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Elementary Derivation of First-Passage Distribution

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Abstract

One of the most intricate formulas relating to a Stochastic process known as Brownian motion specifies the probability density function of the time the process takes to reach, for the first time, a given state. There are several existing derivations thereof (a special case of inverse Gaussian distribution), most of them requiring sophisticated mathematical tools and rather complex reasoning. In this article, we present a simple derivation of the same result, based on elementary arguments and rudimentary mathematics only.

Keywords

Brownian Motion, Random Walk, Gambler's Ruin Problem, Stirling's Formula, Probability Generating Function

1. Introduction

We recall that Brownian motion, denoted $X(t)$, is a time-homogeneous Markovian process of independent increments which has two parameters, d (*drift*) and c (*diffusion coefficient*); the main result states that the distribution of $X(t)$, given that $X(0)=a$, is Normal with the mean of $a+d\cdot t$ and variance of $c\cdot t$ [1]. Graphically, t is typically displayed using the horizontal axis while the values of $X(t)$ are shown on the vertical scale (of the so-called state space).

It is intuitively obvious (and formally proven below) that, when a is positive and d is negative, the process must reach, sooner or later, State 0; it is a major and complex task (see [2] [3]) to prove that the exact time (denoted T) of this event happening for the first time (*i.e.* of the first passage) is a random variable having inverse Gaussian distribution [4]. Simplifying the usual arguments and finding the corresponding probability density function (PDF) of T by *elementary* means is the goal of this article.

This is achieved by first *discretizing* both time (making each of its units into ck^2 discrete time steps) and state space (making each spatial unit into k discrete vertical steps), resulting in a much simpler process called random walk [5]. We can then easily solve the same problem using the *discrete* version to find the probability mass function (PMF) of a similarly defined first-passage time $\tilde{T}$ (now, an integer-valued random variable), and then convert this discrete PMF to the continuous PDF of T by taking the $k \to \infty$ limit.

To underline the importance of first-passage distributions, we mention that they arise naturally whenever one is interested in the time required for a fluctuating quantity to reach a prescribed threshold, including problems in physics, finance, reliability, and biology.

Note that paragraphs and sections starting with the ■ prefix are optional and can be omitted without affecting our derivation of the main result (only in those sections, we occasionally use concepts and formulas which may be deemed less elementary).

2. Gambler's Ruin Problem

The discrete version of the process has the following interesting interpretation: consider a gambler who starts with i monetary units (we call them *dollars*) and keeps on betting $1 on a flip of a *biased* coin (the probability of winning an individual round is $p < \frac{1}{2}$, of losing is $q := 1 - p$) until losing *all* his money (*i.e.* reaching State 0), assuming that his opponent has *infinite* resources. Each round of this game takes one discrete time step, while one dollar is the state-space unit.

To derive a formula for the probability that the game (which ends by the gambler getting broke) will take *exactly* n rounds, we consider all possible paths which take the gambler from State i to State 0 *without* visiting 0 on any previous occasion, and compute their total probability. The individual probabilities are easy: each such path must consist of exactly $\frac{n-i}{2}$ wins and $\frac{n+i}{2}$ losses (note that n must be even/odd when i is even/odd respectively, while $n \ge i$); therefore, it has the probability of $p^{(n-i)/2} q^{(n+i)/2}$.

To count how many such paths are there is more difficult: to avoid visiting State 0 before completing n rounds (or time steps), the *last* step must be a *loss*; we are then left with an easier (yet equivalent) task of establishing the number of paths that connect State i to State 1 in $n-1$ steps, while avoiding State 0. This is done in two steps: ignoring the last condition (*i.e. allowing* visits to 0), the number of such paths is $\binom{n-1}{\frac{n-i}{2}}$, since *any* combination of $\frac{n-i}{2}$ wins and $\frac{n+i-2}{2}$ losses will do. We then need to subtract the number of those paths (from i to 1, in $n-1$ steps) which *have* visited State 0, at least once, prior to reaching State 1.

To count these, we use the following classical reflection principle (or André's reflection method, see [5] [6]), which provides a convenient way of counting paths constrained by a boundary: visualize flipping the *first* part (up to and including the first visit to State 0) of any such path with respect to the time axis, while keeping its *second* part (from that point on) intact (see **Figure 1**).

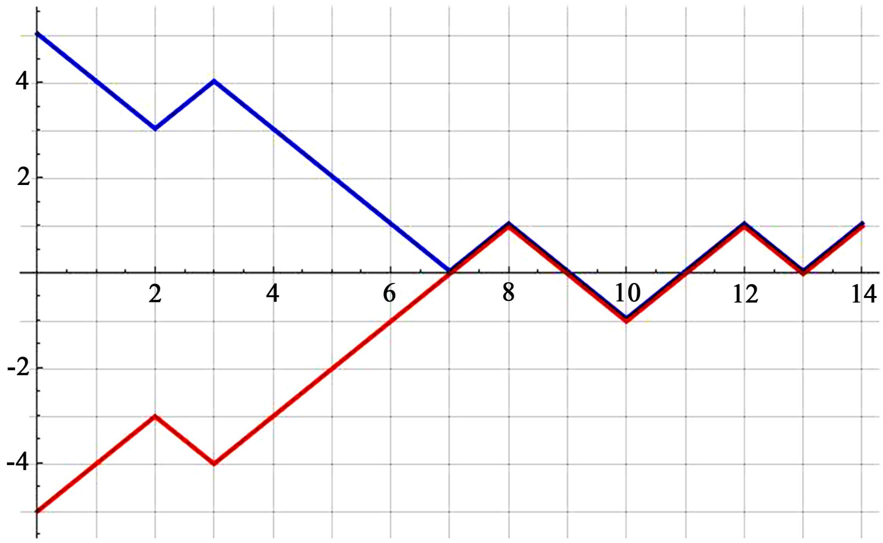

Figure 1. Original (blue) and flipped (red) path.

This results in a path which starts at $-i$ and ends up, after $n-1$ steps, in State 1 (*necessarily* visiting State 0); the important thing is that there is a *one-to-one* correspondence between paths of the newly created (red) set and those of the original (blue) set. Counting the number of paths of the former is easy: this time we need exactly $\frac{n-i-2}{2}$ wins and $\frac{n+i}{2}$ losses; the answer is thus $\binom{n-1}{\frac{n+i}{2}}$. The number of paths contributing to the probability of the game taking exactly n rounds is thus

$$\begin{aligned}\binom{n-1}{\frac{n-i}{2}}-\binom{n-1}{\frac{n+i}{2}}&=\frac{(n-1)!}{\left(\frac{n-i}{2}\right)!\left(\frac{n+i}{2}-1\right)!}-\frac{(n-1)!}{\left(\frac{n-i}{2}-1\right)!\left(\frac{n+i}{2}\right)!}\\&=\frac{(n-1)!}{\left(\frac{n-i}{2}\right)!\left(\frac{n+i}{2}\right)!}\cdot\left(\frac{n+i}{2}-\frac{n-i}{2}\right)=\frac{i}{n}\binom{n}{\frac{n+i}{2}}\end{aligned}\tag{1}$$

The resulting PMF of $\tilde{T}$ is then

$$f(n)=\frac{i}{n}\binom{n}{\frac{n+i}{2}}p^{(n-i)/2}q^{(n+i)/2}\tag{2}$$

when $n=i,i+2,i+4,i+6,i+8,\cdots$ (zero otherwise). This classical first-passage probability for a one-dimensional random walk can be found, in various forms, in standard treatments of random walks and probability theory, such as [5] [7] [8].

■ Note that an alternate way of expressing the same result is

$$f(i+2j)=\frac{\left(\frac{i}{2}\right)_{(j)}\left(\frac{i+1}{2}\right)_{(j)}}{(i+1)_{(j)}\cdot j!}\cdot 4^j p^j q^{i+j} \tag{3}$$

where $j=0,1,2,3,\cdots$ and $w_{(j)}:=w(w+1)(w+2)\cdots(w+j-1)$. This can be shown by replacing n by $i+2j$ in (2) and expanding:

$$\begin{aligned}&\frac{i}{i+2j}\binom{i+2j}{i+j,j}p^j q^{i+j}\\&=\frac{i(i+2j-1)(i+2j-2)\cdots(i+1)i!}{(i+j)(i+j-1)\cdots(i+1)i!\cdot j!}p^j q^{i+j}\\&=\frac{\left(\frac{i+2j-1}{2}\right)\left(\frac{i+2j-2}{2}\right)\cdots\left(\frac{i+1}{2}\right)\frac{i}{2}}{(i+j)(i+j-1)\cdots(i+1)j!}4^j p^j q^{i+j}\end{aligned} \tag{4}$$

■ Probability Generating Function (PGF) of $\tilde{T}$

The goal of this section is to prove that, when $p\le\frac{1}{2}$, reaching State 0 becomes inevitable. This requires evaluating the following PGF of $\tilde{T}$ (obtained by performing the corresponding summation)

$$P(z):=\mathbb{E}\left(z^{\tilde{T}}\right)=\sum_{j=0}^{\infty}f(i+2j)z^{i+2j}=q^i z^i\,{}_2\mathcal{F}_1\left(\frac{i}{2},\frac{i+1}{2};i+1;4pqz^2\right) \tag{5}$$

at $z=1$ (which yields the probability of *ever* reaching State 0) and confirming that the answer is 1. Unfortunately, this form of PGF (containing ${}_2\mathcal{F}_1$, called hypergeometric function [9]) is not very suitable for our purpose (in addition to breaking our promise of using elementary functions only); luckily, there is an alternate way (more in the spirit of the current article) of deriving an *equivalent* version of the same PGF, which is then much easier to evaluate.

To find the alternate formula, we first introduce an *infinite* set of such PGFs (one for each possible value of i), denoting them $P_i(z)$ (where $i=0,1,2,3,\cdots$) and relating them to each other in the following manner: based on what happens in the *next* round of the game (there are just two possibilities) and assuming that we are now in State i, the formula of total probability tells us that

$$\mathbb{E}\left(z^{\tilde{T}_i}\right)=p\cdot\mathbb{E}\left(z^{1+\tilde{T}_{i+1}}\right)+q\cdot\mathbb{E}\left(z^{1+\tilde{T}_{i-1}}\right) \tag{6}$$

where the first term corresponds to *winning* the round (the original $\tilde{T}_i$ then becomes $1+\tilde{T}_{i+1}$, since one time step has been taken and $\tilde{T}_{i+1}$ more steps are still needed to reach State 0); a similar argument applies to the second term (when the round is lost). By the definition of PGF, (6) is equivalent to saying that

$$P_i(z)=pzP_{i+1}(z)+qzP_{i-1}(z) \tag{7}$$

which constitutes an infinite set of difference equations to be solved for all such $P_i(z)$ functions.

We solve this second-order linear difference equation by seeking a solution of the form $P_i(z)=\lambda^i$ [10], where λ must meet

$$\lambda^i = pz\lambda^{i+1} + qz\lambda^{i-1} \tag{8}$$

for all i, implying that λ must be a root of the following characteristic (quadratic, in this case) polynomial

$$\lambda = pz\lambda^2 + qz \tag{9}$$

The superposition principle then implies that a *general* solution to (7) is a *linear combination* of the two individual solutions, namely

$$P_i(z) = A\cdot\lambda_1^i + B\cdot\lambda_2^i$$

where λ_1 and λ_2 are the two roots of (9) and A and B are yet to be established.

To find the A and B values, we need to assume that the opponent starts with $N-i$ dollars and may thus lose the game as well. This means that our solution must also meet the following boundary conditions, namely $P_0(z)=1$ and $P_N(z)=1$, indicating that, upon reaching either State 0 or State N, the game is over and no more rounds are to be played. It is easy to verify that

$$P_i(z) = \frac{\lambda_1^i\left(1-\lambda_2^N\right)-\lambda_2^i\left(1-\lambda_1^N\right)}{\lambda_1^N-\lambda_2^N} \tag{10}$$

where

$$\lambda_{1,2} = \frac{1\pm\sqrt{1-4pqz^2}}{2pz} \tag{11}$$

(roots of the characteristic polynomial) is the solution which meets (7) *and* both boundary conditions [5].

To get the solution we need (the opponent has infinite resources), we simply take the $N\to\infty$ limit of (10), getting

$$P_i(z) = \lambda_2^i = \left(\frac{1-\sqrt{1-4pqz^2}}{2pz}\right)^i \tag{12}$$

since $\lambda_1 > \lambda_2$ when $0\le z\le 1$. Note that (12) is the *only* linear combination of the two individual solutions which does *not* exceed 1 when $0\le z\le 1$ (a *necessary* condition of a PGF); using this condition, we could have proceeded to derive (12) more directly (but somehow less comprehensibly). Also note that we have effectively shown that (12) and (5) are identical functions.

To prove that visiting State 0 sooner or later is inevitable is now quite simple: showing that (12) is equal to 1 when $z=1$ (and $p\le\frac{1}{2}$) follows from

$$\sqrt{1-4pq} = \sqrt{1-4p+4p^2} = \sqrt{(1-2p)^2} = 1-2p \tag{13}$$

Note that, when $p>\frac{1}{2}$,

$$P_i(1)=\left(\frac{1-\sqrt{(1-2p)^2}}{2p}\right)^i=\left(\frac{1-(2p-1)}{2p}\right)^i=\left(\frac{q}{p}\right)^i \tag{14}$$

Since this value is less than 1, the process may *never* reach State 0 and the gambler then keeps on winning indefinitely (with a probability given by, and thus explaining, the deficit).

3. Brownian-Motion Limit

To convert (2) into a PDF of the first-passage time T of a Brownian motion with drift d (assumed to be *negative*, *i.e.* $d<0$), diffusion coefficient $c>0$, and starting in State $a>0$, we proceed (as indicated in the Introduction) as follows:

- change the amount of a single bet to $\frac{1}{k}$, where k is a (large) integer; the initial State i is then replaced by $a\cdot k$,
- play $c\cdot k^2$ rounds of the new game each unit of *real* time t,
- while setting the player's probability of winning each round at

$$p:=\frac{1}{2}+\frac{d}{2ck}\ \left(<\frac{1}{2}\right) \tag{15}$$

This means that, in a single round, the distribution of *change* in the value of $X(t)$, *i.e.* of its *increment* Δ, is

Δ	$-\frac{1}{k}$	$\frac{1}{k}$
Pr	$\frac{1}{2}-\frac{d}{2ck}$	$\frac{1}{2}+\frac{d}{2ck}$

having the expected value of $\frac{d}{c\cdot k^2}$ and variance equal to $\frac{1}{k^2}-\frac{d^2}{c^2k^4}$. This further implies that, at time t (after ck^2t independent rounds have been played) the expected value of the *total* increment is $d\cdot t$, with the corresponding variance of $c\cdot t-\frac{d^2t}{c\cdot k^2}$; Brownian motion is then the limit of this process as $k\to\infty$. Central Limit Theorem tells us that the distribution of $X(t)$ is Normal, with the mean of $a+d\cdot t$ and variance equal to $c\cdot t$.

3.1. Stirling's Formula

To find the corresponding limit of the $\tilde{T}$ distribution, namely of (2), with the goal of getting the PDF of T, we need the following result

$$\ln k!=\left(k+\frac{1}{2}\right)\ln k-k+\frac{1}{2}\ln(2\pi)+\cdots \tag{16}$$

where the ellipsis represents an expression which tends to 0 when $k\to\infty$ [8].

To prove it (following the same reference), we start with

$$k! = \int_0^\infty x^k \mathrm{e}^{-x} \mathrm{d}x \tag{17}$$

(proved easily using by-part integration) which, after the $x = k \cdot y$ substitution, becomes

$$\begin{aligned} k^k \int_0^\infty k \cdot y^k \exp(-ky) \mathrm{d}y &= k^{k+1} \int_0^\infty \exp(-k(y - \ln y)) \mathrm{d}y \\ &= k^{k+1} \mathrm{e}^{-k} \int_0^\infty \exp(-k(y - \ln y - 1)) \mathrm{d}y \end{aligned} \tag{18}$$

Taylor-expanding the integrand's *exponent* at $y = 1$ yields

$$-k \cdot \frac{(y-1)^2}{2} + k \cdot \frac{(y-1)^3}{3} - k \cdot \frac{(y-1)^4}{4} + \cdots \tag{19}$$

while the integrand itself can then be expanded (and subsequently integrated), using the classical Laplace Method [11], as follows

$$\begin{aligned} &\int_0^\infty \exp\left(-k \cdot \frac{(y-1)^2}{2}\right) \cdot \left(1 + k \cdot \frac{(y-1)^3}{3} - k \cdot \frac{(y-1)^4}{4} + k^2 \cdot \frac{(y-1)^6}{2 \cdot 3^2} + \cdots\right) \mathrm{d}y \\ &= \frac{\sqrt{2\pi}}{\sqrt{2\pi k}} \int_{-\sqrt{k}}^\infty \exp\left(-\frac{u^2}{2}\right) \cdot \left(1 + \frac{u^3}{3\sqrt{k}} - \frac{u^4}{4k} + \frac{u^6}{2 \cdot 3^2 k} + \cdots\right) \mathrm{d}u \end{aligned} \tag{20}$$

after the $y = 1 + \frac{u}{\sqrt{k}}$ substitution. Note that the method exponentiates the first term of (19) directly (resulting in a Gaussian peak), while the remaining two terms, being $\frac{1}{\sqrt{k}}$ and $\frac{1}{k}$ small respectively, are expanded accordingly; furthermore, for large k, it is legitimate to change the integral's lower limit to $-\infty$. All we need now are the 0th, 3rd, 4th and 6th moments of the standard Normal distribution; these are equal to 1, 0, 3 and 15 respectively. The value of (20) is thus

$$\sqrt{\frac{2\pi}{k}} \cdot \left(1 - \frac{3}{4k} + \frac{5}{6k} + \cdots\right) = \sqrt{\frac{2\pi}{k}} + \frac{1}{12} \cdot \sqrt{\frac{2\pi}{k^3}} + \cdots \tag{21}$$

For (18), and consequently for (17), this implies

$$k! \approx k^{k+1/2} \mathrm{e}^{-k} \sqrt{2\pi} \cdot \left(1 + \frac{1}{12k} + \cdots\right) \tag{22}$$

which is a re-statement of (16), with the extra $\frac{1}{12k}$ term indicating an error of the formula (ignored in our subsequent $k \to \infty$ limits). Note that this result is readily extended to *non-integer* positive k values when (17) is used as a *definition* of a factorial.

3.2. PDF of First-Passage Time

To find the PDF of the random variable T (continuous version of $\tilde{T}$, defined by $T := \frac{\tilde{T}}{k^2 t}$), all we need is to replace i by ak, n by $ck^2 t$ and p by $\frac{1}{2} + \frac{d}{2ck}$ in (2), further divide by the discrete step of $\frac{2}{ck^2}$ (converted from $\tilde{T}$ to T scale), and take the $k \to \infty$ limit. This becomes easier when done with the $\ln$ of the PMF instead; we start with

$$\begin{aligned}
\ln\binom{n}{\frac{n+i}{2}} &\overset{\text{sub}}{=} \ln\left(ctk^2!\right) - \ln\left(\frac{ctk^2+ak}{2}!\right) - \ln\left(\frac{ctk^2-ak}{2}!\right) \\
&= \left(ctk^2+\frac{1}{2}\right)\underline{\ln\left(ctk^2\right)}\boxed{-ctk^2} + \ln\sqrt{2\pi} \\
&\quad -\left(\frac{ctk^2+ak}{2}+\frac{1}{2}\right)\left[\underline{\ln\left(ctk^2\right)} - \ln 2 + \left(\frac{a}{ctk} - \frac{a^2}{2c^2t^2k^2} + \cdots\right)\right] \\
&\quad \boxed{+\frac{ctk^2+ak}{2}} - \ln\sqrt{2\pi} \\
&\quad -\left(\frac{ctk^2-ak}{2}+\frac{1}{2}\right)\left[\underline{\ln\left(ctk^2\right)} - \ln 2 + \left(-\frac{a}{ctk} - \frac{a^2}{2c^2t^2k^2} + \cdots\right)\right] \\
&\quad \boxed{+\frac{ctk^2-ak}{2}} - \ln\sqrt{2\pi} \\
&\overset{\text{lim}}{=} -\frac{1}{2}\ln\left(ctk^2\right) + \left(ctk^2+1\right)\ln 2 + \frac{a^2}{2ct} - \frac{a^2}{ct} - \ln\sqrt{2\pi} + \cdots
\end{aligned} \tag{23}$$

where the two terms in parentheses ending with $+\cdots$ represent expansions of $\ln\left(1+\frac{a}{ctk}\right)$ and $\ln\left(1-\frac{a}{ctk}\right)$; note that the *total* coefficient of the underscored terms reduces to $-\frac{1}{2}$, while the boxed terms cancel out in full ($\overset{\text{sub}}{=}$ and $\overset{\text{lim}}{=}$ imply: 'after substitution' and 'after taking the $k \to \infty$ limit', respectively).

Secondly

$$\ln\left(\frac{i}{n}\right) \overset{\text{sub}}{=} \ln a - \ln c - \ln t - \ln k \tag{24}$$

and

$$\begin{aligned}
&\frac{n-i}{2}\ln p + \frac{n+i}{2}\ln q \overset{\text{sub}}{=} \frac{ctk^2-ak}{2}\ln\left(\frac{1}{2}+\frac{d}{2ck}\right) + \frac{ctk^2+ak}{2}\ln\left(\frac{1}{2}-\frac{d}{2ck}\right) \\
&\frac{ctk^2-ak}{2}\left(-\ln 2 + \frac{d}{ck} - \frac{d^2}{2c^2k^2} + \cdots\right) + \frac{ctk^2+ak}{2}\left(-\ln 2 - \frac{d}{ck} - \frac{d^2}{2c^2k^2} + \cdots\right) \\
&\overset{\text{lim}}{=} -ctk^2\ln 2 - \frac{ad}{c} - \frac{d^2t}{2c}
\end{aligned} \tag{25}$$

Finally, adding the re-scaling factor of

$$2\ln k + \ln c - \ln 2 \tag{26}$$

yields the following total of the last four results (note that all k-related terms have either cancelled out or had a zero limit)

$$\ln f(t) = -\frac{(a+dt)^2}{2ct} + \ln\sqrt{\frac{a^2}{2\pi ct^3}} \tag{27}$$

This is then easily converted to the PDF of T, namely

$$f(t) = \frac{a}{\sqrt{2\pi c t^3}} \exp\left(-\frac{(a+dt)^2}{2ct}\right) \tag{28}$$

when $t > 0$ (zero otherwise); note that the a and c parameters are positive while d must be negative. The corresponding distribution is called *inverse Gaussian*; it has a mean of $\frac{a}{-d}$, variance equal to $\frac{ac}{-d^3}$, while $3\sqrt{\frac{c}{-ad}}$ is its skewness. When $d = 0$, (28) gets reduced to a special case of Levy distribution (effectively the distribution of Y^{-2}, where Y is Normal with the mean of 0 and variance of $\frac{c}{a^2}$).

Note that $\int_0^\infty f(t)\,\mathrm{d}t = 1$ when the three parameters meet the above conditions; this result is inherited from (2). To find the total probability of (28) when $d > 0$ requires its non-trivial integration; this can be bypassed by taking the $k \to \infty$ limit of (14) or, more easily, of its natural logarithm, thus getting

$$ak\left[\ln\left(\frac{1}{2} - \frac{d}{2ck}\right) - \ln\left(\frac{1}{2} + \frac{d}{2ck}\right)\right] = ak\left(-\frac{2d}{ck} + \cdots\right) \overset{\lim}{=} \frac{-2ad}{c} \tag{29}$$

This implies that the probability of Brownian motion with such parameters *never* reaching State 0 is

$$1 - \exp\left(\frac{-2ad}{c}\right) \tag{30}$$

3.3. ■ Moment Generating Function (MGF) of T

The usual way of deriving (28) is to find the corresponding MGF first (using a long sequence of intricate arguments), and then convert it to the corresponding PDF (yet another difficult task); the advantage of our approach is in bypassing this entirely and obtaining (28) directly from its discrete analog, as done already in this article.

Nevertheless, for completeness, we now show how the same MGF is almost effortlessly derived (again, from its discrete analog) by taking a limit of the PGF of $\tilde{T}$. All we need to do is to replace z in

$$P_i(z) := \mathbb{E}\left(z^{\tilde{T}}\right) = \left(\frac{1 - \sqrt{1 - 4pqz^2}}{2pz}\right)^i \tag{31}$$

by $\exp\left(\frac{\phi}{ck^2}\right)$, since the old time step now takes $\frac{1}{ck^2}$ of the new time units (*i.e.* $T = \frac{\tilde{T}}{ck^2}$), while recalling the definition of

$$M(\phi) := \mathbb{E}\left(\exp(\phi \cdot T)\right) \tag{32}$$

Furthermore, making the usual $i \to ak$ and $p \to \frac{1}{2} + \frac{d}{2ck}$ substitutions and taking the $k \to \infty$ limit (again, it is easier to work with $\ln M(\phi)$ instead) results in

$$\begin{aligned}
&\ln P_i\left(\exp\left(\frac{\phi}{ck^2}\right)\right)\\
&\overset{\text{sub}}{=} ak\left[\ln\left(1-\sqrt{1-\left(1-\frac{d^2}{c^2k^2}\right)\left(1+\frac{2\phi}{ck^2}+\cdots\right)}\right)-\ln\left(1+\frac{d}{ck}\right)-\frac{\phi}{ck^2}\right]\\
&= ak\left[\ln\left(1-\sqrt{\frac{d^2}{c^2k^2}-\frac{2\phi}{ck^2}+\cdots}\right)-\frac{d}{ck}+\cdots\right]\\
&= ak\left(-\sqrt{\frac{d^2}{c^2k^2}-\frac{2\phi}{ck^2}}-\frac{d}{ck}+\cdots\right)\overset{\text{lim}}{=}-\frac{ad}{c}-\frac{a\sqrt{d^2-2c\phi}}{c}
\end{aligned} \tag{33}$$

which agrees with the classical result of [3].

Rather than using this MGF to re-derive (28) (which would be, as already mentioned, quite difficult), we proceed in reverse, *verifying* that we can get the same MGF directly from (28). This happens to be much easier, since it is possible to bypass actual integration and proceed as follows

$$\begin{aligned}
M(\phi) &= \int_0^\infty f(t)\exp(\phi\cdot t)\,\mathrm{d}t\\
&= \frac{a}{\sqrt{2\pi c}}\int_0^\infty t^{-3/2}\exp\left(-\frac{d^2t^2+2dat+a^2-2c\phi t^2}{2ct}\right)\mathrm{d}t\\
&= \exp\left(\frac{-ad-aD}{c}\right)\cdot\frac{a}{\sqrt{2\pi c}}\int_0^\infty t^{-3/2}\exp\left(-\frac{D^2t^2-2Dat+a^2}{2ct}\right)\mathrm{d}t\\
&= \exp\left(\frac{-ad-aD}{c}\right)
\end{aligned} \tag{34}$$

where $D := \sqrt{d^2-2c\phi}$. The last step follows from the fact that

$$\int_0^\infty f(t)\,\mathrm{d}t = \frac{a}{\sqrt{2\pi c}}\int_0^\infty t^{-3/2}\exp\left(-\frac{d^2t^2+2dat+a^2}{2ct}\right)\mathrm{d}t = 1 \tag{35}$$

for *any* negative value of d, including $-D$; (34) thus matches the previous result.

4. Conclusions

We have shown how the first-passage-time distribution of one-dimensional Brownian motion can be obtained from the corresponding problem for a simple discrete random walk. The essential idea is to discretize both space and time, solve the resulting gambler's-ruin problem exactly, and then take a carefully chosen limit in which the spatial step tends to zero while the number of discrete time steps tends to infinity. In this way, a result that is not particularly transparent when derived directly from Brownian motion emerges naturally from a much simpler discrete problem.

The derivation also illustrates the close connection between random walks and Brownian motion. The discrete first-passage probability is obtained by elementary path counting, while the Brownian density follows from the appropriate diffusive scaling. The resulting density is the familiar inverse Gaussian first-passage distribution. The calculation therefore provides not a new first-passage formula, but a relatively direct derivation of a classical result and, in particular, an alternative route for readers who may find derivations based on differential equations or more advanced stochastic-process methods less accessible.

Several extensions of the present approach would be interesting. One natural direction is to consider a second absorbing boundary, leading to first-passage problems in a finite interval. Another is to replace the fixed boundary by a time-dependent one. It would also be interesting to investigate whether analogous discrete constructions can provide useful derivations for first-passage problems associated with other stochastic processes, or for related quantities such as the distribution of the maximum of a random walk or Brownian motion.

The restriction to one spatial dimension is another important limitation of the present treatment. Higher-dimensional first-passage problems are considerably richer, and it is not obvious how far the simple path-counting argument used here can be extended. Exploring discrete approximations to such problems may nevertheless provide an interesting bridge between random-walk methods and the corresponding Brownian-motion results.

Finally, the present derivation establishes the limiting distribution but does not address the rate at which the discrete first-passage distribution approaches its Brownian limit. Quantifying this convergence, and determining how large the number of discrete steps must be for the continuous approximation to be accurate, would be a useful subject for further work.

Thus, although the present calculation is deliberately restricted in scope, it suggests that discrete random walks can provide a particularly transparent starting point for understanding a wider class of first-passage problems.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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