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Riesz Composition Operators on Weighted Bergman Spaces

Yecheng Shi

School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang, China
Email: 09ycshi@sina.cn

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Abstract

For every analytic self-map of the disk, we prove that Rieszness on the standard weighted Bergman spaces A^p_α is independent of the weight $\alpha > -1$ and exponent $1 \le p < \infty$. The proof uses logarithmic convexity in the weight and an interpolation estimate involving H^2. For symbols of valence at most M, we obtain sharp upper and lower bounds comparing the essential spectral radii on H^2 and on Bergman spaces with different weights. These bounds also give equivalence with Rieszness on H^2 for symbols of finite valence. Boundary periodic points with finite angular derivatives give explicit lower bounds. We also apply classical criteria involving Koenigs functions. The examples attain the bounds, give Riesz operators with no compact positive power, and separate the Hardy and Bergman cases for symbols of infinite valence.

Keywords

Composition Operator, Riesz Operator, Weighted Bergman Space, Essential Spectral Radius, Finite Valence, Logarithmic Convexity

1. Introduction

Let $\mathbb{D}$ be the open unit disk and $\mathbb{T} = \partial\mathbb{D}$. For $\alpha > -1$ and $1 \le p < \infty$, let A^p_α denote the space of analytic functions on $\mathbb{D}$ satisfying

$$\|f\|^p_{A^p_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^\alpha \mathrm{d}A(z) < \infty, \tag{1}$$

where $\mathrm{d}A$ is normalized area measure. Every analytic self-map φ of $\mathbb{D}$ induces a bounded composition operator $C_\varphi f = f \circ \varphi$ on A^p_α.

For a bounded operator T on a Banach space X, its essential norm is

$$\|T\|_{\mathrm{e},X} = \inf\{\|T-K\| : K \text{ is compact on } X\}.$$

The operator T is Riesz if its image in the Calkin algebra has spectrum $\{0\}$, equivalently, if

$$r_{\mathrm{e}}(T;X)=\lim_{n\to\infty}\left\|T^n\right\|_{\mathrm{e},X}^{1/n}=0. \tag{2}$$

For an analytic self-map φ, write $\varphi_0(z)=z$ and $\varphi_{n+1}=\varphi\circ\varphi_n$. Since $C_\varphi^n=C_{\varphi_n}$, Rieszness is determined by the essential norms of the operators C_{φ_n}.

Bourdon and Shapiro ([1], Theorems 4.2 and 5.3) constructed a Riesz composition operator on H^2 whose positive powers are all noncompact, and proved that every Riesz composition operator on H^2 has a symbol with an attracting fixed point in $\mathbb{D}$. For univalent symbols with an attracting fixed point, Poggi-Corradini ([2], Corollary 3.6) characterized Rieszness by the absence of finite angular derivatives at boundary fixed points of the iterates. The relation between Hardy integrability of Koenigs functions and the essential spectral radius was studied in [3]-[5]. Conditions on Koenigs domains involving twisted sectors were studied in [6] [7]. MacCluer and Shapiro [8] related angular derivatives to compactness. Shapiro [9] proved the essential norm formula on H^2.

The spectra of composition operators on Hardy and Bergman spaces were studied in [10] [11]. For standard weighted Bergman spaces, Pons ([12], Theorem 4.1) established an exact relation between the essential spectral radii on A_0^2 and A_α^2 for univalent symbols. He also computed the essential spectral radius for $\varphi(z)=z^M$, $M\geq 2$, when $1<p<\infty$ ([12], Lemma 5.1). Pons also related the essential spectral radii on A_α^p and A_α^2 for $1<p<\infty$ ([12], Theorem 3.8). Qian, Wu and Wulan ([13], Lemma 8) proved the same scaling relation for $1\leq p<\infty$, including the endpoint $p=1$.

An analytic self-map φ of $\mathbb{D}$ has valence at most $M\in\mathbb{N}$ if every $w\in\mathbb{D}$ has at most M inverse images under φ, counted with multiplicity. We compare the essential spectral radii on H^2 and A_α^p, and on Bergman spaces with different weights, with explicit constants depending on M. When $M=1$, the bounds recover the exact weight relation in ([12], Theorem 4.1). The lower bounds are attained by $\varphi(z)=z^M$. We use the known scaling relation in p throughout.

For arbitrary analytic symbols, we prove logarithmic convexity in the weight and an interpolation estimate involving H^2. These estimates and the upper comparison show that Rieszness is independent of the weight and exponent without a valence assumption. The proofs use inequalities for sums of powers in generalized Nevanlinna counting functions and the essential norm estimates in [14]. We then apply these comparisons to boundary periodic points and Koenigs functions using the classical results cited above.

We first compare the essential spectral radii on A_α^p and H^2.

Theorem 1.1. *Let $\alpha>-1$, $1\leq p<\infty$, and let φ be an analytic self-map of $\mathbb{D}$. Then*

$$r_{\mathrm{e}}\left(C_\varphi;A_\alpha^p\right)\leq r_{\mathrm{e}}\left(C_\varphi;H^2\right)^{2(\alpha+2)/p}. \tag{3}$$

If φ has valence at most $M\in\mathbb{N}$, then

$$M^{-(\alpha+1)/p} r_e\left(C_\varphi; H^2\right)^{2(\alpha+2)/p} \le r_e\left(C_\varphi; A^p_\alpha\right) \le r_e\left(C_\varphi; H^2\right)^{2(\alpha+2)/p}. \tag{4}$$

Both constants in (4) *are sharp. Consequently, if* φ *has finite valence, then* C_φ *is Riesz on* A^p_α *if and only if it is Riesz on* H^2. *For univalent* φ, *we have*

$$r_e\left(C_\varphi; A^p_\alpha\right) = r_e\left(C_\varphi; H^2\right)^{2(\alpha+2)/p}. \tag{5}$$

For two Bergman weights, we have the following sharp bounds.

Theorem 1.2. *Let* $-1 < \alpha < \beta$, $1 \le p < \infty$, *and let* φ *be an analytic self-map of* $\mathbb{D}$. *Then*

$$r_e\left(C_\varphi; A^p_\beta\right) \le r_e\left(C_\varphi; A^p_\alpha\right)^{(\beta+2)/(\alpha+2)}. \tag{6}$$

If φ *has valence at most* $M \in \mathbb{N}$, *then*

$$M^{-\frac{\beta-\alpha}{p(\alpha+2)}} r_e\left(C_\varphi; A^p_\alpha\right)^{\frac{\beta+2}{\alpha+2}} \le r_e\left(C_\varphi; A^p_\beta\right). \tag{7}$$

By (6), Rieszness on A^p_α implies Rieszness on A^p_β. The converse follows from the next estimate, which also holds without a valence assumption.

Theorem 1.3. *Let* $-1 < \alpha < \beta$, $1 \le p < \infty$, *and let* φ *be an analytic self-map of* $\mathbb{D}$. *Then*

$$r_e\left(C_\varphi; A^p_\alpha\right) \le r_e\left(C_\varphi; H^2\right)^{\frac{2(\beta-\alpha)}{p(\beta+1)}} r_e\left(C_\varphi; A^p_\beta\right)^{\frac{\alpha+1}{\beta+1}}. \tag{8}$$

Equality holds for univalent φ *and for* $\varphi(z) = z^M$, $M \ge 2$.

All exponents in (8) are positive, so the estimate also applies when one of the essential spectral radii is zero.

Corollary 1.4. *For every analytic self-map* φ *of* $\mathbb{D}$, *the following conditions are equivalent*:

(i) C_φ *is Riesz on* A^p_α *for some* $\alpha > -1$ *and* $1 \le p < \infty$;

(ii) C_φ *is Riesz on* A^p_α *for every* $\alpha > -1$ *and* $1 \le p < \infty$.

Rieszness on H^2 *implies these conditions. If* φ *has finite valence, the converse also holds.*

Example 5.3 shows that Rieszness on every A^p_α need not imply Rieszness on H^2 when the symbol has infinite valence.

We also obtain a lower bound for the essential spectral radius when an iterate of φ has a boundary fixed point with a finite angular derivative (Theorem 4.2).

Section 2 collects the estimates for counting functions used in Section 3 to prove the main results. Section 4 treats boundary periodic points and Koenigs functions, and Section 5 gives the examples, followed by concluding remarks.

2. Counting Functions and Essential Norms

For a nonconstant analytic self-map ψ of $\mathbb{D}$ and $s > 0$, let

$$N_{\psi,s}(w) = \sum_{\psi(z)=w} \left(\log \frac{1}{|z|}\right)^s, \quad w \in \mathbb{D} \setminus \{\psi(0)\},$$

where inverse images are counted with multiplicity. The sum is zero when

$w \notin \psi(\mathbb{D})$. Put

$$Q_s(\psi) = \limsup_{|w|\to 1^-} \frac{N_{\psi,s}(w)}{\left(\log\left(1/|w|\right)\right)^s}.$$

For constant ψ, we set $Q_s(\psi) = 0$.

We use the following known formulas for essential norms and essential spectral radii.

Lemma 2.1. *Let* $\alpha > -1$ *and put* $s = \alpha + 2$. *There exist constants* $c_\alpha, C_\alpha > 0$, *depending only on* α, *such that*

$$c_\alpha Q_s(\psi) \le \left\|C_\psi\right\|^2_{\mathrm{e},A^2_\alpha} \le C_\alpha Q_s(\psi) \tag{9}$$

for every analytic self-map ψ *of* $\mathbb{D}$. *Moreover*,

$$\left\|C_\psi\right\|^2_{\mathrm{e},H^2} = Q_1(\psi), \tag{10}$$

and, for $1 \le p < \infty$,

$$r_{\mathrm{e}}\left(C_\psi; A^p_\alpha\right)^p = r_{\mathrm{e}}\left(C_\psi; A^2_\alpha\right)^2. \tag{11}$$

Proof. Shapiro [9] proved (10). The estimate (9) follows from ([14], Theorem 5) with equal source and target parameters; see also ([12], Eq. (2.1)). The norm defined by a logarithmic weight in that theorem is equivalent to our norm, with constants depending only on α. The identity (11), including $p = 1$, is given in ([13], Lemma 8). Indeed, the radial weight $\omega_\alpha(r) = (\alpha+1)\left(1-r^2\right)^\alpha$ satisfies

$$\hat{\omega}_\alpha(r) := \int_r^1 \omega_\alpha(t)\,\mathrm{d}t \asymp (1-r)^{\alpha+1},$$

and therefore belongs to the doubling class $\hat{\mathcal{D}}$ used in that lemma. □

Applying Lemma 2.1 to the iterates of φ gives the limits

$$\begin{aligned} \lim_{n\to\infty} Q_1(\varphi_n)^{1/n} &= r_{\mathrm{e}}\left(C_\varphi; H^2\right)^2, \\ \lim_{n\to\infty} Q_{\alpha+2}(\varphi_n)^{1/n} &= r_{\mathrm{e}}\left(C_\varphi; A^p_\alpha\right)^p. \end{aligned} \tag{12}$$

The constants in (9) are independent of φ_n, and their nth roots tend to 1. For a constant symbol, all quantities in (12) are zero.

Lemma 2.2. *Let* $1 \le s < t$. *For every analytic self-map* ψ *of* $\mathbb{D}$,

$$Q_t(\psi) \le Q_s(\psi)^{t/s}. \tag{13}$$

If ψ *has valence at most* $L \in \mathbb{N}$, *then*

$$Q_t(\psi) \ge L^{1-t/s} Q_s(\psi)^{t/s}. \tag{14}$$

Proof. The assertion is immediate for constant ψ. For nonconstant ψ and w near $\mathbb{T}$ with $w \ne \psi(0)$, set $x_z = \log\left(1/|z|\right)/\log\left(1/|w|\right)$ for $\psi(z) = w$. Since $t/s > 1$,

$$\sum_{\psi(z)=w} x_z^t \le \left(\sum_{\psi(z)=w} x_z^s\right)^{t/s}.$$

For infinitely many inverse images, apply this inequality to finite sums and pass

to the limit. Taking the limit superior as $|w|\to 1^-$ proves (13). If ψ has valence at most L, Hölder's inequality gives

$$\left(\sum_{\psi(z)=w} x_z^s\right)^{t/s} \le L^{t/s-1}\sum_{\psi(z)=w} x_z^t,$$

which proves (14). □

Lemma 2.3. *Let* $1\le s<t$, $0<\theta<1$, *and* $u=(1-\theta)s+\theta t$. *For every analytic self-map* ψ *of* $\mathbb{D}$,

$$Q_u(\psi)\le Q_s(\psi)^{1-\theta}\,Q_t(\psi)^{\theta}. \tag{15}$$

Proof. For constant ψ, the inequality is immediate. Otherwise, use the quantities x_z from the preceding proof. Hölder's inequality gives

$$\sum_{\psi(z)=w} x_z^u \le \left(\sum_{\psi(z)=w} x_z^s\right)^{1-\theta}\left(\sum_{\psi(z)=w} x_z^t\right)^{\theta}.$$

For an infinite fiber, first apply the inequality to finite partial sums and then pass to the limit. Taking the limit superior as $|w|\to 1^-$ proves (15). All quantities are finite: $Q_1(\psi)<\infty$ by (10), and Lemma 2.2 gives $Q_v(\psi)\le Q_1(\psi)^v$ for $v>1$. □

3. Essential Spectral Radius Comparisons

Proof of Theorem 1.1. Put $s=\alpha+2>1$. By Lemmas 2.1 and 2.2,

$$\begin{aligned}\left\|C_{\varphi_n}\right\|_{e,A^2_\alpha}^2 &\le C_\alpha Q_s(\varphi_n)\\ &\le C_\alpha Q_1(\varphi_n)^s\\ &= C_\alpha\left\|C_{\varphi_n}\right\|_{e,H^2}^{2s}.\end{aligned}$$

Since C_α is independent of n, taking nth roots and letting $n\to\infty$ gives

$$r_e\left(C_\varphi;A^2_\alpha\right)\le r_e\left(C_\varphi;H^2\right)^s.$$

The relation (11) proves (3).

Suppose now that φ has valence at most M. Then φ_n has valence at most M^n. By the same lemmas,

$$\begin{aligned}\left\|C_{\varphi_n}\right\|_{e,A^2_\alpha}^2 &\ge c_\alpha Q_s(\varphi_n)\\ &\ge c_\alpha M^{-n(s-1)}Q_1(\varphi_n)^s\\ &= c_\alpha M^{-n(s-1)}\left\|C_{\varphi_n}\right\|_{e,H^2}^{2s}.\end{aligned}$$

Taking nth roots and letting $n\to\infty$, we obtain

$$r_e\left(C_\varphi;A^2_\alpha\right)\ge M^{-(s-1)/2}r_e\left(C_\varphi;H^2\right)^s.$$

Applying (11) gives (4).

For $M=1$, the two bounds in (4) coincide. Example 5.1 gives equality in the lower bound for $\varphi(z)=z^M$, while $\varphi(z)=z$ gives equality in the upper bound. The equivalence of Rieszness follows from (2). □

By ([3], Theorem 5.4), the essential spectral radii on Hardy spaces satisfy $r_e\left(C_\varphi; H^q\right)^q = r_e\left(C_\varphi; H^2\right)^2$ for $1 \le q < \infty$. Thus Theorem 1.1 also holds with H^2 replaced by H^q, provided that $2(\alpha+2)/p$ is replaced by $q(\alpha+2)/p$.

Proof of Theorem 1.2. Put $s = \alpha + 2$ and $t = \beta + 2$, so that $1 < s < t$. Using (13) and (9) with weights β and α, we obtain

$$\begin{aligned} \left\|C_{\varphi_n}\right\|^2_{e,A^2_\beta} &\le C_\beta Q_t(\varphi_n) \\ &\le C_\beta Q_s(\varphi_n)^{t/s} \\ &\le C_\beta c_\alpha^{-t/s} \left\|C_{\varphi_n}\right\|^{2t/s}_{e,A^2_\alpha}. \end{aligned}$$

The constant $C_\beta c_\alpha^{-t/s}$ is independent of n. Hence

$$r_e\left(C_\varphi; A^2_\beta\right) \le r_e\left(C_\varphi; A^2_\alpha\right)^{t/s}.$$

Together with (11), this proves (6).

Suppose that φ has valence at most M. Since φ_n has valence at most M^n, (14) and (9) give

$$\begin{aligned} \left\|C_{\varphi_n}\right\|^2_{e,A^2_\beta} &\ge c_\beta Q_t(\varphi_n) \\ &\ge c_\beta M^{-n(t-s)/s} Q_s(\varphi_n)^{t/s} \\ &\ge c_\beta C_\alpha^{-t/s} M^{-n(t-s)/s} \left\|C_{\varphi_n}\right\|^{2t/s}_{e,A^2_\alpha}. \end{aligned}$$

Taking nth roots and letting $n \to \infty$ yields

$$r_e\left(C_\varphi; A^2_\beta\right) \ge M^{-(t-s)/(2s)} r_e\left(C_\varphi; A^2_\alpha\right)^{t/s}.$$

Since $(t-s)/s = (\beta-\alpha)/(\alpha+2)$, (11) gives (7). Sharpness follows from Example 5.1: z^M attains the lower bound, and the identity map attains the upper bound. □

The lower bounds in Theorems 1.1 and 1.2 use the worst-case estimate that φ_n has valence at most M^n. For a fixed symbol, the valence of its iterates may grow more slowly. Thus sharpness holds over the class of symbols of valence at most M, but need not mean optimality for each individual symbol.

Proposition 3.1. *Let* $-1 < \alpha < \beta$, $0 < \theta < 1$, *and put* $\gamma = (1-\theta)\alpha + \theta\beta$. *For every analytic self-map* φ *of* $\mathbb{D}$ *and every* $1 \le p < \infty$,

$$r_e\left(C_\varphi; A^p_\gamma\right) \le r_e\left(C_\varphi; A^p_\alpha\right)^{1-\theta} r_e\left(C_\varphi; A^p_\beta\right)^\theta. \tag{16}$$

Equality holds for univalent φ *and for* $\varphi(z) = z^M$, $M \ge 2$.

Proof. Put $s = \alpha + 2$, $t = \beta + 2$, and $u = (1-\theta)s + \theta t = \gamma + 2$. Lemma 2.3, applied to $\psi = \varphi_n$, gives

$$Q_u(\varphi_n) \le Q_s(\varphi_n)^{1-\theta} Q_t(\varphi_n)^\theta.$$

Taking nth roots, using (12), and then taking pth roots proves (16).

For univalent φ, equality follows from (5). For $\varphi(z) = z^M$, equality follows from (20). □

The multiplicative form of (16) includes zero essential spectral radii. The next proof uses the same interpolation inequality with the counting exponent 1, corresponding to H^2.

Proof of Theorem 1.3. Set

$$\theta = \frac{\alpha+1}{\beta+1} \in (0,1).$$

Since $\alpha+2 = (1-\theta)+\theta(\beta+2)$, Lemma 2.3 yields

$$Q_{\alpha+2}(\varphi_n) \le Q_1(\varphi_n)^{1-\theta} Q_{\beta+2}(\varphi_n)^{\theta}.$$

Taking n th roots and using (12), we obtain

$$r_e\left(C_\varphi; A^p_\alpha\right)^p \le r_e\left(C_\varphi; H^2\right)^{2(1-\theta)} r_e\left(C_\varphi; A^p_\beta\right)^{p\theta}.$$

Taking p th roots proves (8). For univalent symbols, equality follows by substituting (5) and using $(1-\theta)+\theta(\beta+2) = \alpha+2$. For $\varphi(z) = z^M$, equality follows from (20) and $(\beta+1)\theta = \alpha+1$. □

Proof of Corollary 1.4. Fix $1 \le p < \infty$ and $-1 < \alpha < \beta$. By (6), Rieszness on A^p_α implies Rieszness on A^p_β. Conversely, suppose that $r_e\left(C_\varphi; A^p_\beta\right) = 0$. Since C_φ is bounded on H^2, Theorem 1.3 gives

$$r_e\left(C_\varphi; A^p_\alpha\right) \le r_e\left(C_\varphi; H^2\right)^{\frac{2(\beta-\alpha)}{p(\beta+1)}} r_e\left(C_\varphi; A^p_\beta\right)^{\frac{\alpha+1}{\beta+1}} = 0.$$

This proves independence of the weight without any valence assumption. Equation (11) then gives independence of p. Finally, (3) gives Rieszness on the Bergman spaces whenever C_φ is Riesz on H^2. For symbols of finite valence, (4) gives the converse. □

4. Boundary Periodic Points and Koenigs Functions

Suppose that ψ has a finite angular derivative at $\zeta \in \mathbb{T}$, with angular limit $\psi(\zeta) \in \mathbb{T}$. By the Julia-Carathéodory theorem,

$$\lim_{r\to 1^-} \frac{1-r^2}{1-|\psi(r\zeta)|^2} = \frac{1}{|\psi'(\zeta)|}. \tag{17}$$

Normalized reproducing kernels give the following lower bound for the essential norm.

Proposition 4.1. *Let $\alpha > -1$. If an analytic self-map ψ has a finite angular derivative at $\zeta \in \mathbb{T}$, then*

$$\left\|C_\psi\right\|_{e,A^2_\alpha} \ge |\psi'(\zeta)|^{-(\alpha+2)/2}. \tag{18}$$

Proof. The reproducing kernel of A^2_α is

$$K_w(z) = (1-\overline{w}z)^{-(\alpha+2)}, \quad \|K_w\|_{A^2_\alpha} = \left(1-|w|^2\right)^{-(\alpha+2)/2}.$$

Put $k_w = K_w / \|K_w\|_{A^2_\alpha}$. Since polynomials are dense in A^2_α, $k_w \to 0$ weakly as $|w| \to 1$.

Let B be a compact operator on A_α^2. Then $\left\|B^* k_w\right\|_{A_\alpha^2} \to 0$. Since $C_\psi^* K_w = K_{\psi(w)}$, we have

$$\begin{aligned}\left\|C_\psi - B\right\| &\geq \limsup_{|w|\to 1^-} \left\|\left(C_\psi - B\right)^* k_w\right\|_{A_\alpha^2} \\ &\geq \limsup_{|w|\to 1^-} \left\|C_\psi^* k_w\right\|_{A_\alpha^2} \\ &= \limsup_{|w|\to 1^-} \left(\frac{1-|w|^2}{1-|\psi(w)|^2}\right)^{(\alpha+2)/2}.\end{aligned}$$

Taking the infimum over compact B, then setting $w = r\zeta$ and using (17), gives (18). □

Theorem 4.2. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that for some $m \geq 1$ and $\zeta \in \mathbb{T}$, φ_m fixes ζ in the angular sense and has a finite angular derivative there. Then, for every $\alpha > -1$ and $1 \leq p < \infty$,*

$$r_e\left(C_\varphi; A_\alpha^p\right) \geq \left|\varphi_m'(\zeta)\right|^{-(\alpha+2)/(mp)} > 0. \tag{19}$$

Consequently, if C_φ is Riesz on A_α^p, then φ has an attracting fixed point in $\mathbb{D}$.

Proof. Put $\psi = \varphi_m$ and $d = \psi'(\zeta) > 0$. Since ψ fixes ζ in the angular sense, the chain rule for angular derivatives ([1], Lemma 2.6) gives $\psi_k'(\zeta) = d^k$ for $k \geq 1$. Proposition 4.1 yields

$$\left\|C_{\varphi_{mk}}\right\|_{e,A_\alpha^2} = \left\|C_{\psi_k}\right\|_{e,A_\alpha^2} \geq d^{-k(\alpha+2)/2}.$$

Taking (mk)th roots and letting $k \to \infty$, we obtain

$$r_e\left(C_\varphi; A_\alpha^2\right) \geq d^{-(\alpha+2)/(2m)}.$$

Equation (11) gives (19).

Suppose now that C_φ is Riesz. If φ has no fixed point in $\mathbb{D}$, its Denjoy-Wolff point ζ lies on $\mathbb{T}$, and φ has a finite positive angular derivative at ζ. Taking $m = 1$ in (19) gives a contradiction.

Thus φ fixes some $a \in \mathbb{D}$. If $\left|\varphi'(a)\right| = 1$, the equality case of the Schwarz lemma shows that φ is an elliptic automorphism. Conjugation by a disk automorphism sending a to 0 transforms φ into a rotation and induces a similarity of composition operators. The operator induced by the rotation is unitary on A_α^2, so all its powers have essential norm 1. Hence $r_e\left(C_\varphi; A_\alpha^2\right) = 1$, contrary to the Riesz property. Therefore $\left|\varphi'(a)\right| < 1$. □

If $\zeta_0, \cdots, \zeta_{m-1}$ is a boundary cycle of φ with finite angular derivatives at each point, where $\varphi\left(\zeta_j\right) = \zeta_{j+1}$ and $\zeta_m = \zeta_0$, the angular chain rule gives

$$\left|\varphi_m'(\zeta_0)\right| = \prod_{j=0}^{m-1} \left|\varphi'\left(\zeta_j\right)\right|.$$

Thus the lower bound in Theorem 4.2 can be evaluated from the angular derivatives along the cycle.

For univalent symbols that are not automorphisms, Poggi-Corradini's criterion on H^2 and Theorem 1.1 give the following characterization. The geometric criterion uses twisted sectors, defined in ([1], p. 239) as sets of the form

$$S_\varepsilon(\Gamma)=\bigcup_{w\in\Gamma}\{z\in\mathbb{C}:|z-w|<\varepsilon|w|\},$$

where $\varepsilon>0$ and Γ is a simple arc joining 0 to ∞. A domain contains no twisted sector if it contains no such set for any ε and Γ.

Corollary 4.3. *Let φ be a univalent analytic self-map of $\mathbb{D}$ that is not an automorphism. For $\alpha>-1$ and $1\le p<\infty$, the following are equivalent*:

(i) C_φ *is Riesz on* A^p_α;

(ii) C_φ *is Riesz on* H^2;

(iii) *No iterate of* φ *has a boundary fixed point with finite angular derivative.*

Each of these conditions implies that φ has an attracting fixed point in $\mathbb{D}$. When φ has such a fixed point a, let σ be its univalent Koenigs function, normalized by $\sigma(a)=0$ and $\sigma'(a)=1$. Then conditions (i)-(iii) *are also equivalent to $\sigma\in\bigcap_{0<q<\infty}H^q$ and to the absence of twisted sectors in $\sigma(\mathbb{D})$.*

Proof. The equivalence of (i) and (ii) follows from (5). Condition (i) implies (iii) by Theorem 4.2. Suppose that (iii) holds. If φ had no interior fixed point, its boundary Denjoy-Wolff point would be fixed with a finite positive angular derivative, contradicting (iii). Thus φ has an interior fixed point a. Since φ is univalent and not an automorphism, $0<|\varphi'(a)|<1$. After conjugating a to 0 by a disk automorphism, ([2], Corollary 3.6) gives (ii). Such conjugation induces similar composition operators and preserves boundary periodic points with finite angular derivatives. The equivalence with the conditions on σ and $\sigma(\mathbb{D})$ follows from [4] [5] [7]; see also ([1], Section 6). Membership of the Koenigs function in each Hardy space is preserved by precomposition with a disk automorphism and multiplication by a nonzero constant. □

The criterion in terms of the Koenigs function also holds for symbols of finite valence.

Corollary 4.4. *Let φ have valence at most $M\in\mathbb{N}$. Suppose that $\varphi(a)=a$ for some $a\in\mathbb{D}$, where $0<|\varphi'(a)|<1$, and let σ be its Koenigs function, normalized by $\sigma(a)=0$ and $\sigma'(a)=1$. The following conditions are equivalent*:

(i) C_φ *is Riesz on* A^p_α *for some* $\alpha>-1$ *and* $1\le p<\infty$;

(ii) C_φ *is Riesz on* A^p_α *for every* $\alpha>-1$ *and* $1\le p<\infty$;

(iii) $\sigma\in\bigcap_{0<q<\infty}H^q$.

Proof. The equivalence of (i) and (ii) follows from Corollary 1.4 without using finite valence. Since φ has finite valence, Theorem 1.1 shows that these conditions are equivalent to Rieszness of C_φ on H^2. The equivalence with (iii) follows from the Hardy-space criterion for Koenigs maps in [5]; see also the discussion in ([1], Section 6, p. 244). This criterion applies to arbitrary analytic self-maps fixing 0 with $0<|\varphi'(0)|<1$, including nonunivalent symbols: Rieszness on H^2 is equivalent to membership of the Koenigs function in every H^q, $q>0$. In particular, the univalence assumption used for the geometric-model results in [4] is

not required here. As in the proof of Corollary 4.3, conjugation by a disk automorphism reduces the fixed point a to 0. □

The argument in ([1], Section 5.2) gives the following necessary condition on Bergman spaces, without a valence assumption.

Proposition 4.5. *Let* $\varphi(a)=a$ *for some* $a\in\mathbb{D}$, *and put* $\lambda=\varphi'(a)$, *where* $0<|\lambda|<1$. *Let* σ *be the analytic Koenigs function normalized by* $\sigma(a)=0$ *and* $\sigma'(a)=1$, *so that* $\sigma\circ\varphi=\lambda\sigma$. *If* C_φ *is Riesz on* A^p_α *for some* $\alpha>-1$ *and* $1\le p<\infty$, *then*

$$\sigma^k\in A^p_\alpha\ \ (k\ge 1),\quad \sigma\in\bigcap_{1\le q<\infty}A^q_\alpha.$$

Proof. Fix $k\ge 1$ and put $L_j(f)=f^{(j)}(a)/j!$ for $0\le j\le k$. These are bounded linear functionals on A^p_α. Let C'_φ be the Banach transpose, defined by $\left(C'_\varphi L\right)(f)=L\left(C_\varphi f\right)$. The chain rule gives

$$C'_\varphi L_j=\lambda^j L_j+\sum_{\ell=0}^{j-1}b_{j,\ell}L_\ell,\quad 0\le j\le k.$$

Thus $\operatorname{span}\{L_0,\cdots,L_k\}$ is invariant under C'_φ, and its restriction has the eigenvalue λ^k. Consequently $\lambda^k\in\sigma\left(C'_\varphi\right)=\sigma\left(C_\varphi\right)$. Since C_φ is Riesz and $\lambda^k\neq 0$, there exists $0\neq f\in A^p_\alpha$ with $f\circ\varphi=\lambda^k f$.

Let j be the order of vanishing of f at a. Comparing the first nonzero Taylor coefficients gives $\lambda^j=\lambda^k$, so $j=k$. Write $f(z)=c(z-a)^k+O\left((z-a)^{k+1}\right)$, where $c\neq 0$. The analytic function $h=f-c\sigma^k$ satisfies $h\circ\varphi=\lambda^k h$ and vanishes to order at least $k+1$ at a. If $h\not\equiv 0$ has a zero of order $m>k$ at a, then $\lambda^m=\lambda^k$, a contradiction. Hence $h\equiv 0$ and $\sigma^k=c^{-1}f\in A^p_\alpha$.

As k is arbitrary, $\sigma\in A^{pk}_\alpha$ for every $k\ge 1$. The measure in (1) is finite; given $1\le q<\infty$, choose k with $pk\ge q$ to conclude that $\sigma\in A^q_\alpha$. □

***Remark* 4.6.** *The converse of Proposition* 4.5 *is not asserted. We exclude* $\varphi'(a)=0$ *because the argument uses the nonzero multiplier in Schröder's equation. By Corollary* 1.4, *the conclusion of the proposition holds for every standard Bergman weight*:

$$\sigma\in\bigcap_{\beta>-1}\bigcap_{1\le q<\infty}A^q_\beta.$$

Membership in all these Bergman spaces does not, by itself, imply that $\sigma\in\bigcap_{0<q<\infty}H^q$. Thus Proposition 4.5 supplies a necessary condition and does not extend the equivalence in Corollary 4.4 to arbitrary symbols.

We also obtain a sufficient condition from ([1], Corollary 3.2) and (3). Suppose that φ is univalent, extends continuously to $\overline{\mathbb{D}}$, and has no finite angular derivative at any boundary fixed point. Assume that every $\zeta\in\mathbb{T}$ not fixed by φ has an iterate $\varphi_n(\zeta)$ in $\mathbb{D}$. Then C_φ is Riesz on A^p_α for every $\alpha>-1$ and $1\le p<\infty$.

5. Examples

We use the compactness criterion of MacCluer and Shapiro ([8], Theorem 3.5):

for every analytic self-map ψ of $\mathbb{D}$, every $\alpha > -1$, and every $1 \le p < \infty$, the operator C_ψ is compact on A^p_α if and only if ψ has no finite angular derivative at any point of $\mathbb{T}$. No univalence or finite-valence assumption is required.

Example 5.1. Let $M \ge 2$ be an integer and put $\varphi(z) = z^M$. Pons ([12], Lemma 5.1) computed the essential spectral radius on A^p_α for $1 < p < \infty$. The following calculation gives the formula for $1 \le p < \infty$ directly. For any integer $K \ge 1$ and $w \ne 0$, all K inverse images under z^K have modulus $|w|^{1/K}$. Hence

$$N_{z^K,s}(w) = K^{1-s}\left(\log\left(1/|w|\right)\right)^s, \quad Q_s\left(z^K\right) = K^{1-s} \quad (s \ge 1).$$

Since $\varphi_n(z) = z^{M^n}$, equation (12) yields

$$r_e\left(C_{z^M}; A^p_\alpha\right) = M^{-(\alpha+1)/p}, \quad r_e\left(C_{z^M}; H^2\right) = 1. \tag{20}$$

The lower bound in (4) is attained. Moreover, for $-1 < \alpha < \beta$,

$$M^{-\frac{\beta-\alpha}{p(\alpha+2)}} r_e\left(C_{z^M}; A^p_\alpha\right)^{\frac{\beta+2}{\alpha+2}} = M^{-(\beta+1)/p} = r_e\left(C_{z^M}; A^p_\beta\right).$$

Hence the factors involving M in Theorems 1.1 and 1.2 are sharp. The identity map gives equality in both upper bounds.

Example 5.2. We use Bourdon and Shapiro's construction of a strip with a bulge to obtain an example on weighted Bergman spaces. Let

$$G = \{w \in \mathbb{C} : |w| < 1\} \cup \{w \in \mathbb{C} : 0 < \operatorname{Im} w < 1\}.$$

Choose a conformal map $\sigma : \mathbb{D} \to G$ with $\sigma(0) = 0$, and put $\varphi(z) = \sigma^{-1}\left(\frac{\sigma(z)}{2}\right)$. Since $\frac{1}{2}G \subset G$, the map φ is univalent. Bourdon and Shapiro ([1], Theorem 4.2) proved that C_φ is Riesz on H^2. By Theorem 1.1, it is Riesz on every A^p_α.

We show that no positive power of C_φ is compact on A^p_α. Fix $n \ge 1$ and choose $x > 2^{n+1}$. Then x and $2^{-n}x$ belong to the straight boundary arc $\{w \in \mathbb{R} : w > 1\}$ of G. The boundary of G is analytic near both points. The local boundary extension theorem and Schwarz reflection therefore give conformal extensions of σ and σ^{-1} across the corresponding arcs.

Let $\zeta \in \mathbb{T}$ be the boundary point corresponding to x. Since $\varphi_n = \sigma^{-1} \circ \left(2^{-n}\sigma\right)$, the map φ_n extends analytically across ζ and satisfies

$$\varphi_n(\zeta) \in \mathbb{T}, \quad 0 < \left|\varphi_n'(\zeta)\right| < \infty.$$

The compactness criterion of MacCluer and Shapiro ([8], Theorem 3.5) therefore shows that C_{φ_n} is not compact on A^p_α.

Since $n \ge 1$ was arbitrary, no positive power of C_φ is compact on any A^p_α.

Example 5.3. We give an explicit Blaschke product B such that C_B is compact on every A^p_α but is not Riesz on H^2. Similar constructions with zeros equally spaced on circles appear in ([15], Section 3).

For $j \ge 4$, put

$$N_j = 2^j, \quad r_j = 1 - 2^{-3j/2}, \quad a_{j,k} = r_j e^{2\pi i k/N_j}, \quad 0 \le k < N_j.$$

Since

$$\sum_{j=4}^{\infty}\sum_{k=0}^{N_j-1}\left(1-\left|a_{j,k}\right|\right)=\sum_{j=4}^{\infty}2^{-j/2}<\infty,$$

the product

$$B(z)=z\prod_{j=4}^{\infty}\prod_{k=0}^{N_j-1}\frac{\left|a_{j,k}\right|}{a_{j,k}}\frac{a_{j,k}-z}{1-\overline{a_{j,k}}z}$$

converges uniformly on compact subsets of $\mathbb{D}$. Thus B is inner, $B(0)=0$, and B has infinite valence, since $B^{-1}(0)$ contains all the points $a_{j,k}$.

We claim that B has no finite angular derivative at any point of $\mathbb{T}$. Fix $\zeta\in\mathbb{T}$. For each $j\geq 4$, choose $0\leq k_j<N_j$ such that $\left|\zeta-\mathrm{e}^{2\pi\mathrm{i}k_j/N_j}\right|\leq\frac{\pi}{N_j}$. Then

$$\left|\zeta-a_{j,k_j}\right|\leq\pi 2^{-j}+\left(1-r_j\right)\leq(\pi+1)2^{-j}.$$

Suppose that B has a finite angular derivative at ζ. Since $B\left(a_{j,k_j}\right)=0$, Julia's inequality gives

$$1\leq\left|B'(\zeta)\right|\frac{\left|\zeta-a_{j,k_j}\right|^2}{1-\left|a_{j,k_j}\right|^2}.$$

Consequently,

$$\begin{aligned}\left|B'(\zeta)\right|&\geq\frac{1-r_j^2}{\left|\zeta-a_{j,k_j}\right|^2}\\&\geq\frac{2^{j/2}}{(\pi+1)^2}.\end{aligned}$$

Letting $j\to\infty$ contradicts the assumed finiteness of the angular derivative.

By the compactness criterion of MacCluer and Shapiro ([8], Theorem 3.5), C_B is compact on A_α^p for every $\alpha>-1$ and $1\leq p<\infty$.

Since B is inner and $B(0)=0$, Nordgren's theorem [16] shows that C_B is an isometry on H^2. Every power of C_B is therefore an isometry. An isometry on an infinite-dimensional Hilbert space has essential norm 1. To see this, apply the isometry and any compact operator to an orthonormal sequence. The images under the compact operator converge to zero in norm, while those under the isometry have norm 1. Consequently,

$$r_{\mathrm{e}}\left(C_B;A_\alpha^p\right)=0\quad(\alpha>-1,1\leq p<\infty),\quad r_{\mathrm{e}}\left(C_B;H^2\right)=1.$$

Thus Rieszness on every A_α^p does not imply Rieszness on H^2 for arbitrary analytic self-maps. In particular, the finite valence assumption in the lower bound of (4) cannot be omitted.

6. Concluding Remarks

For every analytic self-map φ of the disk, Rieszness of C_φ on the standard

weighted Bergman spaces A_α^p is independent of $\alpha > -1$ and $1 \le p < \infty$. The proof uses the essential spectral radius comparisons and interpolation in Theorem 1.3, without a valence assumption.

For symbols of valence at most M, the two-sided bounds comparing H^2 and weighted Bergman spaces are sharp over this class, and Rieszness on H^2 and the Bergman spaces is equivalent. Boundary periodic points provide explicit lower bounds. The examples show that a Riesz composition operator need not have a compact positive power and that the Hardy-Bergman equivalence can fail for infinite-valence symbols.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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