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Foundations of Tribonacci Quasigeometric Sequences

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Abstract

A Tribonacci quasigeometric sequence (TQGS) is an integer sequence constrained simultaneously by the exact third-order recurrence and nearest-integer multiplication by the Tribonacci constant τ. The framework introduced here is defined by

$$h_{n+3} = h_{n+2} + h_{n+1} + h_n, \quad h_{n+1} = \mathrm{rnd}(\tau h_n), \quad \tau^3 = \tau^2 + \tau + 1.$$

Writing $\delta_n = h_{n+1} - \tau h_n$ gives the exact discrepancy propagation law

$$\delta_{n+2} = (1 + \tau - \tau^2)\delta_n + (1 - \tau)\delta_{n+1}, \quad |\delta_n| < \frac{1}{2}.$$

These relations establish the basic TQGS compatibility dynamics, an explicit admissibility and non-degeneracy criterion for genuine Tribonacci parent numbers, and the complementary notion of global barrenness. For rounded orbits $b_{n+1} = \mathrm{rnd}(\tau b_n)$, the integer Tribonacci defect

$$E_n = b_n + b_{n+1} + b_{n+2} - b_{n+3} \in \{-1, 0, +1\}$$

separates exact local TQGS compatibility ($E_n = 0$) from the two incompatible defect classes. A nonzero defect is invariant along a barren rounded orbit, so primitive barren roots generate infinite signed barren lineages. Thus the principal contribution is the introduction and development of the TQGS framework itself—its defining compatibility laws, discrepancy dynamics, parent-number criterion, and excluded-state dynamics—with barren integers forming one structural component of that broader theory.

Keywords

Tribonacci Constant, Quasigeometric Sequence, Parent Number, Discrepancy, Barrenness, Primitive Barren Integer, Nearest-Integer Rounding, Invariant Defect, Barren Lineage

1. Introduction

The quasigeometric framework used in this paper extends the additive-multiplicative construction introduced by Mamombe [1], in which an integer sequence is constrained simultaneously by an exact recurrence and a rounded multiplicative law generated by an irrational proportion. The present paper introduces the corresponding third-order construction: the golden-section/Fibonacci pair is replaced by the Tribonacci constant and the Tribonacci recurrence.

Classical work on Tribonacci and third-order recurrences includes Feinberg [2], Shannon and Horadam [3], and Spickerman [4]. More recent developments include generalized Tribonacci sums and polynomials [5] [6], matrix formulations [7], bihyperbolic and generalized third-order extensions [8] [9], probabilistic Binet-type formulations [10], generalized Fibonacci-Tribonacci polynomials [11], weighted Fibonacci and Tribonacci sums [12], matrix identities [13], and multiplicative Diophantine questions involving Tribonacci numbers [14]. The nearest-integer aspect also belongs to the broader tradition of Beatty-type and irrational-distribution problems [15] [16].

The central issue is compatibility. Repeated nearest-integer multiplication by τ always produces an integer orbit, but such an orbit need not satisfy the exact Tribonacci recurrence. Conversely, an exact Tribonacci recurrence need not remain inside the nearest-integer rounding window. The discrepancy formalism measures this compatibility exactly.

A second issue is arithmetic classification. A genuine parent number must be both non-degenerate under the inverse-forward rounded transformation and locally admissible under the Tribonacci recurrence. Integers failing the local compatibility condition are not exceptional parents; they are globally excluded from every strict TQGS. These are called barren integers. The examples 8 and 16 are the first important exclusions.

The paper therefore has two linked contributions. Firstly, it formulates the TQGS framework itself: its defining laws, discrepancy dynamics, admissibility thresholds, and genuine parent-number criterion. Secondly, within that framework, it develops the internal dynamics of the excluded barren set. Primitive barren integers are shown to generate infinite barren lineages under repeated nearest-integer multiplication by τ, governed by a conserved integer defect of sign ±1.

2. Tribonacci Constant and Rounding Convention

Let τ denote the unique real root greater than one of

$$\tau^3 = \tau^2 + \tau + 1 \tag{1}$$

so that

$$\tau = 1.839286755214161\cdots \tag{2}$$

Throughout, nearest-integer rounding is fixed by

$$\mathrm{rnd}(x) = \lfloor x + 1/2 \rfloor \tag{3}$$

Thus rnd(x) = m exactly when

$$m-1/2 \le x < m+1/2 \tag{4}$$

The cubic in (1) has no rational root, so τ is irrational. For any nonzero integer m, τm cannot be a half-integer; at zero the rounding error is zero. Thus, for integer sequences, the half-open interval in (4) is equivalent to the strict absolute-error window used below.

3. Tribonacci Quasigeometric Sequences

Definition 3.1 (Strict TQGS).

An integer sequence $H = (h_n)$ is a strict Tribonacci quasigeometric sequence if it satisfies simultaneously

$$h_{n+3} = h_{n+2} + h_{n+1} + h_n \tag{5}$$

and

$$h_{n+1} = \mathrm{rnd}(\tau h_n) \tag{6}$$

Here the sequence is infinite and indexed by $n \ge 1$, and both defining laws hold for every such index. The first relation is exact and additive; the second is multiplicative before rounding.

Example.

Starting from $h_1 = 1$, rounded multiplication gives $h_2 = 2$ and $h_3 = 4$, and the compatible strict TQGS begins

$$1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, \cdots \tag{7}$$

4. Tribonacci Discrepancy and Threshold Dynamics

Definition 4.1 (Discrepancy).

For consecutive terms define

$$\delta_n = h_{n+1} - \tau h_n \tag{8}$$

The rounded law (6) is equivalent to the fundamental window

$$|\delta_n| < \frac{1}{2} \tag{9}$$

Theorem 4.2 (Exact discrepancy propagation).

Along every exact Tribonacci recurrence,

$$\delta_{n+2} = \left(1 + \tau - \tau^2\right)\delta_n + (1 - \tau)\delta_{n+1} \tag{10}$$

Proof

Write $h_{n+1} = \tau h_n + \delta_n$ and $h_{n+2} = \tau h_{n+1} + \delta_{n+1}$. Substitute these expressions into $\delta_{n+2} = h_{n+3} - \tau h_{n+2}$ and use $h_{n+3} = h_{n+2} + h_{n+1} + h_n$. The coefficient of h_n vanishes by $\tau^3 = \tau^2 + \tau + 1$, leaving (10). □

Numerically,

$$\delta_{n+2} = -0.5436890127\cdots\delta_n - 0.8392867552\cdots\delta_{n+1} \tag{11}$$

The characteristic roots are complex conjugates with common modulus

$\tau^{-\frac{1}{2}} < 1$. Every discrepancy solution therefore decays asymptotically with oscillation. This is spectral stability; it does not imply that the Euclidean norm decreases at every step. The half-unit boundaries in (9) are the exact rounding thresholds.

5. Initial Admissibility and Genuine Parent Numbers

For a positive integer p, define its first rounded successors by

$$h_1 = p,\ \ h_2 = \mathrm{rnd}(\tau p),\ \ h_3 = \mathrm{rnd}(\tau h_2) \tag{12}$$

The exact recurrence requires

$$h_4 = p + h_2 + h_3 \tag{13}$$

whereas rounded multiplication requires $h_4 = \mathrm{rnd}(\tau h_3)$. Define the initial compatibility discrepancy

$$\Delta(p) = p + \mathrm{rnd}(\tau p) + \mathrm{rnd}\big(\tau\,\mathrm{rnd}(\tau p)\big) - \tau\,\mathrm{rnd}\big(\tau\,\mathrm{rnd}(\tau p)\big) \tag{14}$$

Initial admissibility is

$$|\Delta(p)| < \frac{1}{2} \tag{15}$$

The inverse-forward return map is

$$R(p) = \mathrm{rnd}\big(\tau\,\mathrm{rnd}(p/\tau)\big) \tag{16}$$

Definition 5.1 (Genuine Tribonacci parent number).

A positive integer p is a parent number precisely when

$$R(p) \neq p \quad \textit{and} \quad |\Delta(p)| < \frac{1}{2} \tag{17}$$

This definition incorporates non-degeneracy and strict TQGS admissibility at the same level. In particular, an integer that fails (15) is excluded before parent status is assigned.

The parent-number line begins

$$\begin{aligned} &1,3,5,10,12,14,19,21,23,25,27,30,32,34,36,38,41,43,45,47,49,\\ &51,54,56,58,60,62,65,67,69,71,\cdots \end{aligned} \tag{18}$$

6. Barren Integers

Definition 6.1 (Tribonacci-barren integer).

A positive integer b is barren if it cannot occur at any position of any valid strict TQGS.

$$b \in \mathcal{B}_T \Leftrightarrow h_n \neq b \quad \textit{for every strict TQGS } H \textit{ and every } n \geq 1 \tag{19}$$

Theorem 6.2 (Local failure implies global barrenness).

If

$$|\Delta(b)| > \frac{1}{2} \tag{20}$$

then b is barren.

Proof

Suppose b occurred as h_k in a strict TQGS. Both defining laws hold at every index, so the rounded orbit beginning at b must satisfy its first Tribonacci compatibility condition. Its initial discrepancy therefore has absolute value less than 1/2, *contradicting* (20). *Thus b cannot occur in any strict TQGS.* □

The First Exclusions: 8 and 16

For $b = 8$, rounded iteration gives

$$8 \to 15 \to 28 \to 52 \tag{21}$$

The Tribonacci recurrence would instead require

$$8 + 15 + 28 = 51 \tag{22}$$

Hence the rounded and additive laws disagree, and 8 is barren.

For $b = 16$,

$$16 \to 29 \to 53 \to 97 \tag{23}$$

whereas

$$16 + 29 + 53 = 98 \tag{24}$$

Thus 16 is also barren, but on the opposite side of the compatibility defect.

7. Primitive Barren Integers and Barren Lineages

Define the rounded Tribonacci transformation

$$T\tau(m) = \mathrm{rnd}(\tau m) \tag{25}$$

Definition 7.1 (Primitive barren integer).

A barren integer b is primitive if there is no barren positive integer a satisfying $T\tau(a) = b$.

$$b \text{ primitive} \iff b \in B_T \quad \text{and there is no} \quad a \in B_T \quad \text{with} \quad T\tau(a) = b \tag{26}$$

A primitive barren integer is therefore the first member of its barren lineage. This definition distinguishes creation from inheritance. For example, 15 is barren but not primitive because $T\tau(8) = 15$ and 8 is barren.

$$8 \to 15 \to 28 \to 52 \to 96 \to 177 \to 326 \to 600 \to 1104 \to \cdots \tag{27}$$

Thus 8, not 15, is the primitive root of this lineage. Similarly,

$$16 \to 29 \to 53 \to 97 \to 178 \to 327 \to 601 \to 1105 \to \cdots \tag{28}$$

8. Integer Tribonacci Defect

Let b_0 be any positive integer and generate its rounded orbit by

$$b_{n+1} = \mathrm{rnd}(\tau b_n) \tag{29}$$

Define the rounding error

$$\varepsilon_n = b_{n+1} - \tau b_n, |\varepsilon_n| < \frac{1}{2} \tag{30}$$

and define the integer Tribonacci defect

$$E_n = b_n + b_{n+1} + b_{n+2} - b_{n+3} \tag{31}$$

Theorem 8.1 (Defect criterion).

Along a rounded orbit,

$$E_n = 0 \tag{32}$$

if and only if the four consecutive terms b_n, b_{n+1}, b_{n+2}, b_{n+3} satisfy the exact Tribonacci recurrence.

Proof

This is immediate from (31): $E_n = 0$ *is exactly* $b_{n+3} = b_n + b_{n+1} + b_{n+2}$. *Since the orbit already satisfies nearest-integer multiplication by construction,* $E_n = 0$ *is precisely local TQGS compatibility.* □

Lemma 8.2 (Error representation of the defect).

$$E_n = \left(1 + \tau - \tau^2\right)\varepsilon_n + \left(1 - \tau\right)\varepsilon_{n+1} - \varepsilon_{n+2} \tag{33}$$

Proof

Substitute $b_{n+1} = \tau b_n + \varepsilon_n$ *successively into* (31). *The coefficient of* b_n *vanishes by the defining cubic for* τ. □

Lemma 8.3 (Defect trichotomy).

For every rounded orbit,

$$E_n \in \{-1, 0, +1\} \tag{34}$$

Proof

Equation (33), *gives* $|E_n| < \dfrac{\tau^2 - 1}{2} < 1.2$. *Since* E_n *is an integer, only* -1, 0, *and* $+1$ *are possible.* □

Consequently a locally incompatible rounded state has exactly one of two nonzero defect signs:

$$E_n = -1 \quad \text{or} \quad E_n = +1 \tag{35}$$

9. Barren Propagation Theorem

The central result is that the nonzero defect cannot disappear once it occurs.

Lemma 9.1 (Nonzero defect invariance).

If $E_n = d$ *with* $d \in \{-1, +1\}$, *then*

$$E_{n+1} = d \tag{36}$$

Proof

Put $x = \varepsilon_n$ and $y = \varepsilon_{n+1}$. From (33),

$$\varepsilon_{n+2} = -d + \left(1 + \tau - \tau^2\right)x + \left(1 - \tau\right)y \tag{37}$$

Take the integer candidate $c = b_{n+1} + b_{n+2} + b_{n+3} - d$. *Using* $\tau^3 = \tau^2 + \tau + 1$, *its rounding error is*

$$\varepsilon^*_{n+3} = c - \tau b_{n+3} = \left(2 - \tau\right)\left[-d + \left(1 + \tau\right)x + y\right] \tag{38}$$

Put $A = 1 + \tau - \tau^2$ *and* $B = 1 - \tau$. *For fixed defect d, the closure of the feasible region is the polygon defined by* $|x| \le \dfrac{1}{2}$, $|y| \le \dfrac{1}{2}$ *and* $|-d + Ax + By| \le \dfrac{1}{2}$. *The*

candidate error (38) *is affine, so its extrema on this compact polygon occur at vertices. Evaluating those vertices gives* (39) *and* (41). *Both bounds lie strictly inside the nearest-integer window, so the same conclusion holds for the actual open feasible region.*

For $d=-1$, *exact vertex evaluation gives*

$$2-\tau^2/2 \le \varepsilon_{n+3}^* \le (2-\tau)(2+\tau/2) \tag{39}$$

that is, numerically,

$$0.308512116\cdots \le \varepsilon_{n+3}^* \le 0.469225361\cdots \tag{40}$$

Hence $-\frac{1}{2} < \varepsilon_{n+3}^* < \frac{1}{2}$, *so nearest-integer rounding selects the candidate and* $E_{n+1}=-1$.

For $d=+1$, *central symmetry gives*

$$-(2-\tau)(2+\tau/2) \le \varepsilon_{n+3}^* \le -(2-\tau^2/2) \tag{41}$$

or

$$-0.469225361\cdots \le \varepsilon_{n+3}^* \le -0.308512116\cdots \tag{42}$$

Again the candidate lies strictly inside the nearest-integer window, and $E_{n+1}=+1$. *Therefore* $E_{n+1}=E_n$ *whenever* $E_n \neq 0$. □

Theorem 9.2 (Zero-defect invariance and exact admissibility).

If $E_n=0$, *then* $E_{n+1}=0$. *Consequently, a positive integer b is barren if and only if its initial defect* E_0 *is nonzero.*

Proof

Set $d=0$ *in* (38). *The candidate error is* $c-\tau b_{n+3}=(2-\tau)((1+\tau)x+y)$. *Since* $|x|<\frac{1}{2}$ *and* $|y|<\frac{1}{2}$, *the triangle inequality gives*

$$|c-\tau b_{n+3}| \le \frac{(2-\tau)(\tau+2)}{2} = \frac{4-\tau^2}{2} < \frac{1}{2} \tag{42a}$$

Thus $c=b_{n+4}$ *and* $E_{n+1}=0$. *Induction shows that* $E_0=0$ *generates a strict TQGS beginning at b, so b is not barren. Conversely,* $E_0 \neq 0$ *gives* $|\Delta(b)|>\frac{1}{2}$ *and Theorem 6.2 implies barrenness.* □

Theorem 9.3 (Primitive barren propagation).

Let b_0 *be a primitive barren integer and define*

$$b_{n+1}=\mathrm{rnd}(\tau b_n), n\ge 0 \tag{43}$$

Then every b_n *is barren. Moreover,*

$$E_n=E_0\in\{-1,+1\} \quad \text{for all} \quad n\ge 0 \tag{44}$$

Proof

Theorem 9.2 *gives* $E_0\neq 0$. *Lemma* 8.3 *gives* $E_0=\pm 1$, *and Lemma* 9.1 *preserves this value at every step. Theorem* 6.2 *then makes every descendant barren.* □

Barren polarity

The invariant sign in (44) divides primitive barren lineages into two classes.

The lineage rooted at 8 has defect −1:

$$8+15+28-52=-1 \tag{45}$$

$$15+28+52-96=-1 \tag{46}$$

whereas the lineage rooted at 16 has defect +1:

$$16+29+53-97=+1 \tag{47}$$

$$29+53+97-178=+1 \tag{48}$$

Thus the sign

$$\sigma(b_0)=E_0\in\{-1,+1\} \tag{49}$$

is a conserved binary invariant of a barren lineage. We refer to it as the barren polarity.

10. Unit Perturbation and Primitive Barren Creation

Primitive barren roots arise naturally when valid TQGS terms are perturbed by one unit. Let $H=(h_n)$ be a valid parent-rooted TQGS and consider

$$q_n^+=h_n+1,\ \ q_n^-=h_n-1 \tag{50}$$

A perturbed integer is first tested by the initial discrepancy criterion. If it is barren, it must then be tested for a barren predecessor before it can be called primitive.

This second test is essential: a barren perturbation need not create a new lineage.

For example,

$$14+1=15 \tag{51}$$

but 15 is inherited because

$$T\tau(8)=15 \tag{52}$$

By contrast, the following are primitive barren creation events. Table 1 records the producing sequences, perturbations and parent-relative indices:

$$7+1=8,\ \ 17-1=16,\ \ 72+1=73,\ \ 75+1=76 \tag{53}$$

Table 1. Primitive barren roots obtained by unit perturbation.

Primitive root	Producer	Perturbation	Parent-relative index
8	1, 2, 4, 7, 13, …	+1 to 7	4
8	5, 9, 17, 31, …	−1 to 9	2
16	5, 9, 17, 31, …	−1 to 17	3
73	21, 39, 72, 132, …	+1 to 72	3
73	12, 22, 40, 74, …	−1 to 74	4
76	41, 75, 138, …	+1 to 75	2
76	23, 42, 77, 142, …	−1 to 77	3

Table 1 exhibits primitive-root creation at parent-relative indices 2, 3 and 4. A

barren hit at index 1 can instead be an inherited descendant, as 15 demonstrates. These examples do not establish a universal bound on creation indices or universal unit-perturbation stability from index 5. Those statements require a separate proof. The propagation theorem is independent of the perturbation mechanism.

The preceding results separate the strict TQGS state space from its globally excluded complement. Parent numbers seed admissible dynamics; primitive barren integers seed incompatible rounded dynamics. Both use the same nearest-integer transformation, but their local Tribonacci defects are fundamentally different.

$$\text{admissible parent} \to \text{strict TQGS compatibility} \to E = 0 \tag{54}$$

whereas

$$\text{primitive barren root} \to \text{rounded barren lineage} \to E \equiv +1 \text{ or } E \equiv -1 \tag{55}$$

The barren-lineage theorem therefore supplies more than an exclusion test. It gives an internal dynamic on the barren set. Once the orbit enters a nonzero defect class, the defect sign is invariant and the orbit cannot return to local Tribonacci compatibility.

For every positive integer m, $\tau m - m \geq \tau - 1 > \frac{1}{2}$, hence $T_\tau(m) \geq m+1$. Every positive rounded orbit is strictly increasing and unbounded. Consequently, each primitive barren root generates an infinite barren lineage. The lineages rooted at 8 and 16 exhibit the two invariant polarities.

11. Discussion

The distinction between a barren integer and a primitive barren integer is essential. Barrenness is a state-space exclusion property; primitivity is a genealogical property relative to the rounded transformation T_τ. Thus 15, 28, 52, and 96 are barren but are not primitive within the lineage rooted at 8.

The integer defect E_n provides the natural invariant for this genealogy. Its trichotomy $\{-1, 0, +1\}$ is forced jointly by the half-unit rounding window and the algebraic identity for τ. The value zero represents exact local Tribonacci compatibility. The values ±1 represent the only two possible local failures, and Lemma 9.1 shows that either failure is dynamically persistent under rounded multiplication.

Frequency of Certified Barren Integers and Genuine Parents

To quantify the classes, criteria (15) and (17) were evaluated on initial intervals of positive integers. By Theorem 9.2, $|\Delta(p)| > \frac{1}{2}$ characterizes the entire barren set.

Table 2 reports finite-range counts; these computations do not establish an asymptotic density theorem.

For each positive integer in the tested range, compute its first three rounded successors using (25), evaluate the initial defect (31), and test the inverse-forward map (16). A nonzero defect is counted as barren; a zero defect is counted as a genuine parent when the return differs from the initial integer, and otherwise as

admissible but degenerate. Theorem 9.2 makes these classes disjoint and exhaustive. The table counts and the parent list (18) were independently verified using rational bounds for the cubic root separated by 10^{-60}; both endpoints gave the same integer at every rounding decision.

Table 2. Finite-range counts of barren integers and genuine parents.

Range $1 \le p \le N$	Certified barren	Genuine parents	Admissible but degenerate
100	11 (11.0%)	42 (42.0%)	47 (47.0%)
1000	83 (8.3%)	419 (41.9%)	498 (49.8%)
10000	805 (8.05%)	4198 (41.98%)	4997 (49.97%)
100000	8042 (8.042%)	41961 (41.961%)	49997 (49.997%)

The finite-range frequencies stabilize rapidly near 8.0% for integers certified barren by local failure, near 42.0% for genuine parents, and near 50.0% for admissible but inverse-forward degenerate integers. These observations are computational rather than an asymptotic theorem, but they show that barrenness and parenthood are not sparse exceptional phenomena on the tested number line. A rigorous density theorem remains a natural continuation of the TQGS framework.

A treatment of the entire higher-order k-nacci family is found in [17].

12. Conclusions

This paper introduces the Tribonacci quasigeometric sequence framework, in which exact third-order addition and nearest-integer multiplication by the Tribonacci constant are imposed simultaneously. The discrepancy recurrence gives the exact compatibility dynamics, the half-unit thresholds define admissibility, and the inverse-forward condition distinguishes genuine parent numbers from degenerate admissible integers.

Within this framework, integers failing local Tribonacci compatibility are globally barren. Primitive barren integers generate infinite barren lineages under repeated nearest-integer multiplication by τ. The integer Tribonacci defect takes only the values -1, 0, or $+1$, and a nonzero defect is invariant under the rounded transformation. Consequently, every descendant of a primitive barren root is barren, and each lineage carries a conserved polarity ± 1.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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