (23) and (24), the endogenously derived optimal saving rate is given by:

$$s = \left\{ \left(n + \mu + \phi k^* + \delta \right) \Big/ \left[\theta \left(n + \mu + \phi k^* \right) + \delta + \rho \right] \right\} \alpha . \quad (25)$$

If $\rho = 0$, $\phi = 0$, and $\theta = 1$, then the optimal saving rate is,

$$s = \alpha , \quad (26)$$

which is the standard Solow-Swan result in a world of exogenous technical change. That is, the saving rate must be set equal to the income share of capital.[14] However, if $\rho > 0$, $\theta > 1$, and $\phi > 0$, then the optimal saving rate is not only a function of the deep parameters ρ, θ, ϕ, μ, δ, and n, as well as k^*, but must also be set equal to a fraction of capital's income share, with the fraction equal to the term $\left(n + \mu + \phi k^* + \delta \right) \Big/ \left[\theta \left(n + \mu + \phi k^* \right) + \delta + \rho \right]$, given by Equation (25), is the (gross) social marginal product of capital, inclusive of the positive externalities via learning experience associated with capital accumulation in the endogenous growth model. Equivalently put, income going to capital as a share of total output should be a multiple of the amount saved and invested to compensate capital for the additional output generated by endogenous growth and induced learning. A value of π equal to s, implicit in the standard model, would undercompensate capital and thus would be suboptimal from society's point of view.

2. Speed of Adjustment to Equilibrium

This section addresses the question of whether the presence of learning by doing increases the speed of adjustment of the model to its steady-state. The nonlinear system is described by Equations (20) and (21). Linearize this system around the steady-state values c^* and k^*.

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = J \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix} \quad (27)$$

in which J is the Jacobian matrix. Denote by v_1 and v_2 the two given eigenvectors and $\lambda_1 < 0$ and $\lambda_2 > 0$ the two eigenvalues associated with J. Then,

$$\begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix} = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t} . \quad (28)$$

$C_2 = 0$ must hold for $k \to k^*$. $C_2 > 0$ violates the transversality condition; if $C_2 < 0$, then $k \to 0$ in quadrant II, Figure 1, which is also a violation of the transversality condition. Therefore,

[13]The effects of an increase in the depreciation rate of physical capital are similar. An increase in δ shifts the consumption growth curve to the left and the capital intensity growth downward, with the two curves intersecting at a lower equilibrium c^* and k^*.

[14]In Figure 1, this condition is associated with maximum consumption per L at $c^{**} > c^*$.

$$c_t = c^* + e_1^{\lambda t}\left(c_0 - c^*\right), \tag{29}$$

$$k_t = k^* + e_1^{\lambda t}\left(k_0 - k^*\right). \tag{30}$$

Next, define the adjustment ratio,

$$p_t = \frac{g_t - g_0}{g^* - g_0} \tag{31}$$

$g_t = g^* + \alpha\left(\frac{\dot{k}}{k_t}\right)$, g^* is the steady state growth rate of output, and $\frac{\dot{k}}{k_t}$ is given by (27) and (30). The denominator is the distance the growth rate has adjusted by t. Substituting (27) and (30) into (31) solves for t in years required for a fraction p_t of the way from g_0 to g^*, from which **Tables 1-3** are computed.

Tables 1-3 assume an initial annual growth rate of 9 percent.[15] The steady-state annual growth rate ranges from 3 percent to 5.4 percent, depending on the elasticity of intertemporal substitution and the learning coefficient. Szpiro (1986) tested the CRRA utility function used in Equation (13) on the basis of data for 15 industrial countries using property/liability insurance data and found that the CRRA cannot be rejected. His estimate of θ is between 1 and 2.[16] Therefore, each table assumes a particular value for the intertemporal substitution elasticity, from low (0.53) to medium (0.67) to high (0.91), corresponding to θ = 1.9, 1.5, and 1.1. The first columns of the tables show the fraction p_t of adjustment. Subsequent columns show estimates of adjustment speed in years corresponding to various values of the learning coefficient ranging from a zero value (standard model, ϕ = 0 or no learning by doing) to ϕ = 0.004 and ϕ = 0.008. The memorandum items show (1) the values for the endogenous growth component (equal to the product of the learning coefficient and the equilibrium capital intensity), ranging from zero when ϕ = 0, 1.4 percentage points when ϕ = 0.004 and 2.4 two percentage points when ϕ= 0.008; and (2) the optimal saving rate, ranging from 0.1765 when $(1/\theta)$ = 0.53 and ϕ = 0, to 0.2054 when $(1/\theta)$ = 0.91 and ϕ = 0.008. In all numerical simulations, the Cobb-Douglas production function used is $f(k) = k^\alpha$, with the following parameter values: α = 0.3, ρ = 0.04, μ = 0.01, δ = 0.04, and n = 0.02.[17]

Table 1. Estimated adjustment in years to the steady state from initial high growth rate (g_0 = 0.09) and low intertemporal elasticity of substitution ($1/\theta$ = 0.53).

pt	$\varphi = 0$ $g^{Y*} = 0.0300$	$\varphi = 0.004$ $g^{Y*} = 0.0417$	$\varphi = 0.008$ $g^{Y*} = 0.0501$
0.25	1.6	1.4	1.3
0.50	4.2	3.6	3.3
0.75	9.6	7.9	7.0

[15]This is a reasonable initial position in developing countries with relatively small stocks of capital per efficient worker.

[16]Szpiro's (1986) estimate of θ for the United States is 1.19, which implies an estimate of 0.84 for the elasticity of intertemporal substitution.

[17]The computations reported in **Tables 1-3** use Microsoft EXCEL's *Goal Seek* and *Solver* tools.

Continued

0.90	17.8	14.0	12.3
Memorandum item:			
Endogenous growth			
component (ϕk^*)	0.0000	0.0117	0.0201
Saving rate (s^*)	0.1765	0.1735	0.1719

Table 2. Estimated adjustment in years to the steady state from initial high growth rate (g_0 = 0.09) and medium intertemporal elasticity of substitution ($1/\theta = 0.67$).

pt	$\varphi = 0$ $g^{Y^*} = 0.0300$	$\varphi = 0.004$ $g^{Y^*} = 0.0427$	$\varphi = 0.008$ $g^{Y^*} = 0.05196$
0.25	1.4	1.3	1.2
0.50	3.7	3.2	3.0
0.75	8.5	7.1	6.4
0.90	15.7	12.8	11.3
Memorandum item:			
Endogenous growth			
component (ϕk^*)	0.0000	0.0127	0.0219
Saving rate (s^*)	0.1826	0.1851	0.1865

Table 3. Estimated adjustment in years to the steady state from initial high growth rate (g_0 = 0.09) and high intertemporal elasticity of substitution ($1/\theta = 0.91$).

pt	$\varphi = 0$ $g^{Y^*} = 0.0300$	$\varphi = 0.004$ $g^{Y^*} = 0.0437$	$\varphi = 0.008$ $g^{Y^*} = 0.0543$
0.25	1.2	1.1	1.1
0.50	3.1	2.8	2.7
0.75	7.2	6.3	5.8
0.90	13.4	11.3	10.3
Memorandum item:			
Endogenous growth			
component (ϕk^*)	0.0000	0.0138	0.0243
Saving rate (s^*)	0.1891	0.1992	0.2054

Tables 1-3 clearly show that, for each given degree of the elasticity of intertemporal substitution, the presence of learning by doing ($\phi > 0$, as opposed to $\phi = 0$) not only leads to a higher long-run growth rate of per capita output, but also to a faster speed of adjustment to the steady state. Moreover, an increase in the degree of learning by doing (increase in ϕ) contributes to an even faster adjustment speed. The intuitive reason for the latter result is that the learning by doing component of the effective labor growth equation (natural rate) adjusts to any discrepancy between the capital growth equation (warranted rate) and the natural rate. The Solow-Swan model focuses *exclusively* on the adjustment of the warranted rate. The extended model of this paper relies on *both* the adjustments of the warranted and natural rates of growth, so that the speed of adjustment to the steady state (defined by equality between the two rates) is a lot faster. The tables also show that, holding the learning coefficient constant, adjustment to the steady state is faster as the elasticity of intertemporal substitution increases. Looking at the last columns of each table, in which the endogenous component of technical change adds at least 2 percentage points to the steady-state growth rate of output, whereas it takes between 13 and 18 years for the standard model with no learning by doing to reach 90 percent of the time required to reach its steady-state GDP growth path, depending on the elasticity of intertemporal substitution, it would take only between 10 and 12 years for the model with learning by doing to converge to the steady-state growth path.[18]

Figure 5 graphs the adjustment of the annual growth rate of output to its steady-state level when the learning coefficient $\phi = 0.008$ (the endogenous growth component adds 2.4 percentage points to the steady- state growth rate of output) and $(1/\theta) = 0.91$ (high elasticity of intertemporal substitution). The initial and steady-state annual growth rates of output are 9 percent and 5.4 percent, respectively. The initial conditions, shown in Figure 1, are somewhere in quadrant III, are more relevant to the developing world that is initially understocked with capital, and therefore, its assumed initial growth rate (0.090) is above the steady-state level (0.054).[19]

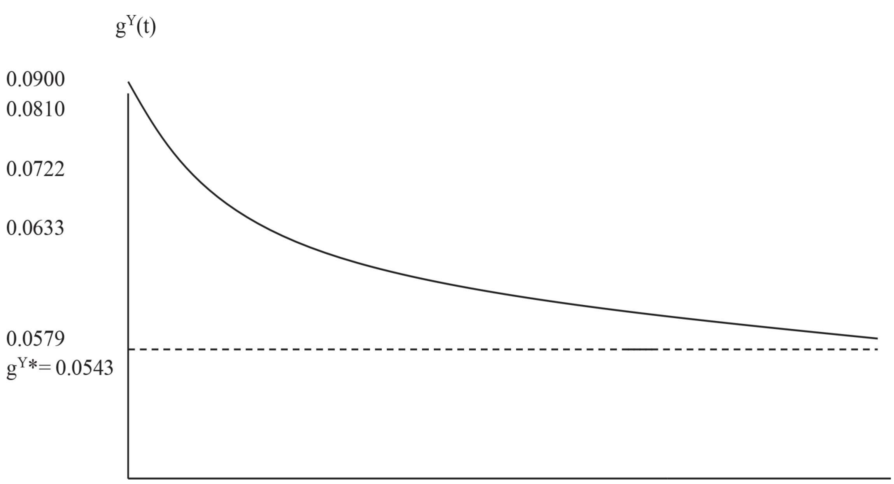

Figure 5. The developing world: adjustment to the optimal steady-state growth path.

[18]That is, 10 years for the high substitution elasticity of 0.91, 11 years for the medium elasticity of 0.67, and 12 years for the low elasticity of 0.53.

[19]The initial growth rate is based on initial $k_0 = 1.54$ and initial $c_0 = 0.8283$, both levels below the steady-state values $k^* = 3.04$ and $c^* = 1.1091$.

At time $t = 0$, the growth rate is 9 percent per annum. After the first year, the growth rate is 8.1 percent; after nearly 3 years, it is 7.2 percent; after 6 years, it is 6.3 percent; and after 10 years, it is 5.8 percent, just 0.4 percentage point off the steady-state growth rate. This is a plausible adjustment speed, since many developing countries (notably in East Asia) have been growing at over 5 percent annually for at least a decade. A numerical simulation for the United States was run, using the following parameter values and initial conditions: $\alpha = 0.3$, $\delta = 0.04$, $\phi = 0.003$,[20] $\mu = 0.01$, $\theta = 1.19$, $\rho = 0.04$, $n = 0.02$; and $g^Y = 0.01$, based on $k_0 = 4.51$ and $c_0 = 1.654$. The estimate for $\theta = 1.19$ owes to Szpiro (1986). The steady- state optimal values are: $g^{Y*} = 0.04$, $k^* = 3.51$, and $c^* = 1.175$. These initial conditions, shown in **Figure 1** somewhere in quadrant I, are more relevant to the United States; owing to its advanced state, it is initially overstocked with capital, and therefore, its assumed initial growth rate (0.01) is below the steady-state level (0.04).

Figure 6 plots the adjustment of the output growth rate toward the steady-state level over time (in years). With its high degree of intertemporal substitution, starting from an initial growth rate of 1 percent, it would take 16 years for the US economy to reach a 3.7 percent annual GDP growth rate, equivalent to 90 percent of the time required to reach its steady-state GDP growth path of 4 percent per annum. This is indeed a reasonable adjustment speed, since the United States (and many other advanced economies) have been growing at this rate for at least this long.

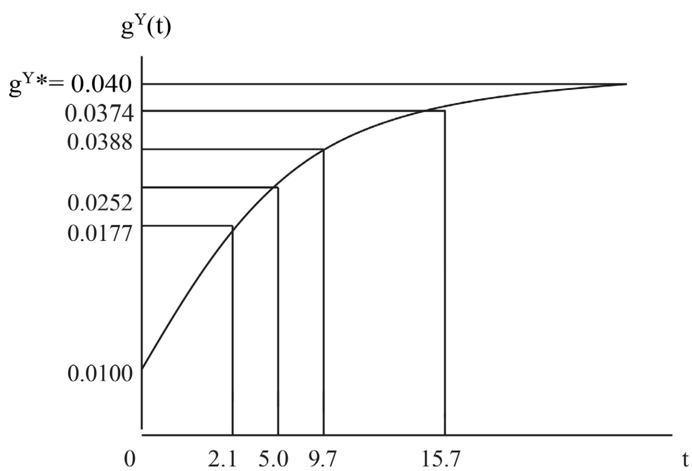

Figure 6. The United States: adjustment to the optimal steady-state growth path.

3. Conclusion

This paper has adopted a modified Arrow learning-by-doing model and embedded it in a Ramsey optimal growth framework. A simple extension of the Solow-

[20]This value adds a percentage point to the steady-state growth rate of output [$\phi k^* = 0.003(3.51)$]. This lower contribution of endogenous growth (compared to its larger contribution in developing countries) partly reflects the existence of a large R&D sector in the United States, not explained endogenously by the model. That is to say, learning by doing via on-the-job training is much larger in developing countries.

Swan growth model in an optimizing framework produces empirically plausible and testable predictions about the per capita output growth effects of changes in preferences, population growth, and public policies that affect the equilibrium capital-labor ratio and, directly or indirectly, any or all the model's parameters, particularly the degree of learning by doing associated with the economy's stock of capital per efficient worker. For a given elasticity of intertemporal substitution, the presence of learning by doing leads to a higher long-run growth rate of per capita output. Holding the learning coefficient constant, adjustment to the steady state is faster as the elasticity of intertemporal substitution increases. The intuitive reason for the latter result is that the learning-by-doing component of the effective labor growth equation (natural rate) adjusts to any discrepancy between the capital growth equation (warranted rate) and the natural rate. The Solow-Swan model focuses *exclusively* on the adjustment of the warranted rate. The extended model of this paper relies on *both* the adjustments of the warranted and natural rates of growth, so that adjustment to the steady state (defined by equality between the two rates) is a lot faster. The implications for growth policy are straightforward. Fiscal and monetary policies that promote investments in physical, human, and intellectual capital in a stable financial environment are keys to a steady and high growth rate of per capita income.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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