

Matrix Boundary Functional for Recurrent Z-Class Transitions in Golden-Section Quasigeometric Sequences

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Abstract

Golden-section quasigeometric sequences possess a Z-class whose admissible same-parity pairs generate an ordered midway-index word. For the r^{th} occurrence of the recurrent transition subword (8, 11, 9), let (a_r, b_r) , (c_r, d_r) , and (e_r, f_r) denote the three consecutive admissible same-parity Z-class pairs corresponding respectively to the indices 8, 11, and 9. Here a_r , c_r and e_r are the lower endpoints, while b_r , d_r and f_r are the corresponding upper endpoints. Their midpoints are $\mu_r = (a_r + b_r)/2$, $\nu_r = (c_r + d_r)/2$, and

$\xi_r = (e_r + f_r)/2$. The transition matrix is $M_r = \begin{pmatrix} (a_r, b_r) & (c_r, d_r) & (e_r, f_r) \\ \mu_r & \nu_r & \xi_r \end{pmatrix}$.

Thus each column is one symbol of the ordered transition, with its generating Z-class pair above and its midpoint below. The inaugural and first recurrent

matrices are $M_1 = \begin{pmatrix} 103/117 & 137/151 & 171/185 \\ 110 & 144 & 178 \end{pmatrix}$,

$M_2 = \begin{pmatrix} 1090/1104 & 1124/1138 & 1158/1172 \\ 1097 & 1131 & 1165 \end{pmatrix}$. The construction therefore produces a family of boundary pairs from the recurrent matrix structure.

Keywords

Quasigeometric Sequences, Golden Section, Z-Class, Parent Numbers, Transition Matrix, Midway-Index Word, Boundary Functional, Fibonacci Recurrence

1. Introduction

Golden-section quasigeometric sequences (QGS) were introduced in the founda-

tional work of Mamombe (2016) [1] and developed further within the Proportiones Perfectus formulation of Mamombe (2017) [2]. In this framework, the Fibonacci additive recurrence is considered together with a nearest-integer golden-section constraint, producing an arithmetic class of quasigeometric sequences with associated parent-number and indexed structures. The present work uses this QGS framework as its starting point.

Within the golden-section QGS, the Z-class is generated by the indexed transformation developed by Mamombe (2017) [2]. Admissible same-parity Z-class pairs determine ordered midpoints whose positions in their originating QGS form a midway-index word. The recurrent local word (8, 11, 9) is the object studied here. Each occurrence is represented by three consecutive admissible Z-class pairs and their three midpoints, naturally producing a 2×3 matrix.

Definition 1 (Z class).

A golden-section quasigeometric sequence H_n is a member of the Z class if $10(h_5, h_6) = (p_1, p_2)$, where P_n is itself a golden-section quasigeometric sequence.

This is the defining membership condition introduced by Mamombe (2017). Within the Z class, when two successive sequences have the same arrangement of odd and even terms, they are symmetrical about a sequence L_n from the family generated by 7, 11, 18, 29, 47, ... The same-parity Z-class pairs used in the present work are admissible pairs selected from this previously defined class; they do not define the Z class itself.

Fibonacci and Lucas recurrence theory is developed classically by Vajda [3] and Koshy [4]. Relations among Beatty sequences, Fibonacci numbers and the golden ratio are discussed by Russo and Schwiebert [5]. Falcón and Plaza [6]-[8] developed k-Fibonacci numbers, their Pascal 2-triangle and associated polynomial theory; Mamombe [9] [10] extended this direction to k-Fibonacci/k-Lucas systems and generalized a:k:m-Fibonacci numbers. Recent work continues to study generalized Fibonacci and Lucas recurrences, arithmetic properties, return words, period structures, representation problems and intersections of recurrence sequences [11]-[26]. The QGS framework used here originates in Mamombe (2016) [1] and Mamombe (2017) [2].

2. Golden-Section QGS and Parent Numbers

Given

$$\varphi = \frac{1 + \sqrt{5}}{2},$$

a golden-section QGS $H = (h_1, h_2, \dots)$ satisfies

$$h_{n+2} = h_{n+1} + h_n$$

together with

$$h_{n+1} = \text{rnd}(\varphi h_n).$$

The Fibonacci QGS indexing convention used here is

$$f_1 = 1, \quad f_2 = 2, \quad f_{n+2} = f_{n+1} + f_n.$$

Let $\mathcal{P}$ denote the set of positive parent numbers, equivalently the positive integers omitted by the rounded golden map $n \mapsto \text{rnd}(\varphi n)$. A parent-number obstruction centre C satisfies

$$C-1 \in \mathcal{P}, \quad C \notin \mathcal{P}, \quad C+1 \in \mathcal{P}.$$

Its associated parent-sandwich boundary is

$$\mathcal{B}(C) = \frac{C-1}{C+1}.$$

3. Z-Class Pairs and the Midway-Index Word

Ordering admissible Z-class pairs by increasing lower endpoint and taking the midpoint of the parent numbers then computing the index of the midway in a valid QGS produces the midway-index word

$$W = 8, 9, 10, 8, 11, 9, 8, 12, 8, 10, 9, 8, 13, \dots$$

The first appearances of successive new midway indices occur at positions 1, 2, 3, 5, 8, 13, 21, 34, ... Moreover, the numbers of internal occurrences of the symbol 8 between the first admissions $10 \rightarrow 11$, $11 \rightarrow 12$, $12 \rightarrow 13$, $13 \rightarrow 14$ and $14 \rightarrow 15$ are 1, 1, 2, 3 and 5. The recurrent word (8, 11, 9) is therefore embedded in a Fibonacci-organized symbolic structure.

3.1. Computational Table of the First 50 Admissible Pairs

Table 1 records the first 50 ordered admissible Z-class pairs, their midpoints, and the corresponding midway indices. It is the direct computational basis for the recurrent (8, 11, 9) matrices used below.

Table 1. First 50 ordered admissible Z-class pairs, midpoints and midway indices.

No.	Admissible pair	Midpoint	Index in QGS
1	27/41	34	8
2	48/62	55	9
3	82/96	89	10
4	103/117	110	8
5	137/151	144	11
6	171/185	178	9
7	192/206	199	8
8	226/240	233	12
9	260/274	267	8
10	281/295	288	10
11	315/329	322	9
12	336/350	343	8
13	370/384	377	13

Continued

14	404/418	411	8
15	425/439	432	9
16	459/473	466	11
17	493/507	500	8
18	514/528	521	10
19	548/562	555	9
20	569/583	576	8
21	603/617	610	14
22	637/651	644	8
23	658/672	665	9
24	692/706	699	10
25	713/727	720	8
26	747/761	754	12
27	781/795	788	8
28	802/816	809	9
29	836/850	843	11
30	870/884	877	8
31	891/905	898	10
32	925/939	932	9
33	946/960	953	8
34	980/994	987	15
35	1014/1028	1021	8
36	1035/1049	1042	9
37	1069/1083	1076	10
38	1090/1104	1097	8
39	1124/1138	1131	11
40	1158/1172	1165	9
41	1179/1193	1186	8
42	1213/1227	1220	13
43	1247/1261	1254	8
44	1268/1282	1275	9
45	1302/1316	1309	10
46	1323/1337	1330	8
47	1357/1371	1364	12
48	1391/1405	1398	8
49	1412/1426	1419	9
50	1446/1460	1453	11

Rows 4 - 6 give the inaugural word (8, 11, 9): $103/117 \rightarrow 110 \rightarrow 8$, $137/151 \rightarrow 144 \rightarrow 11$, and $171/185 \rightarrow 178 \rightarrow 9$. Rows 38 - 40 give the first recurrent occurrence used here: $1090/1104 \rightarrow 1097 \rightarrow 8$, $1124/1138 \rightarrow 1131 \rightarrow 11$, and $1158/1172 \rightarrow 1165 \rightarrow 9$.

3.2. Fibonacci Admission and Return Counts

The first admissions of indices 8, 9, 10, 11, 12, 13, 14 and 15 occur at positions 1, 2, 3, 5, 8, 13, 21 and 34. Counting only internal occurrences of 8 between successive new admissions gives the following table. These Fibonacci return counts are summarized in **Table 2**.

Table 2. Fibonacci return counts of the symbol 8 between successive first admissions.

Admission interval	Positions of internal 8 s	Count
$10 \rightarrow 11$	4	1
$11 \rightarrow 12$	7	1
$12 \rightarrow 13$	9, 12	2
$13 \rightarrow 14$	14, 17, 20	3
$14 \rightarrow 15$	22, 25, 27, 30, 33	5

4. Transition Matrix

For the r^{th} occurrence of the ordered word (8, 11, 9), let q_r be the table position of its first symbol. Write the three consecutive admissible pairs as (a_r, b_r) , (c_r, d_r) , and (e_r, f_r) , with associated midpoints $\mu_r = (a_r + b_r)/2$, $\nu_r = (c_r + d_r)/2$, and $\xi_r = (e_r + f_r)/2$. The columns are fixed by the symbolic order 8, 11, 9.

$$M_r = \begin{pmatrix} (a_r, b_r) & (c_r, d_r) & (e_r, f_r) \\ \mu_r & \nu_r & \xi_r \end{pmatrix}.$$

$$\mu_r = \frac{a_r + b_r}{2}, \nu_r = \frac{c_r + d_r}{2}, \xi_r = \frac{e_r + f_r}{2}.$$

Equivalently, if **Table 1** (or its continuation) is denoted by $T(q) = (P(q), m(q), k(q))$, then every occurrence q_r satisfying $(k(q_r), k(q_r + 1), k(q_r + 2)) = (8, 11, 9)$ generates

$$M_r = \begin{pmatrix} P(q_r) & P(q_r + 1) & P(q_r + 2) \\ m(q_r) & m(q_r + 1) & m(q_r + 2) \end{pmatrix}$$

This table-indexed form makes the matrix reproducible directly from the ordered Z-class data: locate an exact (8, 11, 9) block in the Index column, take the corresponding three admissible pairs as row 1, and take their midpoints as row 2.

Let the r^{th} occurrence of (8, 11, 9) be generated by the ordered admissible pairs $P_r^{(8)} = (a_r^{(8)}, b_r^{(8)})$, $P_r^{(11)} = (a_r^{(11)}, b_r^{(11)})$, and $P_r^{(9)} = (a_r^{(9)}, b_r^{(9)})$. Define

$$m_r^{(j)} = \frac{a_r^{(j)} + b_r^{(j)}}{2}, j \in \{8, 11, 9\}.$$

The transition matrix is

$$M_r = \begin{pmatrix} P_r^{(8)} & P_r^{(11)} & P_r^{(9)} \\ m_r^{(8)} & m_r^{(11)} & m_r^{(9)} \end{pmatrix}.$$

The first row preserves the complete pair geometry, while the second row records the midpoint projection. Accordingly, an occurrence of the symbolic word is represented simultaneously in endpoint and midpoint coordinates.

5. Parent-Number Production by the Transition Matrix

Let $\mathcal{P}$ denote the positive parent-number line, *i.e.* the positive integers omitted by the nearest-integer golden map $n \mapsto \text{rnd}(\varphi n)$. For the present argument it is useful to write the parent numbers in their explicit inhomogeneous Beatty form

$$p_n = \left\lfloor \varphi^2 n - \frac{\varphi}{2} \right\rfloor, n \geq 1.$$

This representation is complementary to $\text{rnd}(\varphi n) = \lfloor \varphi n + 1/2 \rfloor$: the slopes satisfy $\varphi^{-1} + \varphi^{-2} = 1$, while the offsets satisfy $(1/2)\varphi^{-1} - (\varphi/2)\varphi^{-2} = 0$. Hence the two sequences partition the positive integers. In particular, $\mathcal{P} = \{p_n : n \geq 1\}$.

Theorem 1 (Parent-Number Production Theorem).

Let (a, b) be any admissible Z-class pair occurring in a transition matrix M , with $a, b \in \mathcal{P}$ and $b - a = 14$, and let $m = (a + b)/2 = a + 7$. Then the same pair geometry necessarily produces the additional parent numbers

$$a - 2, m - 1 = a + 6, m + 1 = a + 8, b + 2 = a + 16 \in \mathcal{P},$$

while $a - 1$, $m = a + 7$, and $b + 1 = a + 15$ are not parent numbers. Consequently every admissible matrix column produces the three parent-sandwich boundaries

$$\frac{a-2}{a}, \frac{m-1}{m+1}, \frac{b}{b+2}.$$

Proof

Write $a = p_k$ and $\theta = \{\varphi^2 k - \varphi/2\}$. Since $b = a + 14$ is also a parent number, the monotonic parent sequence gives $b = p_{k+5}$. Using $\varphi^2 = 2 + 1/\varphi$, the condition $p_{k+5} - p_k = 14$ is equivalent to $\lfloor \theta + 5/\varphi \rfloor = 4$, hence $\theta \geq 4 - 5/\varphi$. It follows successively that $p_{k-1} = a - 2$, $p_{k+2} = a + 6$, $p_{k+3} = a + 8$, $p_{k+5} = a + 14 = b$, and $p_{k+6} = a + 16 = b + 2$. Because the parent sequence is strictly increasing, the integers lying between each indicated consecutive parent pair—namely $a - 1$, $a + 7 = m$, and $a + 15 = b + 1$ —are omitted from $\mathcal{P}$. Thus the three displayed parent-sandwich boundaries are produced by the admissible pair itself, without a separate parent-number test.

6. Inaugural Transition

The inaugural transition matrix is presented in **Table 3**.

Table 3. Matrix M_1 for the inaugural occurrence of (8, 11, 9).

Layer	8	11	9
Z-class pair	103/117	137/151	171/185
Midpoint	110	144	178

$$M_1 = \begin{pmatrix} 103/117 & 137/151 & 171/185 \\ 110 & 144 & 178 \end{pmatrix}$$

By Theorem 1, each of the three admissible columns of M_1 produces three parent-sandwich boundaries. The inaugural matrix therefore produces

$$\frac{101}{103}, \frac{109}{111}, \frac{117}{119}, \frac{135}{137}, \frac{143}{145}, \frac{151}{153}, \frac{169}{171}, \frac{177}{179}, \frac{185}{187}.$$

7. First Recurrent Transition

The first recurrent transition matrix is presented in **Table 4**.

Table 4. Matrix M_2 for the first recurrent occurrence of (8, 11, 9).

Layer	8	11	9
Z-class pair	1090/1104	1124/1138	1158/1172
Midpoint	1097	1131	1165

$$M_2 = \begin{pmatrix} 1090/1104 & 1124/1138 & 1158/1172 \\ 1097 & 1131 & 1165 \end{pmatrix}$$

By Theorem 1, the three columns of M_2 produce the boundary family

$$\frac{1088}{1090}, \frac{1096}{1098}, \frac{1104}{1106}, \frac{1122}{1124}, \frac{1130}{1132}, \frac{1138}{1140}, \frac{1156}{1158}, \frac{1164}{1166}, \frac{1172}{1174}.$$

8. Return-Position and Transition-Matrix Laws

Two distinct displacement variables are required. The symbolic displacement measures movement in the ordered Z-class table, whereas the arithmetic displacement measures movement of the corresponding matrix coordinates. Keeping these quantities separate makes the construction reproducible without assuming that a translated matrix automatically generates a parent-number boundary.

Lemma 1 (Return-Position Lemma).

Let $k(q)$ denote the midway index in row q of the ordered Z-class table. Define q_1 as the first table position for which $(k(q_1), k(q_1 + 1), k(q_1 + 2)) = (8, 11, 9)$, and recursively define

$$q_{r+2} = \min \{ q > q_r : (k(q), k(q+1), k(q+2)) = (8, 11, 9) \}.$$

Define the symbolic return displacement by

$$\Delta q(r) = q_{r+1} - q_r.$$

Then q_r is uniquely determined by the ordered midway-index word whenever the r -th occurrence exists.

Proof

The set in the recursive definition contains exactly the later starting positions at which the three consecutive midway indices are 8, 11, 9. Taking its minimum selects the next occurrence and excludes intervening permutations such as (9, 11,

8). Thus the recursion enumerates the occurrences in their table order.

Lemma 2 (Matrix Reconstruction Lemma).

Let $T(q) = (P(q), m(q), k(q))$ denote the admissible pair, midpoint and midway index at table position q . For an occurrence beginning at q_r , the transition matrix is

$$M_r = \begin{pmatrix} P(q_r) & P(q_r+1) & P(q_r+2) \\ m(q_r) & m(q_r+1) & m(q_r+2) \end{pmatrix}$$

For the Z-class local geometry displayed by the computed $(8, 11, 9)$ occurrences, let $v_r = m(q_r+1)$ be the central midpoint. Then the matrix takes the one-coordinate form

$$M_r = \begin{pmatrix} (v_r - 41, v_r - 27) & (v_r - 7, v_r + 7) & (v_r + 27, v_r + 41) \\ v_r - 34 & v_r & v_r + 34 \end{pmatrix}$$

Proof

Each admissible pair has endpoint separation 14, hence each pair is seven units on either side of its midpoint. In a $(8, 11, 9)$ block the three consecutive midpoints are separated by 34, so they are $v_r - 34$, v_r and $v_r + 34$. Substituting ± 7 about each midpoint gives the six displayed endpoints. Thus q_r fixes the three table rows, and the central midpoint v_r fixes every integer coordinate of M_r .

Definition 4 (Arithmetic matrix displacement).

For successive occurrences define

$$\Delta v(r) = v_{r+1} - v_r = m(q_{r+1} + 1) - m(q_r + 1).$$

This identity is the exact bridge between symbolic return positions and arithmetic matrix translations.

Corollary 3.1 (Transition-Matrix Translation Law).

For two successive occurrences in the same local Z-class geometry, every endpoint and midpoint is translated by $\Delta v(r)$. Hence

$$M_{r+1} = M_r + \Delta v(r),$$

where scalar addition to a pair entry translates both of its endpoints. Equivalently,

$$\begin{aligned} a_{r+1} &= a_r + \Delta v(r), b_{r+1} = b_r + \Delta v(r), c_{r+1} = c_r + \Delta v(r), \\ d_{r+1} &= d_r + \Delta v(r), e_{r+1} = e_r + \Delta v(r), f_{r+1} = f_r + \Delta v(r), \\ \mu_{r+1} &= \mu_r + \Delta v(r), v_{r+1} = v_r + \Delta v(r), \xi_{r+1} = \xi_r + \Delta v(r). \end{aligned}$$

8.1. Boundary Production under Matrix Reconstruction

The translation law reconstructs the coordinates of each successive M_r . Once its three columns are identified as admissible separation-14 parent pairs, Theorem 1 applies directly to each column and produces the associated parent numbers and parent-sandwich boundaries.

For a column with endpoints (a, b) and midpoint $m = a + 7$, the produced parent numbers are $a - 2$, a , $m - 1$, $m + 1$, b , and $b + 2$, and the intervening obstruction centres are $a - 1$, m , and $b + 1$.

Thus matrix reconstruction and boundary production are deterministic stages of the same construction; boundary extraction does not require an independent parent-number test.

The logical order is therefore: the return word determines q_r ; q_r determines M_r ; the admissible separation-14 pair geometry invokes Theorem 1; and Theorem 1 produces the complete boundary family.

Theorem 2 (Reproducible Transition-Boundary Theorem).

For every occurrence q_r of (8, 11, 9) in the ordered midway-index word, the three table rows q_r , $q_r + 1$, $q_r + 2$ uniquely determine a transition matrix M_r . On the computed Z-class domain the matrix has the one-coordinate form of Lemma 3, and successive matrices are related by the arithmetic displacement $\Delta v(r)$. Because each matrix column is an admissible separation-14 parent pair, Theorem 1 produces three parent-sandwich boundaries per column and hence the complete nine-member boundary family of M_r .

Proof

Uniqueness of q_r follows from Lemma 2. The table-indexed construction and one-coordinate reconstruction follow from Lemma 3, and the translation relation follows from Corollary 3.1. Applying Theorem 1 to each of the three admissible columns then produces the corresponding parent numbers and three boundary triples. No independent boundary test is required.

8.2. Computed Displacement Spectrum

For the computed boundary-centre sequence

$$v_r = 144, 1131, 1741, 2118, 2728, 3338, 3715, 4325, 5312, 5922, 6299, 6909, 7896, \dots,$$

the arithmetic displacements are

$$\Delta v(r) = 987, 610, 377, 610, 610, 377, 610, 987, 610, 377, 610, 987, \dots$$

With $f_1 = 1$ and $f_2 = 2$, the observed values are $f_{15} = 987$, $f_{14} = 610$ and $f_{13} = 377$. Hence, on this computed return domain,

$$\Delta v(r) \in \{f_{13}, f_{14}, f_{15}\} = \{377, 610, 987\}.$$

This is an observed displacement spectrum on the computed domain; the theorem does not require an unproved closed recurrence for the sequence of symbolic return gaps $\Delta q(r)$ or arithmetic gaps $\Delta v(r)$.

8.3. Return-Word Context

The Fibonacci organization of first admissions and internal returns invites comparison with classical return-word theory. Huang and Wen [16] studied return words of the Fibonacci sequence, while Justin and Vuillon [27] developed return-word results for Sturmian and episturmian words. The present midway-index word is not assumed to be Sturmian. These works provide a combinatorial context for the observed Fibonacci return counts without adding a classification not established by the QGS construction.

9. Conclusion

A 2×3 matrix representation of the recurrent Z-class word (8, 11, 9) yields a reproducible parent-number production and boundary framework for golden-section quasigeometric sequences. The Return-Position Lemma determines the starting table position q_i of each occurrence; the Matrix Reconstruction Lemma determines M_i ; and the Parent-Number Production Theorem shows that every admissible separation-14 column necessarily generates additional parent numbers and three parent-sandwich boundaries. Accordingly, each transition matrix produces a nine-member boundary family without a separate parent-number test. The inaugural matrix reproduces 117/119 among its generated boundaries, while the first recurrent reproduces 1130/1132, which boundaries have been used in physical correspondences [2]. On the computed domain, the arithmetic displacement spectrum is $\{377, 610, 987\} = \{f_{13}, f_{14}, f_{15}\}$. The resulting formulation links symbolic recurrence, Z-class matrix geometry, parent-number production, and boundary formation through explicit reproducible maps.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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