

Classification of Computational Methods for High Powers of Matrices with Pedagogical Reflections

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Abstract

The computation of high powers of matrices is a core component of linear algebra courses and a pervasive cognitive bottleneck for students. To address common teaching challenges such as fragmented methodological systems, the absence of systematic selection strategies, and frequent misapplication of underlying principles, this paper categorizes seven widely used computational methods into three modules: structural simplification, similarity transformation, and polynomial degree reduction. Their unified mathematical essence is revealed through the lenses of linear space decomposition and polynomial annihilation theory. A progressive decision flowchart is constructed based on the cost-based recommendation logic of computational operations, guiding students to select methods in ascending order of information acquisition cost. Furthermore, targeted teaching improvement strategies are proposed from three dimensions: thematic integration, condition discrimination, and integration of computational thinking, targeting four typical cognitive misconceptions.

Keywords

High Powers of Matrices, Method Classification, Solution Selection Decision, Pedagogical Strategies

1. Introduction

As the central object of study in linear algebra, matrices serve as a critical bridge between algebraic theory and engineering applications, including reachability analysis in graph theory, steady-state solution of Markov chains, and discrete-system modeling. The computation of high powers of matrices spans multiple teach-

ing chapters—matrix operations, eigenvalue theory, Jordan canonical form, and others—and features broad knowledge coverage, diverse solution methods, and a high degree of problem-solving sophistication.

In teaching practice, students generally face three difficulties. First, difficulty in memorization: they fail to establish logical connections among various formulas, resulting in fragmented knowledge. Second, difficulty in method selection: without systematic selection strategies, they often resort to blind trial and error. Third, difficulty in principle comprehension: vague understanding of the scope of applicability leads to frequent errors, such as misusing the binomial theorem or ignoring diagonalizability prerequisites.

Most mainstream textbooks adopt a structure of method enumeration followed by illustrative examples [1] [2]. While this ensures content completeness, it weakens the logical relationships among methods and the analysis of their applicable scenarios. Existing pedagogical studies [3]-[7] mainly focus on enumerating specific computational techniques and summarizing relevant skills, but lack a unified analytical framework and an operable decision-making workflow. This gap makes it difficult to guide students toward systematic problem-solving thinking.

Against this background, this paper systematically reconstructs the computational methods for high powers of matrices from the perspective of mathematics education. By extracting their unified mathematical essence, constructing a progressive selection decision process, and proposing targeted pedagogical improvement strategies based on four typical cognitive misconceptions, this paper aims to help students build a complete methodological system and improve both their problem-solving efficiency and core mathematical literacy.

2. Classification of Computational Methods and Their Mathematical Essence

Unless otherwise specified, all matrices discussed in this paper are defined over the complex field $\mathbb{C}$. According to different core simplification paths, the common methods for computing high powers of matrices can be divided into three categories: structural simplification, similarity transformation, and polynomial degree reduction. These three categories simplify power operations from three dimensions—the inherent structure of the matrix, the transformation to a similarity canonical form, and the degree reduction via annihilating polynomials.

2.1. Structural Simplification Methods

Structural simplification methods exploit special structural features of matrices to avoid full element-wise multiplication, achieving the highest computational efficiency in applicable scenarios.

(1) Inductive Recursion Method

By computing the first few powers of a matrix, one can summarize the pattern of element variation and derive the general formula for the n th power, which is then rigorously proved by mathematical induction [8]. This method applies to

scenarios where matrix elements follow arithmetic, geometric, or combinatorial patterns as the exponent increases. Taking a 3rd-order Jordan block as an example:

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

By computing J^2 and J^3 , one can inductively derive the general formula for the n th power:

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

(2) Decomposition Method

The matrix is decomposed into the sum or product of matrices with special algebraic properties, and the computation is simplified using those properties. It mainly includes two scenarios:

- **Rank-1 matrix decomposition:** If $\text{rank}(A) = 1$, then A can be written as $A = \alpha\beta^T$, where α is a column vector and β^T is a row vector. By the associative law of matrix multiplication,

$$A^n = (\beta^T \alpha)^{n-1} A,$$

where $\beta^T \alpha$ is a scalar, which greatly reduces the computational cost.

- **Nilpotent decomposition:** If the matrix can be written as the sum of a scalar matrix and a strictly upper (or lower) triangular matrix, i.e., $A = kE + U$ with U nilpotent satisfying $U^m = 0$. Since scalar matrices commute with every matrix, the binomial theorem gives

$$A^n = \sum_{i=0}^{m-1} \binom{n}{i} (kE)^{n-i} U^i.$$

The expansion contains only finitely many nonzero terms, so no full high powers of U need to be computed.

(3) Matrix Block Method

If $A = \text{diag}(A_1, A_2, \dots, A_k)$ is a block diagonal matrix, then its power satisfies

$$A^n = \text{diag}(A_1^n, A_2^n, \dots, A_k^n).$$

This method reduces the problem of computing a high power of a large matrix to computing powers of several smaller sub-blocks, achieving a significant dimension reduction.

2.2. Similarity Transformation Methods

The core principle of similarity transformation methods is that similar matrices share identical power properties. If there exists an invertible matrix P such that $P^{-1}AP = B$, then $A^n = PB^nP^{-1}$. By transforming a general matrix into a simpler

canonical form, the power computation is substantially simplified.

(4) Diagonalization Method

If A is diagonalizable (*i.e.*, for each eigenvalue, its algebraic multiplicity equals its geometric multiplicity), then there exists an invertible matrix X such that $X^{-1}AX = \Lambda$, where Λ is a diagonal matrix whose diagonal entries are the eigenvalues of A . In this case,

$$A^n = X\Lambda^n X^{-1}.$$

The power of a diagonal matrix is obtained by exponentiating each diagonal entry separately. This method is the most standard and generally applicable approach for diagonalizable matrices.

(5) Jordan Canonical Form Method

Every square matrix over the complex field is similar to a Jordan canonical form. Let $P^{-1}AP = J$, where J is a Jordan matrix. Then

$$A^n = PJ^n P^{-1}.$$

The power of a Jordan matrix can be computed via the block diagonal method combined with the general formula for powers of Jordan blocks. This is a universal method for square matrices and is particularly useful for non-diagonalizable (defective) matrices.

2.3. Polynomial Degree Reduction Methods

Polynomial degree reduction methods use annihilating polynomials to convert high powers of a matrix into low-degree matrix polynomials. They are particularly favorable in manual computation scenarios where an annihilating polynomial can be easily obtained, whereas constructing similarity-transformation matrices incurs heavy computational burden.

(6) Cayley-Hamilton Theorem Method

Let the characteristic polynomial of an $n \times n$ matrix A be $f(\lambda) = |\lambda E - A|$. By the Cayley-Hamilton theorem, $f(A) = 0$. For any positive integer m , polynomial division with remainder gives,

$$\lambda^m = q(\lambda)f(\lambda) + r(\lambda),$$

where $r(\lambda) = 0$ or $\deg(r(\lambda)) < n$. Substituting A yields $A^m = r(A)$, thus reducing the degree.

(7) Minimal Polynomial Method

The minimal polynomial is the monic annihilating polynomial of least degree, and using it for degree reduction is more efficient than using the characteristic polynomial. Let $m(\lambda)$ be the minimal polynomial of A . Then $m(A) = 0$, and by the same division algorithm, A^m reduces to a matrix polynomial of degree less than $\deg(m(\lambda))$. This method is particularly advantageous when the characteristic polynomial has eigenvalues of high algebraic multiplicity.

2.4. Unified Mathematical Essence of the Method System

Although the foregoing seven computational methods differ in form and applica-

tion scenarios, their mathematical essence can be uniformly attributed to two core principles: the theory of linear space decomposition and the theory of matrix annihilating polynomials, together with several simplification techniques based on the properties of matrix operations.

From the geometric perspective, the diagonalization method, the Jordan canonical form method, and the matrix block method all rely on direct-sum decompositions of linear spaces. The diagonalization and Jordan methods use similarity invariance to represent the matrix on a basis of eigenvectors or generalized eigenvectors, transforming it into the simplest possible structure (diagonal or Jordan form) for easier power computation. The block method can either use similarity transformation to achieve a block diagonal form or directly exploit an inherent block diagonal structure to divide and conquer.

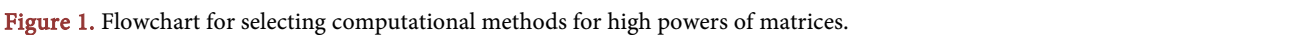
From the algebraic perspective, the Cayley-Hamilton theorem and the minimal polynomial method are grounded in the fundamental fact that every square matrix admits a nonzero annihilating polynomial. Indeed, all $n \times n$ matrices form a finite-dimensional linear space, so every matrix has an annihilating polynomial. Through algebraic homomorphism, matrix polynomial operations can be embedded in the quotient ring $F[\lambda]/\langle m(\lambda) \rangle$, where the high-degree monomial λ^m modulo any annihilating polynomial is equivalent to a low-degree remainder $r(\lambda)$. The characteristic polynomial used in the Cayley-Hamilton method is a special case of an annihilating polynomial.

It should be particularly pointed out that the rank-one decomposition and nilpotent-type decomposition do not fall into the category of direct-sum decomposition of spaces. Instead, they represent more fundamental algebraic decompositions and operation-simplification mechanisms. The rank-one decomposition exploits the associativity of matrix multiplication to reduce high-order matrix powers to scalar powers. By contrast, the nilpotent-type decomposition takes advantage of the special property that scalar matrices commute with all matrices, and truncates infinite sums to finite sums via the binomial theorem. These two methods achieve extremely high efficiency when the conditions for their applicability are satisfied; nevertheless, they are essentially ingenious applications of operational rules rather than spatial-structure decompositions.

Finally, the inductive-recursive method is a fundamental heuristic tool. It can be used to derive recurrence relations from annihilating polynomials, or to obtain closed-form expressions for the general term directly from the computed sequence of matrix powers. It is usually employed as an auxiliary technique in conjunction with other methods.

Thus, the various methods for computing high powers of matrices are not independent of one another; rather, they are unified by two core principles: space decomposition, which dictates the form of simplification, and annihilating polynomials, which enable degree reduction. Complemented by a range of operational techniques, these methods collectively form the underlying logical framework for matrix-power computation.

3.1. Progressive Selection Decision Workflow



head than similarity transformation. Third, similarity transformation methods with higher cost are utilized when polynomial approaches are infeasible or computationally burdensome. The matrix is transformed to diagonal or Jordan canonical form via eigenvector or generalized eigenvector matrices, which involves solving linear systems and computing matrix inverses. Finally, binary exponentiation, also called the square-and-multiply algorithm, serves as a universal fallback with guaranteed feasibility for any square matrix, with a time complexity of $O(\log n)$ matrix multiplications.

These computational pathways are not mutually exclusive. For example, eigenvectors can in principle be derived from a known characteristic polynomial. Nevertheless, the decision order is arranged by increasing information acquisition cost rather than logical dependence between methods. Students are advised to attempt methods in the prescribed order and terminate the process once a computationally tractable approach yields the result.

3.2. Comparative Analysis of Calculation Examples

To further illustrate the practical performance differences among methods, this subsection takes the same matrix and computes its high power using both the diagonalization method and the Cayley-Hamilton theorem method, comparing their computational paths and efficiency.

Example 1. Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix},$$

compute A^n .

Method 1: Diagonalization Method

The eigenvalues of A are $\lambda_1 = 5$ (simple) and $\lambda_2 = -1$ (double). The corresponding eigenvectors are

$$\xi_1 = (1, 1, 1)^T, \quad \xi_2 = (-1, 1, 0)^T, \quad \xi_3 = (-1, 0, 1)^T.$$

The double eigenvalue has two linearly independent eigenvectors, so A is diagonalizable. Let

$$X = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then $A = X\Lambda X^{-1}$. Therefore,

$$A^n = X\Lambda^n X^{-1} = \frac{1}{3} \begin{pmatrix} 5^n + 2(-1)^n & 5^n - (-1)^n & 5^n - (-1)^n \\ 5^n - (-1)^n & 5^n + 2(-1)^n & 5^n - (-1)^n \\ 5^n - (-1)^n & 5^n - (-1)^n & 5^n + 2(-1)^n \end{pmatrix}.$$

Method 2: Cayley-Hamilton Theorem Method

The characteristic polynomial of A is

$$f(\lambda) = (\lambda - 5)(\lambda + 1)^2 = \lambda^3 - 3\lambda^2 - 9\lambda - 5. \text{ Assume}$$

$$\lambda^n = q(\lambda)f(\lambda) + a\lambda^2 + b\lambda + c.$$

Substituting the eigenvalues and, for the multiple root, its derivative equation, we solve for

$$\begin{cases} a = \frac{5^n + (6n-1)(-1)^n}{36}, \\ b = \frac{5^n - (12n+1)(-1)^n}{18}, \\ c = \frac{5^n + (35-30n)(-1)^n}{36}. \end{cases}$$

By the Cayley-Hamilton theorem, $f(A) = 0$, so $A^n = aA^2 + bA + cE$. Substitution yields exactly the same result as the diagonalization method.

The two methods produce identical results but differ clearly in performance. The diagonalization method has a clear procedure and verifiable intermediate steps, but for large matrices it requires heavy computational cost of matrix inversion. The Cayley-Hamilton method reduces the problem to solving a linear system whose size depends only on the matrix order, not on the exponent, thus showing a distinct advantage for very high powers.

For quantitative comparison, numerical tests were performed on the example matrix with $n = 100$ in a Python environment. The diagonalization method takes an average of approximately 1.2×10^{-4} seconds per execution, while the Cayley-Hamilton method takes about 3.4×10^{-5} seconds per execution, making it approximately 3.6 times faster than the diagonalization method. As the exponent increases further, the relative efficiency ratio between the two methods remains essentially stable.

4. Typical Cognitive Misconceptions and Teaching Improvement Strategies

4.1. Analysis of Four Typical Cognitive Misconceptions

Drawing on teaching practice and graded assignment data collected over two consecutive semesters of the 2024-2025 academic year (with approximately 90 first-year undergraduate students per semester), we conducted a statistical analysis of errors from about 360 answer sheets across three assessment categories: assignments, unit tests, and final examinations. Our analysis reveals that high-frequency errors in high-order matrix power computation predominantly fall into the following categories.

(1) Misapplication of the Binomial Theorem

Students arbitrarily write $A = B + C$ and directly apply the scalar-field binomial theorem:

$$A^n = \sum_{i=0}^n \binom{n}{i} B^i C^{n-i},$$

which leads to incorrect results.

The error stems from uncritically transferring scalar arithmetic properties to matrices, ignoring the necessary condition for binomial expansion: $BC = CB$. Matrix multiplication is not generally commutative.

(2) Confusion of Block Matrix Power Conditions

Students often mistakenly assume that the power of any block matrix equals the block-wise power of its sub-blocks. This confuses general block matrices with block diagonal matrices—only block diagonal matrices allow independent power computation for each sub-block.

(3) Coefficient Errors with Multiple Roots in the Cayley-Hamilton Method

When the characteristic polynomial has multiple roots, students simply substitute the eigenvalue into the remainder equation without considering derivative constraints, resulting in insufficient independent equations. The root cause is failing to recognize that for a k -fold eigenvalue, the polynomial equality must also hold for its first $k-1$ derivatives, which provide additional constraints.

(4) Blind Application of Diagonalization Without Checking Prerequisites

Students take diagonalization as a universal formula without verifying whether the algebraic multiplicity equals the geometric multiplicity for each eigenvalue, lacking awareness of defective matrices.

4.2. Teaching Improvement Strategies

Targeting the above misconceptions and knowledge fragmentation, we propose the following teaching improvement measures, aligned with the methodological system of matrix power computation.

(1) Implement Thematic Integrated Teaching

Break the chapter boundaries of textbooks and design a thematic review course on “Computation of High Powers of Matrices”. Adopt a problem-driven teaching approach: present matrices of different structures, encourage students to explore multiple solutions, and guide them to generalize the three major categories of methods—structural simplification, similarity transformation, and polynomial degree reduction. By elucidating their unified mathematical essence and constructing mind maps, help students establish the step-by-step reasoning: “check structure first, then consider the power, and finally select the canonical form.”

(2) Strengthen Condition Discrimination Through Counterexample Teaching

Adhere to the principle of “conditions before procedures.” For each method, first clarify its applicability boundary. For example, when introducing binomial expansion, emphasize the commutativity requirement. Use the four typical misconceptions as authentic teaching material, display actual student errors, and guide students to identify the root causes independently. Design targeted counterexamples and compare correct versus incorrect solutions to reinforce students’ awareness of prerequisites, thereby reducing principle-based errors at the source.

(3) Integrate Computational Thinking with Engineering Applications

In line with the requirements of new engineering education, integrate computational thinking into teaching. Supplement with basic programming exercises using tools such as MATLAB and Python, and let students experience the efficiency differences among algorithms through runtime comparisons. Meanwhile, introduce simple engineering application cases to enhance students' learning motivation and knowledge transfer ability.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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