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A Class of Semimodules with a Free Basis of Cardinality 2

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Abstract

A class of semimodules with a free basis of cardinality 2 is investigated. We present the semiring form for the existence of a basis of cardinality 1 in such semimodules, and construct semirings satisfying the condition by means of congruences. In addition, we discuss the freeness of bases of cardinality 1, and prove that $\left\{(c,d)^{\mathrm{T}}\right\}$ forms a free basis of $V_2(S)$ if and only if $ca+db=1$.

Keywords

Semiring, Quotient Semirings, Free Semimodules, Free Basis

1. Introduction

As a natural generalization of modules over rings, the semimodules over semirings have been extensively investigated in the literature. The study of their structures has fundamental significance for understanding linear systems over semirings, matrix theory, and the design of corresponding algorithms (see, e.g., [1]-[5]). In semimodule theory, the study of bases has always been a core problem. A semimodule is said to be free if every element of the semimodule can be uniquely linearly represented by a generating set, which is called a free basis. In 2014, Shu [6], Tan [7] proved that for free semimodules over a commutative semiring S, all free bases share the same cardinality, which equals the minimum cardinality of any generating set; moreover, every basis with the minimum cardinality is a free basis. In 2016, Tan [8] showed that $V_n(S)$ cannot be generated by fewer than n elements. However, this conclusion does not necessarily hold for noncommutative semirings. In 2025, Shu [9] discussed the cardinality of free semimodules over noncommutative semirings. In the same year, Shu [10] also explored the properties of semirings with invariant basis number (IBN). So interesting problems appear: over what class of semirings do there exist free semimodules admitting free bases of varying cardinalities? And is a basis of minimum cardinality necessarily not a free basis?

This paper focuses on semimodules with a free basis of cardinality 2. The main problem addressed is: over which semirings can such semimodules have a basis of cardinality 1? To answer this question, we characterize the conditions that a semiring must satisfy when a semimodule with a free basis of cardinality 2 admits a basis of cardinality 1. Furthermore, we construct explicit examples of such semirings via congruences, which confirms that such semirings are structurally existent rather than purely vacuous abstract assumptions. In addition, we examine the freeness of bases of cardinality 1 in this class of semimodules. We prove that $\left\{(c,d)^{\mathrm{T}}\right\}$ is a free basis of $V_2(S)$ if and only if $ca+db=1$. This result provides an algebraic criterion for the transition from free bases of cardinality 2 to those of cardinality 1, and reveals the intrinsic connection between the corresponding combinatorial invertibility conditions in semirings and the properties of free bases.

The paper is organized as follows. Section 2 presents some basic concepts and preliminaries used throughout the paper. Section 3 investigates the form of semirings in which a semimodule with a free basis of cardinality 2 admits a basis of cardinality 1, provides the corresponding congruence construction, and further discusses the freeness of the basis of cardinality 1.

2. Definition

We present some necessary definitions definitions in this section. For more details, we refer the reader to Refs. [8] [11] [12] and references therein. For notational convenience, we denote by $\mathbb{N}$ the set of all positive integers, and by $\underline{n}$ the set $\{1,\cdots,n\}$ for n in $\mathbb{N}$.

Definition 2.1 [9] [11] *A semiring is a nonempty set* S *on which equipped with two binary operations of addition and multiplication, satisfying the following axioms:*

1) $(S,+,0)$ *is a commutative monoid;*

2) $(S,\cdot,1)$ *is a monoid;*

3) *The equalities* $r\cdot(s+t)=r\cdot s+r\cdot t$ *and* $(s+t)\cdot r=s\cdot r+t\cdot r$ *hold for all* $r,s,t\in S$*;*

4) $0\cdot r=r\cdot 0=0$ *holds for all* $r\in S$*;*

5) $0\neq 1$*.*

A semiring S is called commutative if $r\cdot r'=r'\cdot r$ for all $r,r'\in S$.

Example 2.2 [11] *Consider the set of all binary relations on a fixed set* X*,*

$$\mathrm{Rel}(X)=\{R\mid R\subseteq X\times X\},$$

on which two operations have been defined. The addition is given by the union of relations: $R+S=R\cup S$*. The multiplication is given by the composition of relations:* $R\cdot S=R\circ S=\{(x,z)\mid\exists y\in X:(x,y)\in R\text{ and }(y,z)\in S\}$*. Define the special* *elements: Additive identity* $0=\varnothing$ *which is the empty relation; Multiplicative identity* $1=I_X=\{(x,x)\mid x\in X\}$ *which is the identity relation on* X*.*

It is easy to verify that $(\mathrm{Rel}(X),\cup,\circ,\varnothing,I_X)$ *satisfies the definition of a semiring. This semiring is called the relation semiring, which is also referred to as the semiring of relation algebras. Taking* $X=\mathbb{N}$*, we obtain the infinite-element relation semiring* $\mathrm{Rel}(\mathbb{N})$*.*

Definition 2.3 [11] *Let* $(S,+_S,\cdot_S)$ *and* $(F,+_F,\cdot_F)$ *be semirings. A map* $\varphi:S\to F$ *is called a semiring homomorphism if and only if for all* $a,b\in S$*, the following hold:*

$$\varphi(a+_S b)=\varphi(a)+_F\varphi(b);\ \varphi(a\cdot_S b)=\varphi(a)\cdot_F\varphi(b);\ \varphi(0_S)=0_F;\ \varphi(1_S)=1_F.$$

The kernel of a semiring homomorphism φ *is defined as*

$$\ker\varphi=\{(a,b)\in S\times S\mid\varphi(a)=\varphi(b)\}.$$

Definition 2.4 [10] *A commutative monoid* $(M,+,\mathbf{0})$ *together with a scalar multiplication* $(s,\boldsymbol{m})\mapsto s\boldsymbol{m}$ *from* $S\times M$ *to* M *is a left semimodule over a semiring* S *(a left* S*-semimodule) if and only if the following identities hold for all* λ,μ *in* S *and* $\boldsymbol{m},\boldsymbol{m}'$ *in* M*:*

1) $(\lambda\mu)\boldsymbol{m}=\lambda(\mu\boldsymbol{m})$;
2) $\lambda(\boldsymbol{m}+\boldsymbol{m}')=\lambda\boldsymbol{m}+\lambda\boldsymbol{m}'$;
3) $(\lambda+\mu)\boldsymbol{m}=\lambda\boldsymbol{m}+\mu\boldsymbol{m}$;
4) $1\boldsymbol{m}=\boldsymbol{m}$;
5) $\lambda\mathbf{0}=\mathbf{0}=0\boldsymbol{m}$.

Similarly, we have the definition of a right S-semimodule. Semimodules are also called semilinear spaces in some literatures, see, e.g., [6] [12].

In what follows, unless otherwise stated, S-semimodules always mean left semimodules.

Definition 2.5 [8] *Let A be a nonempty subset of an S-semimodule M. If every element of M can be expressed as a linear combination of elements from A, then A is called a generating set of M. If for each* $a\in A$*,* a *cannot be expressed as a linear combination of the remaining elements of A, then A is called a basis of M. If every element of M has a unique representation as a linear combination of elements of A, then A is called a free basis of M. A generating set consisting of a single element is still called a basis of M.*

Example 2.6 [9] *Let S be a semiring. For each* $n\geq 1$*, let*

$$V_n(S)=\{(a_1,a_2,\cdots,a_n)^{\mathrm{T}}:a_i\in S,i\in\underline{n}\},$$

where $(a_1,a_2,\cdots,a_n)^{\mathrm{T}}$ *denotes the transpose of* $(a_1,a_2,\cdots,a_n)$*. Define*

$$\boldsymbol{x}+\boldsymbol{y}=(x_1+y_1,x_2+y_2,\cdots,x_n+y_n)^{\mathrm{T}},$$
$$r*\boldsymbol{x}=(r\cdot x_1,r\cdot x_2,\cdots,r\cdot x_n)^{\mathrm{T}}$$

for all $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}$*,* $\boldsymbol{y}=(y_1,y_2,\cdots,y_n)^{\mathrm{T}}\in V_n(S)$ *and all* $r\in S$*. Then* $V_n(S)$ *is an S-semimodule, where the zero element* $\mathbf{0}_{n\times1}=(0,0,\cdots,0)^{\mathrm{T}}$*.*

When $n=2$*,* $V_2(S)=\{(a_1,a_2)^{\mathrm{T}}:a_i\in S,i\in\underline{n}\}$ *is an S-semimodule, and*

$\left\{\boldsymbol{e}_1 = (1,0)^{\mathrm{T}}, \boldsymbol{e}_2 = (0,1)^{\mathrm{T}}\right\}$ *is a free basis of* $V_2(S)$.

3. Main Results

In what follows, we discuss whether there exists a semiring S such that the S-semimodule M has a free basis of cardinality 2 and also admits a basis of cardinality 1.

Lemma 3.1 [9] *Let* M_1 *and* M_2 *be two free S-semimodules. Then* $M_1 \cong M_2$ *if and only if there exist free bases with the same cardinality in* M_1 *and* M_2, *separately.*

By Lemma 3.1, we know that for an S-semimodule M, if M has a free basis of cardinality n, then M is isomorphic to the semimodule $V_n(S)$. Therefore, the semimodule $V_2(S)$ considered below is of general significance.

Theorem 3.2 *In the S-semimodule* $V_2(S)$, *the singleton* $\left\{(c,d)^{\mathrm{T}}\right\}$ *is a basis of* $V_2(S)$ *if and only if there exist* $a,b \in S$ *satisfying* $ac = bd = 1$ *and* $ad = bc = 0$.

Proof. In the semimodule $V_2(S)$, there exist $a,b \in S$ such that $ac = bd = 1$ and $ad = bc = 0$ if and only if

$$a \cdot (c,d)^{\mathrm{T}} = (ac,ad)^{\mathrm{T}} = (1,0)^{\mathrm{T}};$$

$$b \cdot (c,d)^{\mathrm{T}} = (bc,bd)^{\mathrm{T}} = (0,1)^{\mathrm{T}}.$$

This holds if and only if the vectors $(1,0)^{\mathrm{T}}$ and $(0,1)^{\mathrm{T}}$ can be expressed as linear combinations of $(c,d)^{\mathrm{T}}$, which is equivalent to that $\left\{(c,d)^{\mathrm{T}}\right\}$ is a basis of $V_2(S)$.

Theorem 3.3 *Let* $(S,+,\cdot)$ *be a finite set equipped with binary operations of addition and multiplication, where* 0 *is the additive zero element and* 1 *is the multiplicative identity element. If exist* $a,b,c,d \in S$ *satisfy*

$$ac = 1,\ bd = 1,\ ad = bc = 0,$$

then S cannot form a semiring.

Proof. Define a set theoretic map $\varphi_a : S \to S$, by $\varphi_a(x) = ax$. For any $s \in S$, we have

$$s = 1 \cdot s = (ac)s = a(cs) = \varphi_a(cs),$$

so φ_a is surjective. Since S is a finite set, every surjective map is injective.

Direct computation gives

$$\varphi_a(d) = ad = 0,\quad \varphi_a(bc) = a(bc) = a \cdot 0 = 0.$$

By the injectivity of φ_a, we obtain $d = bc$. Combining with the condition $bc = 0$, we deduce that $d = 0$.

Substituting into $bd = 1$, we get

$$1 = bd = b \cdot 0 = 0,$$

which contradicts $0 \neq 1$.

In what follows, we first construct an infinite semiring by means of the universal property, and then impose a congruence relation on the quotient set satisfies the equalities $ac = bd = 1$, $ad = bc = 0$. In this way we obtain the desired quotient semiring.

Start with the set $\{a,b,c,d\}$. Define the multiplication for the generated semiring by concatenation of strings over this set, and adjoin the empty string 1 representing the multiplicative identity. For example, $ab \cdot ad = abad$. In this manner, the set $\{a,b,c,d\}$ generates a multiplicative monoid *G*.

Take all finite formal sums of elements in the monoid *G*, and denote the enlarged set by E. That is, every $x \in E$ can be written as

$$x = \sum_i x_i, \quad x_i \in G.$$

The addition and multiplication on E are defined as follows:

- **Addition**: Formal addition, which is commutative and associative. The additive zero element 0 is the empty sum.
- **Multiplication**: Expand by the distributive law, and perform string concatenation from the monoid for each term:

$$(x_1 + x_2)(y_1 + y_2) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2.$$

The set E equipped with such addition and multiplication is exactly the semiring generated by the generating set $\{a,b,c,d\}$.

Remark 3.4 *In the semiring E, only equalities derivable from the definition of a semiring hold. Two formal sums whose equality cannot be derived from the definition are regarded as distinct elements in E. For example, we do not have* $ac = 1$, $bc = cb$, *and so on. Therefore, if* $x_1 \cdot y_1 = x_2 \cdot y_2$, *we have* $x_1 = x_2$ *and* $y_1 = y_2$. *In the semiring E at this stage,* ac *is merely a monomial obtained by concatenation and is not equal to* 1*; similarly,* ad *is not equal to* 0.

To satisfy the requirements of Proposition 3.2, we now introduce a semiring congruence relation to enforce the desired equalities, and construct the quotient semiring via this congruence relation.

Within $E \times E$, we first set

$$R_0 = \{(ac,1),(bd,1),(ad,0),(bc,0)\}.$$

We construct the set R as follows. First, we require $R_0 \subseteq R$. Next, R is closed under addition and multiplication: for all $x \in E$ and $(u,v) \in R$, we have $(x+u, x+v)$, $(u+x, v+x)$, (xu, xv), $(ux, vx) \in R$. Finally, for every $x \in E$, we add (x,x) to *R*; for every $(x,y) \in R$, we add (y,x) to *R*; and for any $(x,y),(y,z) \in R$, we add (x,z) to *R*.

Proposition 3.5 *The relation R constructed by the above procedure is the smallest semiring congruence containing* $\{(ac,1),(bd,1),(ad,0),(bc,0)\}$.

Proof. First, the construction of R guarantees that R is an equivalence relation. Let (x,x'), $(y,y') \in R$. Then we have $(x+y, x+y') \in R$ and $(x+y', x'+y') \in R$. By transitivity, $(x+y, x'+y') \in R$. Similarly, we obtain $(xy, x'y') \in R$. Hence R satisfies the conditions for a semiring congruence: it is an equivalence relation compatible with addition and multiplication. Therefore, R is a semiring congruence on *E*.

Let

$$\mathcal{C} = \{\rho \subseteq E \times E \mid \rho \text{ is a semiring congruence on } E \text{ and } R_0 \subseteq \rho\}.$$

Take arbitrary $\rho \in \mathcal{C}$, *i.e.*, ρ is a semiring congruence on E with $R_0 \subseteq \rho$. Clearly, for all $x \in E$ and all $(u,v) \in R_0$, we have

$$(x+u, x+v), (u+x, v+x), (xu, xv), (ux, vx) \in \rho.$$

This shows that every semiring congruence containing R_0 must contain *R*. Consequently, *R* is the smallest semiring congruence containing R_0.

Denote the equivalence classes by

$$a_S = [a],\ b_S = [b],\ c_S = [c],\ d_S = [d],$$

and let $1_S = [1]$ be the multiplicative identity and $0_S = [0]$ be the additive zero element of the quotient semiring.

Verify the conditions under the operations on the quotient semiring:

$$a_S c_S = [a] \cdot [c] = [ac] = [1] = 1_S,$$

$$b_S d_S = [b] \cdot [d] = [bd] = [1] = 1_S,$$

$$a_S d_S = [ad] = [0] = 0_S,$$

$$b_S c_S = [bc] = [0] = 0_S.$$

Since the semiring E is generated by $\{a,b,c,d\}$, the quotient semiring S is naturally generated by a_S, b_S, c_S, d_S. Thus the quotient semiring $S = E/R$ is a semiring satisfying all the required conditions.

Now arises a critical question: whether the quotient semiring forces the equality $0_S = 1_S$? To answer this question, we have the following proposition.

Theorem 3.6 *Let E be the semiring generated by the elements* $\{a,b,c,d\}$ *as described above. Denote the natural projection by* $\pi : E \to S = E/R$. *For any semiring* F, *if* $\varphi : E \to F$ *is a semiring homomorphism satisfying*

$$\varphi(ac) = 1_F,\ \varphi(bd) = 1_F,\ \varphi(ad) = 0_F,\ \varphi(bc) = 0_F,$$

then there exists a unique semiring homomorphism $\bar{\varphi} : S \to F$ *such that the diagram commutes:* $\varphi = \bar{\varphi} \circ \pi$ (see **Figure 1**).

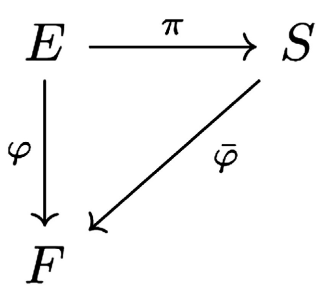

Figure 1. Commutative diagram.

Proof. By the definition of the natural projection, for each $s \in S$, there exists some $x \in E$ with $s = \pi(x)$. Define the map

$$\overline{\varphi}(s) := \varphi(x).$$

For the homomorphism φ, we have the base relation pairs:

$$(ac,1) \in R_0 : \varphi(ac) = 1_F = \varphi(1);\ (bd,1) \in R_0 : \varphi(bd) = 1_F = \varphi(1);$$

$$(ad,0) \in R_0 : \varphi(ad) = 0_F = \varphi(0);\ (bc,0) \in R_0 : \varphi(bc) = 0_F = \varphi(0).$$

Clearly, $(u,v) \in \ker\varphi$ holds for all $(u,v) \in R_0$. Since R is the smallest semiring congruence containing R_0, and $\ker\varphi$ is a semiring congruence containing R_0, we obtain $R \subseteq \ker\varphi$. Consequently, $\varphi(u) = \varphi(v)$ for every $(u,v) \in R$. Therefore, for any $s \in S$, if $x, y \in E$ satisfy $s = \pi(x) = \pi(y)$, then $(x,y) \in R$, which yields $\varphi(x) = \varphi(y)$. This shows that there exists a unique element $t \in F$ such that $\overline{\varphi}(s) = t$, so the map $\overline{\varphi}$ is well-defined.

We now verify that $\overline{\varphi}$ is a semiring homomorphism and that the diagram commutes.

Take arbitrary $u \in E$ and set $s = \pi(u)$. Then

$$(\overline{\varphi} \circ \pi)(u) = \overline{\varphi}(\pi(u)) = \overline{\varphi}(s) = \varphi(u).$$

Let $s_1, s_2 \in S$. There exist $u_1, u_2 \in E$ such that $s_1 = \pi(u_1)$, $s_2 = \pi(u_2)$. Since π preserves addition and φ is a semiring homomorphism,

$$\begin{aligned}\overline{\varphi}(s_1 + s_2) &= \overline{\varphi}(\pi(u_1 + u_2)) = \varphi(u_1 + u_2) = \varphi(u_1) + \varphi(u_2) \\ &= \overline{\varphi}(\pi(u_1)) + \overline{\varphi}(\pi(u_2)) = \overline{\varphi}(s_1) + \overline{\varphi}(s_2).\end{aligned}$$

$$\begin{aligned}\overline{\varphi}(s_1 \cdot s_2) &= \overline{\varphi}(\pi(u_1 \cdot u_2)) = \varphi(u_1 \cdot u_2) = \varphi(u_1) \cdot \varphi(u_2) \\ &= \overline{\varphi}(\pi(u_1)) \cdot \overline{\varphi}(\pi(u_2)) = \overline{\varphi}(s_1) \cdot \overline{\varphi}(s_2).\end{aligned}$$

Moreover,

$$\overline{\varphi}(0_S) = \overline{\varphi}(\pi(0_E)) = \varphi(0_E) = 0_F,\ \overline{\varphi}(1_S) = \overline{\varphi}(\pi(1_E)) = \varphi(1_E) = 1_F.$$

Hence $\overline{\varphi} : S \to F$ is a semiring homomorphism that makes the diagram commute.

Now suppose there is another semiring homomorphism $\overline{\varphi}' : S \to F$ satisfying $\varphi = \overline{\varphi}' \circ \pi$. For any $s \in S$, there exists $u \in E$ such that $s = \pi(u)$. Then

$$\overline{\varphi}'(s) = \overline{\varphi}'(\pi(u)) = (\overline{\varphi}' \circ \pi)(u) = \varphi(u) = (\overline{\varphi} \circ \pi)(u) = \overline{\varphi}(\pi(u)) = \overline{\varphi}(s).$$

Since s is arbitrary, $\overline{\varphi} = \overline{\varphi}'$. That is, the homomorphism $\overline{\varphi}$ making the diagram commute is unique.

Thus the universal property holds.

By Proposition 3.6, suppose that $\pi(0) = \pi(1)$ holds in the semiring S. Then

$$0_F = \varphi(0) = \overline{\varphi}(\pi(0)) = \overline{\varphi}(\pi(1)) = \varphi(1) = 1_F.$$

This contradicts the fact that F is a semiring. Therefore, as long as we can find a semiring F satisfying the given equalities, we must have $0_S \neq 1_S$ in $S = E/R$.

Example 3.7 *Let* $F = \mathrm{Rel}(\mathbb{N})$ *be the semiring of binary relations on* $\mathbb{N}$. *Clearly,* $0_F = \varnothing \neq 1_F = I_F$. *Take the following elements in F:*

$$a_F = \{(i,2i) \mid i \in \mathbb{N}\}, \quad b_F = \{(i,2i+1) \mid i \in \mathbb{N}\},$$
$$c_F = \{(2i,i) \mid i \in \mathbb{N}\}, \quad d_F = \{(2i+1,i) \mid i \in \mathbb{N}\}.$$

Recall that the relational multiplication is defined as

$$a_F \cdot c_F = \{(i,j) \mid \exists n \in \mathbb{N} : (i,n) \in a_F \text{ and } (n,j) \in c_F\}.$$

For every $i \in \mathbb{N}$*, set* $n = 2i$ *and* $j = i \in \mathbb{N}$*. Then* $a_F \cdot c_F = \{(i,i) \mid i \in \mathbb{N}\} = I_{\mathbb{N}} = 1_F$.

Similarly, $b_F \cdot d_F = 1_F$*. When computing* $a_F \cdot d_F$ *(or* $b_F \cdot c_F$*), for any* $i \in \mathbb{N}$ *there exists no* $n \in \mathbb{N}$ *such that* $(i,n) \in a_F$ *and* $(n,j) \in d_F$ *(or* $(i,n) \in b_F$ *and* $(n,j) \in c_F$*). Consequently,* $a_F \cdot d_F = \varnothing = 0_F$ *and* $b_F \cdot c_F = \varnothing = 0_F$*.*

Define the homomorphism $\varphi : E \to F$ *by*

$$\varphi(a) = a_F,\ \varphi(b) = b_F,\ \varphi(c) = c_F,\ \varphi(d) = d_F,\ \varphi(0) = 0_F,\ \varphi(1) = 1_F.$$

We obtain

$$\varphi(ac) = a_F \circ c_F = 1_F, \quad \varphi(bd) = b_F \circ d_F = 1_F,$$
$$\varphi(ad) = a_F \circ d_F = 0_F, \quad \varphi(bc) = b_F \circ c_F = 0_F.$$

Thus F satisfies the required conditions. From the commutative-diagram argument, we conclude that $0_S \neq 1_S$ *holds in* $S = E/R$*.*

In fact, every binary relation is isomorphic to a Boolean matrix. Consider the set U of infinite-dimensional Boolean matrices whose entries are either 0 or 1, with at most finitely many ones in each row and each column. Matrix addition on U is given by $X + Y = (x_{ij} \vee y_{ij})$*, and matrix multiplication* $Z = (z_{ij}) = XY$ *is defined by* $z_{ij} = \mathrm{V}_k x_{i,k} \wedge y_{k,j}$*. That is, addition corresponds to logical OR and multiplication corresponds to logical AND. The zero element is the zero matrix O, and the multiplicative identity is the identity matrix I, where* $x_{ii} = 1$ *and all other entries are 0. One can verify that U is a semiring, called the infinite-dimensional Boolean matrix semiring.*

Take four matrices in U:

$$A = (a_{ij}), \text{ where } a_{i,2i} = 1 \text{ and all other entries are } 0;$$
$$B = (b_{ij}), \text{ where } b_{i,2i+1} = 1 \text{ and all other entries are } 0;$$
$$C = (c_{ij}), \text{ where } c_{2i,i} = 1 \text{ and all other entries are } 0;$$
$$D = (d_{ij}), \text{ where } d_{2i+1,i} = 1 \text{ and all other entries are } 0.$$

We have

$$AC = I,\ BD = I,\ AD = O,\ BC = O.$$

This shows that the infinite-dimensional Boolean matrix semiring U is also a semiring satisfying the conditions.

The quotient semiring we constructed is an abstract construction based on the existence provided by the universal property. It is an infinite semiring, so we cannot write down an exhaustive list of its elements. In contrast to finite semirings, for the infinite quotient semiring *S*, the map $\varphi_a : S \to S, x \mapsto ax$ is surjective but not injective. There may exist two distinct elements d and bc such that $\varphi_a(d) = \varphi_a(bc) = 0$. This does not force $d = 0$.

By Proposition 3.2, the vector $\{\boldsymbol{v} = (c,d)^{\mathrm{T}}\}$ is a basis of $V_2(S)$. But is $\{\boldsymbol{v}\}$ a free basis of $V_2(S)$?

Theorem 3.8 *Let S be a semiring, and suppose that there exist elements* a,b,c,d *in* S *such that* $ac = bd = 1$ *and* $ad = bc = 0$. *In the semimodule* $V_2(S)$, *if* $\exists u,v,w \in S$ *such that* $(cu+dv)w = 1$, *then* $\{(c,d)^{\mathrm{T}}\}$ *is a free basis of* $V_2(S)$.

Proof. Suppose $s_1, s_2 \in S$ satisfy $s_1(c,d)^{\mathrm{T}} = s_2(c,d)^{\mathrm{T}}$, *i.e.*

$$\begin{cases} s_1c = s_2c & (1) \\ s_1d = s_2d & (2) \end{cases}$$

Multiply both sides of (1) on the right by u to obtain $s_1dv = s_2dv$. Multiply both sides of (2) on the right by v to obtain $s_1dv = s_2dv$. Add the two equalities and then multiply both sides on the right by w:

$$s_1(cu+dv)w = s_2(cu+dv)w.$$

Hence $s_1 = s_2$, so $\{(c,d)^{\mathrm{T}}\}$ is a free basis.

Theorem 3.9 *Let S be a semiring, and suppose that there exist elements* a,b,c,d *in S such that* $ac = bd = 1$ *and* $ad = bc = 0$. *In the semimodule* $V_2(S)$, $\{(c,d)^{\mathrm{T}}\}$ *is a free basis of* $V_2(S)$ *if and only if* $ca + db = 1$.

Proof. Consider $ca + db \in S$. We compute:

$$\begin{aligned}(ca+db)\cdot(c,d)^{\mathrm{T}} &= (ca+db)\cdot(c,d)^{\mathrm{T}} \\ &= \left((ca+db)c,(ca+db)d\right)^{\mathrm{T}} \\ &= \left(c(ac)+d(bc),c(ad)+d(bd)\right)^{\mathrm{T}} \\ &= (c\cdot 1 + d\cdot 0, c\cdot 0 + d\cdot 1)^{\mathrm{T}} \\ &= 1\cdot(c,d)^{\mathrm{T}}.\end{aligned}$$

When $ca + db = 1$, the condition of Proposition 3.8 is satisfied, so $\{(c,d)^{\mathrm{T}}\}$ is a free basis of $V_2(S)$.

When $ca + db \neq 1$, there exist distinct elements $ca + db$ and 1 such that $(ca+db)\cdot(c,d)^{\mathrm{T}} = 1\cdot(c,d)^{\mathrm{T}}$. That is, $(c,d)^{\mathrm{T}}$ admits at least two distinct representations. Therefore $\{(c,d)^{\mathrm{T}}\}$ is not a free basis of $V_2(S)$.

It should be noted that if $ca + db \neq 1$, then there do not exist $u,v,w \in S$ satisfying $(cu+dv)w = 1$; otherwise we would get a contradiction to the fact that $\{(c,d)^{\mathrm{T}}\}$ is not a free basis.

In our constructed quotient semiring $S = E/R$, one cannot deduce $ca + db = 1$ from the given hypotheses. Thus in general $\{(c,d)^{\mathrm{T}}\}$ in $V_2(S)$ is a non-free basis of cardinality one. Nevertheless, the following example shows that semirings with $ca + db = 1$ also exist.

Example 3.10 *In the relation semiring* $F = \mathrm{Rel}(\mathbb{N})$ *from Example* 3.7, *we have*

$$ca = \{(2i,2i) \mid i \in \mathbb{N}\};\ db = \{(2i+1,2i+1) \mid i \in \mathbb{N}\}.$$

Here ca is the identity relation on even numbers and db is the identity relation on odd numbers. Hence $ca+db=1_F$*, so* $\left\{(c,d)^{\mathrm{T}}\right\}$ *is a free basis of* $V_2(F)$*.*

Now within $F=\mathrm{Rel}(\mathbb{N})$*, set*

$$a'=\left\{(i,3i)\mid i\in\mathbb{N}\right\},\quad b'=\left\{(i,3i+1)\mid i\in\mathbb{N}\right\},$$
$$c'=\left\{(3i,i)\mid i\in\mathbb{N}\right\},\quad d'=\left\{(3i+1,i)\mid i\in\mathbb{N}\right\}.$$

One can easily verify that $a'c'=b'd'=1$ *and* $a'd'=b'c'=0$*. Consequently, for every* $(x,y)^{\mathrm{T}}\in V_2(T)$*, we have* $(x,y)^{\mathrm{T}}=(xa'+yb')(c',d')^{\mathrm{T}}$*. Moreover,*

$$c'a'=\left\{(3i,3i)\mid i\in\mathbb{N}\right\};\ d'b'=\left\{(3i+1,3i+1)\mid i\in\mathbb{N}\right\}.$$

Thus $c'a'+d'b'\neq 1_F$ *(since* $3i+2$ *is not contained in* $c'a'+d'b'$*). Therefore* $\left\{(c',d')^{\mathrm{T}}\right\}$ *is not a free basis of* $V_2(F)$*.*

This illustrates that the semimodule $V_2(F)$ *over the relation semiring F possesses both free bases of cardinality one and non-free bases of cardinality one.*

4. Conclusions

This paper investigates when the semimodule $V_2(S)$ admits a singleton basis. We characterize necessary-and-sufficient conditions on S and prove that such semirings must be infinite. Constructing a congruence over the free semiring generated by $\{a,b,c,d\}$ yields an infinite quotient semiring, whose non-triviality is confirmed via the relation semiring $\mathrm{Rel}(\mathbb{N})$. We also obtain the criterion that $(c,d)^{\mathrm{T}}$ is a free basis of $V_2(S)$ if and only if $ca+db=1$. Our examples show that $V_2(S)$ over a fixed semiring may possess both free and non-free singleton bases.

For commutative semirings, free semimodules have free bases of fixed cardinality, and every minimal generating set is a free basis. These properties fail in the non-commutative setting: one semimodule can simultaneously carry a cardinality-2 free basis and a cardinality-1 basis, so ordinary singleton bases must be distinguished from free ones. This reveals essential distinctions among ring-module theory, commutative-semiring semimodule theory and non-commutative-semiring semimodule theory, and our results enrich the basis theory for non-commutative semimodules. Future work may consider free semimodules whose free basis has cardinality $n\geq 3$, and explore conditions for the existence of bases with different cardinalities in higher-dimensional cases.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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