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Gradient Estimates and Liouville-Type Theorems of Positive Solutions to $\Delta_{p_1,\cdots,p_r} v + f(v) = 0$ on Riemannian Manifolds

Xinyu Fan

Mathematical Science Research Center, Chongqing University of Technology, Chongqing, China
Email: 2891354505@qq.com

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Abstract

In this paper, we study gradient estimates and Liouville-type theorems for positive solutions of the nonlinear elliptic equation $\Delta_{p_1,\cdots,p_r} v + f(v) = 0$ on complete Riemannian manifolds, where $\Delta_{p_1,\cdots,p_r} v = \operatorname{div}\left(\sum_{i=1}^{r} |\nabla v|^{p_i-2} \nabla v\right)$. Assuming that the sectional curvature satisfies $\mathrm{Sec} \geq -K$, with $K \geq 0$, we construct suitable auxiliary functions and employ the Bernstein method, the maximum principle, and Hessian comparison to derive local upper bounds for $|\nabla v|/v$. The exponent hypothesis is a positive structural gap β, expressed through the weighted mean and variance of the exponents. The bounds depend on the dimension, the curvature lower bound, and the structural parameters. For global solutions in the stated regularity class, we obtain $|\nabla v|^2 / v^2 \leq (n-1)K/\beta$. Consequently, when the sectional curvature is nonnegative and the exponents and the nonlinear term satisfy the stated conditions, every global positive solution is constant. The manifold M is assumed to be connected throughout the paper.

Keywords

$(p_1,\cdots,p_r)$-Laplacian, Gradient Estimate, Liouville-Type Theorem

1. Introduction

The classical Liouville theorem states that any bounded harmonic function defined on $\mathbb{R}^2$ must be constant. In 1975, Yau [1] proved that if M is a complete Riemannian manifold whose Ricci curvature satisfies $\mathrm{Ric} \geq 0$, then any positive

harmonic function on M must be constant. Cheng and Yau [2] further used the Bochner formula and the maximum principle to establish the Cheng-Yau gradient estimates, which have had an important influence on geometric partial differential equations.

In 1981, Gidas and Spruck [3] proved the nonexistence of positive entire solutions of the Lane-Emden equation

$$\Delta u + u^{\alpha} = 0$$

in $\mathbb{R}^n$ when $n > 2$ and $1 \le \alpha < (n+2)/(n-2)$. Subsequently, Serrin and Zou [4] proved Liouville-type theorems for positive solutions of

$$\Delta_p v + f(v) = 0 \tag{1.1}$$

in $\mathbb{R}^n$, under conditions including $1 < p < n$, nonnegativity and subcriticality of f, and the existence of $q > p$ such that $f(t) \ge t^{q-1}$ for sufficiently large t. Here a nonnegative function f is subcritical if there is a constant $0 < \gamma_0 < np/(n-p) - 1$ such that

$$f'(t) \le \gamma_0 \frac{f(t)}{t}, \quad t > 0.$$

Cuccu, Mohammed, and Porru [5] investigated (1.1) in $\mathbb{R}^n$, allowing f to change sign. They obtained a Cauchy-Liouville theorem under the condition

$$f'(t) \le (p-1)\frac{n+1}{n-1}\frac{f(t)}{t}, \quad t > 0.$$

Many results along these lines for semilinear and quasilinear equations have been obtained (see [6]-[16]).

Kotschwar and Ni [17] studied positive p-harmonic functions on Riemannian manifolds with a lower sectional curvature bound by a maximum principle argument. Wang and Zhang [18] used a Nash-Moser iteration argument to obtain gradient estimates for positive p-harmonic functions under a lower Ricci curvature bound. This approach has also been developed for various quasilinear elliptic equations. One can see [19] for related results and a survey. The two curvature assumptions appearing above are not interchangeable, and this limits how far our estimate can be compared with [18]. Wang and Zhang obtain their local gradient estimate for p-harmonic functions under a lower *Ricci* curvature bound, because their Nash-Moser iteration needs only mean-value and volume comparison, both of which do follow from $\mathrm{Ric} \ge -K$. Our Bernstein argument instead differentiates the distance function twice: the estimates (3.19) are produced by contracting the pointwise matrix inequality for $\mathrm{Hess}\, q$ against the anisotropic endomorphism Θ.

The Hessian comparison theorem supplying that matrix inequality is the single place where we use a lower bound on the *sectional* curvature, since a Ricci lower bound controls only the trace Δd and not the individual tangential eigenvalues of $\mathrm{Hess}\, d$ that Θ detects. Replacing $\mathrm{Sec} \ge -K$ by $\mathrm{Ric} \ge -K$ in Theorem 1.1 is therefore an open problem rather than a matter of presentation, and it is the essential difference between our method and the iteration method of [18]. Stated pre-

cisely: a lower Ricci curvature bound alone does not provide the matrix comparison used here. Laplacian comparison yields only the trace bound $\Delta d \le (n-1)c_K(d)$, whereas the contraction in (3.19) requires the tangential operator inequality $\operatorname{Hess} d \le c_K(d)(g - dd \otimes dd)$, which is available under $\operatorname{Sec} \ge -K$.

This describes a limitation of the present argument only; we do not claim that the estimate itself fails under $\operatorname{Ric} \ge -K$, and a different method could still reach it there.

More general quasilinear elliptic equations have also been studied. Zeng, Ban, Dong, and Ma [20] considered the operator

$$\Delta_{p_1,\cdots,p_r} v := \operatorname{div}\left(\sum_{i=1}^{r} |\nabla v|^{p_i - 2} \nabla v \right) \tag{1.2}$$

and investigated the Liouville problem for positive solutions of $\Delta_{p_1,\cdots,p_r} v + f(v) = 0$ in $\mathbb{R}^n$ by the Bernstein method.

For quasilinear second-order elliptic equations on general Riemannian manifolds, obtaining gradient estimates and corresponding Liouville theorems remains an active problem. Building on these studies, this paper extends the Bernstein argument to the mixed operator (1.2) on complete Riemannian manifolds. A direct comparison with [20], the Euclidean counterpart of the present work, makes the contribution more precise than the statement above that we merely extend the Bernstein argument.

In [20] the operator (1.2) acts on $\mathbb{R}^n$, where no curvature term can appear: the distance function is smooth away from the origin, $\operatorname{Hess} d^2 = 2dx \otimes dx + 2d \operatorname{Hess} d$ is replaced by the flat identity $\Delta d^2 = 2n$, and the whole cutoff computation is exact. Three assumptions and one conclusion change once we pass to a complete manifold. First, the hypothesis $\operatorname{Sec} \ge -K$ is genuinely new: the bounds (3.19) for $\mathcal{L}q$ and $\Theta(\nabla q, \nabla q)$ come from the Hessian comparison theorem, which is exactly the step where sectional curvature enters. Second, the exponent restriction of [20] is replaced by the structural gap β in (1.4), and every constant in Theorem 1.1 depends on $p_1, \cdots, p_r$ only through β. Third, the maximum point must be handled across the cut locus, which forces the Calabi support argument at the end of Step 2 and has no counterpart in the Euclidean proof. Finally the conclusion is stronger in form: [20] yields a Liouville-type conclusion directly, whereas we first obtain the quantitative local bound (1.11) and the curvature-dependent global bound (1.13), and deduce the Liouville statement in Corollary 1.5 only afterwards. Setting $K = 0$ recovers the Euclidean situation and forces v to be constant immediately. Convention on connectedness. The manifold M is assumed to be connected throughout this paper, consistently with the Abstract. The convention matters: the gradient bound (1.13) controls only $|\nabla v|/v$, so without connectedness it forces v to be constant on each connected component separately, with possibly different constants on different components. For instance, on the disjoint union of two copies of $\mathbb{R}^n$ with $f \equiv 0$, the function equal to $c_1 > 0$ on the first copy and $c_2 > 0$ on the second is a global positive solution of $\Delta_{p_1,\cdots,p_r} v + f(v) = 0$ that is not constant on M.

Let $n \ge 2$, $r \ge 1$, and $1 < p_1 < \cdots < p_r$, with the ordering condition understood

as vacuous when $r=1$. Define

$$S(t)=\sum_{i=1}^{r} t^{p_i-2},\quad w_i(t)=\frac{t^{p_i-2}}{S(t)},\quad \mu(t)=\sum_{i=1}^{r}(p_i-1)w_i(t),\quad t>0. \tag{1.3}$$

Set

$$\sigma^2(t)=\sum_{i=1}^{r} w_i(t)\left(p_i-1-\mu(t)\right)^2,\quad b(t)=\frac{\mu(t)^2}{n-1}-\sigma^2(t),\quad \beta=\inf_{t>0} b(t). \tag{1.4}$$

From Lemma 2.1, the exponent hypothesis (1.6) used below is a sufficient condition for

$$\beta>0. \tag{1.5}$$

We have

Theorem 1.1 *Let* $\left(M^n,g\right)$ *be a complete Riemannian manifold without boundary with* $\mathrm{Sec}\geq -K$, *where* $K\geq 0$. *Assume*

$$\frac{p_r-1}{p_1-1}<\frac{\sqrt{n}+1}{\sqrt{n}-1}. \tag{1.6}$$

and $f\in C^1\left((0,\infty)\right)$. *Suppose either*

$$f(s)\geq 0,\quad f'(s)\leq\frac{n+1}{n-1}(p_1-1)\frac{f(s)}{s}\quad \textit{for every}\quad s>0, \tag{1.7}$$

or

$$f(s)\leq 0,\quad f'(s)\leq\frac{n+1}{n-1}(p_r-1)\frac{f(s)}{s}\quad \textit{for every}\quad s>0, \tag{1.8}$$

Let $\Omega\subset M$ *be open and let* $v>0$ *be a weak solution of*

$$\Delta_{p_1,\cdots,p_r}v+f(v)=0,\quad \Delta_{p_1,\cdots,p_r}v=\mathrm{div}\left(S\left(|\nabla v|\right)\nabla v\right), \tag{1.9}$$

with

$$v\in C^1(\Omega)\cap C^3(\Omega_*),\quad \Omega_*=\left\{x\in\Omega:|\nabla v(x)|>0\right\}. \tag{1.10}$$

The flux at a critical point is defined by its continuous value zero. Equation (1.9) *is imposed weakly on all of* Ω, *meaning*

$$\int_\Omega S\left(|\nabla v|\right)\langle\nabla v,\nabla\zeta\rangle\,\mathrm{d}V_g=\int_\Omega f(v)\zeta\,\mathrm{d}V_g\quad \textit{for every}\quad \zeta\in C_c^\infty(\Omega),$$

where the integrand containing the flux is interpreted by continuity at critical points. On Ω_*, *the equation holds classically.*

For every $0<\varepsilon<1$, *there is a finite constant* $C_\varepsilon=C_\varepsilon(n,p_1,\cdots,p_r,\beta)>0$ *such that, whenever* $B(o,R)\subset\Omega$,

$$\sup_{B(o,R/2)}\frac{|\nabla v|^2}{v^2}\leq\frac{C_\varepsilon}{R^2}+\frac{C_\varepsilon\sqrt{K}}{R}+\frac{(n-1)K}{(1-\varepsilon)\beta}. \tag{1.11}$$

In particular, for a constant $C=C(n,p_1,\cdots,p_r,\beta)$,

$$\sup_{B(o,R/2)}\frac{|\nabla v|^2}{v^2}\leq C\left(\frac{1}{R^2}+K\right). \tag{1.12}$$

If $\Omega = M$, *then*

$$\frac{|\nabla v|^2}{v^2} \le \frac{(n-1)K}{\beta} \quad \text{everywhere on} \quad M. \tag{1.13}$$

We clarify the role of the exponent hypothesis (1.6). The proof of Theorem 1.1 never uses (1.6) as such; it uses only positivity of the structural gap, $\beta > 0$. Indeed the pointwise inequality (3.3) holds with $2(1-\varepsilon)\beta$ as the zeroth-order coefficient, the constant Γ_ε in (3.4) is finite as soon as $\beta > 0$, and the cutoff constant C_ε defined in (3.17) involves the exponents only through β.

Hence (1.6) may be replaced by the weaker assumption $\beta > 0$, stated in (1.5), and with that substitution the statement of Theorem 1.1 matches the emphasis of the Abstract. We keep the explicit ratio condition because it is readily verified from the exponents alone and, by Lemma 2.1, it is also necessary for $\beta > 0$ when $r = 2$. For clarity we separate the three roles involved. The ratio condition (1.6) is an explicit sufficient test for $\beta > 0$, used nowhere else in the proof. The quantity that actually governs all constants is the structural gap β of (1.4), and Theorem 1.1 remains valid if (1.6) is replaced by $\beta > 0$.

Lemma 2.1 proves that (2.2) is also necessary for $\beta > 0$ when $r = 2$; no necessity statement is asserted there for $r \ge 3$, and necessity in that range remains open.

Remark 1.2. For $r = 1$, one has $\mu = p - 1$, $\sigma^2 = 0$, and $\beta = (p-1)^2/(n-1)$. Thus (1.13) becomes

$$\frac{|\nabla v|^2}{v^2} \le \frac{(n-1)^2 K}{(p-1)^2}.$$

This agrees with the global p-harmonic estimate of Kotschwar and Ni [17], after matching their curvature convention. We spell out that match. In [17] the curvature hypothesis is written with a squared parameter, namely $\mathrm{Sec} \ge -\kappa^2$, and the global estimate for a positive p-harmonic function there reads $|\nabla v|^2/v^2 \le (n-1)^2 \kappa^2/(p-1)^2$.
Under our convention $\mathrm{Sec} \ge -K$ the conversion is $\kappa^2 = K$, *i.e.* $\kappa = \sqrt{K}$.

Since (1.13) with $r = 1$ and $\beta = (p-1)^2/(n-1)$ gives the coefficient $(n-1)^2 K/(p-1)^2$, the two statements agree after this substitution.

Remark 1.3. For $r = 1$, the coefficient is attained when $f = 0$: on the upper half-space model of hyperbolic space with $\mathrm{Sec} = -1$,

$$g = y^{-2}\left(\sum_{j=1}^{n-1} \mathrm{d}x_j^2 + \mathrm{d}y^2\right), \quad v = y^{(n-1)/(p-1)},$$

one has

$$\Delta_p v = 0$$

and

$$|\nabla v|^2/v^2 = (n-1)^2/(p-1)^2.$$

The example above settles sharpness only in the single-exponent case $r = 1$,

where the variance σ^2 vanishes identically and $\beta=(p-1)^2/(n-1)$ is explicit. For genuinely mixed exponents, that is $r\geq 2$ with $p_1<\cdots<p_r$, we do not know whether the constant $(n-1)/\beta$ in (1.13) is sharp: the hyperbolic model above does not adapt to the mixed operator, because there the weights w_i depend on $|\nabla v|$ and no explicit non-constant entire solution of $\Delta_{p_1,\ldots,p_r}v=0$ on a space form is known for which $|\nabla v|^2/v^2$ attains $(n-1)K/\beta$ with $K=1$. Sharpness in the mixed case is therefore left as an open problem.

Remark 1.4. We record how much of the smoothness assumption (1.10) is automatic. The operator has $(p_1,\cdots,p_r)$-growth with each $p_i>1$, so a weak solution of (1.9) is locally $C^{1,\alpha}$ for some $\alpha\in(0,1)$; we refer to Tolksdorf [21] and DiBenedetto [22] for the model degenerate case, and to Marcellini [23] for the general theory under non standard growth. These results cover every exponent $p_i>1$, so nothing deteriorates when some exponents lie below two, where the operator is singular rather than degenerate; the critical set Ω_* can even be empty there, and the conclusion then rests on the continuous flux convention described after (1.10). The same theory also gives $C^{2,\alpha}_{\mathrm{loc}}$ regularity on Ω_*, where the equation is uniformly elliptic with locally Hölder coefficients.

What regularity does not provide is continuity of ∇v up to and across the critical set together with the third order differentiability used in Step 1, so the assumption (1.10) is retained as a genuine additional hypothesis. To make the scope explicit: the regularity (1.10) is an assumption imposed throughout the argument. The conclusions of Theorem 1.1 and Corollary 1.5 apply to weak solutions in this class, and the proof given here does not establish (1.10) for arbitrary weak solutions. The preceding remark only identifies which part of (1.10) is supplied by existing regularity theory; any claim that a strictly broader class of weak solutions satisfies (1.10) would require a regularity theorem with matching hypotheses, and no such extension is claimed in this paper.

As a direct corollary of Theorem 1.1, we have the following Liouville-type result.

Corollary 1.5. *Under the hypotheses of Theorem* 1.1, *if* $\Omega=M$ *and* $\mathrm{Sec}\geq 0$, *then* v *is constant, and its constant value* $c>0$ *must satisfy* $f(c)=0$. *Consequently, if* f *has no positive zero, no global positive weak solution satisfying* (1.10) *exists. Under the connectedness convention above,* v *is constant on all of* M. *Without it, the same proof shows that* v *equals a constant* $c_j>0$ *with* $f(c_j)=0$ *on each connected component* M_j *of* M, *the constants possibly differing between components.*

The two sign and differential conditions on f are consistent with the conditions used in [5] [20].

Organization. In section 2 we prove several lemmas, which will be used in the proof of Theorem 1.1 in section 3.

2. Preliminaries

In this section we prove several lemmas which will be used in the proof of Theo-

rem 1.1.

Lemma 2.1. *Write* $m_- = p_1 - 1 > 0$ *and* $m_+ = p_r - 1$. *A sufficient condition for* (1.5) *is*

$$(p_r - p_1)^2 < \frac{4(p_1 - 1)(p_r - 1)}{n-1}. \tag{2.1}$$

Equivalently,

$$\frac{p_r - 1}{p_1 - 1} < \frac{\sqrt{n}+1}{\sqrt{n}-1}. \tag{2.2}$$

Under this condition, $\beta \geq \beta_0 > 0$, *where*

$$\beta_0 = \begin{cases} \dfrac{m_-^2}{n-1}, & \dfrac{m_+}{m_-} \leq \dfrac{n+1}{n-1}, \\ m_- m_+ - \dfrac{n-1}{4n}(m_- + m_+)^2, & \dfrac{m_+}{m_-} > \dfrac{n+1}{n-1}. \end{cases} \tag{2.3}$$

For $r = 1$, *every* $p_1 > 1$ *is admissible, and the exact gap is* $\beta = (p_1 - 1)^2 / (n-1)$. *For* $r = 2$, *condition* (2.1) *is also necessary for* $\beta > 0$.

Proof. Let $x_i = p_i - 1 \in [m_-, m_+]$. Since $(x_i - m_-)(m_+ - x_i) \geq 0$, summation with weights w_i gives

$$\sigma^2 \leq (\mu - m_-)(m_+ - \mu).$$

Hence

$$b(t) \geq Q(\mu(t)), \quad Q(s) = \frac{n}{n-1} s^2 - (m_- + m_+) s + m_- m_+. \tag{2.4}$$

The vertex is $s_* = (n-1)(m_- + m_+)/(2n)$. It always lies below m_+. If $m_+ / m_- \leq (n+1)/(n-1)$, it lies at or below m_-, so the minimum of Q on $[m_-, m_+]$ is $m_-^2/(n-1)$. Otherwise the minimum is $m_- m_+ - (n-1)(m_- + m_+)^2/(4n)$. This latter expression is positive exactly when (2.1) holds; elementary algebra also gives (2.2). This proves the asserted sufficient condition and bound.

For $r = 1$, the variance is zero. For $r = 2$, $w_2 / w_1 = t^{p_2 - p_1}$ traverses $(0, \infty)$, so μ traverses (m_-, m_+). The variance bound above is then an equality. If (2.1) fails, the minimum of Q is attained at an interior point and is nonpositive. Thus $\beta > 0$ fails. □

On Ω_*, put

$$\begin{aligned} &t = |\nabla v|, \ e = \frac{\nabla v}{t}, \ a(t) = \mu(t) - 1, \ \phi = \frac{t^2}{v^2}, \\ &\Theta = I + a(t) e \otimes e, \ \mathcal{L}u = \operatorname{tr}(\Theta \operatorname{Hess} u). \end{aligned} \tag{2.5}$$

The eigenvalues of Θ are $1, \cdots, 1, \mu(t)$, and $m_- \leq \mu(t) \leq m_+$. In particular, Θ is positive definite, including when some or all of the exponents are less than two.

Lemma 2.2. *Let* $n \geq 2$, $p_i > 1$, *and* v *be differentiable at a point where* $\nabla v \neq 0$. *The symmetric endomorphism*

$$\Theta = I + \frac{\sum_{i=1}^r (p_i - 2) |\nabla v|^{p_i - 2}}{\sum_{i=1}^r |\nabla v|^{p_i - 2}} \frac{\nabla v \otimes \nabla v}{|\nabla v|^2}$$

is positive definite with respect to g.

Proof. In the direction $e=\nabla v/|\nabla v|$, its eigenvalue is $\mu=\sum_i (p_i-1)w_i>0$. On the orthogonal complement of e, all eigenvalues are one.

Lemma 2.3. *At a point where* $t=|\nabla v|>0$, *choose an orthonormal frame with* $e_1=\nabla v/t$, *and write* $v_{ij}=\mathrm{Hess}\, v(e_i,e_j)$. *With* $a(t)=\mu(t)-1$, *one has*

$$(1+a)v_{11}^2+\sum_{l=2}^{n}v_{ll}^2\geq\left(\frac{n}{n-1}+a\right)v_{11}^2-\frac{2}{n-1}v_{11}\Delta v+\frac{(\Delta v)^2}{n-1}. \tag{2.6}$$

Proof. The Cauchy-Schwarz inequality gives $\sum_{l=2}^{n}v_{ll}^2\geq(\Delta v-v_{11})^2/(n-1)$. Expanding the square proves (2.6). □

3. Proofs of Theorem 1.1

In this section, we employ the Bernstein method to prove Theorem 1.1. We divide the argument into three steps.

Step 1. A pointwise differential inequality

Define the coefficient functions

$$\begin{aligned}&A(t)=-|3-\mu(t)|,\quad B(t)=2b(t),\quad C=\frac{2}{n-1},\\&D(t)=\frac{n+1}{n-1}\mu(t)-1,\quad E(t)=2(b(t)-\mu(t)),\quad F(t)=2(b(t)+\mu(t)).\end{aligned} \tag{3.1}$$

Here $B(t)\geq 2\beta>0$ and $F(t)=B(t)+2\mu(t)>0$.

Lemma 3.1. *At a point of* Ω_*, *choose an orthonormal frame with* $e_1=e$. *Let* $\phi_i=e_i(\phi)$, $\eta=\phi_1/\phi$, *and* $z=f(v)/(tS(t))$. *Then*

$$\begin{aligned}\frac{\mathcal{L}\phi}{\phi}\geq{}&(2-a)\frac{\sum_{l=2}^{n}\phi_l^2}{2\phi^2}+B\phi+Cz^2+D\eta z+E\eta\sqrt{\phi}+\frac{F}{4}\eta^2\\&+\frac{2}{S(t)}\left[\frac{n+1}{n-1}\mu(t)\frac{f(v)}{v}-f'(v)\right]+2\mathrm{Ric}(e,e).\end{aligned} \tag{3.2}$$

Consequently, for each $0<\varepsilon<1$,

$$\frac{\mathcal{L}\phi}{\phi}\geq 2(1-\varepsilon)\beta\phi-\Gamma_\varepsilon\frac{|\nabla\phi|^2}{2\phi^2}-2(n-1)K, \tag{3.3}$$

where one may take

$$\Gamma_\varepsilon=\sup_{t>0}\left[|3-\mu(t)|+\frac{D(t)^2}{2C}+\frac{E(t)^2}{2\varepsilon B(t)}\right]<\infty. \tag{3.4}$$

Proof. Differentiating the weights gives

$$tw_i'(t)=(p_i-1-\mu(t))w_i(t),\quad ta'(t)=\sigma^2(t).$$

Let $h=\mathrm{Hess}\, v(e,e)=v_{11}$, $U=\sum_{l=2}^{n}v_{1l}^2$, and $F_0=f(v)/S(t)$. The equation on Ω_* is

$$\Delta v+ah=-F_0. \tag{3.5}$$

For completeness, expansion of the mixed operator in the adapted frame first

gives

$$\sum_{i=1}^{r} t^{p_i-2}\Delta v + \sum_{i=1}^{r}(p_i-2)t^{p_i-2}v_{11} = -f(v). \tag{3.6}$$

Dividing by $S(t)$ gives (3.5).

To differentiate this identity, use the smooth local vector field $e = \nabla v/t$, before imposing the pointwise adapted frame. We have

$$e(t) = h, \quad \nabla_e e = \frac{1}{t}\sum_{l=2}^{n} v_{1l}e_l, \quad e(h) = v_{11;1} + \frac{2U}{t},$$

where $v_{ij;k} = \left(\nabla_{e_k}\mathrm{Hess}\, v\right)(e_i, e_j)$ denotes the covariant derivative of the Hessian. Also $S'(t)/S(t) = a(t)/t$. Differentiating (3.5) therefore gives

$$(\Delta v)_1 + av_{11;1} = -\frac{2a}{t}U - \frac{\sigma^2}{t}h^2 + \frac{aF_0}{t}h - \frac{f'(v)t}{S(t)}. \tag{3.7}$$

Equivalently, before normalizing, the differentiated equation reads

$$\begin{aligned}&\sum_{i=1}^{r} t^{p_i-2}(\Delta v)_1 + \sum_{i=1}^{r}(p_i-2)t^{p_i-2}v_{11;1}\\ &= -2\sum_{i=1}^{r}(p_i-2)t^{p_i-3}\sum_{l=2}^{n}v_{1l}^2 - \sum_{i=1}^{r}(p_i-2)t^{p_i-3}h\Delta v\\ &\quad -\sum_{i=1}^{r}(p_i-2)(p_i-3)t^{p_i-3}h^2 - \sum_{i=1}^{r}(p_i-2)t^{p_i-3}h^2 - f'(v)t.\end{aligned} \tag{3.8}$$

If

$$c(t) = \sum_{i=1}^{r}(p_i-2)(p_i-3)w_i(t),$$

then $\sigma^2 = c + a - a^2$. Substituting $\Delta v = -ah - F_0$ into (3.8) therefore recovers (3.7). The first and second covariant derivatives of ϕ are

$$\phi_i = \frac{2}{v^2}\sum_{l=1}^{n} v_{il}v_l - \frac{2t^2}{v^3}v_i, \tag{3.9}$$

and

$$\phi_{ij} = \frac{2}{v^2}\sum_{l=1}^{n}\left(v_{il}v_{jl} + v_l v_{il;j}\right) - \frac{4}{v^3}\sum_{l=1}^{n}\left(v_j v_{il}v_l + v_i v_{jl}v_l\right) - \frac{2t^2}{v^3}v_{ij} + \frac{6t^2}{v^4}v_i v_j. \tag{3.10}$$

At the adapted point, $v_1 = t$ and $v_l = 0$ for $l \geq 2$, so

$$\phi_{11} = \frac{2t}{v^2}v_{11;1} + \frac{2}{v^2}\sum_{l=1}^{n} v_{1l}^2 - \frac{10t^2}{v^3}h + \frac{6t^4}{v^4}. \tag{3.11}$$

The Bochner formula gives

$$\Delta\phi = \frac{2t}{v^2}(\Delta v)_1 + \frac{2}{v^2}\left|\mathrm{Hess}\, v\right|^2 - \frac{8t^2}{v^3}h - \frac{2t^2}{v^3}\Delta v + \frac{6t^4}{v^4} + \frac{2}{v^2}\mathrm{Ric}(\nabla v, \nabla v). \tag{3.12}$$

Combining (3.11) and (3.12) gives

$$\begin{aligned}\mathcal{L}\phi &= \frac{2t}{v^2}\left[(\Delta v)_1 + av_{11;1}\right] + \frac{2}{v^2}\left(\left|\mathrm{Hess}\, v\right|^2 + a\sum_{l=1}^{n} v_{1l}^2\right)\\ &\quad -\frac{2t^2}{v^3}\left[\Delta v + (4+5a)h\right] + 6\mu\frac{t^4}{v^4} + \frac{2}{v^2}\mathrm{Ric}(\nabla v, \nabla v).\end{aligned} \tag{3.13}$$

Lemma 2.3 and the symmetry of the Hessian give

$$\begin{aligned}|\mathrm{Hess}\, v|^2 + a\sum_{l=1}^{n} v_{1l}^2 &\geq \mu h^2 + (2+a)U + \frac{(\Delta v - h)^2}{n-1} \\ &= \mu h^2 + (2+a)U + \frac{(\mu h + F_0)^2}{n-1}.\end{aligned} \tag{3.14}$$

Combining (3.7)-(3.14) yields

$$\begin{aligned}\mathcal{L}\phi \geq \frac{2}{v^2}\left[(2-a)U + (b+\mu)h^2 + DF_0 h + \frac{F_0^2}{n-1} - \frac{f'(v)t^2}{S(t)}\right] \\ -\frac{2t^2}{v^3}\left[-F_0 + 4\mu h\right] + 6\mu\frac{t^4}{v^4} + \frac{2}{v^2}\mathrm{Ric}(\nabla v, \nabla v).\end{aligned} \tag{3.15}$$

Finally,

$$\phi_1 = \frac{2th}{v^2} - \frac{2t^3}{v^3}, \quad \phi_l = \frac{2tv_{1l}}{v^2}\,(l \geq 2), \quad h = \frac{t^2}{v} + \frac{v^2\phi_1}{2t}.$$

Substituting these expressions into (3.15), dividing by ϕ, and collecting terms gives (3.2) with exactly the coefficients in (3.1).

The same coefficients can be written in the weighted notation as

$$\begin{aligned}B &= 2\left[\frac{n}{n-1}a^2 - c + \frac{1}{n-1} + \frac{3-n}{n-1}a\right], \\ E &= 2\left[\frac{n}{n-1}a^2 - c + \frac{2-n}{n-1} + \frac{4-2n}{n-1}a\right], \\ F &= 2\left[\frac{n}{n-1}a^2 - c + \frac{n}{n-1} + \frac{2}{n-1}a\right].\end{aligned} \tag{3.16}$$

The reaction term in brackets in (3.2) is nonnegative. Indeed, when $f \geq 0$, use $\mu \geq p_1 - 1$ and (1.7); when $f \leq 0$, use $\mu \leq p_r - 1$ and (1.8). We display the two comparisons separately, because multiplying an inequality by $f(v)/v$ reverses its direction when $f(v)/v < 0$. Recall $p_1 - 1 \leq \mu(t) \leq p_r - 1$ and $v > 0$.

If $f \geq 0$, then $f(v)/v \geq 0$, so multiplication preserves the direction and gives $\mu(t)\, f(v)/v \geq (p_1 - 1)\, f(v)/v$. Combining this with (1.7) yields

$$\frac{n+1}{n-1}\mu(t)\frac{f(v)}{v} - f'(v) \geq \frac{n+1}{n-1}(p_1 - 1)\frac{f(v)}{v} - f'(v) \geq 0,$$

where the last inequality is exactly (1.7) evaluated at $s = v$. If $f \leq 0$, then $f(v)/v \leq 0$, multiplication reverses the inequality, and now the upper bound $\mu(t) \leq p_r - 1$ is the useful one, giving again $\mu(t)\, f(v)/v \geq (p_r - 1)\, f(v)/v$. With (1.8) this yields

$$\frac{n+1}{n-1}\mu(t)\frac{f(v)}{v} - f'(v) \geq \frac{n+1}{n-1}(p_r - 1)\frac{f(v)}{v} - f'(v) \geq 0.$$

In both cases the bracket in (3.2) is nonnegative, and only after this verification do we discard the reaction term.

Also

$$(2-a)\frac{\sum_{l=2}^{n}\phi_l^2}{2\phi^2} \geq A\frac{|\nabla\phi|^2}{2\phi^2}.$$

Discard the nonnegative reaction term and $F\eta^2/4$, and use

$$Cz^2 + D\eta z \geq -\frac{D^2}{4C}\eta^2, \quad E\eta\sqrt{\phi} \geq -\varepsilon B\phi - \frac{E^2}{4\varepsilon B}\eta^2.$$

Since $\eta^2 \leq |\nabla\phi|^2/\phi^2$, $B \geq 2\beta$, and $\mathrm{Ric}(e,e) \geq -(n-1)K$, these inequalities prove (3.3).

The constant in (3.4) is finite: $\mu \in [m_-, m_+]$, $0 < b \leq m_+^2/(n-1)$, and $B \geq 2\beta$. Thus all the numerators are uniformly bounded and the denominators are bounded away from zero.

Step 2. The auxiliary function and distance estimates

Proof of Theorem 1.1. Fix $0 < \varepsilon < 1$, and put

$$T = n + p_r - 2, \quad \Lambda = \max\{1, p_r - 1\}, \quad C_\varepsilon = \frac{8T + 48\Lambda + 16\Gamma_\varepsilon}{(1-\varepsilon)\beta}. \tag{3.17}$$

First prove a point estimate at the center of any ball $\overline{B(y,\rho)} \subset \Omega$. Let

$$d(x) = d(y,x), \quad q = d^2, \quad \psi = \rho^2 - q, \quad H = \psi^2\phi.$$

From this point to the end of Step 2 the centre $y \in \Omega$ and the radius $\rho > 0$ are fixed once and for all, with $\overline{B(y,\rho)} \subset \Omega$. We make this explicit because the argument below maximises over a compact set: the inclusion guarantees that v, and hence ϕ, is defined and continuous on the whole closed ball, and that the cutoff $\psi = \rho^2 - q$ is smooth enough there, so $H = \psi^2\phi$ attains a finite maximum on $\overline{B(y,\rho)}$.

The centre estimate is first proved under the stronger local condition $\overline{B(y,\rho)} \subset \Omega$. Completeness makes this closure compact, and the assumed regularity (1.10) makes v, hence $\phi = |\nabla v|^2/v^2$, continuous on it; the cutoff $\psi = \rho^2 - q$ is defined and continuous on $\overline{B(y,\rho)}$ as well, so $H = \psi^2\phi$ attains its maximum there. We stress that this closure condition is an extra local hypothesis and does not follow from $B(y,\rho) \subset \Omega$ alone, since v need not extend continuously to boundary points outside Ω. Every ball used in Step 3 satisfies this closure condition, as verified at the end of Step 3; alternatively one may restrict to radii $0 < \rho < \mathrm{dist}(y, \partial\Omega)$, which leaves the principal estimate unchanged.

Completeness implies that the closed ball is compact. By (1.10), ϕ is continuous, so H attains its maximum. It vanishes on the boundary. If its maximum is zero, then ϕ vanishes on the ball and the desired estimate is immediate. Otherwise a maximum point x_* lies in the interior and belongs to Ω_*.

Suppose first that q is smooth at x_*. The maximum principle for the positive definite matrix Θ gives

$$\frac{\nabla\phi}{\phi} = \frac{2\nabla q}{\psi}, \quad \mathcal{L}H \leq 0.$$

At this point the complete product expansion is

$$\mathcal{L}H = \psi^2 \mathcal{L}\phi - 2\psi\phi\mathcal{L}q - 6\phi\Theta(\nabla q, \nabla q).$$

Consequently,

$$\frac{\mathcal{L}\phi}{\phi} \le \frac{2\mathcal{L}q}{\psi} + \frac{6\Theta(\nabla q, \nabla q)}{\psi^2}. \tag{3.18}$$

Define

$$c_K(s) = \begin{cases} \sqrt{K}\coth\left(\sqrt{K}s\right), & K > 0, \\ 1/s, & K = 0. \end{cases}$$

The Hessian comparison theorem under $\mathrm{Sec} \ge -K$ gives $\mathrm{Hess}\, d \le c_K(d)\left(g - \mathrm{d}d \otimes \mathrm{d}d\right)$ away from the center and cut locus. This is the same type of comparison used in ([17], Lemma 2.2). Here $\mathrm{d}d$ denotes the differential of the distance function.

In particular, away from the center and cut locus,

$$\mathrm{Hess}\, q(X,Y) = 2X(d)Y(d) + 2d\,\mathrm{Hess}\, d(X,Y), \quad \Delta q = 2 + 2d\Delta d,$$

and $q_{11} = 2(e_1 d)^2 + 2d\,\mathrm{Hess}\, d(e_1, e_1)$.

Since $dc_K(d) \ge 1$,

$$\mathrm{Hess}\, q = 2\mathrm{d}d \otimes \mathrm{d}d + 2d\,\mathrm{Hess}\, d \le 2dc_K(d)g \le 2\left(1 + \sqrt{K}\rho\right)g,$$

where $s\coth s \le 1 + s$ was used. At the center, $\mathrm{Hess}\, q = 2g$, so the last bound also holds there. Since Θ is positive definite by Lemma 2.2, contracting the matrix inequality with Θ gives

$$\mathcal{L}q \le 2T\left(1 + \sqrt{K}\rho\right), \quad \Theta(\nabla q, \nabla q) \le 4\Lambda\rho^2. \tag{3.19}$$

Indeed, $\mathrm{tr}\,\Theta = n + \mu - 1 \le T$ and the largest eigenvalue of Θ is at most Λ. Moreover, at x_*,

$$\frac{|\nabla\phi|^2}{2\phi^2} = \frac{2|\nabla q|^2}{\psi^2} = \frac{8d^2}{\psi^2} \le \frac{8\rho^2}{\psi^2}. \tag{3.20}$$

Combining (3.3) and (3.18)-(3.20) and multiplying by $\psi(x_*)^2$, we obtain

$$\begin{aligned} 2(1-\varepsilon)\beta H(x_*) &\le 4T\left(1 + \sqrt{K}\rho\right)\psi(x_*) + \left(24\Lambda + 8\Gamma_\varepsilon\right)\rho^2 + 2(n-1)K\psi(x_*)^2 \\ &\le \left[4T + 24\Lambda + 8\Gamma_\varepsilon + 4T\sqrt{K}\rho\right]\rho^2 + 2(n-1)K\rho^4. \end{aligned} \tag{3.21}$$

We next justify this calculation if x_* lies on the cut locus of y. Let $r_* = d(y, x_*) > 0$, choose a minimizing unit-speed geodesic γ from y to x_*, and put $y_\delta = \gamma(\delta)$ for $0 < \delta < r_*$. Calabi's upper support for the distance is

$$d_\delta(x) = \delta + d(y_\delta, x).$$

It is smooth near x_*, satisfies $d_\delta \ge d$, and has $d_\delta(x_*) = r_*$. In a sufficiently small neighborhood, $d_\delta < \rho$, so

$$H_\delta = \left(\rho^2 - d_\delta^2\right)^2 \phi \le H, \quad H_\delta(x_*) = H(x_*).$$

Thus H_δ also has a local maximum at x_*. Perform the calculation there with

$q_\delta = d_\delta^2$. Hessian comparison from y_δ gives

$$\operatorname{Hess} d_\delta(x_*) \le c_K(r_* - \delta)\left(g - \mathrm{d}d_\delta \otimes \mathrm{d}d_\delta\right).$$

Since $r_* c_K(r_* - \delta) \ge 1$, it follows that

$$\mathcal{L} q_\delta(x_*) \le 2T r_* c_K(r_* - \delta).$$

All gradient bounds above remain valid at x_*, with $|\nabla q_\delta|^2 = 4r_*^2$. Letting $\delta \downarrow 0$ recovers (3.21). This also covers the cut locus without differentiating the nonsmooth distance function itself.

Step 3. Completion of the gradient estimate.

At the center of the ball, the maximality of $H(x_*)$ gives

$$\rho^4 \phi(y) = H(y) \le H(x_*).$$

Divide (3.21) by $2(1-\varepsilon)\beta\rho^4$ to obtain

$$\phi(y) \le \frac{2T + 12\Lambda + 4\Gamma_\varepsilon}{(1-\varepsilon)\beta\rho^2} + \frac{2T\sqrt{K}}{(1-\varepsilon)\beta\rho} + \frac{(n-1)K}{(1-\varepsilon)\beta}. \tag{3.22}$$

Now fix $y \in B(o, R/2)$. The closed ball $\overline{B(y, R/2)}$ is contained in $B(o, R)$, since $d(o, y) < R/2$. Apply (3.22) with $\rho = R/2$ and then take the supremum over these y. The choice (3.17) dominates both resulting cutoff coefficients and proves (1.11). Indeed, every ball used in Step 3 satisfies the closure condition required in Step 2:

If $z \in \overline{B(y, R/2)}$ and $y \in B(o, R/2)$, then $d(o, z) \le d(o, y) + d(y, z) < R/2 + R/2 = R$, hence $\overline{B(y, R/2)} \subset B(o, R) \subset \Omega$.

An equally valid alternative is to restrict Step 2 to radii $0 < \rho < \operatorname{dist}(y, \partial\Omega)$, which makes the closure condition automatic without changing the centre estimate (3.22) or the principal estimate.

Choosing $\varepsilon = 1/2$ and using $\sqrt{K}/R \le \left(K + R^{-2}\right)/2$ proves (1.12).

If v is global, first keep ε fixed and let $R \to \infty$ in the center estimate. This gives $\phi(y) \le (n-1)K/\left((1-\varepsilon)\beta\right)$. Only after this limit do we let $\varepsilon \downarrow 0$, obtaining (1.13). The possible divergence of C_ε as $\varepsilon \downarrow 0$ is harmless in this order of limits. □

4. Concluding Remark

In this paper, we establish gradient estimates and Liouville-type theorems for positive solutions to certain nonlinear elliptic equations on complete Riemannian manifolds with sectional curvature bounded from below. Our argument relies on the Bernstein technique, maximum principle and Hessian comparison. The obtained estimates are consistent with the global p-harmonic estimates of Kotschwar and Ni. It remains an interesting problem to extend these results to manifolds with only Ricci curvature lower bound.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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