



Equilibrium Current Dynamics in Nonlinear Electrical Systems Using a Cusp Catastrophe Model

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Abstract

The behavior of nonlinear electrical systems is often characterized by abrupt transitions, multiple operating states, and hysteresis phenomena that conventional linear models cannot adequately describe. This paper presents a cusp catastrophe model for analyzing the equilibrium dynamics of electrical current in nonlinear electrical systems. The proposed model is based on the quartic potential function, $U(I) = \frac{I^4}{4} + \frac{V}{2}I^2 + RI$ where the state variable I denotes the electric current, while the control parameters V and R represent the applied voltage and electrical resistance, respectively. The equilibrium states are obtained from the stationary condition of the potential function, leading to the cubic equation $I^3 + VI + R = 0$. The discriminant of this equation is employed to determine the number and stability of equilibrium currents, revealing three distinct operating regimes corresponding to one equilibrium, multiple equilibria, and fold bifurcations. The cusp boundary $4V^3 + 27R^2 = 0$ separates regions of unique and multiple steady-state solutions in the control-parameter space. The proposed catastrophe-theoretic framework provides a qualitative understanding of sudden current transitions, bistability, and hysteresis arising from variations in voltage and resistance. The results demonstrate that cusp catastrophe theory offers a powerful mathematical framework for studying nonlinear electrical behavior and may contribute to the design, stability analysis, and control of switching circuits, nonlinear electronic devices, and emerging intelligent electrical systems.

Subject Areas

Theoretical Physics

Keywords

Cusp Catastrophe, Nonlinear Electrical Systems, Equilibrium Current, Bifurcation Analysis, Potential Function, Voltage, Resistance, Hysteresis

1. Introduction

Nonlinear electrical systems are fundamental components of modern engineering applications, including power electronics, switching circuits, nonlinear semiconductor devices, permissive systems, and intelligent electrical networks [1]-[6]. Unlike linear circuits, nonlinear systems frequently exhibit complex dynamic behaviors such as multiple equilibrium states, hysteresis, sudden jumps, and bifurcation phenomena [7]-[15]. These behaviors cannot always be explained adequately using classical linear circuit theory, thereby motivating the development of mathematical models capable of describing abrupt qualitative changes in system behavior [16]-[21].

Catastrophe theory, introduced by René Thom, provides a rigorous mathematical framework for studying discontinuous transitions in systems whose governing parameters change continuously. Among the seven elementary catastrophes, the cusp catastrophe is one of the most widely applied because it describes systems controlled by two independent parameters and naturally explains bistability, hysteresis, and sudden state transitions. Consequently, cusp catastrophe theory has found applications in mechanics, biology, economics, engineering, psychology, epidemiology, and numerous other scientific disciplines where nonlinear phenomena arise [22] [23].

In electrical engineering, nonlinear devices often possess multiple operating points under identical external conditions. Small variations in applied voltage, resistance, or other circuit parameters may produce abrupt switching between distinct equilibrium states. Such behavior is characteristic of tunnel diodes, memristors, nonlinear semiconductor devices, and other nonlinear electronic components [24] [25].

This paper develops a cusp catastrophe model for investigating equilibrium current dynamics in nonlinear electrical systems. The model is formulated through the quartic potential function

$$U(I) = \frac{I^4}{4} + \frac{V}{2}I^2 + RI,$$

where I represents the electrical current, V denotes the applied voltage, and R represents the electrical resistance. The equilibrium currents correspond to the stationary points of the potential function, obtained by solving the cubic equation $I^3 + VI + R = 0$. The discriminant of the cubic equation provides a complete characterization of the number of equilibrium states and identifies the critical conditions under which fold bifurcations occur. Furthermore, the cusp boundary

$4V^3 + 27R^2 = 0$ partitions the parameter space into regions exhibiting unique and multiple equilibrium currents, thereby offering a geometric interpretation of nonlinear electrical behavior.

The proposed model extends the application of catastrophe theory to equilibrium current analysis by interpreting the control parameters as physically meaningful electrical quantities. Rather than relying solely on numerical simulations, the catastrophe-theoretic framework reveals the qualitative mechanisms responsible for sudden current switching, bistability, and hysteresis. This perspective provides valuable insight into the stability and control of nonlinear electrical systems and complements traditional methods of nonlinear circuit analysis [26]-[28].

The main contributions of this paper are summarized as follows:

- 1) A new cusp catastrophe formulation for *equilibrium current dynamics* in nonlinear electrical systems.
- 2) A physical interpretation of the catastrophe control parameters in terms of applied voltage and electrical resistance.
- 3) An analytical characterization of *equilibrium currents* using the discriminant of the governing cubic equation.
- 4) A geometric description of fold bifurcations and cusp boundaries that separate different operating regimes.
- 5) A qualitative framework for explaining bistability, hysteresis, and abrupt current transitions in nonlinear electrical devices.

2. The Canonical Cusp Catastrophe

Consider the potential function $U(x) = \frac{x^4}{4} + \frac{a}{2}x^2 + bx$, where x is the state variable, a and b are control parameters. The equilibrium points satisfy $\frac{dU}{dx} = x^3 + ax + b = 0$. Thus, equilibria are the roots of the cubic equation $x^3 + ax + b = 0$ [9]-[14]. See **Figure 1** for the canonical cusp catastrophe.

2.1. Number of Equilibria

The discriminant is $\Delta = -4a^3 - 27b^2$.

- If $\Delta < 0$, there is one real equilibrium.
- If $\Delta = 0$, two equilibria merge at a fold bifurcation.
- If $\Delta > 0$, there are three real equilibria.

The cusp boundary is therefore $4a^3 + 27b^2 = 0$.

2.2. Stability of Equilibria

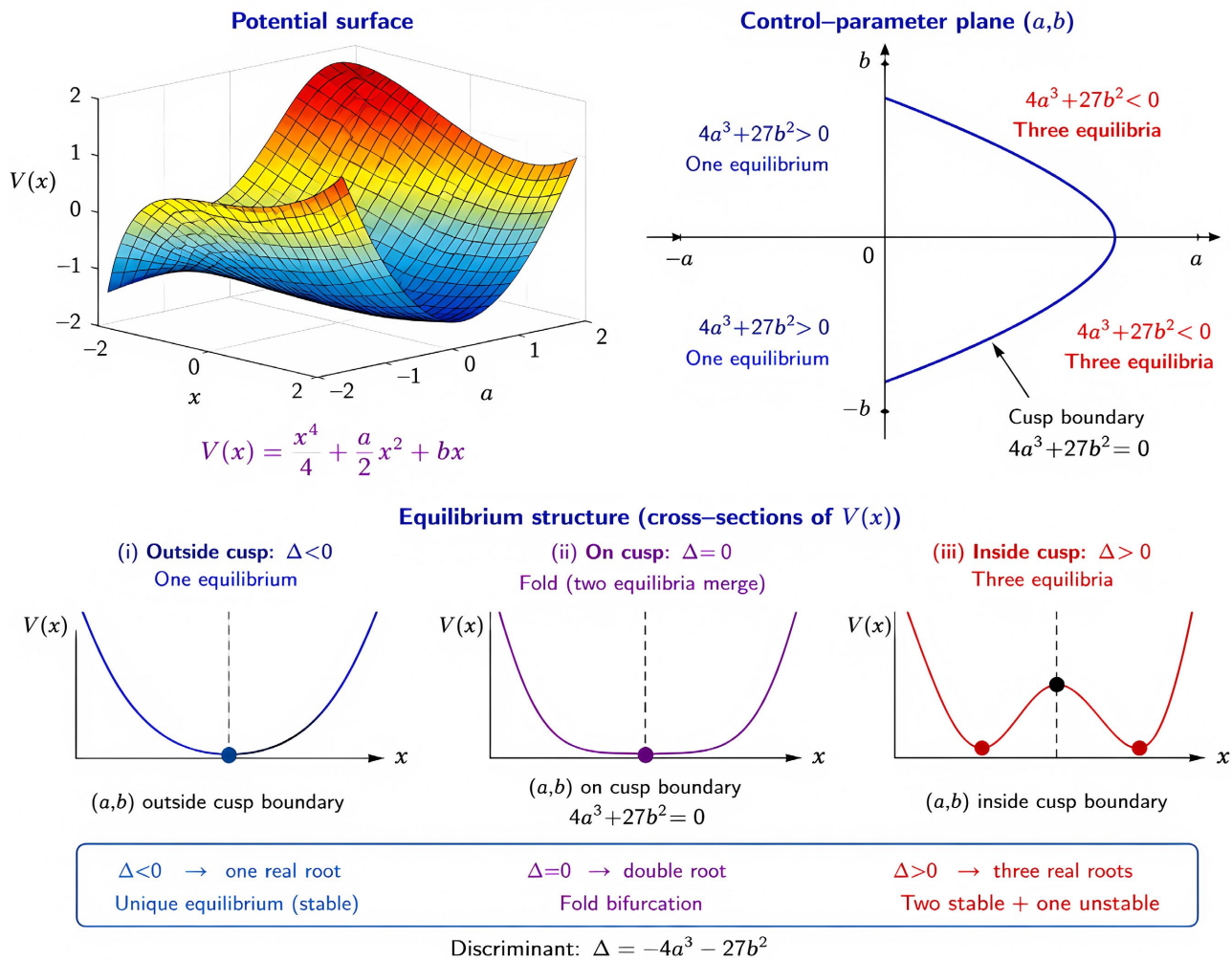
For the gradient system is locally stable x^* , an equilibrium $\dot{x} = -(x^3 + ax + b)$

when $\left. \frac{d}{dx}(x^3 + ax + b) \right|_{x=x^*} = 3(x^*)^2 + a > 0$.

Inside the cusp region ($\Delta > 0$):

- the lower branch is stable;

CUSP CATASTROPHE



Notation: The cusp catastrophe possesses a bifurcation structure governed by the discriminant of the stationary cubic equation. Outside the cusp region, the system admits a unique equilibrium, whereas inside the cusp region, three equilibria exist: two stable branches separated by an unstable branch. Fold bifurcations occur on the cusp boundary, where stable and unstable equilibria coalesce and disappear, giving rise to abrupt state transitions and hysteresis phenomena.

Figure 1. Canonical cusp catastrophe.

- the middle branch is unstable;
- the upper branch is stable.

Hence the system exhibits bistability.

What Causes a Catastrophe?

As a control parameter changes continuously, a stable equilibrium branch may reach a fold point where it disappears.

At that instant:

- 1) the current stable state no longer exists;
- 2) the system cannot remain nearby;
- 3) it jumps suddenly to another stable branch.

This abrupt transition is called a catastrophe.

2.3. Hysteresis: Suppose $a < 0$ Is Fixed and b Is Slowly Increased

- The system follows one stable branch.
- At the upper fold, that branch vanishes.
- The state jumps to the other stable branch.

If b is then decreased, the jump back occurs at a different fold point.

Thus, the forward and backward paths differ, producing hysteresis.

3. Cusp Catastrophe Model for an Electrical System

Consider an electrical system in which the state variable I represents the electric current, while the control parameters V and R represent the applied voltage and the electrical resistance, respectively. The potential function can model the system, see **Figure 2**.

$$U(I) = \frac{I^4}{4} + \frac{V}{2}I^2 + RI,$$

where:

- I is the *electric current* flowing through the circuit (state variable),
- V is the *applied voltage*, acting as the primary control parameter,
- R is the *electrical resistance*, serving as the secondary control parameter.

The potential function represents the energy landscape of the electrical system.

CUSP CATASTROPHE MODEL FOR EQUILIBRIUM CURRENT DYNAMICS IN NONLINEAR ELECTRICAL SYSTEMS

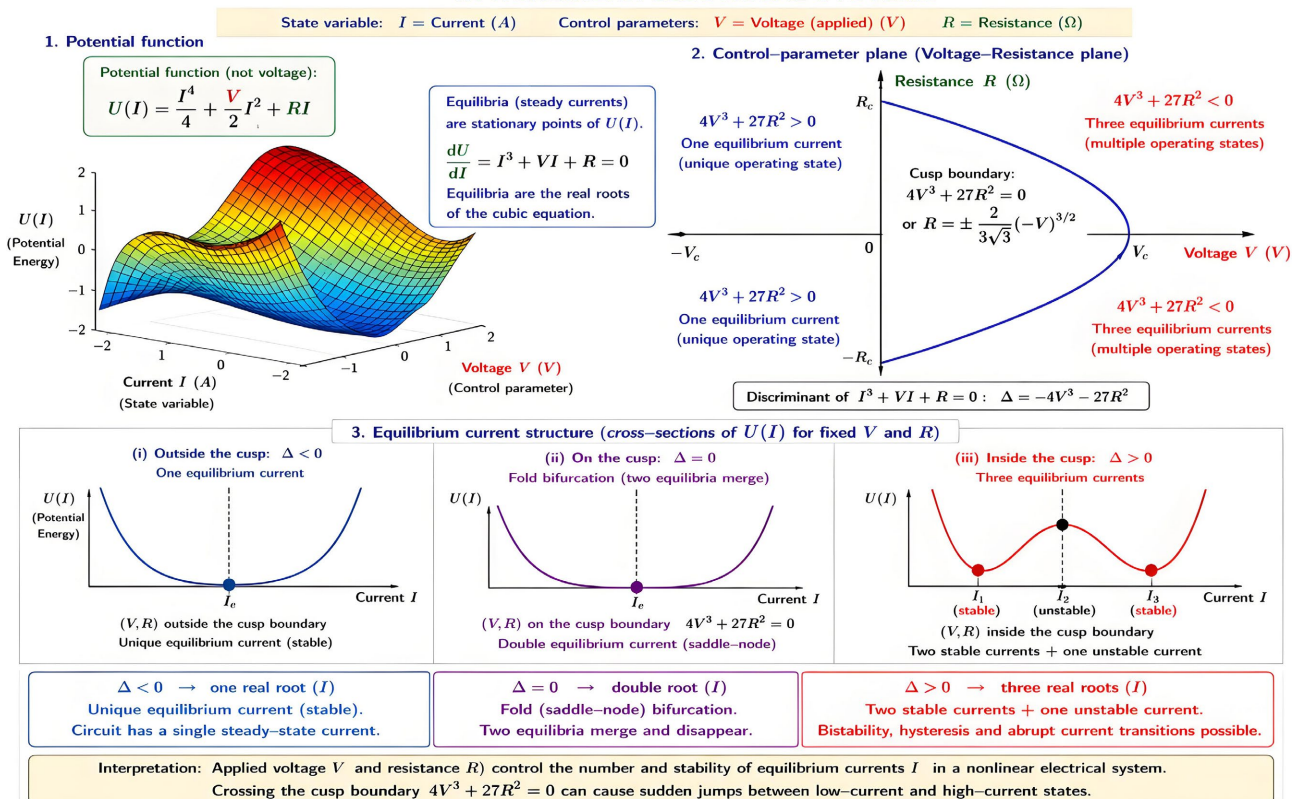


Figure 2. Cusp catastrophe model for equilibrium current dynamics in nonlinear electrical systems.

Stable operating conditions correspond to local minima of this potential, while unstable conditions correspond to local maxima.

3.1. Equilibrium Current

The equilibrium states of the electrical system are obtained by setting the derivative of the potential function equal to zero: $\frac{dU}{dx} = I^3 + VI + R = 0$. Hence, the equilibrium currents satisfy the cubic equation $I^3 + VI + R = 0$.

Each real solution of this equation represents a possible steady-state current for the specified values of the applied voltage and resistance.

3.2. Number of Equilibrium Currents

The number of equilibrium currents depends on the discriminant of the cubic equation, $\Delta = -4V^3 - 27R^2$. Three different operating regimes are possible.

1) Single Equilibrium ($\Delta < 0$)

When $\Delta < 0$, the cubic equation has *one real solution*. Physically, the electrical circuit possesses a unique equilibrium current for the given voltage and resistance. The system responds smoothly to parameter variations, and no abrupt transitions occur.

2) Fold (Saddle-Node) Bifurcation ($\Delta = 0$)

When $\Delta = 0$, two equilibrium currents coincide and merge into a single point. This condition corresponds to a *fold bifurcation*, where a stable and an unstable operating state collide and disappear. Near this point, a very small change in either the *applied voltage* or the *resistance* can produce a sudden change in the circuit current.

3) Three Equilibria ($\Delta > 0$)

When $\Delta > 0$, the cubic equation possesses three distinct real solutions. In this regime, the circuit admits multiple possible equilibrium currents under the same operating conditions. Typically, two of these equilibria are stable while the intermediate one is unstable. Consequently, the system may exhibit bistability, where the observed current depends on the previous operating history or the initial conditions.

4) Cusp Catastrophe Boundary

The transition between these operating regimes is determined by the cusp catastrophe boundary, defined by $4V^3 + 27R^2 = 0$. This curve, which is called the bifurcation set, divides the control-parameter plane (V, R) into regions with different numbers of equilibrium currents. See **Figure 3** for the bifurcation set.

From an electrical engineering perspective:

- Outside the cusp region, each combination of applied voltage and resistance produces a unique steady-state current.
- Inside the cusp region, multiple equilibrium currents coexist, allowing the circuit to switch abruptly between different current levels.
- On the cusp boundary, the system reaches a critical condition where infinitesimal changes in voltage or resistance can trigger a sudden jump in the current.

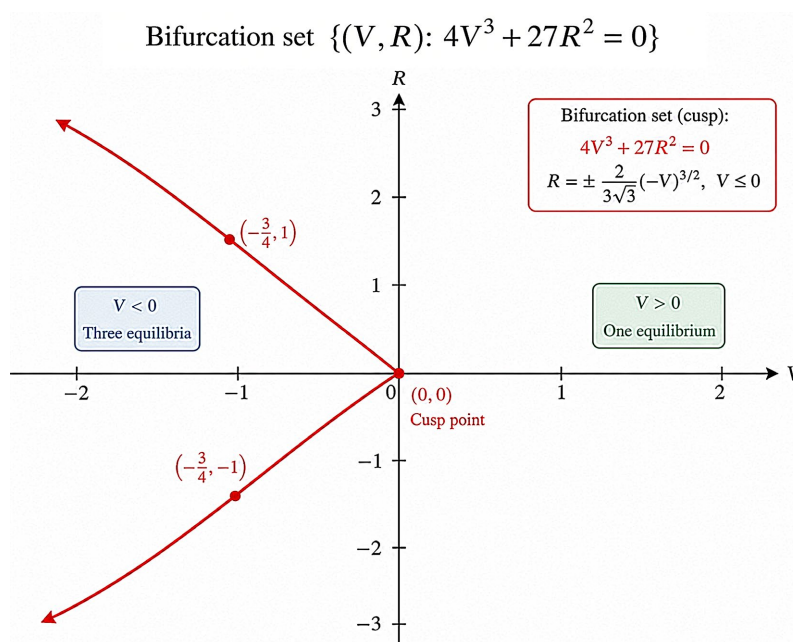


Figure 3. Bifurcation set.

Figure 2 illustrates the application of the cusp catastrophe model to analyze the equilibrium behavior of current in a nonlinear electrical system. In this model, the **state variable** is the electrical current I , whereas the **control parameters** are the applied voltage V and the electrical resistance R . The figure demonstrates how variations in these two electrical parameters influence the number and stability of equilibrium current states.

The bifurcation set $\mathcal{B} = \{(V, R) \in \mathbb{R}^2 : 4V^3 + 27R^2 = 0\}$ is the boundary in the control parameter space (V, R) , where V denotes the voltage and R denotes the current. This curve separates regions in which the electrical system exhibits qualitatively different equilibrium behaviors.

4. Physical Interpretation

Interpreting the cusp catastrophe model in terms of current, voltage, and resistance provides a qualitative description of nonlinear electrical behavior. The *applied voltage* (V) drives the system, while the *electrical resistance* (R) influences its operating characteristics. Their interaction determines whether the circuit has a single stable operating current or multiple possible current states. Near the cusp boundary, gradual variations in voltage or resistance may result in abrupt current switching, hysteresis, and bistable operation—phenomena commonly observed in nonlinear electrical devices such as switching circuits, tunnel diodes, memristive systems, and other nonlinear electronic components.

5. Physical Implications of the Model for Nonlinear Electrical Systems

One of the most important *physical implications* of the model is the prediction of

bistability. When the discriminant satisfies $\Delta = -4V^3 - 27R^2 > 0$, the system possesses three equilibrium currents. In practical electrical systems, this typically corresponds to two stable operating currents separated by an *unstable equilibrium*. Depending on the initial operating conditions or the previous state of the system, the electrical circuit may settle into either of the stable current levels. Such behavior is frequently observed in nonlinear switching devices and memory-based electronic components.

The *cusp catastrophe model* also explains the occurrence of hysteresis. As the applied voltage or the resistance changes gradually, the current does not necessarily follow a unique path. Instead, the system may remain in one *stable equilibrium* until a critical point is reached, after which it *suddenly jumps* to another equilibrium state. When the control parameter is reversed, the return transition occurs at a different critical value, forming a *hysteresis loop*. This phenomenon is common in nonlinear semiconductor devices, magnetic circuits, permissive systems, and switching electronics.

Another important implication is the existence of critical transition points. The cusp boundary, $4V^3 + 27R^2 = 0$, represents the set of parameter values at which two equilibrium currents merge through a *fold bifurcation*. At these critical conditions, extremely small variations in voltage or resistance may produce disproportionately large changes in the current. Such sensitivity is particularly relevant in the design of electrical protection systems, fault detection algorithms, and adaptive control strategies.

The model further illustrates how voltage and resistance jointly determine the stability of electrical operation. An increase in the applied voltage may drive the system toward regions where multiple equilibria exist, while variations in resistance modify the location of the cusp boundary and the accessibility of stable operating states. Consequently, voltage and resistance should not always be considered independently when analyzing nonlinear electrical behavior, since their interaction governs the overall stability structure.

6. Conclusions and Future Research

This paper has presented a cusp catastrophe model for investigating equilibrium current dynamics in nonlinear electrical systems. By representing the electric current as the state variable and the applied voltage and electrical resistance as control parameters, the proposed model establishes a direct relationship between nonlinear circuit behavior and catastrophe theory.

The cubic equation determines the equilibrium currents

$$I^3 + VX + R = 0,$$

whose discriminant completely characterizes the number of possible operating states. The analysis demonstrates that the electrical system may exhibit one equilibrium, multiple equilibria, or fold bifurcations depending on the values of the voltage and resistance. Furthermore, the cusp boundary

$$4V^3 + 27R^2 = 0$$

separates regions of unique and multiple equilibrium currents, providing a clear geometric interpretation of nonlinear electrical behavior.

The proposed framework successfully explains several important physical phenomena observed in nonlinear electrical systems, including bistability, hysteresis, critical transitions, and sudden current switching. Unlike purely numerical approaches, the cusp catastrophe model provides analytical insight into the qualitative changes in system behavior resulting from continuous parameter variations. Consequently, the model contributes to a deeper understanding of stability mechanisms in nonlinear electrical circuits and offers an effective mathematical tool for predicting catastrophic operating transitions.

From an engineering standpoint, the model may assist in the design and analysis of nonlinear electronic devices, switching circuits, intelligent electrical networks, and emerging energy systems where abrupt state changes are either undesirable or intentionally utilized.

Although the present study focuses on equilibrium analysis, several promising research directions remain open.

7. Future Work May Include

- Extending the model to time-dependent nonlinear electrical systems described by ordinary or partial differential equations.
- Incorporating capacitive and inductive elements to investigate transient and oscillatory electrical behavior.
- Developing fractional-order cusp catastrophe models for electrical systems exhibiting memory effects and anomalous dynamics.
- Studying stochastic catastrophe models that account for measurement noise, parameter uncertainty, and environmental disturbances.
- Applying the proposed framework to permissive circuits, nonlinear semiconductor devices, tunnel diodes, Josephson junctions, and smart-grid technologies.
- Investigating multi-parameter catastrophe models, such as butterfly or swallowtail catastrophes, to describe more complex electrical phenomena involving additional control variables.
- Designing catastrophe-based control strategies capable of preventing undesirable switching or exploiting bistability in intelligent electrical systems.
- Validating the theoretical predictions through numerical simulations and experimental investigations on practical nonlinear electrical circuits.

Overall, the cusp catastrophe model provides a mathematically rigorous and physically meaningful framework for understanding nonlinear electrical phenomena. Its ability to relate equilibrium structure, bifurcation behavior, and catastrophic transitions makes it a promising analytical tool for future research in nonlinear circuit theory, electrical engineering, and intelligent dynamical systems.

Conflicts of Interest

The author declares no conflicts of interest.

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