

Higher-Order k-Nacci Quasigeometric Systems

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How to cite this paper: Mamombe, L. (2026) Higher-Order k-Nacci Quasigeometric Systems. *Open Journal of Discrete Mathematics*, 16, 77-88. <https://doi.org/10.4236/ojdm.2026.164007>

Received: September 5, 2026

Accepted: October 12, 2026

Published: October 15, 2026

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Abstract

We herein introduce and develop a general theory of higher-order k -nacci quasigeometric systems by imposing simultaneously the k -step additive recurrence $h_{n+k} = h_{n+k-1} + h_{n+k-2} + \dots + h_n$ and nearest-integer multiplication by the dominant k -nacci root α_k , $h_{n+1} = \text{rnd}(\alpha_k h_n)$, $\alpha_k^k = \alpha_k^{k-1} + \dots + \alpha_k + 1$. For the rounded orbit $x_0 = n$ and $x_{j+1} = \text{rnd}(\alpha_k x_j)$, admissibility is measured by the finite k -step discrepancy

$$D_k(n) = x_k - \sum_{j=0}^{k-1} x_j.$$

Using the exact root identity $\alpha_k + \alpha_k^{-k} = 2$, the rounded dynamics reduce to the integer correction map

$$R_k(n) = 2n - c_k(n), \quad c_k(n) = \left\lfloor \alpha_k^{-k} n + 1/2 \right\rfloor,$$

which yields the exact correction-sum formula

$$D_k(n) = n - \sum_{j=0}^{k-1} c_k(x_j).$$

Hence n is a genuine k -nacci parent number precisely when $D_k(n) = 0$. The same representation gives k -step recovery and discrepancy conservation,

$$\text{rnd}\left(\frac{R_k^k(n)}{\alpha_k^k}\right) = n, \quad D_k(R_k(n)) = D_k(n).$$

Consequently, barrenness is inherited along forward rounded orbits. Primitive barren integers are the barren members of the explicit no-predecessor sequence

$$q_k(m) = \left\lfloor \left\lfloor \alpha_k / (\alpha_k - 1) \right\rfloor m - 1 / \left\lfloor 2(\alpha_k - 1) \right\rfloor \right\rfloor, \quad m \geq 1,$$

and every barren integer belongs to a unique forward orbit rooted at one such

primitive generator. The formulas hold for arbitrary fixed $k \geq 3$; Tribonacci, Tetranacci, Pentanacci, and higher-order examples illustrate the general theory.

Keywords

Quasigeometric Sequence, k-Nacci Sequence, Multinacci Constant, Nearest-Integer Rounding, Parent Number, Barren Integer, Beatty-Type Sequence, Discrepancy, Symbolic Dynamics

1. Introduction

For a fixed integer $k \geq 2$, the k -generalized Fibonacci (k -nacci) recurrence is

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad n \geq 2, \quad (1)$$

with the conventional initial data $F_1^{(k)} = 1$ and $F_0^{(k)} = F_{-1}^{(k)} = \cdots = F_{2-k}^{(k)} = 0$. Thus $k = 2$ gives Fibonacci, $k = 3$ Tribonacci, $k = 4$ Tetranacci, and so forth. This terminology and recurrence are standard in the k -generalized Fibonacci literature [1]-[4].

The corresponding characteristic polynomial is $x^k - x^{k-1} - \cdots - x - 1$. It has a unique real root α_k in $(1, 2)$, the dominant k -nacci (multinacci) constant. Quasigeometric sequence (QGS) theory imposes an additional arithmetic condition: consecutive integer terms must also be related by nearest-integer multiplication by α_k . In the golden-section case, this dual additive-quasigeometric framework was developed in earlier QGS work [5]-[7].

Recent work shows that higher-order Fibonacci recurrences remain an active subject across algebraic, combinatorial, matrix, convolution, and Diophantine directions. Özdemir and Karakaya developed matrix identities for k -generalized Fibonacci numbers [8], while Bérczes, Pink, and Young studied Cullen and Woodall numbers in k -generalized Fibonacci sequences [9]. Frontczak and Gryszka obtained general convolution sums whose specializations include Tribonacci, Tetranacci, and Pentanacci numbers [10]. Billore, Patel, and Makwana developed extended generalized Fibonacci and Tribonacci polynomials with Binet-type, generating-function, and classical identity formulas [11]. Recent Tribonacci-specific work includes the Fibonacci-product problem of Luca, Odjoumani, and Togbé [12], the generalized Wythoff/Trithoff construction of Chen *et al.* [13], and Gryszka's self-convolution formula [14].

The higher-order literature has also continued to develop at the level of explicit recurrence structure. Gryszka treated sums of squares for Tribonacci and Tetranacci sequences with arbitrary initial conditions [15]; Bates *et al.* gave a new combinatorial interpretation of partial sums of m -step Fibonacci numbers [16]; Mane derived new expressions for k -generalized Fibonacci numbers using Fuss-Catalan numbers [17]; and Karakaya and Özdemir obtained further matrix identities for generalized Fibonacci and Tribonacci numbers [18]. Demirkol Ozkaya studied

Tribonacci numbers represented as products of three Fibonacci numbers [19], while De la Sen analyzed boundedness, convergence, oscillation, and stability in generalized Fibonacci-type recurrences [20]. These recent developments complement the present QGS problem: rather than deriving further identities inside the additive recurrence alone, the present paper asks when the additive k -step law is compatible with nearest-integer multiplication by the dominant multinacci root.

The higher-order extension creates a compatibility problem. An orbit generated by nearest-integer multiplication need not satisfy the k -step additive recurrence. We call a positive integer admissible when the two rules agree over the first k steps and barren when they do not. The purpose of this paper is to develop an exact arithmetic description of this obstruction, distinguish primitive from inherited barrenness, and describe the resulting parent-number set.

2. Definitions and the k -Nacci QGS

Fix $k \geq 3$. Let α_k be the unique real root in $(1, 2)$ of

$$\alpha_k^k = \alpha_k^{k-1} + \alpha_k^{k-2} + \cdots + \alpha_k + 1, \quad (2)$$

equivalently,

$$\alpha_k^k - \alpha_k^{k-1} - \alpha_k^{k-2} - \cdots - \alpha_k - 1 = 0. \quad (3)$$

Definition 2.1 (strict k -nacci QGS).

An integer sequence $H = (h_0, h_1, \dots)$ is a strict k -nacci quasigeometric sequence when, at every applicable index n , it satisfies both

$$h_{n+k} = h_{n+k-1} + h_{n+k-2} + \cdots + h_{n+1} + h_n, \quad (4)$$

and

$$h_{n+1} = \text{rnd}(\alpha_k h_n), \quad (5)$$

where rnd denotes nearest-integer rounding.

Lemma 2.2 (absence of rounding ties).

For every positive integer n , $\alpha_k n$ is not a half-integer.

Proof

The polynomial in (3) is monic with integer coefficients. Its real root α_k is irrational for $k \geq 2$. If $\alpha_k n = m + 1/2$ for integers $m, n > 0$, then $\alpha_k = (2m+1)/(2n)$ would be rational, a contradiction. Hence nearest-integer rounding is unambiguous. $\square$

Define the rounded map

$$R_k(n) = \text{rnd}(\alpha_k n) = \lfloor \alpha_k n + 1/2 \rfloor. \quad (6)$$

For $n \in \mathbb{N}$ set $x_0 = n$ and $x_{j+1} = R_k(x_j)$. The local k -step additive discrepancy is given by Equation (7):

$$D_k(n) = x_k - \sum_{j=0}^{k-1} x_j \quad (7)$$

Definition 2.3 (admissible and barren).

A positive integer n is admissible when $D_k(n) = 0$ and barren when

$D_k(n) \neq 0$. In parent-number terminology, the admissible integers form the genuine k -nacci parent-number set.

3. The Exact Correction Map

Lemma 3.1 (root identity).

For every $k \geq 2$,

$$\alpha_k + \alpha_k^{-k} = 2. \quad (8)$$

Proof

Multiply (2) by $\alpha_k - 1$. The right-hand side telescopes, giving $\alpha_k^{k+1} - 2\alpha_k^k + 1 = 0$. Division by α_k^k yields $\alpha_k - 2 + \alpha_k^{-k} = 0$, which is (8). $\square$

Set $\beta_k = \alpha_k^{-k} = 2 - \alpha_k$. Then

$$R_k(n) = \text{rnd}(2n - \beta_k n). \quad (9)$$

Because $2n$ is an integer and no half-integer tie occurs, $\text{rnd}(M - y) = M - \lfloor y + 1/2 \rfloor$ for integer M . Hence

$$R_k(n) = 2n - \lfloor \beta_k n + 1/2 \rfloor. \quad (10)$$

Define the integer-valued correction

$$c_k(n) = \lfloor \beta_k n + 1/2 \rfloor \in \mathbb{Z}. \quad (11)$$

Therefore the rounded map has the exact form

$$R_k(n) = 2n - c_k(n). \quad (12)$$

This establishes explicitly both the integer-valued nature of c_k and the exact passage from nearest-integer rounding to the correction representation.

4. Finite Arithmetic Admissibility Criterion

Let $x_0 = n$ and iterate

$$x_{j+1} = 2x_j - c_k(x_j), \quad j \geq 0. \quad (13)$$

Lemma 4.1 (iteration formula).

For every $j \geq 1$, we state Equation (14):

$$x_j = 2^j n - \sum_{i=0}^{j-1} 2^{j-1-i} c_k(x_i) \quad (14)$$

Proof

The case $j = 1$ is (13). If (14) holds for j , substitute it into $x_{j+1} = 2x_j - c_k(x_j)$; the powers and summation indices shift exactly to the stated formula for $j + 1$. $\square$

Theorem 4.2 (correction-sum discrepancy formula).

For every positive integer n , we state Equation (15):

$$D_k(n) = n - \sum_{j=0}^{k-1} c_k(x_j) \quad (15)$$

Proof

Substitute (14) into

$$D_k(n) = x_k - \sum_{j=0}^{k-1} x_j$$

The coefficient of n is

$$2^k - \sum_{j=0}^{k-1} 2^j = 2^k - (2^k - 1) = 1$$

Fix $i \in \{0, \dots, k-1\}$. The correction $c_k(x_i)$ occurs in x_k with coefficient -2^{k-1-i} and in

$$-\sum_{j=0}^{k-1} x_j$$

with total coefficient

$$+ \sum_{j=i+1}^{k-1} 2^{j-1-i} = 2^{k-1-i} - 1$$

Their sum is -1 . Summing over i gives (15). □

Corollary 4.3 (finite admissibility test).

We state Equation (16):

$$n = \sum_{j=0}^{k-1} \left\lfloor \beta_k x_j + \frac{1}{2} \right\rfloor \quad (16)$$

A positive integer n is admissible if and only if Equation (16) holds, where $x_0 = n$ and $x_{j+1} = 2x_j - \left\lfloor \beta_k x_j + 1/2 \right\rfloor$. Thus, admissibility is decided by exactly k rounded iterations.

5. Recovery and Discrepancy Conservation

Lemma 5.1 (k -step recovery).

For every $n \in \mathbb{N}$, including $n = 1$,

$$\text{rnd}\left(R_k^k(n)/\alpha_k^k\right) = n. \quad (17)$$

Proof

Write $x_{j+1} = \alpha_k x_j + \varepsilon_j$, where $|\varepsilon_j| < 1/2$ by Lemma 2.2. Iteration gives Equation (18):

$$\frac{x_k}{\alpha_k^k} = n + \sum_{j=0}^{k-1} \frac{\varepsilon_j}{\alpha_k^{j+1}} \quad (18)$$

Dividing (2) by α_k^k gives

$$\sum_{r=1}^k \alpha_k^{-r} = 1$$

Therefore we have Equation (19):

$$\left| \sum_{j=0}^{k-1} \frac{\varepsilon_j}{\alpha_k^{j+1}} \right| < \frac{1}{2} \sum_{r=1}^k \alpha_k^{-r} = \frac{1}{2} \quad (19)$$

The inequality is strict because every $|\varepsilon_j| < 1/2$. Hence x_k/α_k^k lies strictly within one half-unit of n , so nearest-integer rounding gives n . □

Since $R_k^k(n) = x_k$, Equation (17) may also be written $\text{rnd}(x_k/\alpha_k^k) = n$.

Theorem 5.2 (discrepancy conservation).

For every $n \in \mathbb{N}$,

$$D_k(R_k(n)) = D_k(n). \quad (20)$$

Proof

Let $x_j = R_k^j(n)$. Then $D_k(R_k(n)) - D_k(n) = x_{k+1} - 2x_k + x_0$. By (12), $x_{k+1} - 2x_k = -c_k(x_k)$. From the recovery lemma and (10) applied backward to the recovered integer, $c_k(x_k) = x_0$. Hence the difference is zero. Equivalently, the same identity follows directly by applying the correction-sum formula to the shifted orbit. $\square$

Corollary 5.3 (orbit invariance).

For every $j \geq 0$, $D_k(R_k^j(n)) = D_k(n)$. Therefore admissibility, barrenness, discrepancy sign, and discrepancy magnitude are constant along a forward rounded orbit.

6. Barren Inheritance and Discrepancy Classes

Theorem 6.1 (barren inheritance).

If n is barren, then $R_k^j(n)$ is barren for every $j \geq 0$.

Proof

If $D_k(n) \neq 0$, Corollary 5.3 gives $D_k(R_k^j(n)) = D_k(n) \neq 0$. $\square$

For $d \in \mathbb{Z}$ define Equation (21):

$$\mathcal{B}_{k,d} = \{n \in \mathbb{N} : D_k(n) = d\} \quad (21)$$

Then $R_k(\mathcal{B}_{k,d}) \subseteq \mathcal{B}_{k,d}$. The parent-number set is $\mathcal{B}_{k,0}$; every class with $d \neq 0$ is barren. Thus forward-orbit inclusion follows directly from discrepancy conservation, not from a numerical test.

7. Primitive Barren Generators and Predecessors

Lemma 7.1 (uniqueness of a positive predecessor).

For any $y \in \mathbb{N}$ there is at most one $x \in \mathbb{N}$ satisfying $R_k(x) = y$.

Proof

$R_k(x) = y$ is equivalent to $y - 1/2 < \alpha_k x < y + 1/2$. Thus x lies in an open interval of length $1/\alpha_k < 1$. Any interval of length strictly less than one contains at most one integer. $\square$

Lemma 7.2 (strict growth).

For every $n \in \mathbb{N}$ and $k \geq 3$, $R_k(n) > n$.

Proof

Since $\alpha_k > 3/2$ for $k \geq 3$, $(\alpha_k - 1)n > 1/2$ for $n \geq 1$. Hence $\alpha_k n > n + 1/2$, and nearest-integer rounding gives $R_k(n) \geq n + 1$. $\square$

Definition 7.3.

A barren integer is inherited barren if it has a positive predecessor under R_k . A

primitive barren integer is a barren integer with no positive predecessor. The phrase “no predecessor” is therefore not a third barren category; it is the defining dynamical property that distinguishes primitive from inherited barrenness.

If a barren y has a predecessor x , discrepancy conservation gives $D_k(x) = D_k(y) \neq 0$; hence x is barren. Thus, for barren integers, “no barren predecessor” and “no positive predecessor” are equivalent without circularity.

8. Explicit Enumeration of the No-Predecessor Sequence

The image of R_k is the shifted Beatty-type sequence

$$\mathcal{A}_k = \{\lfloor \alpha_k n + 1/2 \rfloor : n \geq 1\}. \quad (22)$$

Let $\mathcal{N}_k = \mathbb{N} \setminus \mathcal{A}_k$. These are exactly the positive integers with no positive R_k -predecessor. The floor-function notation and standard integer-part identities used in the enumeration below follow the classical treatment in [21]. For $y \geq 1$, the number of image values not exceeding y is

$$A_k(y) = \lfloor (y + 1/2)/\alpha_k \rfloor, \quad (23)$$

because $\lfloor \alpha_k n + 1/2 \rfloor \leq y$ is equivalent to $n < (y + 1/2)/\alpha_k$, and the boundary cannot be integral by irrationality. Hence the complement counting function is

$$N_k(y) = y - \lfloor (y + 1/2)/\alpha_k \rfloor. \quad (24)$$

Theorem 8.1 (closed form for no-predecessor integers).

Let $q_k(m)$ be the m -th positive integer omitted from $\mathcal{A}_k$. Then

$$q_k(m) = \left\lfloor \left[\alpha_k / (\alpha_k - 1) \right] m - 1 / \left[2(\alpha_k - 1) \right] \right\rfloor, \quad m \geq 1. \quad (25)$$

Proof

Put $\gamma = \alpha_k / (\alpha_k - 1)$ and $\delta = -1 / [2(\alpha_k - 1)]$. For $q = \lfloor \gamma m + \delta \rfloor$, the inequalities $q \leq \gamma m + \delta < q + 1$ are equivalent, after multiplication by $\alpha_k - 1$, to

$$(\alpha_k - 1)q + 1/2 \leq \alpha_k m < (\alpha_k - 1)(q + 1) + 1/2. \quad (26)$$

Rearranging these inequalities yields $N_k(q) = m$ and $N_k(q - 1) = m - 1$. Hence q is the least positive integer y for which $N_k(y) = m$, i.e. the m -th omitted integer. Irrationality excludes equality at the floor boundaries. $\square$

Corollary 8.2.

Every primitive barren generator belongs to $\{q_k(m) : m \geq 1\}$.

The term “Beatty-type” is used deliberately: (22) and (25) are inhomogeneous irrational floor sequences, but no claim of a classical homogeneous Beatty complementary pair is required.

9. Primitive Barren Generator Theorem

For $m \geq 1$ set $y_{m,0} = q_k(m)$, and recursively

$$y_{m,j+1} = 2y_{m,j} - \lfloor \beta_k y_{m,j} + 1/2 \rfloor, \quad j = 0, \dots, k-1. \quad (27)$$

Define

$$I_k(m) = \begin{cases} 1 & \text{if } q_k(m) \neq \sum_{j=0}^{k-1} \lfloor \beta_k y_{m,j} + 1/2 \rfloor \\ 0 & \text{otherwise} \end{cases} \quad (28)$$

Theorem 9.1 (primitive barren generator classification).

The complete primitive barren generator set is

$$\mathcal{P}_k = \{q_k(m) : I_k(m) = 1, m \geq 1\}. \quad (29)$$

Proof

By Theorem 8.1, every $q_k(m)$ has no positive predecessor. By Corollary 4.3, $I_k(m) = 1$ is exactly equivalent to $D_k(q_k(m)) \neq 0$, i.e. barrenness. Thus (29) selects precisely the barren integers with no predecessor. $\square$

The index set $\{m : I_k(m) = 1\}$ is determined by a finite correction itinerary. This motivates, but does not prove, possible connections with mechanical words or other symbolic-dynamical codings.

10. Exhaustion of the Barren Set

Theorem 10.1 (Barren Orbit Exhaustion Theorem).

Every barren integer belongs to a unique forward orbit generated by a primitive barren integer given by Equation (30):

$$\mathcal{B}_k = \coprod_{p \in \mathcal{P}_k} \{R_k^j(p) : j \geq 0\} \quad (30)$$

Proof

Let $n \in \mathcal{B}_k$. If n has no predecessor, then $n \in \mathcal{P}_k$. Otherwise its unique predecessor n_{-1} exists and is barren by discrepancy conservation. By Lemma 7.2, $n = R_k(n_{-1}) > n_{-1}$, so every backward step strictly decreases the positive integer. The process therefore terminates at a barren integer p with no predecessor, and $n = R_k^j(p)$ for some $j \geq 0$. If two primitive forward orbits intersected, uniqueness of predecessors (Lemma 7.1) would force their backward chains to coincide all the way to the same primitive root. Hence the union is disjoint. $\square$

Corollary 10.2 (true parent-number set).

The genuine k -nacci parent-number set is

$$\mathcal{G}_k = \mathbb{N} \setminus \mathcal{B}_k. \quad (31)$$

11. Numerical Examples and Reproducible Computation

All numerical examples below were recomputed from the defining polynomial and the exact integer formulas (6), (7), and (16). High-precision arithmetic is used only to evaluate α_k safely away from rounding boundaries; the classification itself is integer-valued. The terminology Tribonacci, Tetranacci, and Pentanacci follows the standard m -step hierarchy used in recent recurrence literature [10] [13] [15]–[18].

k	System	α_k (approx.)	First barren (D_k)
3	Tribonacci	1.8392867552142	8 (+1)
4	Tetranacci	1.9275619754829	11 (-1)
5	Pentanacci	1.9659482366455	23 (-1)
6	Hexanacci	1.9835828434243	39 (-1)
7	Heptanacci	1.9919641966050	47 (-1)

For $k = 3$, the no-predecessor sequence begins

$$1, 3, 5, 8, 10, 12, 14, 16, 19, 21, 23, 25, 27, 30, 32, \dots \quad (32)$$

The first primitive barren generators include 8 and 16, with $D_3(8) = +1$ and $D_3(16) = -1$. The forward orbit of 8 is

$$8 \rightarrow 15 \rightarrow 28 \rightarrow 52 \rightarrow 96 \rightarrow \dots \quad (33)$$

Direct verification uses $R_3(n) = \lfloor \alpha_3 n + 1/2 \rfloor$: $R_3(8) = 15$, $R_3(15) = 28$, $R_3(28) = 52$, and $R_3(52) = 96$. Thus 15 is barren but not primitive. For $k = 4$ the first primitive barren values are 11, 16, 24, 36, 45, ...; for $k=5$ they begin 23, 27, 32, 34, 36, ...

12. Near-Doubling Structure at High Order

Identity (8) gives

$$\alpha_k = 2 - \alpha_k^{-k}. \quad (34)$$

Hence

$$R_k(n) = \text{rnd}(2n - n\alpha_k^{-k}). \quad (35)$$

The correction $c_k(n)$ vanishes exactly when $n\alpha_k^{-k} < 1/2$, equivalently

$$n < \alpha_k^k / 2. \quad (36)$$

The first nonzero correction therefore occurs at the least integer n satisfying $n > \alpha_k^k / 2$; equality cannot occur because α_k is irrational. This threshold concerns the correction map only and does not by itself determine the first barren integer, because admissibility depends on the complete k -step correction sum.

13. Beatty-Type and Symbolic-Dynamical Interpretation

The no-predecessor candidates form the explicit inhomogeneous floor sequence (25), while actual primitive barrenness is selected by the finite discrepancy criterion (28). This places the construction near classical work on complementary floor sequences and codings of irrational maps [22]–[26]. The present results require only the explicit counting and complement formulas proved above. They do not require the stronger assertion that the primitive-generator indices form a classical Beatty sequence or a Sturmian word.

Possible substitution, morphic, or mechanical-word descriptions of $\{m: I_k(m) = 1\}$ are therefore proposed directions for further study. They are logically down-

stream of, and are not used in, the classification, recovery, conservation, or exhaustion theorems.

14. Discussion

Three notions are now sharply separated. A no-predecessor integer is a gap in the image of R_k . A primitive barren integer is a no-predecessor integer with nonzero k -step discrepancy. An inherited barren integer is a forward image of a primitive barren generator. Discrepancy conservation explains why barrenness persists along the forward orbit.

The correction representation also yields a practical finite classification procedure. Parent-number admissibility is determined intrinsically from k rounded iterations, while Theorem 8.1 restricts the search for primitive generators to an explicitly enumerated no-predecessor sequence. The symbolic-dynamical remarks concern possible additional organization of the already-defined indicator sequence and do not enter any proof of the preceding theorems.

15. Conclusion

For every fixed $k \geq 3$, the dominant k -nacci root converts nearest-integer multiplication into an exact almost-doubling map with an integer correction. This identity yields a finite discrepancy formula, k -step recovery, discrepancy conservation, inheritance of barrenness, explicit enumeration of all integers without predecessors, classification of primitive barren generators, and a decomposition of the barren set into disjoint forward orbits. These are analytic statements for arbitrary fixed $k \geq 3$. The numerical lists for $k = 3, \dots, 7$ are illustrations. Possible Beatty-complementarity refinements and symbolic-dynamical descriptions are not assumed by the proofs and remain separate structural questions.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix: Reproducible Classification Procedure

1) Fix $k \geq 3$ and compute α_k as the dominant real root of $x^k - x^{k-1} - \dots - x - 1 = 0$ to sufficient precision.

2) Set $\beta_k = 2 - \alpha_k = \alpha_k^{-k}$.

3) Enumerate no-predecessor candidates $q_k(m)$ from (25).

4) For each candidate $n = q_k(m)$, set $x_0 = n$ and iterate

$$x_{j+1} = 2x_j - \left\lfloor \beta_k x_j + 1/2 \right\rfloor \quad \text{for } j = 0, \dots, k-1.$$

5) Compute

$$D_k(n) = n - \sum_{j=0}^{k-1} \left\lfloor \beta_k x_j + \frac{1}{2} \right\rfloor$$

6) Record n as primitive barren if $D_k(n) \neq 0$. Its entire inherited branch is $\{R_k^j(n) : j \geq 0\}$.

7) Numerical precision must be increased whenever $\beta_k x_j + 1/2$ lies close to an integer. Lemma 2.2 guarantees that exact ties do not occur.