



Generalized Topological Dynamics via Λ_θ -Open and Λ_δ -Open Sets: Transitivity, Mixing, Minimality, and Chaoticity

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Abstract

This paper investigates recent advances in topological dynamics by applying modified open-set topologies to analyze key system behaviors, such as transitivity, minimality, chaos, and mixing. By relaxing standard topological constraints through Λ_θ -open and Λ_δ -open sets, these frameworks offer crucial insights into system dynamics where classical transitivity fails. Within this context, we introduce and analyze Λ_θ -transitivity and Λ_δ -transitivity, evaluating their fundamental properties alongside existing generalizations. Additionally, we propose Λ_θ -mixing and Λ_δ -minimality—variants of θ -mixing and δ -minimality—and establish their interrelationships with other established modes of transitivity, minimality, and chaoticity. The primary contribution of this work lies in establishing a strict hierarchical framework that demonstrates how dynamical properties translate across topologies of varying granularities and establishes sufficient conditions under which generalized forms imply classical properties.

Subject Areas

Mathematics

Keywords

Λ_δ -Transitivity, θ -Transitivity, δ -Transitivity, α -Minimality, Generalized Transitivity

1. Introduction

In this paper, we introduce and define two new classes of topological transitive maps, called Λ_θ -transitive and Λ_δ -transitive and investigate their core structural

properties. The author extended classical dynamical properties such as transitivity, minimality, and mixing, which were established in the foundational works of Akin [1] and Auslander [2]. These classical notions were later extended by the author to generalized dynamical properties using θ -type and δ -type topologies [3]-[5]. Subsequent studies explore hierarchical dynamics, mixing, exactness, and chaos across θ -, α - [5] [6], and classical systems [7] [8], with related extensions in [9]. The notions of θ -continuous maps, θ -open sets, θ -closed sets, θ -closure, irresolute maps, and H -closed spaces were introduced by [10]-[15]. For these sets and maps, we introduce the notions of Λ_θ -transitive maps and Λ_θ -minimal maps. Classical concepts of transitivity, minimality, and topological mixing were first established by [16]-[21].

While standard topological dynamics relies on the traditional topology τ , relaxed open-set configurations (such as θ -open and δ -open topologies) allow systems with irregular or discontinuous dynamics to be analyzed under generalized transitivity. However, a significant theoretical gap exists in understanding how dynamical properties behave when switching between these generalized topologies. By constructing the finer Λ_θ - and Λ_δ -topologies, this paper bridges this gap. We establish a unified hierarchy connecting Λ_* -dynamics to classical $*$ -dynamics ($*$ $\in \{\theta, \delta\}$), detailing how mixing implies transitivity in generalized spaces, and proving that stronger topology structures (τ_{Λ_δ}) preserve chaotic and transitive behaviors in coarser ones (τ_{Λ_θ}).

We denote the Λ^* -interior and the Λ^* -closure of a subset A of X by $\Lambda^*\text{Int}(A)$ and $\Lambda^*\text{Cl}(A)$, respectively, where $*$ $\in \{\theta, \delta\}$. By a space X , we mean a topological space (X, τ) . A point $x \in X$ is called a Λ^* -adherent point of A if $A \cap \Lambda^*\text{Cl}(V) \neq \emptyset$ for every Λ^* -open set V containing x . The set of all Λ^* -adherent points of $A \subseteq X$ is called the Λ^* -closure of A , denoted $\Lambda^*\text{Cl}(A)$. A subset A is Λ^* -closed if $A = \Lambda^*\text{Cl}(A)$. The complement of a Λ^* -closed set is a Λ^* -open set; the family of all Λ^* -open sets forms a topology on X , denoted by τ_{Λ^*} (the Λ^* -topology). Therefore, when $*$ $= \theta$, the Λ_θ -topology is coarser than τ and stronger than the θ -topology.

2. Preliminaries and Definitions

In this section, we review and introduce the foundational notions of Λ_* -open sets for $*$ $\in \{\theta, \delta\}$, along with their corresponding dynamical properties including Λ_* -transitivity, Λ_* -mixing, Λ_* -minimality, and Λ_* -chaoticity.

Definition 2.1 (Λ_* -open set): For $*$ $\in \{\theta, \delta\}$, a subset A of a topological space X is called a Λ_* -set if it coincides with the intersection of all $*$ -open supersets of A . A subset A is called Λ_* -closed if $A = L \cap C$, where L is a Λ_* -set and C is a $*$ -closed set. The complement of a Λ_* -closed set is called a Λ_* -open set.

Notation 2.2 (Λ_* -density): For $*$ $\in \{\theta, \delta\}$, a subset A of X is called Λ_* -dense in X if $\Lambda_*\text{Cl}(A) = X$.

Definition 2.3: A function $f : X \rightarrow X$ is called Λ_* -irresolute if the inverse image of each Λ_* -open set is a Λ_* -open set in X .

Definition 2.4: Let (X, τ) be a topological space and $f : X \rightarrow X$ be a Λ_* -irresolute map. The map f is called Λ_* -**transitive** if, for every pair of non-empty Λ_* -open sets U and V in X , there exists an integer $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$.

Definition 2.5: A Λ_* -irresolute map $f : X \rightarrow X$ is called topologically Λ_* -**mixing** if, given any non-empty Λ_* -open subsets $U, V \subseteq X$, there exists $N \geq 1$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$.

Definition 2.6: A system (X, f) is called Λ_* -**minimal** if all orbits are Λ_* -dense; that is, for all $x \in X$, the orbit $O_f(x)$ is Λ_* -dense in X .

Definition 2.7: The map $f : X \rightarrow X$ is Λ_* -**chaotic** if f is Λ_* -transitive on X and the set $P(f)$ of periodic points of f is Λ_* -dense in X .

3. Main Results on Λ_* -Generalized Dynamical Properties

Having established the structural topology of Λ_* -open sets, we now analyze how these modified open sets govern global dynamical behaviors. The following results explore two primary directions: first, the implication from Λ_* -dynamics to standard $\star$ -dynamics under single topology constructions; second, the preservation of dynamical properties (transitivity, periodic density, and chaoticity) across nested topologies $(\tau_{\Lambda_\theta} \subseteq \tau_{\Lambda_\delta})$.

Proposition 3.1: Λ_θ -Minimal implies θ -Minimal.

Proof: Assume (X, f) is Λ_θ -minimal. By definition, for every $x \in X$, the orbit $O(x, f)$ is Λ_θ -dense. Let V be any arbitrary non-empty θ -open set in X . Since $\tau_\theta \subseteq \tau_{\Lambda_\theta}$, V is also a non-empty Λ_θ -open set. By Λ_θ -minimality, $O(x, f) \cap V \neq \emptyset$. Because this holds for all non-empty θ -open sets V , $O(x, f)$ is θ -dense for every $x \in X$. Hence, (X, f) is θ -minimal. ■

Theorem 3.2: For $\star \in \{\theta, \delta\}$, since $\tau_\star \subseteq \tau_{\Lambda_\star}$, every $\Lambda_\star$ -transitive map is $\star$ -transitive.

Proof: Assume f is $\Lambda_\star$ -transitive. Let U, V be non-empty $\star$ -open sets. Since $\tau_\star \subseteq \tau_{\Lambda_\star}$, U and V are non-empty $\Lambda_\star$ -open sets. By $\Lambda_\star$ -transitivity, there exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. Therefore, f is $\star$ -transitive. ■

Proposition 3.3: For $\star \in \{\theta, \delta\}$, every $\Lambda_\star$ -mixing map is $\Lambda_\star$ -transitive.

Proof: Assume f is $\Lambda_\star$ -mixing. Let U, V be non-empty $\Lambda_\star$ -open sets. By definition of $\Lambda_\star$ -mixing, there exists $N \geq 1$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$. Taking $n = N$ yields $f^N(U) \cap V \neq \emptyset$. Hence, f is $\Lambda_\star$ -transitive. ■

Proposition 3.4: If $\tau_{\Lambda_\theta} \subseteq \tau_{\Lambda_\delta}$ and the set $P(f)$ of all periodic points of f is τ_{Λ_δ} -dense in X , then $P(f)$ is τ_{Λ_θ} -dense in X .

Proof: Assume $P(f)$ is τ_{Λ_δ} -dense in X . Then every non-empty Λ_δ -open set intersects $P(f)$. Let U be a non-empty Λ_θ -open subset of X . Since $\tau_{\Lambda_\theta} \subseteq \tau_{\Lambda_\delta}$, U is also a non-empty Λ_δ -open set. Thus, $U \cap P(f) \neq \emptyset$, which implies that $P(f)$ is τ_{Λ_θ} -dense in X . ■

Theorem 3.5: If $\tau_{\Lambda_\theta} \subseteq \tau_{\Lambda_\delta}$, then Λ_δ -transitivity $\Rightarrow \Lambda_\theta$ -transitivity.

Proof: Assume f is Λ_δ -transitive. Let U, V be non-empty Λ_θ -open sets. Since $\tau_{\Lambda_\theta} \subseteq \tau_{\Lambda_\delta}$, U and V are non-empty Λ_δ -open sets. By Λ_δ -transitivity, there exists

$n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. Thus, f is Λ_θ -transitive. ■

Theorem 3.6: If $\tau_{\Lambda_\theta} \subseteq \tau_{\Lambda_\delta}$, then Λ_δ -chaoticity $\Rightarrow \Lambda_\theta$ -chaoticity.

Proof. By Theorem 3.5, Λ_δ -transitivity implies Λ_θ -transitivity whenever $\tau_{\Lambda_\theta} \subseteq \tau_{\Lambda_\delta}$. Furthermore, by Proposition 3.4, Λ_δ -density of $P(f)$ implies Λ_θ -density of $P(f)$. Since both conditions for chaoticity (transitivity and dense periodic orbits) transfer from τ_{Λ_δ} to τ_{Λ_θ} , Λ_δ -chaoticity implies Λ_θ -chaoticity. ■

4. Discussion

The results presented in this study demonstrate that modified open-set topologies provide a useful framework for extending the analysis of dynamical systems beyond the restrictions imposed by classical topology. In particular, the introduction of Λ_θ -open and Λ_δ -open sets makes it possible to investigate dynamical behaviors through weaker or modified notions of openness. This leads naturally to the corresponding concepts of Λ_θ -transitivity and Λ_δ -transitivity, which provide additional ways of characterizing the interaction between the topology of the underlying space and the behavior of a dynamical system.

The analysis of Λ_θ -transitivity and Λ_δ -transitivity shows that these properties can capture dynamical interactions that may not be detected by ordinary transitivity. In a classical topological dynamical system (X, f) , transitivity is determined by the existence of sufficiently rich intersections between iterates of nonempty open sets. When the family of open sets is modified, the same dynamical principle can be examined under a different topological structure. Consequently, the modified notions may provide a broader perspective on orbit behavior, especially in systems for which classical topological conditions are too restrictive.

The relationship between the modified transitivity properties and other fundamental dynamical concepts is also significant. Transitivity, mixing, minimality, and chaos describe different aspects of long-term system behavior, and their interaction under modified topologies can reveal structural relationships that are not immediately apparent in the classical setting. In particular, mixing generally imposes stronger intersection requirements than transitivity, whereas minimality concerns the density of individual orbits. The use of Λ_θ -open and Λ_δ -open sets therefore provides a framework for examining how these properties change when the underlying notion of openness is generalized.

Another important aspect of the proposed framework is its relationship with existing generalizations of transitivity. Rather than replacing classical topological dynamics, Λ_θ -transitivity and Λ_δ -transitivity can be viewed as extensions that permit the investigation of dynamical behavior under alternative topological conditions. The resulting hierarchy of properties can help determine which classical implications remain valid and which require additional assumptions. This is particularly relevant when comparing different generalized notions of openness, since inclusion relationships among the corresponding classes of open sets can induce relationships among their associated dynamical properties.

The findings also suggest that modified topological structures can be useful in distinguishing between different degrees of dynamical complexity. A system that fails to satisfy classical transitivity may nevertheless satisfy an appropriately generalized transitivity condition. Such a distinction is mathematically meaningful because it indicates that the apparent lack of dynamical interaction may result from the restrictive choice of topology rather than from the intrinsic behavior of the system.

Nevertheless, the present framework is primarily theoretical. Further investigation is required to establish broader classes of examples and counterexamples, determine sharper relationships among Λ_θ , Λ_δ , and classical dynamical properties, and identify conditions under which generalized transitivity implies minimality, mixing, or various forms of chaos. These questions may provide useful directions for subsequent research.

5. Conclusions

In this paper, we have formalized a generalized framework for topological dynamics utilizing Λ_θ -open and Λ_δ -open topologies. By systematically constructing these spaces, we resolved how relaxed topological structures affect the fundamental dynamical notions of transitivity, mixing, minimality, and chaoticity.

Our main findings establish a clear two-tiered hierarchy. First, in a single space, stronger generalized dynamical properties preserve their weaker classical analogues: Λ_* -mixing strictly implies Λ_* -transitivity, and Λ_* -transitivity (or minimality) implies standard $*$ -transitivity (or minimality) for $*$ $\in \{\theta, \delta\}$. Second, across comparative spaces where $\tau_{\Lambda_\theta} \subseteq \tau_{\Lambda_\delta}$, dynamic complexity is monotonic: systems exhibiting Λ_δ -transitivity, Λ_δ -dense periodic orbits, or Λ_δ -chaoticity automatically inherit their corresponding Λ_θ counterparts.

These results clarify how dynamic sensitivity transfers across generalized open-set structures, offering a viable analytical toolkit for non-standard systems where ordinary open sets fail to capture orbit mixing. Future research may extend these Λ_* -topological frameworks into measure-theoretic and symbolic dynamics, particularly exploring whether stronger conditions (such as Devaney chaoticity or topological entropy) exhibit similar transfer properties across nested generalized topologies. Furthermore, this study developed a modified topological framework for investigating fundamental properties of dynamical systems using Λ_θ -open and Λ_δ -open sets. Within this framework, the concepts of Λ_θ -transitivity and Λ_δ -transitivity were introduced and analyzed in relation to classical transitivity and other generalized dynamical properties.

The results indicate that modifying the underlying notion of openness provides additional flexibility for describing orbit interactions and dynamical behavior. In particular, generalized transitivity can identify forms of dynamical interaction that may not be represented adequately by classical transitivity. The proposed framework therefore extends the scope of topological dynamics while preserving the essential idea of studying the repeated interaction of sets under the action of a

dynamical map.

The study also emphasizes the importance of examining the relationships among transitivity, mixing, minimality, and chaos within generalized topological settings. Inclusion relationships between different classes of open sets can imply corresponding relationships among their dynamical properties, providing a systematic way to compare classical and generalized forms of dynamical behavior.

Overall, Λ_θ -open and Λ_δ -open topologies provide a useful mathematical setting for extending topological dynamics. The concepts of Λ_θ -transitivity and Λ_δ -transitivity advance generalized dynamical theory and provide a basis for further investigations into mixing, minimality, chaoticity, and other forms of dynamical complexity. Future work may focus on establishing stronger implication theorems, constructing nontrivial examples and counterexamples, and applying these generalized topological structures to specific classes of dynamical systems.

Conflicts of Interest

The author declares no conflicts of interest.

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