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On the Worldwide Trade Dynamics: Some Fractional Aspects, Extremal Trade Constellations and the Therewith Corresponding Entropy, Restructuring the Trade Environment, and Fundamental Relations

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Abstract

This paper applies fractional diffusion, the CTRW concept, entropy, and small-world network concepts to global export-import dynamics. It estimates space- and time-fractality orders from trade data, developing a varying order interpretation, and discusses several stylized scenarios for protection, clustering, liberalization, and finance-trade restructuring. The main claim is that global trade dynamics exhibit a subdiffusive character and that entropy changes are connected with fractality order and network structure.

Keywords

Space-Time Fractional Diffusion of Varying Order, Entropy on Fractal Structures, Morse Potential, Small-World Networks, The Albert-László-Barabási Model, Topologically Induced Diffusion

1. Introduction

Export-Import activities take place in a complex environment, in which antagonism between national economies, geopolitical events, and the exploitation trends of industrialized countries dominate the product and services exchange frame. While in earlier times export-import activities took place in the form of exchange based on win-win relations, nowadays, the export-import framework is more complicated. Technological advances, financial superiority, and geopolitical relations could in this context have a strong impact on these dynamics. Further, re-imports and the indirect implication of low-wage countries in the export-import processes complicate the framework. However, we can state here that the grade or intensity of export-import relations between economies is an important imprint of the worldwide exploitation process, avoiding giving a quantitative definition of the globalization grade based on export-import statistics. Figure 1 shows that there is no strong correlation in the data between worldwide trade volume and the production of goods and services. This fact is not surprising, as the export-import relations have their own dynamics in an economic sense. This fact induces some very important results. In ideal circumstances, the evolution of export-import values should be in an immediate correlation with the GNP evolution. That means that important movements in the balance of trade of some countries could occur. Additionally, the share of export-import volume as % to the GNP becomes unstable. These are but indications of significant changes in the behavior of players in the economic exploitation process worldwide. Export-import relations implement some specific characteristics, which are important by analyzing these dynamics. They take place in an extremely inhomogeneous environment, in which the term "national economy" needs an exact definition. Exports are products of firms where national conditions and norms have a strong impact on their cost structure. Among others, they are characterized by:

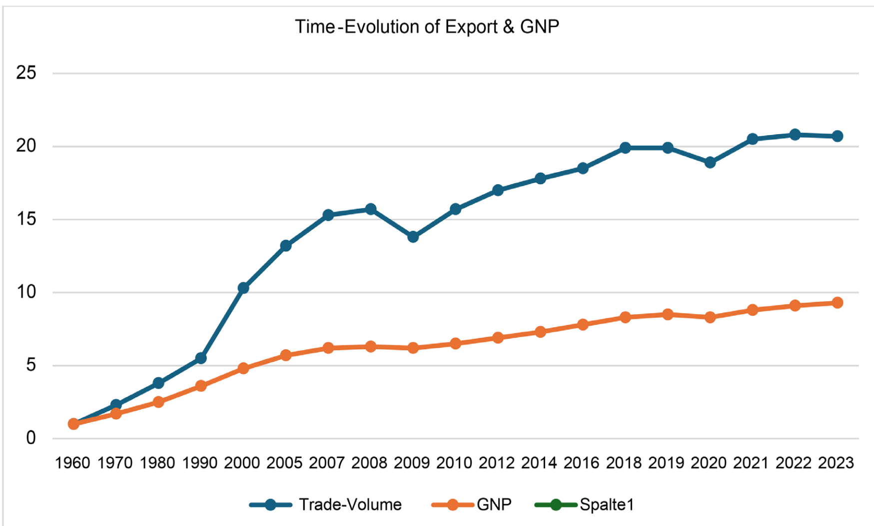

Figure 1. Increase (%) of the trade volume and GNP worldwide in real values. Index: 1960 = 1. Notation: The above graph has been extracted from "Bundeszentrale für politische Bildung 2025 (http://www.bpb.de/). The data are based on statistics of the" World Trade Organization (WTO), Statistical Review 2023. The dimension of export volume and GNP is money-equivalent ($) in real values by constant prices worldwide. The index (variable) in Figure 1 is dimensionless (%).

1. There are some countries that have more export connections than others. They are termed as "hubs" in the corresponding network. This is mostly the case in industrialized economies, like the USA, Germany, Japan, and others.

2. Some economies export goods and services that are in some way more important than others (for example, products of high technology and tourist services). In the case of critical constellations, one could likely abstain more from holidays than from technology know-how.

3. When imports are not well-balanced by suitable exports, this fact could lead to inequalities and criticality in the corresponding dynamics.

4. Economies that are strong and export-oriented react more sensitively in the case of changes in the global economic environment.

5. Stable export-import relations require stable geopolitical constellations, where the latter is rather an unusual event.

6. Export-import volume can be strongly influenced by changes in currency relations and trade customs.

The above characteristics establish a particular frame for the dynamics that are in focus. The fractality orders reflect somehow the criticality respecting the therewith corresponding entropy. Thus, it is important to refer to the changes of the fractality orders, as fractality and entropy are closely connected. In this sense, we bother the question of how the fractality orders could turn up, alter (vary), and what their limits are. In this context, we develop an additive point of view analyzing the fractality-order variation, based on space aspects (Markov processes). Observing some extreme constellations on the trade dynamics, we suppose that these scenarios could be realistic in a fast-changing global economic environment. The description and analysis of the above themes being in focus need a short introduction into the theory of fractional diffusion in finance. In a special part of this article, we assess the impact of customs on the export-import volume by means of a specific Euler scheme.

2. The Fractality Orders of the Export-Import Dynamics Worldwide Estimation of the Fractality Order

Why do we observe dynamics from the fractional point of view? We may answer this question with simple arguments. If we are interested in detecting or analyzing dynamics concerning their criticality or stability, the fractional calculus is an effective tool to do so. Applications of fractional calculus on economic dynamics are a particular domain that needs specific explanation. According to this fact, we introduce the fundamentals of this mathematical field in order to better understand the article's aims and proceedings. The CTRW concept is a fundamental tool for supporting and analyzing economic dynamics.

According to the CTRW theory, the basic tool to analyze dynamics as an evolution of a random walk of a particle in space and time, consists of estimating the probability density of the values of the space jumps and the corresponding survival times on the specific jump values. The CTRW concept (Continuous Time Random Walk) developed by Montroll and Weiss is a very often used and effective method to analyze dynamics observing their diffusivity (Montroll & Weiss, 1965). Now, before doing these, we must refer to some important properties and characteristics of the CTRW model of Montroll & Weiss, which are associated with the statistical sources used to extract dynamics parameters like the fractality order. Often the statistical rows are related to discrete space and time points (fitting) in which much can matter as tick-by-tick data fails to exist (Jeon & Metzler, 2012). Assuming that economic dynamics are surely fractal, and considering the fact that trade evolution needs enough time to change significantly, we can help by these statistical difficulties by conjecturing a Gaussian, a LogNorm distribution, or a linear relation between two successively connected fixed values. Thus, the estimation of the fractality orders observing aggregate values is a risky undertaking. However, the autocorrelation method of Kenneth Falconer (used in this work) allows us to gain a good result for this problem as the diffusivity category of the observing dynamics is much more important than the exact values of the fractality. The data in **Figure 1** are frequently not homogeneous, as some of them refer to a time interval of a decade (10 years), and some in one year. As we need a fixed (constant) time interval (yearly values), we linearize or estimate stochastically gained values over a decade using random data based on Gauss statistics. For the concrete problem, we apply a method of Kenneth Falconer (Falconer, 2003), to estimate, at first, the space fractality order using yearly values for the trade volume. The procedure is based on an autocorrelation analysis of time series. K. Falconer calls the graph of a data series the image of a fractal function. At first, we can estimate the box dimension of the graph using the relation:

$$D_{box}\left(graph\, f\right) = 2 - \lim_{h \to 0} \frac{\log\left(C(0) - C(h)\right)}{2\log h} \tag{2.1}$$

whereby the terms $C(h), C(0)$ are defined within the autocorrelation analysis procedure. By the assumption that the dynamics being in focus exhibits a fractional behavior we get for the power-spectrum $S(\omega)$ the relations

$$S(\omega) \approx \frac{c}{\omega^{-\alpha}}, \quad C(0) - C(h) = bh^{a-1} \tag{2.2}$$

By the hypothesis $c = b = 1$ and $C(0) - C(h) = ch^{4-2D_{box}}$ we obtain the identities

$$4 - 2D_{box} = \alpha - 1 \quad \text{or} \quad 4 - 2D_{box} + 1 = \alpha \tag{2.3}$$

The exponent α obtained above has the dimension of a space fractality order, being assumed that the process itself is a space-time fractional, as opposed to D_{box}, which possesses a pure geometrical property. Now our goal is to extrapolate the time-fractional exponent of the observing data, the space-fractional one being already estimated. Due to this task, there exists among others a procedure using an empirical relationship between the quadratic displacement of a variable in dynamics and the elapsed time in a fractional sense. The relationship to this reads as

$$\left\langle X^2 \right\rangle \equiv Kt^{\beta/\alpha} \tag{2.4}$$

with:

$\left\langle X^2 \right\rangle$ = Quadratic displacement of the variable X

K = normalization factor

β = time-fractional order (exponent)
α = space-fractional order (exponent)
t = elapsed physical time
We obtain from (4) the relation

$$\ln\left(\left\langle X^2\right\rangle\right) = \ln(K) + \frac{\beta}{\alpha} * \ln(t) \tag{2.5}$$

In the case of α = known, we can obtain the β value and vice versa. The above way is of pure empirical character and often verified by investigating time series. A more sophisticated possibility consists of supposing the existence of the subordination relationship of the form $p(x,t) = t^{-\frac{\beta}{\alpha}} f_x\left(\frac{X}{t^{\beta/\alpha}}\right)$ with $f_X = f(x)$ = function of the fractional image, $p(x,t)$ being a probability density function (PDF) in space and time. Multiplying the above relation on both sides by $t^{\beta/\alpha}$ we obtain

$$p(x,t)\,t^{\beta/\alpha} = f_x\left(\frac{X}{t^{\beta/\alpha}}\right) \tag{2.6}$$

Equation (6) allows extrapolating the order β while we know the corresponding "α", using a specific statistical treatment for maximal correlation. Analyzing statistical data of the trade volume over time 1960-2024, we have estimated a time-tractality order of $\beta \sim 0.3$ and a space-fractality order of $\alpha \sim 1.7$, which are risky. These data allow us to classify, but surely the export-import dynamics as a subdiffusive phänomenon. It must be noted that the space-fractality order is relatively low, and the time-fractality order relatively high. The ranges of the orders are $\beta \in (0,1)$ and $\alpha \in (0,2)$. (low values of the orders correspond to high fractality) temporal exponent β (**Figure 2**).

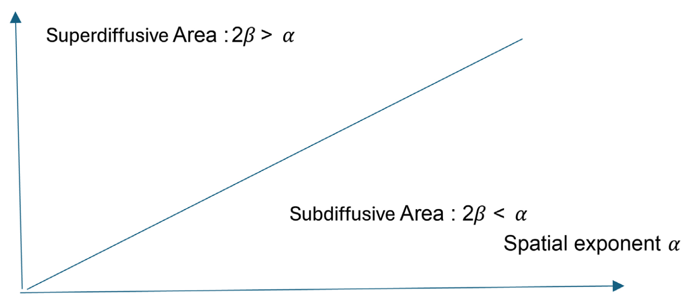

Figure 2. Shema of characterizing the diffusivity of dynamics due to shape α,β. $\beta \in (0,1)$, $\alpha \in (0,2)$.

3. Basics of Fractional Concepts in Finance

The main aim of this article consists not only of classifying the frame of dynamics of the export-import activities in the context of fractional diffusion. We are much more interested in analyzing changes of field variables, which are subjected to important transformations in the economic environment. This allows us to define limit constellations on the corresponding dynamics. In the concrete case of the worldwide export-import dynamics, we assume to observe a subdiffusive process coexisting with Levy flights. Changes in the space- and time-order are in this context of essential importance. In order to deliver meaningful results, we prefer to give a short illustration of the above-mentioned terms and concepts.

3.1. The CTRW Concept in Finance

The CTRW concept developed by Montrol and Weiss considers a walker who stays fixed at a space point X_1 during a waiting time of random length T_1, and then makes a space step to space point X_2 of length $X_2 - X_1$, where on remains fixed for waiting time T_2, before realizing the next step in space and time. The fokus of interest are the PDF's of the lengths $X_{j+1} - X_j$ for $j = 1$ to n, and $T_{i+1} - T_i$ for $i = 1$ to n. We assume PDFs for waiting times and jump lengths are statistically independent. Let be $y(t)$ = the probability density for the waiting times. The corresponding cumulative distribution $\Phi(t)$ for $y(t)$ can be defined as

$$\Phi(t) = P_r(T \le t) = \int_0^t \psi(\tau)\,\mathrm{d}\tau, \psi(\tau) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \tag{3.1}$$

The function $\Phi(t)$ is known in the renewal theory as the failure probability at time t. Corresponding to $\Phi(t)$ we define

$$\Psi(t) = P_r(T > t) = \int_0^\infty \psi(\tau)\,\mathrm{d}\tau = 1 - \int_0^t \psi(\tau)\,\mathrm{d}\tau \tag{3.2}$$

called the probability of no events at or before the time t. There holds

$$\psi(t) = -\frac{\mathrm{d}\Psi(t)}{\mathrm{d}t}$$

Defining the function $m(t)$ as the count of events in the interval $(0,t]$ in the form $m(t) = N(t) = \sum_{K=1}^{\infty} P_r(t_k \le t)$ we gain the formula

$$m(t) = \Phi(t) + \int_0^t m(t-\tau) * \psi(\tau)\,\mathrm{d}\tau \tag{3.3}$$

The formula (3) is known in statistical mechanics as the renewal equation. This formula reads in the laplace domain as

$$\tilde{m}(s) = \frac{\tilde{\psi}(s)}{s\left(1-\tilde{\psi}(s)\right)} \tag{3.4}$$

$$\text{with } \tilde{\Phi}(s) = \frac{\tilde{\psi}(s)}{s} \text{ and } \tilde{\Psi}(s) = \frac{1}{s} - \frac{\tilde{\psi}(s)}{s} \tag{3.5}$$

The relation (5) could be thought of as a convolution of the functions $f_k = \frac{\mathrm{d}F_k(t)}{\mathrm{d}t}$, assuming that for t_1 holds $f_1(t) = \psi(t)$, and for t_2, $f_2(t) = (\psi * \psi)(t)$, we can formulate $y^{*k}(t) = f_k(t)$.

3.2. The Compound Renewal Process and the Memory Function (Gorenflo et al., 2000b; Mainardi et al., 2000; Tarasova et al., 2018)

Proceeding further, we combine the probability law of the waiting times with the probability law of the jump length. In this way, we define the probability $p(x,t)$ to be the walker at the time t at the position X. This convolution leads after algebraic circumtations to the following relation in the combined laplace-fourier domain

$$\hat{\tilde{p}}(k,s)=\frac{1-\tilde{\psi}(s)}{s}\frac{1}{1-\tilde{\psi}(s)\hat{w}(k)} \tag{3.6}$$

where $\psi(t)$ = waiting times density, and $w(x)=\int_{-\infty}^{x} w(\xi)\mathrm{d}\xi = P_r(X<x)$

Relation (1) can be rewritten in

$$\hat{\tilde{p}}(k,s)=\frac{1-\tilde{\psi}(s)}{s}+\tilde{\psi}(s)*\hat{w}(k)\hat{\tilde{p}}(k,s) \tag{3.7}$$

Taking the inverse laplace- and fourier-transformation on (2) simultaneously, we get

$$p(x,t)=\delta(x)*\Psi(t)+\int_0^t\left\{\int_{-\infty}^{\infty} w(x-x')p(x',t')\mathrm{d}x'\right\}\psi(t-\tau)\mathrm{d}\tau \tag{3.8}$$

Relation (3) represents the probability of finding the walker at time t and position x starting from the point $p(x,0)=\delta(x)$, where $\psi(t)$ and $w(x)$ are nonnegative probability density functions for the waiting times and jump-length correspondigly. In order to explain and define the term 'memory', related to (3), we rewrite equation (1) as $s*\hat{\tilde{p}}(k,s)(1-\tilde{\psi}(s)\hat{w}(k))=1-\tilde{\psi}(s)$ that leads after algebraic manipulations to

$$s*(1-\tilde{\psi}(s))/(s\tilde{\psi}(s))-1/(s\tilde{\psi}(s))=\hat{\tilde{p}}(k,s)(\hat{w}(k)-1)-1/s \tag{3.9}$$

Defining

$$(1-\tilde{\psi}(s))/(s\tilde{\psi}(s))=\tilde{\nu}(s) \tag{3.10}$$

we can gain the relation

$$\tilde{\nu}(s)*\left(s\hat{\tilde{p}}(k,s)-1\right)=(\hat{w}(k)-1)*\hat{\tilde{p}}(k,s) \tag{3.11}$$

Taking the inverse-fourier-laplace transform finally on (6), we arrive at the celebrated version of the CTRW-process, which has the structure of a master equation, namely

$$\int_0^t \nu(t-\tau)\mathrm{d}\tau * p(x,\tau)\mathrm{d}\tau=-p(x,t)+\int_{-\infty}^{\infty} w(x-x')p(x',t)\mathrm{d}x' \tag{3.12}$$

The function $\nu(t)$ is called the memory function of the CTRW-process, which has a direct Impact on the dynamics progress. Additionally, the relation (7) allows for introducing time-fractal diffusion-aspects on the CTRW-process. If for example $\tilde{\nu}(s)$ #constant, we observe a non-markovian process by choosing $\nu(t)=\frac{t^{-\beta}}{\Gamma(1-\beta)}$ This relation reeds in the laplace-domain like $\tilde{\nu}(s)=\frac{1}{s^{1-\beta}}$ and equation (7) turns out as

$$\frac{\partial^{\beta}}{\partial t^{\beta}}=-p(x,t)+\int_{-\infty}^{+\infty} p(x,t)w(x-x')\mathrm{d}x' \tag{3.13}$$

In this case, the waiting-time probability density takes the form

$\psi(t) = -\frac{\mathrm{d}}{\mathrm{d}t} E_\beta\left(-t^\beta\right)$, where $E_\beta\left(-t^\beta\right)$ represents the mittag-leffler function of order β, with $0 < \beta < 1$. The survival probability $\Psi(t) = E_\beta\left(-t^\beta\right)$ is the probability of no-events at or before the instant t.

3.3. The Transition to the Space- and Time-Fractional Diffusion (Scalas et al., 2000; Gorenflo & Mainardi, 2008; Scalas et al., 2005)

Equation (8) of chapter B.2 can be transformed to the form called "hydrodynamic limit". This is the transition from the time-fractional master equation to a space-time fractional diffusion equation, by using properly designed scaling arguments. This can be obtained by approximating the fourier transform of the jump PDF, and the laplace transform of the waiting-time PDF, with the help of the following relations

$$\hat{w}(k) \sim 1 - |k|^\alpha,\ k \to 0,\ 0 < \alpha < 2 \tag{3.14}$$

$$\tilde{\psi}(s) \sim 1 - s^\beta,\ s \to 0,\ 0 < \beta < 1 \tag{3.15}$$

Using the scaling factor $\mu^{-1/\beta} h^{\alpha/\beta}$, after diverse algebraic circuntaces we are leading to the fractional diffusion equation

$$\frac{\partial^\beta}{\partial t^\beta} p_0(x,t) = \frac{\partial^\alpha}{\partial t^\alpha} p_0(x,t),\ p_0(x,t) = \delta(x) \tag{3.16}$$

For the above fundamental fractional differential equation exists a lot of articles in the corresponding Literature.

3.4. The Scaling Properties of the Fractional Diffusion

The fundamental Solutions of (B.3. Equation 3) depend primarily on the scaling of the parameters α, β. We observe by $\alpha = 2$, $\beta = 1$ the case of normal diffusion. By $0 < \alpha < 2$, $\beta = 1$ the case of space-fractional diffusion, and by $\alpha = 2$, $0 < \beta < 1$ the case of time-fractional diffusion. Finally, we note the specific composition rule, observing the space-time fractional diffusion as an integral involving the solution-functions (green functions) corresponding to space-fractional and time-fractional diffusion.

Denoting the fundamental solution of the space-time fractional diffusion equation as $p_{\alpha,\beta}(x,t)$ we gain of the above relations the formula

$$\hat{\tilde{p}}_{\alpha,\beta}(k,s) = \frac{s^{\beta-1}}{s^{\beta+}|k|^\alpha} \text{ and } \hat{p}_{\alpha,\beta}(k,t) = E_\beta\left(-|k|^\alpha t^\beta\right) \tag{3.17}$$

Using the similarity property of the fundamental solution, we can rewrite the fundamental solution in the form

$$p_{\alpha,\beta}(ax,bt) = b^{-\gamma} p_{\alpha,\beta}\left(ax/b^\gamma, t\right), \text{ with } \gamma = b/a \tag{3.18}$$

Consequently we get

$$p_{\alpha,\beta}(x,t) = t^{-\gamma} K_{\alpha,\beta}\left(x/t^\gamma\right),\ \gamma = b/a \tag{3.19}$$

where

$$K_{\alpha,\beta}(u) = 1/(2\pi) * \int_{-\infty}^{\infty} \mathrm{e}^{-iqu} E_\beta\left(-|q|^\alpha\right) \mathrm{d}q \tag{3.20}$$

For $\alpha = 1, \beta = 2$ we gain the standard diffusion equation. That means

$$\hat{p}_{2,1}(k,t) = \exp\left(-k^2 t\right) \to p_{2,1}(x,t) = t^{-1/2} \frac{1}{2\pi} \exp\left(\frac{-x^2}{4t} t^{-1/2}\right) \frac{1}{2\pi} \exp\left(\frac{-x^2}{4t}\right) \tag{3.21}$$

For $0 < \alpha < 2, \beta = 1$ we deal with the space, fractional diffusion equation, which reads

$$\hat{p}_{\alpha,1}(k,t) = \exp\left(-|k|^\alpha t\right) \text{ and } p_{\alpha,1}(x,t) = t^{-1/\alpha} L_\alpha\left(\frac{x}{t^{1/\alpha}}\right) \tag{3.22}$$

$L_\alpha(x)$ denotes here the Levy stable PDFs of index α.

For $\alpha = 2, 0 < \beta < 1$ Equation (4) reduces to $\hat{p}_{2,\beta}(k,t) = E_\beta\left(-k^2 t^\beta\right)$

The function $M_{\beta/2}$ denotes the so-called wright-type function of index $\beta/2$. It is defined in the whole complex plane and introduced by F.Mainard. The wright function reads as

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma\left[-\nu n + (1-\nu)\right]}, \text{ with } 0 < \nu < 1, z \in \mathrm{C} \tag{3.23}$$

(See for (Gorenflo et al., 2000b))

Now we refer to a composition rule, which allows to express the general green function (fundamental solution) of the space-time fractional diffusion equation that is the central subject of our work. The fundamental solution (Mainardi et al., 2007) in this case in the laplace-fourier domain can be rewritten

$$\hat{\tilde{p}}_{\alpha,\beta}(k,s) = \frac{s^{\beta-1}}{s^\beta + |k|^\alpha} = s^{\beta-1} \int_0^\infty \exp\left(-r\left(s^\beta + |k|^\alpha\right)\right) \mathrm{d}r \tag{3.24}$$

Equation (8) can be interpreted as

$$\hat{\tilde{p}}_{\alpha,\beta}(k,s) = 2\int_0^\infty p_{\alpha,1}(k,r) * p_{2,2\beta}(r,s) \mathrm{d}r \tag{3.25}$$

And by inversion

$$p_{\alpha,\beta}(x,t) = 2\int_0^\infty p_{\alpha,1}(x,r)\, p_{2,2\beta}(r,t) \mathrm{d}r \qquad \text{(B.4. Equation 10)}$$

or

$$p_{\alpha,\beta}(x,t) = t^{-\beta} \int_0^\infty r^{-1/\alpha} L_\alpha\left(x/r^{1/\alpha}\right) M_\beta\left(r/t^\beta\right) \tag{3.26}$$

4. Varying the Space- and Time-Fractality Order of Dynamics

4.1. A Space-Based Position

Inverse problems are usually hard problems. We propose first a space-based method in order to interpret fractional order changes on the dynamics, which are in focus. It is worth mentioning that fractional order refers to a particular time-window (range). Of essential interest is the following question: Gaining a specific order-value within a time-horizon, can we state that this result is an average-value of particular sub-results if we subdivide this time-range? We could answer that it depends on the method, which is used (see also …). In this context, we propose first a space-based aspect of view. Due to this, we mention that important geopolitical events, customs changes, and parity relations (shifting) could have such a rigorous and fast impact on export-import dynamics. Such events let rise the dynamics of Markov-character. After a Markov chain, which takes place in a characteristic time, follows a phase of adaptation to the new conditions within an appropriate time, which determines the behavior of the time-fractional order. Physically and in the sense of fractional mathematics on dynamics, we deal with a short phase of Markov character and a longer one of subordination. Of course, it is very difficult to localise the characteristic times in which Markov-events take place.

Subordinated superdiffusive dynamics (See for Brockmann, 2003). Assuming fractional orders of varying structure and remaining by the spatial Hypothesis, it is physically meaningful to observe Markov processes, which occur in an operational time being a function of the physical time. The operational time is in this case, a stochastic process in its own right. The above subordinated scenario occurs in many physical situations and specifically in Trade Dynamics after important geopolitical Events or technological inventions. In operational times, the Dynamics can be observed as an usual Langevin Ansatz. The appropriate formula reads as $d(X(\tau)) = (X(\tau))d\tau + dW(\tau)$, with $\tau =$ operational time. This formulation means that the parent Process X(t) is superdiffusive if the operational time could be caused (described) by a power Law. In the case of $F \equiv 0$, one can show that X(t) $= L_{\mu}(\tau)$, where $\mu =$ spatial fractional order. That means the Dynamics in physical time can be observed as a stable Levy process in operational time. The latter result is extremely remarkable. Both the time-motivated and spatial-motivated approaches to the existence of fractional orders of varying character in specific Dynamics lead to the same result. In limes the Dynamics can be characterized as a Levy stable Law.

Before going deeper into these details, we will refer to a time-based method bordering the variation of fractality order, proposed by Yuri Luchko in his article "modeling of financial processes with a space-time fractional diffusion equation of varying order, 2016".

4.2. A Time-Based Interpretation for Modeling Financial Processes with a Space-Time Fractional Diffusion Equation of Varying Order (Yuri Luchko's Position, See for Details (Luchko, 2016))

According to this theory, we consider a sequence of times $t_0 < t_1 < \cdots < t_n$, where in every time interval t_i the dynamics of a process is described by a space-time fractional diffusion equation of the form

$$\left({}_{t_i}^{*}D_t^{\beta_i} - \mu\left[{}^{\theta_i}D_x^{\Omega\beta_i}\right]\right) * g_i\left(x,t\right) = 0 \tag{4.1}$$

where ${}_{t_i}^{*}D_t^{\beta_i}$ = time-fractional derivative operator (caputo type) of order β, ${}^{\theta}D_x^{\Omega\beta}$ = space-fractional derivative operator (Riemann-Liouville type) of order

$\Omega\beta$, with $\frac{\alpha}{\beta}=\Omega$. α denotes here the actual value of the space-order, where $0<\beta<1$, $0<\alpha<2$. $\mu>0$ represents a scaling factor. Further holds $\Omega>0$, and $i=0,1,2,\cdots,n-1$. The system is equipped with the following initial conditions: $g_0(x,t_0)=f(x)$, $g_i(x,t_i)=g_{i-1}(x,t_i)$. For financial applications the case $\alpha_i>1$ and $\theta_i=\alpha_i-2$ is usually the most interesting and relevant one.

Whereas the parameter β of the equation (1) influences the behavior of the solution in the short time-interval $[t_i,t_{i+1}]$, the Factor $\frac{\alpha}{\beta}=\Omega$ determines the long-term one of the dynamics. Thus, due to this theory, the fundamental solution of the above initial-value problem can be represented in the form

$$G(x,t)=\frac{1}{t^{\Omega}}F\left(\frac{x}{t^{\Omega}}\right) \tag{4.2}$$

That means, the evolution of the process can be described by the relations

$$g(x,t)=f(x)*G_0(x,t-t_0),\ \text{for } t\in[t_0,t_1] \tag{4.3}$$

$$g(x,t)=f(x)*G_0(x,t_1-t_0)*\cdots*G_{i-1}(x,t_i-t_{i-1})*G_i(x,t-t_i). \tag{4.4}$$

When all t_i are equal 1, and all $\theta_i=\theta$, then the model is reduced to a space-fractional diffusion equation, with a stable fundamental solution $G(x,t)$. When at least one $\beta_i\#1$, the diffusion becomes non-Markovian. That means memory effects are not negligible. Because of the memory, the model exhibits a complex behavior for short times, driven by the space-time fractional diffusion equation. But for time scales going over characteristic time intervals, the resulting distribution converges to a stable distribution described by a space-fractional diffusion. As we are interested in this article for the entropy production in the observed processes, we deal further with the shannon entropy S defined by the formula $S(t)=-\int_R p(x,t)\ln(p(x,t))\mathrm{d}x$. The entropy production formula reads as $R(t)=\frac{\mathrm{d}}{\mathrm{d}t}S(t)$. Because of the relation $G(x,t)=\frac{1}{t^{\beta/a}}*F\left(\frac{x}{t^{\beta/\alpha}}\right)$), the shannon entropy can be presented as follows:

$$\begin{aligned} S(t)&=-\int\frac{1}{t^{\Omega}}F\left(\frac{x}{t^{\Omega}}\right)*\ln\left(\frac{1}{t^{\Omega}}*F\left(\frac{x}{t^{\Omega}}\right)\right)\mathrm{d}x \\ &=-\int F\left(\frac{x}{t^{\Omega}}\right)*\ln\left(\frac{x}{t^{\Omega}}\right)\frac{1}{t^{\Omega}}\mathrm{d}x-\ln\left(\frac{1}{t^{\Omega}}\right)\int F\left(\frac{x}{t^{\Omega}}\right)\frac{1}{t^{\Omega}}\mathrm{d}x \\ &=S(1)+\Omega\ln(t) \end{aligned} \tag{4.5}$$

The fundamental solution of the space-time fractional diffusion equation "reads as $G(x,t)=\frac{1}{t^{\beta/a}}*F\left(\frac{x}{t^{\beta/\alpha}}\right)$". That means by assuming a constant Ω, the entropy becomes decreasing. We try to define the shannon entropy, for the specific dynamics starting from the composition rule for the green function presented in the article (Luchko, 2016).

4.3. Entropy Production Due to the Composition Rule for the Green Function with $0<\beta<1$ (See also Mainardi et al., 2007)

Subdiffusive dynamics, according to the above article, can be represented with the help of the following relation:

$$G_{\alpha,\beta}^{\theta}(x,t)=2\int_0^{\infty}G_{\alpha,1}^{\theta}(x,u)*G_{2,2\beta}^{\theta}(u,t)\mathrm{d}u \tag{4.6}$$

with $G_{\alpha,1}^{\theta}(x,t)=t^{-1/\alpha}L_{\alpha}^{\theta}\left(\frac{x}{t^{1/\alpha}}\right)$ and $G_{2,2\beta}^{\theta}(x,t)=\frac{t^{-\beta}}{0.5}M_{\beta}\left(\frac{|x|}{t^{\beta}}\right)$

The relation (1) means that subdiffusive dynamics can be represented as a convolution of a Levy-stable distribution and an M-function (of the wright type) with the orders α and β correspondigly. This rule, known as the composition rule for the green functions with $0<\beta<1$, allows us to define the shannon entropy in this case in the following form:

$$S=-\int_R u^{-1/\alpha}L_{\alpha}^{\theta}\left(\frac{x}{u^{1/\alpha}}\right)*\frac{t^{-\beta}}{0.5}M_{\beta}\left(\frac{u}{t^{\beta}}\right)*\log\left[u^{-1/\alpha}L_{\alpha}^{\theta}\left(\frac{x}{u^{1/\alpha}}\right)*\frac{t^{-\beta}}{0.5}M_{\beta}\left(\frac{|u|}{t^{\beta}}\right)\right]\mathrm{d}u \tag{4.7}$$

Now, we observe the entropy in limes, which means for $x\pm\infty$. In this limes counts the following relations $L_{\alpha}^{\theta}(x)=O\left(|x|^{-(\alpha+1)}\right)$ and $0.5M_{\beta}(|x|)\sim Ax^{\alpha}\mathrm{e}^{-bx^{c}}$, with appropriate defined parameters A,b,c. Inserting the above formulas in (2) we get

$$\begin{aligned}S=&-\int_R u^{-1/\alpha}*\frac{1}{u^{1/\alpha}}x^{-(1+\alpha)}\frac{0.5}{t^{\beta}}Au^{\alpha}*\mathrm{e}^{-bu^{c}}\\&*\log\left[u^{-1/\alpha}*\frac{1}{u^{1/\alpha}}x^{-(1+\alpha)}\frac{0.5}{t^{\beta}}Au^{\alpha}*\mathrm{e}^{-bu^{c}}\right]\mathrm{d}u\end{aligned} \tag{4.8}$$

Relation (3) can be rewriten in

$$S=-\int_R x^{-(1+\alpha)}\frac{0.5}{t^{\beta}}Au^{\alpha}*\mathrm{e}^{-bu^{c}}*\log\left[x^{-(1+\alpha)}\frac{0.5}{t^{\beta}}Au^{\alpha}*\mathrm{e}^{-bu^{c}}\right]\mathrm{d}u \tag{4.9}$$

The solution of the relation (4) for a given range of $u(*)$ can help to estimate the entropy production in space and time in dependence on the orders α, β.

In order to gain some fundamental relations in the entropy production, we observe at first the levy-stable law $G_{\alpha,1}^{\theta}(x,t)=\frac{1}{t^{\alpha}}L_{\alpha}^{\theta}\left(\frac{x}{t^{\alpha}}\right)$, which can be also levy strictly stable PDF evolving in time. For $0<\alpha<2$, the levy stable kernel $L_{a}^{\theta}(x)$ reads as $L_{a}^{\theta}(x)=Q\left(|x|^{-(\alpha+1)}\right)$. With parametrization of the time t, we are leading to the following **Figure 3**, which represents the entropy change of the levy stable PDF in dependence on the changes of the order a (Prehl, 2010), in which one can read details about the entropy production on the space-fractional levy structures.

Important result: On subdiffusive dynamics, which can be represented with the help of the composition rule of two green functions, an **increase** in the order α of the corresponding levy stable law leads to a **decrease** in the resulting entropy. This fact may have a strong impact on sozioökonomic relations.

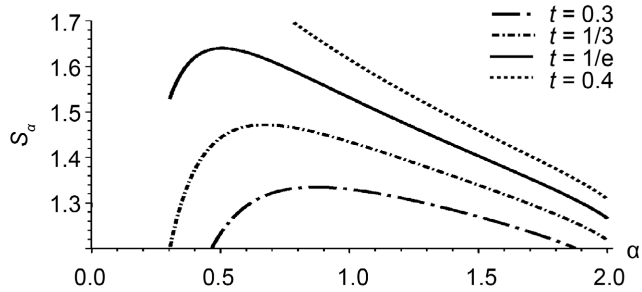

Figure 3. Entropy change vs. space-fractality order α. (Source: "Diffusion on fractals and space-fractional diffusion equations", by Prehl, 2010).

5. Some Important Limit-Szenarios on the Export-Import Dynamics

The term "export" or "import" is meaningful only in connection with the existence of national economies. In principle, a change in goods and services takes place in worldwide markets, between firms, subjected to specific national conditions. Thus, subsidies and other procedures, or customs and currency parities, can hide the real cost-structure of goods and services. It is therefore of high economic interest to ask what could happen when a strong ambience of protection dominates the global export-import dynamics. On the contrary, it could be imaginable an economic climate where the change of goods is quite liberalized. It is also imaginable that the case in which an oligopolistic structure of trade-clusters, like the European Union, dominates the global economic environment. However, these scenarios are more of a theoretical nature and not a practical effort, as this is first a result of political "will" and decisions. The cause for this is very simple. Between the national economics are big differences regarding their specific competitiveness. Thus, players are trying to find a fair exchange relation. Market-specific conditions complicate this problem additionally. We can state here that the above mentioned extreme scenarios are realizable only by political restrictions, and willingness, or by important geopolitical events. In order to gain some analytical results about the entropy, which corresponds to the above extremal scenarios, we refer to the promotion of D. Brockmann (See for details (Brockmann, 2003)), in which, focusing on superdiffusive aspects, he analyzes some very interesting cases of diffusion on small-world networks that are very relevant for the export-import relations.

5.1. The Theory of Topologically Induced Diffusion. The Corresponding Aspects of the Export-Import Dynamics

The above theory is based on the assumption that in a network with stochastic connections between the monomers, changes between monomers take place because of the differences in the potentials of the monomers that are observing, or by win-win exchanges. Referring to the export-import dynamics, we can identify the changes in the chemical space as changes in the export-import balance volume for each monomer. It is worth noting here that this theory analyzes and describes dynamics from an antagonistic point of view. Nevertheless, the above theory is very relevant and actual according to the exchange frame between industrialized economies and the less developed world. The main Instrument analyzing the corresponding dynamics is the stochastic master equation. The conditional probability density function of finding a monomer at site X at time t is given by the equation $\partial_t p(x,t) = \int \mathrm{d}y \left[w(x \mid y,t) p(y,t) - w(y \mid x,t) p(x,t) \right]$ by the initial Condition

$$p(x,t) = p(x,t \mid x_0, t_0) \tag{5.1}$$

The master equation defines the time-dependent rate $w(x \mid y,t)$ of making a transition from a monomer y to a monomer x within a short time interval Δt at time t. From our economic point of view, that means a jump from a monomer y to a monomer x, corresponding to a specific export-import value-change. The transition rate is given by the relation $y \to x$: $w(x \mid y,t) \propto \frac{1}{\tau} \exp\left(-\beta\left[V(x) - V(y)\right]/2\right)$. The relation points out that the probability of the jumb $y \to x$ depends on the potential difference $V(x) - V(y)$ and decreases as the target potential $V(x)$ increases. This assumption is characterized through an economic reality in antagonistic environments, and especially allows us to observe some some interest scenarios on the global export-import relations (For mathematical details, (Brockmann, 2003)). In the next scenarios, we undertake the following identifications in correspondence to terms within the theory of the "topologically induced diffusion".

Monomer → Coressponds to a specific national economy.

Polymer → Corresponds to a chain of linearly connected national economies.

Transition → An export activity from a source national economy to a target one, by a specific volume.

Rigid chains → A topology (structure) of single economies or groups of them, in which trade relations are very restricted as a consequence of political decisions, or protection efforts. A trade exchange activity takes place only between monomers, which are similar in their chemical space (economical exchange conditions).

Coiled polymers → A topology of single economies or groups, in which the connections (linking) between monomers are weak. Trade activities can be realized only under specific conditions. The polymer-topology is relatively flexible.

Topological super diffusion-concept → Complet flexible polymers with long-range connections on all scales. protection efforts, which could constrain exchange events, fail. This frame could be characterized economically as liberal. The competitiveness between national economies dominates the trade volume.

Potential differences → Every national economy and the to it subjected firms and organizations are characterized through a grade of competitiveness that means to achieve a specific economic performance pro time-unity and capital. Physically, this ability is given as economical force. Mathematically, this is possible by the differentiation of the potential, which is given in diverse forms. In an economical sense it is usual to describe the potential with the form $V(x) = x^r$, where

$2 > r > 0$. This formulation describes the evolution of the economic performance in space. However, there are differences in the parameter r between single economies. Indeed, the ability to achieve a specific economic performance could be extracted with the help of the evaluation of competitiveness criteria using an appropriate rank-scala.

5.1.1. Scenario 1: Protection Efforts Dominate the Global Trade Dynamics

From the point of view of the "topologically induced diffusion", only nearest neighbor hopping takes place. The master equation reads as

$$\partial_t p = \exp(-\beta V/2) * \frac{\partial p}{\partial x^2}\left(\exp(\beta V/2)\right) p(x,t) - p(x,t) * \exp(\beta V/2) * \frac{\partial p}{\partial x^2}\left(\exp(-\beta V)\right)$$

However, in order to estimate and interpret the corresponding entropy, we refer to the jump rate probability density. It reads $w(x \mid y) = \frac{1}{\tau}\exp\left(-\beta\left[V(x) - V(y)\right]/2\right) * \rho(|x-y|)$, whereby the function $\rho(|x-y|)$ represents the accessibility distance in the observing network. In our case we can $\rho(|x-y|)$ interpret as the degree of difficulty to reach (link) the monomer x in a geoeconomical sense. The function $\rho(|x-y|)$ is from a general point of view an inverse power law form for $|l| \gg \alpha$, which reads $\rho(l) = \frac{\mu\alpha^{\mu}}{2}|l|^{-(1+\mu)}$ with $|l| \geq \alpha$, and $0 < \mu \leq 2$. α represents here the minimal jumb length. The prefactor $\frac{\mu\alpha^{\mu}}{2}$ helps to normalize $\rho(l)$ to unity. In the case of rigid chains at which the jumbs do not exceed the intermonomer distance-value α, the corresponding entropy can be gain from the formula $\exp\left(-\beta\left[V(x) - V(y)\right]/2\right) * \log\left(\exp\left(-\beta\left[V(x) - V(y)\right]/2\right)\right)$. If we respect in the above relation only the differences $V(x) - V(y)$ we get the following **Figure 4**.

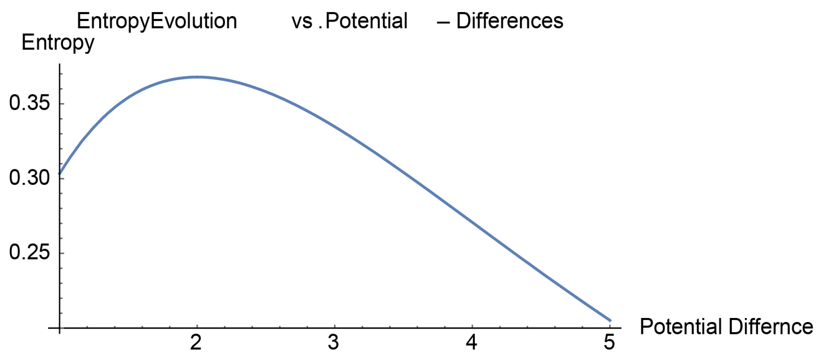

Figure 4. Entropy evolution vs. potential differences ($V(x) - V(y)$).

The entropy evolution in rigid chains is not linear and is characterized by a maximum peak at X = 2, varying the potential-difference. After passing this maximum, the entropy decreases with increasing potential difference. That means high values of potential differences lead to lower entropy. Physically, this is not a paradox but at first a theoretical result.

In the real economic world, a national economy will undertake protection efforts if its trade balance is gradually very negative or the imports are quite outside of the win-win exchange frame. However, changes in trade volume caused by abrupt custom changes and currency parities are connected with huge trade-volume changes. Indeed, such phases lead to new stability points.

5.1.2. Scenario 2: An Oligopolistic Structure of Specific Economic Clusters (Subjected to Specific Trade Agreements) Dominates the Economical Envinronment (Coiled Polymers)

This scenario describes a constellation, which is characterized by a few economic clusters (oligopolistic structure) where each of the clusters is subjected to a specific trade agreement. However, between the clusters the antagonism dominates their trade activities. This scenario may get realizable in the future. Of significant interest is the question of whether specific cluster-topologies like the E.U. can remain as a stable construct in the global economic environment. However, this question depends strongly on the agreement conditions and goals to which the cluster members are subjected and committed. Due to the theory of "topologically induced superdiffusion" in the case of coiled polymers, we get the same results as by the rigid chains.

5.1.3. Scenario 3: Topological Superdiffusion

This situation corresponds to a network architecture where, between monomers, scale-free long-range transitions can take place. In an economic sense, this constelletion is conform to a of high degree globalized world, where customs, monetary parities, and national standards lose their relevance. In a limited situation, antagonism occurs only between firms and organizations worldwide. The possibility of protecting economic interests with the help of customs, monetary parities, so far, fails gradually. The probability density for the jumb rate counts now like $w(x \mid y) = \frac{1}{\tau}\exp\left(-\beta\left[V(x)-V(y)\right]/2\right) * \rho\left(\left|x-y\right|\right)$ with $\rho(l) = \rho\left(\left|x-y\right|\right) = \frac{\mu\alpha^{\mu}}{2}\left|l\right|^{-(1+\mu)}$ and $\left|l\right| \geq \alpha$, $0 < \mu \leq 2$, α = minimum jumb length. The corresponded entropy that we can gain by the formula

$$E = -\exp\left(-\beta\left[V(x)-V(y)\right]/2\right) * \frac{\mu\alpha^{\mu}}{2}\left|l\right|^{-(1+\mu)} * \log\left(\exp\left(-\beta\left[V(x)-V(y)\right]/2\right) * \frac{\mu\alpha^{\mu}}{2}\left|l\right|^{-(1+\mu)}\right)$$

(See the following **Figure 5**).

Keeping l konstant, we can plot the corresponding entropy versus changes of the potential differences and the scaling factor (fractality order) μ. We see that the decrease of μ corresponds to a nonlinear trend with a peak of the entropy evolution. On the contrary, and keeping μ konstant, an increase of the potential difference leads linearly to a decrease of the related entropy. These results are conform to the theory of entropy production in fractal structures. (Figure 6)

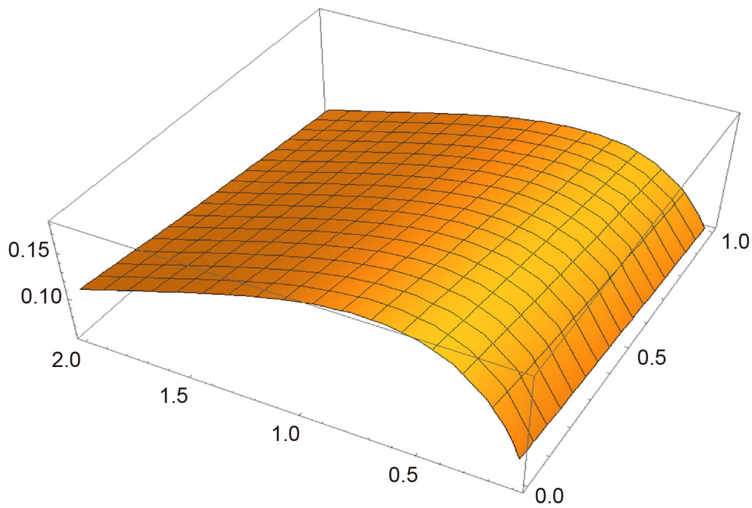

Figure 5. Entropy evolution vs. potential differences, and fractality paramer μ, with $V(x)-V(y)\in(0,1)$ and $\mu\in(0.1,2)$.

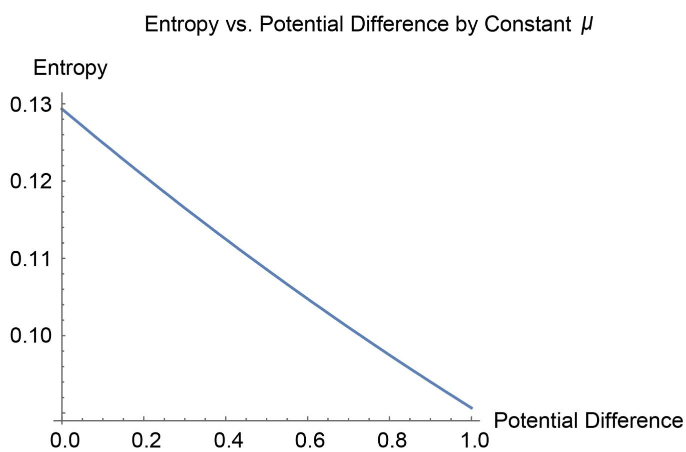

Figure 6. Entropy evolution vs. $V(x)-V(y)$ and constant fraktality $\mu=0.5$.

Remarks: a. The dynamics of rigid chains and coiled polymers can be modeled with the help of an ordinary Fokker-Planck operator, where deterministic gradient dynamics in a potential V are subjected to a stochastic gaussian white noise (see (Brockmann, 2003)). This is no longer the case for topological superdiffusion, where the ansatz is of pure stochastic nature (master equation). The above results are valid by the assumption that the changes in the fractality order of the export-import dynamics can be modelled with the help of successive following markov processes, as the sum of levy-stable densities results also in a stable levy density. At this stage, this is still an open question.

b. In the case of the "topologically induced superdiffusion", the modelling of $\left[V(x)-V(y)\right]$ cannot be described with the help of simple potential differences. A generalization of the form $\Delta V_c = cV(x)-(1-c)V(y)$ is needed, where $c \in [0,1]$. The choice of c is of great significance for adequate modelling. Remarkable is the fact that changes of the fractality order have a stronger impact on the entropy evolution as the differences in the potential specification. In an economic sense, high fractality corresponds to higher entropy following the physical law. In contrast, the impact of potential differences is weak and restricted. That means that stable exchange conditions between economies and economic clusters could lead faster to lower entropy.

6. Transformation from an Oligopolistic to a Polypolistic Structure in the Global Player Frame (Real Scenario)

6.1. Classifying the Behavior of Export-Import Activities, in the Framework of Small-World Networks (Amaral et al., 2000)

Export-import relations can be recast in a small-world network schema that allows a more detailed analysis of the corresponding behavior. The network structure emerges as a result of the connectivity structure between the network vertices. Scale-free networks emerge in the context of preferential attachment and growth. The export-import dynamics could be categorized generally as scale-free networks, which decay with a power law form. Indeed, a weak impact (bias) from preferential attachment on the dynamics leads to a stretched exponential decay, which limits the growth and the number of hubs. This group is the most met in the praxis (Amaral et al., 2000). It is worthy to note here that the connectivity structure of networks is strongly related to the theory of critical phenomena. We preferred to refer to this scenario using the model of Albert-László Barabási, as it allows us to focus on some details more effectively.

It is well known that the finance sector (banks, other finance institutions) influences the export-import activities, playing a specific role in these dynamics. A transformation in this sector from an oligopolistic structure to a polypolistic one could have an immense impact on the trade volume. That means decisions about trade activities are carried out by more centers. These dynamics could be observed with the help of "small-world networks." To this aim, export-import activities should be considered from another point of view. Export or import is carried out by enterprises (firms) and organizations, which are subjected to the specific taxation principles and activation conditions from a certain national economy. This fact authorizes us to define the term "export." An export good is a product of a certain firm or organization, produced under specific national conditions, and is consumed in another country, named import, also under specific conditions. This allows us to observe groups of export goods between national economies as "edges" and "nodes" in a specific small-world network, which establishes the topology of a network. For example, a new export good, which is imported in three other countries, represents in the corresponding network the generation of a new node with three edges. Indeed, the linking of old (existing) nodes is not a pure stochastic process but a result of preferential attachment. In this context, already existing hubs, politically favorable candidates, and other objects with attractive financial efforts are chosen for linking more likely.

6.2. The Albert-László Barabási Model (https://networksciencebook.com/)

According to the theory of the Albert-László Barabási model, the generation of new nodes and the preferential attachment are two unconditional requirements to get scale-free networks. Observing the export-import activities not only from the nationality point of view but as a product group exchange event, we can imagine (construct) a more complicated scale-free network. The Albert-László Barabási model assumes that every new node attaches to the nodes with the highest degree distribution (hubs). (The interested reader can see details of this theory at networksciencebook.com). This algorithmic processing has an immense impact on the arising network topology. The degree probability of the nodes gets out in a power law form. Now, before detecting the structure of the degree probability of real economic networks, we want to refer to two extreme categories of small-world networks. In the first case, we assume the total absence of preferential attachment. This leads to a logarithmic increase of the $p_i(k)$ (degree distribution), which has a much slower growth than the power law one. Consequently, the image of the degree probability is characterized by an exponential form like $p(k)=\frac{e}{m}\mathrm{e}^{\frac{-K}{m}}$. The faster decay of the degree probability eliminates the generation of hubs and the scale-free structure of the network topology.

In the contrary case, we assume the absence of growth, incorporating only the preferential attachment aspect. In this case, the number of nodes remains constant. The degree probability has the form $p(k)=N/2*t$ and no stationarity characterizes the process. For a large *t*, every node is attached to all the other nodes of the network. The process converges to a complete graph. These two extreme cases point out the necessity to incorporate both aspects (coexistence of growth and preferential attachment) in order to get scale-free topologies in the network architecture. Now we must note that the growth of the cumulative probability $p(k)$ with $\pi(k)=\sum_{i=0}^{k}\Pi(k_i)$ is of linear form, if simple attachment algorithms are assumed. Numerous investigations on real networks point out to the fact that $p(k)$ could be rather interpreted in the form $p(k)=k^{\alpha}$, where α can get the values $\alpha=1$, or $\alpha<1$, or $\alpha>1$. Every case has its own dynamic meaning. For $\alpha=1$, we refer to the typical Albert Laszlo Barabasi model with a specific preferential attachment algorithm. For the cases $\alpha<1$ and $\alpha>1$, we must refer to more details as they correspond rather to real economic small world networks. For $\alpha<1$ (Sublinearity), we deal with a preferential attachment characterized by a weak bias, which does not allow for generating scale-free probability topologies (Not enough sufficient). The decay of the probability in this case becames like

$p_k \approx k^{-\alpha} * \exp\left[\frac{-2\mu(\alpha)}{k(1-\alpha)}\right] * k^{1-\alpha}$ (streched exponential) where $\mu(\alpha) = Bias$. In the case of $\alpha > 1$, we refer to the superlinear preferential attachment. The behavior of new nodes tends to link highly connected nodes, which results in the emergence of super hubs. We emphasize here that the sublinearity is a rather dominant property of real economic networks in which the preferential attachment gets a weak form. The degree distribution follows in this context a streched exponential behavior leading to fewer and smaller hubs than in the scale-free case. At this stage, we return to the results concerning the fractality order and its changes in the export-import dynamics. The subdiffusive character of the export-import dynamics allows us to represent this for large times (asymptotically) with the help of a levy-stable PDF. Due to $G_{\alpha,1}^{\theta}(x,t) = t^{-1/\alpha} * L_{\alpha}^{\theta}\left(\frac{x}{t^{1/\alpha}}\right)$, for $-\infty < x < \infty$. All the extremal stable PDF's with $0 < \alpha < 1$ are one sided, the support beeing $\mathcal{R}_0^+$ if $\theta = -\alpha$ and $\mathcal{R}_0^-$ if $\theta = +\alpha$, where extremal PDF's, are stable PDF's with extremal value of the skewness parameter (see details of this theory in the fundamental solution of the space-time fractional diffusion equation, by Mainardi et al., 2000). The following Figure 7 points out the changes of the corresponding entropy vs. fractality-order changes, due to the formula

$$p_k \approx k^{-\alpha} * \exp\left[\frac{-2\mu(\alpha)}{k(1-\alpha)}\right] * k^{1-\alpha}$$

We can see that entropy increases with higher fractality values (higher fractality corresponds to lower α values).

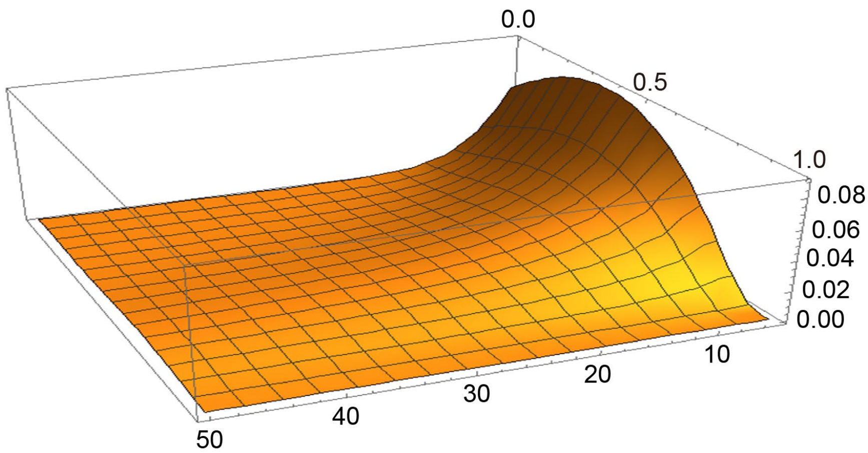

Figure 7. Entropy vs. fractality order $\alpha \in (0,1)$ and k, with $2\mu(\alpha) = 0.4$.

The case of the absence of preferential attachment:

In this case, the links between nodes are chosen stochastically. This results in an exponential degree of probability of the form $e/m * \exp(-k/m)$. The corresponding entropy reads as

$$E = -e/m * \exp(-k/m) * [\log(m) - 1 + k/m]$$

The following graphs express the above relations. It is worthy to note that the entropy decreases significantly if k increases. (Figure 8, Figure 9)

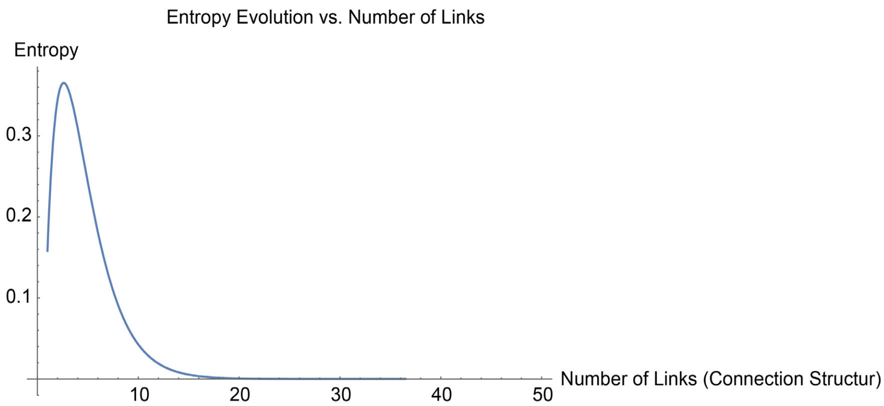

Figure 8. Entropy evolution vs. K (degree Distr.) by konstant m.

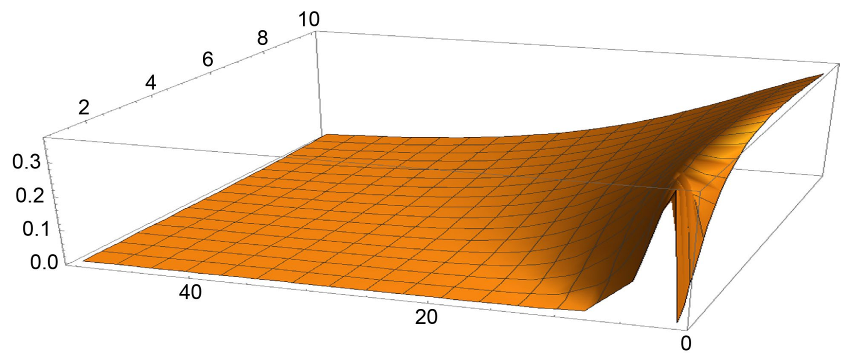

Figure 9. Entropy vs. K, m in absence of preferential attachment, with $m \in (0,10)$ and $k \in (0,50)$.

7. Significant Results

The impact of the linearity-exponent of the degree probability on the entropy production is similar to that of the fractality order on dynamics. That means higher fractality induces higher entropy. Has this paradoxical behavior consequences in socio-economic and geopolitical events? It is evident that by economic decisions, preferential attachment is an indispensable tool to get advantageous results. The choice of existing hubs is an inevitable step to get win-win constellations in an economic sense. This problem is a real optimization problem by specific conditions. In this context, the theme of varying fractality order of the export-import dynamics could be approached by help of the "small-world networks" paradigm. Example: If we intend politically to incorporate in the global economic environment through decision-making, we can not avoid the generation of hubs. If we wish to get then globally better and fairer results for all the players, we must first concentrate on the role and behavior of the hubs, concerning a lower entropy production.

8. Conclusion

The export-import dynamics are an imprint of the worldwide economic exploitation process. In this context, they can send a signal of forthcoming conflicts or criticality. The fractality orders and the therewith corresponding entropy on these dynamics can surely provide an important criterion in this sense. We have seen that changes in the space- and time-fractality order in the case of subdiffusive dynamics lead in limes to a levy stable structure. Using the concept of the "topologically induced diffusion", we can state that the impact of the fractality order on the entropy production is more significant than the impact caused by potential differences. This fact points out the importance of stable exchange conditions between national economies or economic clusters in order to get lower entropy values. The fact that in limes, subdiffusive dynamics degenerate in the form of Levy stable probability densities emphasizes the necessity to focus on the course of the space-fractality order in the time evolution. Due to the article, the fundamental solution of the space-time fractional diffusion equation of Mainardi, Luchko, and Pagnini, the limes form of a levy stable distribution results in $L_{\alpha}^{\theta}(x) = O\left(|x|^{-(\alpha+1)}\right), \quad x \to +(-)\infty$. This formula is connected with higher entropy by lower values of α ($\alpha < 1$), where α denotes the space-fractality order.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Mathematical Appendix

Annex: The Impact of Customs on the Trade Volume

According to this goal, we used a method applied by entsar Abdel Rehim (Abdel-Rehim & Gorenflo, 2005; Abdel-Rehim, 2005).

For the corresponding simulations the cumulative distribution function $\Psi^*(t) = \frac{1}{1+\Gamma(1-\beta)t^\beta}$, $t \geq 0$, $0 < \beta < 1$, seems to be adequate to substitute the asymptotic behavior of the Mittag-Leffler function. This allows us to use the frmula $T = \left(\frac{1}{\Gamma(1-\beta)}\left(\frac{1}{r^*} - 1\right)\right)^{1/\beta}$ in order to generate the jumps of the waiting time T. It holds $r^* = 1 - r$, where r is uniformly distributed in $[0,1)$. Further counts $r^* = \Psi^*(T)$. For the space-jumps we use the probability density Function $w^*(x) = \frac{\alpha}{2}\frac{|x|^{\alpha-1}}{\left(1+|x|^\alpha\right)^2}$, applied by Gorenflo & Mainardi who modified a method of Chechkin & Gonchar. The space jump X can be gained on this way by the formula $X = \left(\frac{1}{2(1-z)} - 1\right)^{1/\alpha}$, if $z \geq 0.5$ and $X = \left(-\frac{1}{2z} - 1\right)^{1/\alpha}$ for $z < 0.5$, where z is uniformaly distributed in $(0,1]$

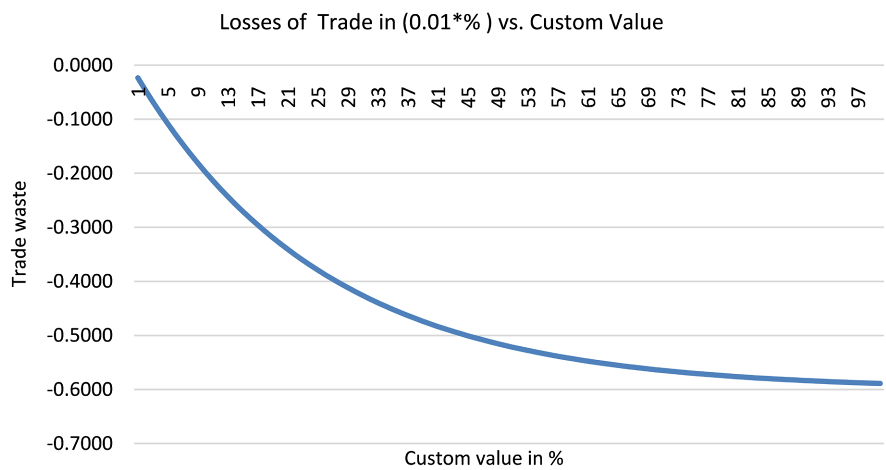

Figure A1. Normed Morse Potential. Losses of Trade Volume vs. Custom Values in 0.01*%.

For the simulation, we found it convenient to involve the Morse potential, of which a normalized version can be seen in the above graph (it means how much of the trade volume is getting lost in 0.01% if we raise a specific value (in %) of customs on it). That means a vertical value (trade wastes) of -0.4 corresponds to 40%. We apply for estimating the impact of customs raises on import goods a specifically modified Euler scheme in its fractional version. According to this scheme, the solution can be gained with the help of the following relations.

$\mathrm{d}X(\tau) = -\frac{V'(X(\tau))}{n}\mathrm{d}\tau + \mathrm{d}L_\mu(\tau)$, which leads to the Algorithm

$X(\tau_i) = X(\tau_{i-1}) - \frac{V'(X(\tau_{i-1}))}{n} * \Delta\tau + \Delta\tau^{1/\mu}\xi_i$, where

$V(X)$ is the involved Potential, $L_\mu(\tau)$ = symmetric levy-stable motion, ξ_i is a specific formula, $\Delta\tau$ is the fractional time jump and $i = 1, 2, \cdots$. This method has been proposed by Magdziard and Coworkers (Weron et al., 2008), as a solution of the fractional Fokker -Planck differential equation

$$\frac{\partial p(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}V'(x) + D\frac{\partial^\mu}{\partial |x|^\mu}\right]p(x,t).$$

${}_0D_t^{-\alpha} = J^\alpha$ denotes the Riemann-Liouville fractional integral operator with $0 < \alpha < 1$ (order of time fractality) and $0 < \mu < 2$ (order of space fractality). Due to magdziard the solution of the fractional Fokker-Planck Ansatz is equal to the probability density function of the subordinated process $Y(t) = X(S_t)$, whereby the parent process $X(\tau)$ is defined as the solution of the stochastic differenttial equation (Magdziarz & Weron, 2007)

$$\mathrm{d}X(\tau) = -\frac{V'(X(\tau))}{n}\mathrm{d}\tau + \mathrm{d}L_\mu(\tau)$$

We have applied in this article both proposals (that of Abdel-Rehim and this of Magdziard and Coworkers), which led to similar resolutions. In this context, we have estimated that the trade volume will approximately get 64% of the trade volume without customs. That means a loss of 35% could be realized in trade volum if customs between 0% and 100% on this would arise. This result is risky and has a tolerance range of a minimum of 10%. The result depends strongly on the probability densities used to generate random data on the arising customs.