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Normalized Solutions to a Planar Schrödinger-Poisson System with Ring-Shaped Potentials and Inhomogeneous Interactions

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Abstract

This paper studies the constrained minimization problem for the Schrödinger-Poisson system with a ring-shaped potential and a logarithmic convolution term in $\mathbb{R}^2$. The energy functional is made well-defined by introducing a weighted Sobolev space $\mathcal{X}$ and decomposing the logarithmic kernel. Define the critical constant $a^* = |Q|_2^2$, where Q is the unique positive radially symmetric solution of the two-dimensional scalar field equation $-\Delta u + u = u^3$. Under suitable assumptions on the interaction coefficient $K(x)$, the existence and nonexistence of minimizers for the minimization problem $e(\gamma, a)$ are systematically studied. The main results show that minimizers exist when $a < a^*$ and do not exist when $a > a^*$ or $a = a^*$ (with $\gamma > 0$). Furthermore, for $\gamma > 0$, the asymptotic behavior of minimizers is characterized as $a \nearrow a^*$.

Subject Areas

Mathematical Analysis

Keywords

Schrödinger-Poisson System, Normalized Solution, Logarithmic Convolution, Ringed-Shaped Potential

1. Introduction and Main Results

In this paper, we consider the following Schrödinger-Poisson system

$$\begin{cases} i\psi_t - \Delta\psi + V(x)\psi - \phi\psi = aK(x)|\psi|^{p-2}\psi & \text{in } \mathbb{R}^N \times \mathbb{R} \\ -\Delta\phi = \gamma|\psi|^2 & \text{in } \mathbb{R}^N \times \mathbb{R} \end{cases}, \tag{1.1}$$

where i is the imaginary unit, $\psi:\mathbb{R}^N\times\mathbb{R}\to\mathbb{C}$ represents a time-dependent wave function, $\phi:\mathbb{R}^N\times\mathbb{R}\to\mathbb{C}$ stands for an internal potential for a nonlocal self-interaction of the wave function ψ, $V(x)$ is an external potential, $K(x)$ is a spatially inhomogeneous interaction, $a\in\mathbb{R}$, $\gamma\neq 0$, $p>2$ and $N\geq 2$. System (1.1) describes the dynamics of a quantum particle or condensate under the combined influence of an external potential $V(x)$, a local nonlinearity and a self-consistent potential generated by the particle density itself.

Under the usual ansatz $\psi(x,t)=e^{i\lambda t}u(x)$, $\lambda\in\mathbb{R}$ and then employing the fundamental solution of $-\Delta$ and disregarding the harmonic component, system (1.1) reduces to the integro-differential equation

$$-\Delta u+\left(V(x)-\lambda\right)u-\gamma\left(G_N*u^2\right)u=aK(x)|u|^{p-2}u\ \text{ in }\mathbb{R}^N, \tag{1.2}$$

where G_N denotes the Green's function of $-\Delta$ given by

$$G_N(x):=\begin{cases}\dfrac{1}{N(N-2)\omega_N}|x|^{2-N}, & N\geq 3\\ -\dfrac{1}{2\pi}\ln|x|, & N=2\end{cases}$$

and $\omega_N>0$ represents the volume of the unit ball in $\mathbb{R}^N$.

Equation (1.2) has been extensively studied in recent decades due to its rich physical content and mathematical challenges. In terms of physics, Paredes et al. in [1] systematically analyzed the physical background of this model. Employing variational methods and numerical simulations, they revealed the crucial role of the nonlocal term in polaron models and self-gravitating systems. From a mathematical perspective, for the case $N\geq 3$ has been widely investigated. For example, this equation for the case $N=3$ with $\gamma>0$ corresponds to the well-known Choquard-Pekar equation which is of great significance in the study of polaron models and self-gravitating matter. Further analysis and conclusions regarding Equation (1.2) with $N\geq 3$ can be found in [2] [3] and the references therein. For the case $N=2$, the logarithmic integral kernel $G_2(x)$ is sign-changing and presents singularities as $|x|$ approaches to zero and infinity. Therefore, the approaches dealing with higher dimensional cases seem difficult to be adapted to the case $N=2$. Wang and Zhang [4] investigated the mass subcritical Schrödinger-Poisson system with logarithmic potentials and via refined energy estimates and blow-up analysis, established the existence of minimizers for any $\rho\in(0,\infty)$ as well as the limiting behavior and uniqueness of positive minimizers as $\rho\to\infty$. In [5], the authors developed a novel variational framework in the standard Sobolev space $H^1(\mathbb{R}^2)$ by approximating the logarithmic kernel.

We are interested in the external potential

$$V(x)=\left(|x|-A\right)^2,$$

where $A>0$ is a fixed parameter. This potential has a ring shape and takes its minimum value on the circle $|x|=A$. Ring-shaped traps have attracted considerable interest in ultracold atomic physics and Bose-Einstein condensation due to their unique topological properties, one can find more details in [6] [7]. It is also interesting to discuss mathematically how the ring shape potentials affect the behavior of BEC. Guo in [8] analyzed the properties of L^2-normalized minimizers for a two dimensional Bose-Einstein condensate with attractive interaction and ring-shaped potential. However, due to the presence of logarithmic convolution and non-uniform interaction forces in this paper, we need to employ new analytical tools and methods.

The coefficient $K(x)$ in the nonlinear term $aK(x)|\psi|^{p-2}\psi$ represents a spatially inhomogeneous interaction strength. In Bose-Einstein condensates, $K(x)$ can be controlled by applying a non-uniform magnetic field near a Feshbach resonance. In nonlinear optics, $K(x)$ corresponds to a material's position-dependent Kerr nonlinearity, the influence of spatially inhomogeneous nonlinearities in Schrödinger-Poisson systems has been studied in various contexts, including two-dimensional Bose-Einstein condensates with inhomogeneous attractive interactions $0<m(x)\le 1$ [9], the ground states of mass critical Schrödinger equations with spatially inhomogeneous nonlinearities in $\mathbb{R}^2$ [10].

The main aim of the present paper is to study the normalized solutions for the elliptic system (1.1) with $V(x)=(|x|-A)^2$, $p=4$ and $N=2$. For this purpose, we consider the constraint minimizers of the following variational problem

$$e(\gamma,a):=\inf_{u\in P}E_{\gamma,a}(u), \tag{1.3}$$

where the energy functional

$$\begin{aligned}E_{\gamma,a}(u):=&\frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u(x)|^2+(|x|-A)^2u^2(x)\right]\mathrm{d}x\\&+\frac{\gamma}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y-\frac{a}{4}\int_{\mathbb{R}^2}K(x)u^4(x)\mathrm{d}x.\end{aligned} \tag{1.4}$$

Throughout the paper, we impose the following assumptions on the inhomogeneous interaction $K(x)$:

(K_1) $K(x)\in C^{0,\alpha}(\mathbb{R}^2)$ with values in $(0,1]$;

(K_2) There exists constants $\beta>0$, $b\ge 2$ and $C>\beta^{-b}$, such that $1-K(x)\le C|x|^b$ for any $|x|<\beta$;

(K_3) $\lim\limits_{|x|\to\infty}|\nabla K(x)|$ exists.

The research of normalized solutions to system (1.1) presents challenges. Compared with the planar Schrödinger-Newton equations studied in [11], the present system features a ring-shaped potential $V(x)=(|x|-A)^2$ and a spatially inhomogeneous interaction $K(x)$, which lead to new phenomena such as the symmetry breaking of the minimizer and a refined characterization of its blow-up profile. The ring-shaped potential $V(x)$ attains its minimum along a closed curve, which breaks radial symmetry and creates a degenerate manifold of minimizers. In addition, the spatially inhomogeneous interaction $K(x)$ introduces local variations in the interaction strength. These variations can compete with the nonlocal Poisson term. This competition complicates the application of standard variational methods.

On account of the slow decay and singular nature of the logarithmic kernel, the functional $E_{\gamma,a}$ does not admit a well-defined realization on the standard Sobolev space $H^1(\mathbb{R}^2)$ with respect to the norm $\|u\|_1 := \left[\int_{\mathbb{R}^2}\left(|\nabla u|^2+u^2\right)\mathrm{d}x\right]^{\frac{1}{2}}$ for any $u\in H^1(\mathbb{R}^2)$. In addition, the kernel is unbounded both as $|x|\to\infty$ and near the origin, calling for a careful decomposition and refined estimates. To overcome these obstacles, we conduct the analysis in an appropriate weighted Sobolev space and split the logarithmic kernel into its positive and negative parts. In light of the difficulties described above, we now provide the essential analytic framework and definitions. First, we adopt the idea from [12] [13] and introduce the following weighted Sobolev space

$$\mathcal{X} := \left\{u\in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2}|x|^2 u^2(x)\,\mathrm{d}x<\infty\right\}$$

endowed with the norm

$$\|u\| := \left\{\int_{\mathbb{R}^2}\left[|\nabla u|^2+\left(1+|x|^2\right)u^2\right]\mathrm{d}x\right\}^{\frac{1}{2}},\ u\in\mathcal{X}.$$

We then define the constraint set

$$P := \left\{u\in\mathcal{X} : |u|_2=1\right\},$$

where $|\cdot|_q$ stands for the usual Lebesgue norm on $L^q(\mathbb{R}^2)$ for $q\in[1,\infty]$. As shown in ([14], Lemma 3.1), the embedding

$$\mathcal{X}\hookrightarrow L^q(\mathbb{R}^2) \text{ is compact for any } q\in[2,\infty). \tag{1.5}$$

We claim that $\int_{\mathbb{R}^2}(|x|-A)^2u^2(x)\,\mathrm{d}x<\infty$ for any $u\in\mathcal{X}$. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^2}(|x|-A)^2u^2(x)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^2}|x|^2u^2(x)\,\mathrm{d}x-2A\int_{\mathbb{R}^2}|x|u^2(x)\,\mathrm{d}x+A^2\int_{\mathbb{R}^2}u^2(x)\,\mathrm{d}x\\
&\le\int_{\mathbb{R}^2}|x|^2u^2(x)\,\mathrm{d}x-2A\left(\int_{\mathbb{R}^2}|x|^2u^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}u^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}+A^2\int_{\mathbb{R}^2}u^2(x)\,\mathrm{d}x\\
&<\infty.
\end{aligned}$$

Next, we perform the decomposition of the logarithmic function. This strategy allows us to treat the negative part by means of the Hardy-Littlewood-Sobolev inequality, while the positive part is controlled via weighted estimates. Inspired by [5] [12] [13] [15], we perform that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln(1+|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y\\
&\quad-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+|x-y|^{-1}\right)u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y\\
&:=D_1(u)-D_2(u).
\end{aligned}\tag{1.6}$$

Denote

$$|u|_* := \left(\int_{\mathbb{R}^2} |x|^2 u^2(x)\,dx\right)^{\frac{1}{2}},\ u \in \mathcal{X}.$$

Noticing that

$$\ln(1+|x-y|) \le |x-y| \le |x|+|y|,\ \forall x, y \in \mathbb{R}^2,$$

by Hölder's inequality we then deduce that

$$D_1(u) \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (|x|+|y|) u^2(x) u^2(y)\,dxdy \le 2|u|_2^3 |u|_*,\ u \in \mathcal{X}. \tag{1.7}$$

In virtue of the Hardy-Littlewood-Sobolev inequality (cf. [16]):

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u(x)||v(y)|}{|x-y|}\,dxdy \le C|u|_{\frac{4}{3}} |v|_{\frac{4}{3}},\ u, v \in L^{\frac{4}{3}}(\mathbb{R}^2), \tag{1.8}$$

there exists a constant $C > 0$ such that

$$D_2(u) \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u^2(x)u^2(y)}{|x-y|}\,dxdy \le C|u|_{\frac{8}{3}}^4,\ u \in L^{\frac{8}{3}}(\mathbb{R}^2). \tag{1.9}$$

It follows from (1.5)-(1.9) that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u^2(x) u^2(y)\,dxdy$ is well defined on $\mathcal{X}$. Together with (K_1), the energy functional $E_{\gamma,a}$ is hence well defined on $\mathcal{X}$.

If the external potential $V(x) = (|x|-A)^2$ in (1.4) is ignored and $K(x) \equiv 1$, it was shown in [12] that $e(\gamma,a)$ possesses minimizers if $\gamma > 0$, $0 < a < a^*$, or $\gamma < 0$, $a \le 0$. Here $a^* := |Q|_2^2$ and $Q = Q(|x|) > 0$ is the unique positive radially symmetric solution of the following scalar field equation (cf. [17]):

$$-\Delta u + u = u^3,\ u \in H^1(\mathbb{R}^2). \tag{1.10}$$

Furthermore, Q decays exponentially as $|x| \to \infty$ in the sense that

$$|Q(x)|, |\nabla Q(x)| = O\left(|x|^{-\frac{1}{2}} e^{-|x|}\right) \text{ as } |x| \to \infty. \tag{1.11}$$

It has been proved in [18] that the above function Q is an achievement function of the equality in the following classical Gagliardo-Nirenberg inequality:

$$\int_{\mathbb{R}^2} u^4\,dx \le \frac{2}{|Q|_2^2} \int_{\mathbb{R}^2} |\nabla u|^2\,dx \int_{\mathbb{R}^2} u^2\,dx,\ u \in H^1(\mathbb{R}^2). \tag{1.12}$$

From (1.10) and (1.12), we can get that

$$\int_{\mathbb{R}^2} |\nabla Q|^2\,dx = \int_{\mathbb{R}^2} Q^2\,dx = \frac{1}{2}\int_{\mathbb{R}^2} Q^4\,dx. \tag{1.13}$$

By means of these results, we establish the existence and nonexistence of minimizers for (1.3).

Theorem 1.1. *Let* $Q(x) = Q(|x|)$ *be the unique positive solution of* (1.10) *and* $a^* := |Q|_2^2$.

1) If $a < a^*$, then there exists at least one minimizer of $e(\gamma,a)$ for $\gamma \ne 0$;

2) If $a > a^*$, then there is no minimizer of $e(\gamma,a)$ for $\gamma \ne 0$ and

$e(\gamma,a)=-\infty$ for $\gamma\neq 0$;

3) If $a=a^*$, then there is no minimizer of $e(\gamma,a)$ for $\gamma>0$ and $e(\gamma,a^*)=-\infty$ for $\gamma>0$.

This theorem provides a complete picture of the constrained minimization problem for $e(\gamma,a)$ under the joint effects of the ring-shaped potential, the logarithmic kernel, and the spatially inhomogeneous interaction. The argument is carried out in the weighted Sobolev space $\mathcal{X}$, where the logarithmic singularity is handled via a positive-negative splitting. The critical constant $a^*=|Q|_2^2$, derived from the Gagliardo-Nirenberg and Young inequalities, allows us to derive a uniform lower bound for the energy functional on the constraint set P. The compact embedding $\mathcal{X}\hookrightarrow L^p(\mathbb{R}^2)$ then yields strong convergence of minimizing sequences, and the existence of a minimizer follows by weak lower semicontinuity. In the parameter regimes where minimizers cannot exist, we construct a suitable family of test functions u_τ and verify that $e(\gamma,a)\leq E_{\gamma,a}(u_\tau)\to-\infty$ as $\tau\to\infty$, thereby excluding the possibility of minimizers. The proposed framework thus successfully resolves the two major difficulties—the non-radial nature of the ring trap and the singular long-range behavior of the convolution kernel—and yields a sharp existence/non-existence criterion for $K(x)$-dependent interactions.

Theorem 1.2. *Assume that* $u_{\gamma,a}\in P$ *is a positive minimizer of* $e(\gamma,a)$ *for* $\gamma>0$ *and* $0<a<a^*$. *Then we have*

$$\lim_{a\nearrow a^*} e(\gamma,a)=-\infty \quad \text{for } \gamma>0,$$

$$\lim_{a\nearrow a^*}\left[\frac{8\pi(a^*-a)}{\gamma a^*}\right]^{\frac{1}{2}} u_{\gamma,a}\left(\left[\frac{8\pi(a^*-a)}{\gamma a^*}\right]^{\frac{1}{2}} x+x_{\gamma,a}\right)=\frac{Q(x)}{\sqrt{a^*}} \quad \text{in } \mathcal{X}\cap L^\infty(\mathbb{R}^2),$$

where $x_{\gamma,a}$ is the unique maximum point of $u_{\gamma,a}$ as $a\nearrow a^*$ and there exists some constant C_0 satisfying

$$\lim_{a\nearrow a^*}\frac{|x_{\gamma,a}|-A}{(a^*-a)^{\frac{1}{2}}}=C_0.$$

Theorem 1.2 presents the location of the maximum point of the scaled minimizer under the influence of the ring-shaped potential. It reveals the symmetry-breaking mechanism induced by the ring-shaped potential and the inhomogeneous interaction on the limiting behavior. To determine the limit of $x_{\gamma,a}$, we proceed by contradiction. The ratio $\dfrac{|x_{\gamma,a}|-A}{\varepsilon_{\gamma,a}}$ must converge to some finite constant C_0. Conversely, if this ratio were unbounded or tended to infinity, the contribution of the ring-shaped potential term would become too large and positive, contradicting the fact that $e(\gamma,a)\to-\infty$. A key tool in this argument is the mean value theorem. Then, owing to the translation non-invariance of the norm on $\mathcal{X}$, the scaled minimizer $v_{\gamma,a}$ is not uniformly bounded as $a\nearrow a^*$. We remedy this by adapting techniques from [19]. Finally, obtaining detailed information about the maximizer $x_{\gamma,a}$ as $a \nearrow a^*$ requires new ideas, because the term $\int_{\mathbb{R}^2}(|x|-A)^2 u_{\gamma,a}^2 \mathrm{d}x$ is not translation invariant and $\lim_{a\nearrow a^*} e(\gamma,a) = -\infty$ when $\gamma > 0$.

2. Existence and Nonexistence of Minimizers

In this section, we shall give the proof of Theorem 1.1 on the existence and nonexistence of minimizers for $e(\gamma,a)$.

Proof of Theorem 1.1. 1. We first prove the existence of minimizers for $e(\gamma,a)$ when $\gamma \neq 0$ and $a < a^*$. Following the Gagliardo-Nirenberg inequality from [18]: for any function $u \in H^1(\mathbb{R}^2)$ with $|u|_2 = 1$, we have

$$|u|_q \le \left(\frac{q}{2|Q_q|_2^{q-2}}\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2}|\nabla u|^2 \mathrm{d}x\right)^{\frac{q-2}{2q}}, \quad q \ge 2, \tag{2.1}$$

where Q_q is the unique positive solution of the following elliptic equation

$$-\frac{q-2}{2}\Delta u + u = u^{q-1}, \quad u \in H^1(\mathbb{R}^2).$$

Equation (1.9) together (2.1) then yields the existence of a constant $C > 0$ such that

$$D_2(u) \le C\left(\int_{\mathbb{R}^2}|\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}, \quad \forall\, u \in P. \tag{2.2}$$

With the above facts at hand, we prove Theorem 1.1 (1) through the following four cases.

Case 1: $\gamma > 0$ and $0 < a < a^*$. By Young's Inequality, we have that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^2}|x|u^2(x)\mathrm{d}x \le \varepsilon\int_{\mathbb{R}^2}|x|^2 u^2(x)\mathrm{d}x + \frac{1}{4\varepsilon}\int_{\mathbb{R}^2}u^2(x)\mathrm{d}x, \tag{2.3}$$

taking $\varepsilon = \frac{1}{4A}$, then by (1.4), (1.6), (1.12), (2.2), (2.3) and assumption (K_1), we get that for $u \in P$.

$$E_{\gamma,a}(u) \ge \frac{1}{2}\left(1-\frac{a}{a^*}\right)\int_{\mathbb{R}^2}|\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2}|x|^2 u^2 \mathrm{d}x - C\left(\int_{\mathbb{R}^2}|\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} - \frac{A^2}{2}. \tag{2.4}$$

We have shown that $E_{\gamma,a}(u)$ admits a lower bound. Take $\{u_n\} \subset P$ to be a minimizing sequence associated with $e(\gamma,a)$, one finds that the two integrals $\int_{\mathbb{R}^2}|\nabla u_n|^2 \mathrm{d}x$ and $\int_{\mathbb{R}^2}|x|^2 u_n^2 \mathrm{d}x$ are bounded uniformly over n from (2.4). Because $|u_n|_2 = 1$, this immediately implies that $\{u_n\}$ is bounded in $\mathcal{X}$. Utilizing (1.5), we can find $u \in \mathcal{X}$ satisfying

$$u_n \rightharpoonup u \text{ in } \mathcal{X} \text{ and } u_n \to u \text{ in } L^q(\mathbb{R}^2) \text{ for any } q \in [2,\infty) \text{ as } n \to \infty \tag{2.5}$$

which implies $u \in P$. Moreover, it follow from [20] that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u_n^2(x)u_n^2(y)\,\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y. \quad (2.6)$$

By the weak lower semicontinuity, for assumption (K_1) we have Applying Hölder inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}K(x)u_n^4(x)\,\mathrm{d}x-\int_{\mathbb{R}^2}K(x)u^4(x)\,\mathrm{d}x\right|\\&\le\left|\int_{\mathbb{R}^2}u_n^2\left(u_n^2-u^2\right)\mathrm{d}x\right|+\left|\int_{\mathbb{R}^2}u^2\left(u_n^2-u^2\right)\mathrm{d}x\right|\\&\le|u_n+u|_3|u_n-u|_3|u_n|_6+|u_n+u|_3|u_n-u|_3|u|_6\\&\le C|u_n-u|_3\to 0\ \text{ as } n\to\infty,\end{aligned}$$

it implies that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^2}K(x)u_n^4(x)\,\mathrm{d}x=\int_{\mathbb{R}^2}K(x)u^4(x)\,\mathrm{d}x.$$

and

$$\int_{\mathbb{R}^2}\left[|\nabla u|^2+(|x|-A)^2u^2\right]\mathrm{d}x\le\liminf_{n\to\infty}\int_{\mathbb{R}^2}\left[|\nabla u_n|^2+(|x|-A)^2u_n^2\right]\mathrm{d}x.$$

We conclude from the above that

$$e(\gamma,a)\le E_{\gamma,a}(u)\le\liminf_{n\to\infty}E_{\gamma,a}(u_n)=e(\gamma,a),$$

which implies that $e(\gamma,a)=E_{\gamma,a}(u)$. Thus u is a minimizer of $e(\gamma,a)$.

Case 2: $\gamma>0$ and $a\le 0$. By (1.4), (1.6), (2.2), (2.3) and assumption (K_1), we obtain that for $u\in P$,

$$E_{\gamma,a}(u)\ge\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^2}|x|^2u^2\mathrm{d}x-C\left(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}-\frac{A^2}{2},$$

which implies that $E_{\gamma,a}(u)$ is bounded from below. Repeating the argument from Case 1, we conclude that $e(\gamma,a)$ admits at least one minimizer for $\gamma>0$ and $a\le 0$.

Case 3: $\gamma<0$ and $0<a<a^*$. Applying Young's inequality, we deduce from (1.7) that for any $\epsilon>0$,

$$D_1(u)\le 2\epsilon|u|_*^2+\frac{1}{2\epsilon}|u|_2^6,\ u\in\mathcal{X}. \quad (2.7)$$

Taking $\epsilon=-\frac{\pi}{\gamma}$, from (1.4), (1.6), (1.12), (2.7) and assumption (K_1) we obtain that for $u\in P$,

$$E_{\gamma,a}(u)\ge\frac{1}{2}\left(1-\frac{a}{a^*}\right)\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^2}(|x|-A)^2u^2\mathrm{d}x-\frac{1}{4}\int_{\mathbb{R}^2}|x|^2u^2\mathrm{d}x-\frac{\gamma^2}{16\pi^2}.$$

With $\varepsilon=\frac{1}{8A}$ in (2.3), we arrive at

$$E_{\gamma,a}(u)\ge\frac{1}{2}\left(1-\frac{a}{a^*}\right)\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x+\frac{1}{8}\int_{\mathbb{R}^2}|x|^2u^2\mathrm{d}x-\frac{3}{2}A^2-\frac{\gamma^2}{16\pi^2}, \quad (2.8)$$

which yields that $E_{\gamma,a}(u)$ is bounded from below. Similar to Case 1, one can choose a minimizing sequence $\{u_n\} \subset P$ and $\lim_{n\to\infty} E_{\gamma,a}(u_n) = e(\gamma,a)$, it follows that we obtain the equivalent conclusion (2.5). Using the same argumentation method as in [20], we can obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u_n^2(x) u_n^2(y)\,dxdy = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u^2(x) u^2(y)\,dxdy. \quad (2.9)$$

Then following proof is similar to that of Case 1.

Case 4: $\gamma < 0$ and $a \le 0$. By (1.4), (1.6), (1.7), (2.3) and assumption (K_1), we get that for $u \in P$,

$$E_{\gamma,a}(u) \ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{8}\int_{\mathbb{R}^2} |x|^2 u^2 dx - \frac{3}{2}A^2 - \frac{\gamma^2}{16\pi^2},$$

which means that $E_{\gamma,a}(u)$ is bounded from below. The following proof is similar to that of Case 3.

2. We next prove the nonexistence of minimizers of $e(\gamma,a)$ for $\gamma \ne 0$, $a > a^*$, or $\gamma > 0$, $a = a^*$. Consider the test function

$$u_\tau(x) := \frac{\tau}{|Q|_2} Q(\tau x), \ \tau > 0.$$

Clearly, $u_\tau \in P$ for all $\tau > 0$. It follows from (1.13) that for $\gamma \ne 0$, $a > a^*$, or $\gamma > 0$, $a = a^*$,

$$\begin{aligned} e(\gamma,a) &\le E_{\gamma,a}(u_\tau) \\ &= \frac{\tau^2}{4a^*}\int_{\mathbb{R}^2}\left[1 - \frac{a}{a^*}k\left(\frac{x}{\tau}\right)\right] Q^4(x)\,dx + \frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2 Q^2 dx + \frac{A^2}{2} - \frac{\gamma}{8\pi}\ln\tau \quad (2.10) \\ &\quad - \frac{A}{a^*\tau}\int_{\mathbb{R}^2}|x| Q^2 dx + \frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y| Q^2(x) Q^2(y)\,dxdy. \end{aligned}$$

Case 1: $\gamma \ne 0, a > a^*$. On account of $K(x) \in C^{0,\alpha}(\mathbb{R}^2)$ and assumption (K_2), we obtain that $\lim_{x\to 0} K(x) = 1$. Therefore

$$\lim_{\tau\to\infty}\int_{\mathbb{R}^2}\left[1 - \frac{a}{a^*}k\left(\frac{x}{\tau}\right)\right] Q^4(x)\,dx = \int_{\mathbb{R}^2}\left(1 - \frac{a}{a^*}\right) Q^4(x)\,dx = 2(a^* - a),$$

combining with (1.11) and (2.10) we have $e(\gamma,a) \to -\infty$ as $\tau \to \infty$.

Case 2: $\gamma > 0, a = a^*$. On account of assumption (K_2), (1.11) and (2.10), we get

$$\begin{aligned} e(\gamma,a) &\le E_{\gamma,a}(u_\tau) \\ &= \frac{\tau^2}{4a^*}\int_{\mathbb{R}^2}\left[1 - k\left(\frac{x}{\tau}\right)\right] Q^4(x)\,dx + \frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2 Q^2 dx + \frac{A^2}{2} \\ &\quad - \frac{A}{a^*\tau}\int_{\mathbb{R}^2}|x| Q^2 dx - \frac{\gamma}{8\pi}\ln\tau + \frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y| Q^2(x) Q^2(y)\,dxdy \\ &\le \frac{C}{4a^*}\tau^{2-b}\int_{\mathbb{R}^2}|x|^b Q^4(x)\,dx + \frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2 Q^2 dx + \frac{A^2}{2} - \frac{A}{a^*\tau}\int_{\mathbb{R}^2}|x| Q^2 dx \\ &\quad - \frac{\gamma}{8\pi}\ln\tau + \frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y| Q^2(x) Q^2(y)\,dxdy \to \infty \ \text{as}\ \tau \to \infty. \end{aligned}$$

The above discussion means that $e(\gamma,a)$ has no minimizer when $\gamma \neq 0$ with $a > a^*$, or when $\gamma > 0$ with $a = a^*$. Moreover, $e(\gamma,a) = -\infty$ for $\gamma \neq 0$ with $a > a^*$ and $e(\gamma,a^*) = -\infty$ for $\gamma > 0$. This establishes the proof of Theorem 1.1.

3. Limiting Behavior of Minimizers

In this section, we prove Theorem 1.2 on the limiting behavior of positive minimizers for $e(\gamma,a)$ as $a \nearrow a^*$, where $\gamma > 0$ is fixed. Standard variational theory yields the Euler-Lagrange equation:

$$\begin{aligned}&-\Delta u_{\gamma,a} + (|x|-A)^2 u_{\gamma,a} + \frac{\gamma}{2\pi}\int_{\mathbb{R}^2} \ln|x-y| u_{\gamma,a}^2(y)\mathrm{d}y u_{\gamma,a} \\ &= \lambda_{\gamma,a} u_{\gamma,a} + aK(x) u_{\gamma,a}^3 \quad \text{in } \mathbb{R}^2,\end{aligned} \tag{3.1}$$

where $\lambda_{\gamma,a} \in \mathbb{R}$ is the associated Lagrange multiplier and satisfies that

$$\lambda_{\gamma,a} = 2e(\gamma,a) + \frac{\gamma}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u_{\gamma,a}^2(x) u_{\gamma,a}^2(y)\mathrm{d}x\mathrm{d}y - \frac{a}{2}\int_{\mathbb{R}^2} K(x) u_{\gamma,a}^4 \mathrm{d}x. \tag{3.2}$$

We first give the limit of $e(\gamma,a)$ when $\gamma \neq 0$.

Lemma 3.1. $\lim_{a\nearrow a^*} e(\gamma,a) = -\infty$ *for* $\gamma > 0$.

Proof. We now examine the limit $\lim_{a\nearrow a^*} e(\gamma,a)$. Assuming $0 < a < a^*$, we set $\tau = (a^* - a)^{-\frac{1}{2}}$ in (2.10) and find that $\tau \to \infty$ as $a \nearrow a^*$. Due to (K_2), we have $K(x) \geq 1 - |x|^b$, then

$$\begin{aligned} e(\gamma,a) &\leq E_{\gamma,a}(u_\tau) \\ &= \frac{\tau^2}{4a^*}\int_{\mathbb{R}^2}\left[1 - \frac{a}{a^*}\left(1 - \left|\frac{x}{\tau}\right|^b\right)\right] Q^4(x)\mathrm{d}x + \frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2 Q^2 \mathrm{d}x + \frac{A^2}{2} \\ &\quad - \frac{A}{a^*\tau}\int_{\mathbb{R}^2}|x| Q^2 \mathrm{d}x - \frac{\gamma}{8\pi}\ln\tau + \frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y| Q^2(x) Q^2(y)\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{4a^*}\int_{\mathbb{R}^2} Q^4 \mathrm{d}x + C\frac{a}{a^*}\tau^{2-b}\int_{\mathbb{R}^2}|x|^b Q^4 \mathrm{d}x + \frac{1}{2a^*\tau^2}\int_{\mathbb{R}^2}|x|^2 Q^2 \mathrm{d}x + \frac{A^2}{2} \\ &\quad - \frac{A}{a^*\tau}\int_{\mathbb{R}^2}|x| Q^2 \mathrm{d}x - \frac{\gamma}{8\pi}\ln\tau + \frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y| Q^2(x) Q^2(y)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

It implies that $\lim_{a\nearrow a^*} E_{\gamma,a}(u_\tau) = -\infty$ for $\gamma > 0$. Therefore, $\lim_{a\nearrow a^*} e(\gamma,a) = -\infty$ when $\gamma > 0$.

We derive some estimates for positive minimizers.

Lemma 3.2. *Let* $u_{\gamma,a}$ *be a positive minimizer of* $e(\gamma,a)$ *for* $\gamma > 0$ *and* $a \in (0,a^*)$, *define*

$$\varepsilon_{\gamma,a} := \left(\int_{\mathbb{R}^2}|\nabla u_{\gamma,a}|^2 \mathrm{d}x\right)^{-\frac{1}{2}} \quad \text{and} \quad v_{\gamma,a}(x) := \varepsilon_{\gamma,a} u_{\gamma,a}(\varepsilon_{\gamma,a}x + x_{\gamma,a}) \quad \text{in } \mathbb{R}^2, \tag{3.3}$$

where $x_{\gamma,a}$ is a global maximum point of $u_{\gamma,a}$. Then we have [(1)]

(1) $\varepsilon_{\gamma,a} > 0$ satisfies that for any $\gamma > 0$,

$$\varepsilon_{\gamma,a} \to 0 \text{ and } \lambda_{\gamma,a}\varepsilon_{\gamma,a}^2 \to -1 \text{ as } a \nearrow a^*; \tag{3.4}$$

(2) There exists a constant $\alpha > 0$, independent of $\gamma > 0$ and $a \in (0, a^*)$, such that

$$\int_{B_2(0)} v_{\gamma,a}^2(x)\,dx \ge \alpha > 0 \text{ as } a \nearrow a^*; \tag{3.5}$$

(3) $v_{\gamma,a} > 0$ satisfies that for any $\gamma > 0$,

$$v_{\gamma,a}(x) \to v_0(x) := \frac{Q(|x|)}{\sqrt{a^*}} \text{ in } H^1(\mathbb{R}^2) \text{ as } a \nearrow a^*, \tag{3.6}$$

where $Q(x) > 0$ is the unique positive solution of (1.10).

Proof. 1. We first show that $\varepsilon_{\gamma,a} \to 0$ as $a \nearrow a^*$. By (1.4), (1.12), (2.2), (3.3) and assumption (K_1), we calculate that

$$\begin{aligned} e(\gamma,a) &= E_{\gamma,a}(u_{\gamma,a}) \\ &\ge \frac{a^*-a}{2a^*}\int_{\mathbb{R}^2}|\nabla u_{\gamma,a}|^2\,dx - C\left(\int_{\mathbb{R}^2}|\nabla u_{\gamma,a}|^2\,dx\right)^{\frac{1}{2}} \\ &\ge -C\varepsilon_{\gamma,a}^{-1}. \end{aligned}$$

Together with Lemma 3.1, we obtain $\lim\limits_{a\nearrow a^*}\varepsilon_{\gamma,a} = 0$.

Then, we proof that

$$\lambda_{\gamma,a}\varepsilon_{\gamma,a}^2 \to -1 \text{ as } a \nearrow a^*. \tag{3.7}$$

Indeed, we deduce from (1.6) and (3.3) that

$$\begin{aligned} \varepsilon_{\gamma,a}^2 e(\gamma,a) = {} & \frac{1}{2} + \frac{\varepsilon_{\gamma,a}^2}{2}\int_{\mathbb{R}^2}(|x|-A)^2 u_{\gamma,a}^2\,dx + \frac{\gamma\varepsilon_{\gamma,a}^2}{8\pi}\left[D_1(u_{\gamma,a}) - D_2(u_{\gamma,a})\right] \\ & - \frac{a\varepsilon_{\gamma,a}^2}{4}\int_{\mathbb{R}^2}K(x)u_{\gamma,a}^4\,dx. \end{aligned} \tag{3.8}$$

Since $\lim\limits_{a\nearrow a^*}\varepsilon_{\gamma,a} = 0$, we derive from (2.2) and (3.3) that

$$0 \le \varepsilon_{\gamma,a}^2 D_2(u_{\gamma,a}) \le C\varepsilon_{\gamma,a} \to 0 \text{ as } a \nearrow a^*.$$

It means

$$\lim_{a\nearrow a^*}\varepsilon_{\gamma,a}^2 D_2(u_{\gamma,a}) = 0. \tag{3.9}$$

Due to (1.12) and (3.3), we have

$$\begin{aligned} \varepsilon_{\gamma,a}^2\int_{\mathbb{R}^2}K(x)u_{\gamma,a}^4\,dx &\le \varepsilon_{\gamma,a}^2\int_{\mathbb{R}^2}u_{\gamma,a}^4\,dx \\ &\le \varepsilon_{\gamma,a}^2\frac{2}{a^*}\int_{\mathbb{R}^2}|\nabla u_{\gamma,a}|^2\,dx \\ &= \frac{2}{a^*}. \end{aligned} \tag{3.10}$$

Combining (3.8) with (3.9) and (3.10), we get that

$$\begin{aligned} & \liminf_{a\nearrow a^*}\varepsilon_{\gamma,a}^2 e(\gamma,a) \\ & \ge \liminf_{a\nearrow a^*}\left[\frac{a^*-a}{2a^*} + \frac{\varepsilon_{\gamma,a}^2}{2}\int_{\mathbb{R}^2}(|x|-A)^2 u_{\gamma,a}^2\,dx + \frac{\gamma\varepsilon_{\gamma,a}^2}{8\pi}D_1(u_{\gamma,a})\right] \ge 0. \end{aligned} \tag{3.11}$$

With the fact $\lim_{a\nearrow a^*} e(\gamma,a)=-\infty$, from (3.8)-(3.10) we obtain that

$$\begin{aligned}0&\ge \limsup_{a\nearrow a^*}\varepsilon_{\gamma,a}^2 e(\gamma,a)\\&=\limsup_{a\nearrow a^*}\left[\frac{1}{2}\left(1-\frac{a\varepsilon_{\gamma,a}^2}{2}\int_{\mathbb{R}^2}K(x)u_{\gamma,a}^4\mathrm{d}x\right)+\frac{\varepsilon_{\gamma,a}^2}{2}\int_{\mathbb{R}^2}\left(|x|-A\right)^2u_{\gamma,a}^2\mathrm{d}x\right.\\&\quad\left.+\frac{\gamma\varepsilon_{\gamma,a}^2}{8\pi}D_1\left(u_{\gamma,a}\right)\right]\\&\ge\limsup_{a\nearrow a^*}\frac{1}{2}\left(1-\frac{a}{a^*}\right)=0.\end{aligned}\tag{3.12}$$

This further gives that

$$\lim_{a\nearrow a^*}\varepsilon_{\gamma,a}^2\int_{\mathbb{R}^2}K(x)u_{\gamma,a}^4\mathrm{d}x=\frac{2}{a^*}.\tag{3.13}$$

Using the similar argument of (3.13), we then derive from (3.11) and (3.12) that

$$\begin{aligned}&\lim_{a\nearrow a^*}\varepsilon_{\gamma,a}^2 e(\gamma,a)=0,\ \lim_{a\nearrow a^*}\varepsilon_{\gamma,a}^2\int_{\mathbb{R}^2}\left(|x|-A\right)^2u_{\gamma,a}^2\mathrm{d}x=0,\\&\lim_{a\nearrow a^*}\varepsilon_{\gamma,a}^2 D_1\left(u_{\gamma,a}\right)=0.\end{aligned}$$

Combining this with (3.2), (3.9) and (3.13), we then derive that the claim (3.7) holds true.

2. We now prove (3.5). Due to (3.1) and (3.3), we obtain that $v_{\gamma,a}$ satisfies

$$\begin{aligned}&-\Delta v_{\gamma,a}+\varepsilon_{\gamma,a}^2\left(\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|-A\right)^2v_{\gamma,a}+\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(y)\mathrm{d}y v_{\gamma,a}\\&=\lambda_{\gamma,a}\varepsilon_{\gamma,a}^2v_{\gamma,a}+aK\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right)v_{\gamma,a}^3-\frac{\gamma}{2\pi}\varepsilon_{\gamma,a}^2\ln\varepsilon_{\gamma,a}v_{\gamma,a}\quad\text{in }\mathbb{R}^2.\end{aligned}\tag{3.14}$$

We deduce from (3.3) that

$$\left\|v_{\gamma,a}\right\|_1^2=\varepsilon_{\gamma,a}^2\int_{\mathbb{R}^2}\left|\nabla u_{\gamma,a}\right|^2\mathrm{d}x+\int_{\mathbb{R}^2}u_{\gamma,a}^2\mathrm{d}x=2.\tag{3.15}$$

Using the similar argument of [20] that

$$\begin{aligned}&\int_{\mathbb{R}^2}\ln\left(1+|x-y|^{-1}\right)v_{\gamma,a}^2(y)\mathrm{d}y\\&\le\int_{\mathbb{R}^2}\frac{v_{\gamma,a}^2(y)}{|x-y|}\mathrm{d}y\le C\left\|v_{\gamma,a}\right\|_1^2=2C\ \text{ uniformly for any } x\in\mathbb{R}^2\end{aligned}\tag{3.16}$$

and

$$\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(y)\mathrm{d}y\ge -2C\ \text{ for any } x\in\mathbb{R}^2,\tag{3.17}$$

where $C>0$ is independent of $\gamma>0$ and $a\in\left(0,a^*\right)$. Therefore, we infer from (3.4), (3.14), (3.17) and and assumption (K_1) that

$$-\Delta v_{\gamma,a}+\frac{5}{9}v_{\gamma,a}\le a^*K\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right)v_{\gamma,a}^3\le a^*v_{\gamma,a}^3\ \text{ as } a\nearrow a^*.\tag{3.18}$$

Because $x_{\gamma,a}$ is a global maximum point of $u_{\gamma,a}$, the same holds for $v_{\gamma,a}$ at the origin 0. This implies $-\Delta v_{\gamma,a}(0)\ge 0$ and from (3.18) we infer

$\frac{5}{9}v_{\gamma,a}(0) \le a^* v_{\gamma,a}^3(0)$ as $a \nearrow a^*$. Thus, for a sufficiently close to a^*, there exists $\kappa > 0$ (independent of $\gamma > 0$ and $a \in (0, a^*)$) such that $v_{\gamma,a}(0) \ge \kappa > 0$. Turning to the De Giorgi-Nash-Moser theory and invoking (3.15) alongside (3.18), we secure a constant $C > 0$, also independent of $\gamma > 0$ and $a \in (0, a^*)$, with the property that

$$\left(\int_{B_2(0)} v_{\gamma,a}^2(x)\mathrm{d}x\right)^{\frac{1}{2}} \ge C \max_{x \in B_1(0)} v_{\gamma,a}(x) \ge C\kappa =: \sqrt{\alpha} > 0 \text{ as } a \nearrow a^*.$$

3. We now prove (3.6). By (3.15), $\{v_{\gamma,a}\}$ is uniformly bounded in $H^1(\mathbb{R}^2)$. Hence, up to a subsequence, we have $v_{\gamma,a} \rightharpoonup v_0$ in H^1, $v_{\gamma,a} \to v_0$ in L^q_{loc} ($q \in [2, \infty)$) and a.e. as $a \nearrow a^*$. From (3.14) we obtain

$$-\Delta v_0 + v_0 = a^* v_0^3 \text{ in } \mathbb{R}^2. \tag{3.19}$$

Fatou's lemma and (3.5) give $0 < |v_0|_2 \le 1$. By the Brézis-Lieb lemma and (3.13), we get that

$$\begin{aligned}
&1 = |v_{\gamma,a}|_2^2 = |v_0|_2^2 + |v_{\gamma,a} - v_0|_2^2 + o(1) \text{ as } a \nearrow a^*, \\
&1 = |\nabla v_{\gamma,a}|_2^2 = |\nabla v_0|_2^2 + |\nabla v_{\gamma,a} - \nabla v_0|_2^2 + o(1) \text{ as } a \nearrow a^*, \\
&\frac{2}{a^*} \leftarrow |v_{\gamma,a}|_4^4 = |v_0|_4^4 + |v_{\gamma,a} - v_0|_4^4 + o(1) \text{ as } a \nearrow a^*.
\end{aligned}$$

It then follows from (1.12) that

$$\begin{aligned}
0 &= \lim_{a \nearrow a^*}\left(|\nabla v_{\gamma,a}|_2^2 - \frac{a}{2}|v_{\gamma,a}|_4^4\right) \\
&\ge \frac{a^*}{2}\left(|v_0|_2^{-2} - 1\right)|v_0|_4^4 + \lim_{a \nearrow a^*}\left(1 - |v_{\gamma,a} - v_0|_2^2\right)|\nabla v_{\gamma,a} - \nabla v_0|_2^2 \ge 0,
\end{aligned} \tag{3.20}$$

so $|v_0|_2 = 1$ and $|\nabla v_{\gamma,a} - \nabla v_0|_2 \to 0$. Hence $v_{\gamma,a} \to v_0$ in H^1. Then (3.20) yields $|\nabla v_0|_2^2 = \frac{a^*}{2}|v_0|_4^4$, thus v_0 solves $-\Delta v_0 + v_0 = a^* v_0^3$ in $\mathbb{R}^2$. By the strong maximum principle, $v_0 > 0$ in $\mathbb{R}^2$. By uniqueness of positive solutions (up to translation), $v_0(x) = \frac{1}{\sqrt{a^*}} Q(|x - y_0|)$ for some $y_0 \in \mathbb{R}^2$. Additionally, since the origin 0 is the global maximum point of $v_{\gamma,a}$, it is also a global maximum point of v_0. Note that $Q(x) = Q(|x|)$ is radially symmetric and strictly decreasing in $|x|$. This further implies that $y_0 = 0$, then

$$v_{\gamma,a}(x) \to v_0(x) = \frac{Q(|x|)}{\sqrt{a^*}} \text{ in } H^1(\mathbb{R}^2) \text{ as } a \nearrow a^*.$$

□

Lemma 3.3. *Let* $v_{\gamma,a}(x)$ *be given by* (3.3) *and* $x_{\gamma,a}$ *is a global maximum point of* $u_{\gamma,a}$. *Then there exists a large constant* $R > 0$ *such that for any* $\gamma > 0$,

$$\left|v_{\gamma,a}(x)\right|\le Ce^{-\frac{2}{3}|x|}\ \text{ and }\ \left|\nabla v_{\gamma,a}(x)\right|\le Ce^{-\frac{1}{2}|x|}\ \text{ uniformly for }\ |x|\ge R\ \text{ as }\ a\nearrow a^*,\tag{3.21}$$

where the constant $C>0$ is independent of $a\in(0,a^*)$.

Proof. Applying the De Giorgi-Nash-Moser theory together with (3.6), (3.15) and (3.18), we obtain

$$v_{\gamma,a}(x)\in L^\infty(\mathbb{R}^2)\ \text{ and }\ v_{\gamma,a}(x)\to 0\ \text{ as }\ |x|\to\infty\ \text{ uniformly in }\ a\nearrow a^*.\tag{3.22}$$

Combining this with (3.18), it then yields that there exists a large constant $R>0$ which independent of $\gamma>0$ and $a\in(0,a^*)$ such that

$$-\Delta v_{\gamma,a}+\frac{4}{9}v_{\gamma,a}\le 0\ \text{ uniformly for }\ |x|\ge R\ \text{ as }\ a\nearrow a^*.\tag{3.23}$$

Using the comparison principle to (3.23), we then derive that

$$\left|v_{\gamma,a}(x)\right|\le Ce^{-\frac{2}{3}|x|}\ \text{ uniformly for }\ |x|\ge R\ \text{ as }\ a\nearrow a^*,\tag{3.24}$$

where the constant $C>0$ is independent of $\gamma>0$ and $a\in(0,a^*)$.

Denoting $x_{\gamma,a}:=\left(x_{\gamma,a}^1,x_{\gamma,a}^2\right)\in\mathbb{R}^2$, we infer from (3.14) that

$$\begin{aligned}&-\Delta\frac{\partial v_{\gamma,a}}{\partial x_j}+\varepsilon_{\gamma,a}^2\left(\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|-A\right)^2\frac{\partial v_{\gamma,a}}{\partial x_j}+2\varepsilon_{\gamma,a}^3\left(\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|-A\right)\frac{\left(\varepsilon_{\gamma,a}x_j+x_{\gamma,a}^j\right)}{\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|}v_{\gamma,a}\\&+\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(y)\mathrm{d}y\frac{\partial v_{\gamma,a}}{\partial x_j}+\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\frac{x_j-y_j}{|x-y|^2}v_{\gamma,a}^2(y)\mathrm{d}y\,v_{\gamma,a}\\&=\lambda_{\gamma,a}\varepsilon_{\gamma,a}^2\frac{\partial v_{\gamma,a}}{\partial x_j}+3a\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right)v_{\gamma,a}^2\frac{\partial v_{\gamma,a}}{\partial x_j}+a\frac{\partial K\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right)}{\partial x_j}\varepsilon_{\gamma,a}v_{\gamma,a}^3\\&\quad-\frac{\gamma}{2\pi}\varepsilon_{\gamma,a}^2\ln\varepsilon_{\gamma,a}\frac{\partial v_{\gamma,a}}{\partial x_j}\ \text{ in }\mathbb{R}^2.\end{aligned}\tag{3.25}$$

Multiplying (3.25) by $\frac{\partial v_{\gamma,a}}{\partial x_j}$, we get that

$$\begin{aligned}&-\frac{1}{2}\Delta\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2+\left|\nabla\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2+\varepsilon_{\gamma,a}^2\left(\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|-A\right)^2\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2\\&+2\varepsilon_{\gamma,a}^3\left(\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|-A\right)\frac{\left(\varepsilon_{\gamma,a}x_j+x_{\gamma,a}^j\right)}{\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|}v_{\gamma,a}\frac{\partial v_{\gamma,a}}{\partial x_j}\\&+\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(y)\mathrm{d}y\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2+\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\frac{x_j-y_j}{|x-y|^2}v_{\gamma,a}^2(y)\mathrm{d}y\,v_{\gamma,a}\frac{\partial v_{\gamma,a}}{\partial x_j}\\&=\lambda_{\gamma,a}\varepsilon_{\gamma,a}^2\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2+3aK\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right)v_{\gamma,a}^2\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2\\&\quad+a\frac{\partial K\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right)}{\partial x_j}\varepsilon_{\gamma,a}v_{\gamma,a}^3\frac{\partial v_{\gamma,a}}{\partial x_j}-\frac{\gamma}{2\pi}\varepsilon_{\gamma,a}^2\ln\varepsilon_{\gamma,a}\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2\ \text{ in }\mathbb{R}^2.\end{aligned}\tag{3.26}$$

With the fact in [20]

$$\int_{\mathbb{R}^2}\frac{u_n^2(y)}{|x-y|}\mathrm{d}y\le C\|u_n\|_{H^1(\mathbb{R}^2)}^2 \text{ holds for any } x\in\mathbb{R}^2 \tag{3.27}$$

and assumption (K_3), we obtain from (3.24) that

$$2\varepsilon_{\gamma,a}^3\left|\left(|\varepsilon_{\gamma,a}x+x_{\gamma,a}|-A\right)\frac{\left(\varepsilon_{\gamma,a}x_j+x_{\gamma,a}^j\right)}{|\varepsilon_{\gamma,a}x+x_{\gamma,a}|}\right|\left|v_{\gamma,a}\frac{\partial v_{\gamma,a}}{\partial x_j}\right|$$

$$\le\left|\left(|\varepsilon_{\gamma,a}x+x_{\gamma,a}|-A\right)\right|^2|v_{\gamma,a}|^2+\varepsilon_{\gamma,a}^6\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2\le C\mathrm{e}^{-|x|}+\varepsilon_{\gamma,a}^6\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2,$$

$$\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\frac{|x_j-y_j|}{|x-y|^2}v_{\gamma,a}^2(y)\mathrm{d}y\left|v_{\gamma,a}\frac{\partial v_{\gamma,a}}{\partial x_j}\right|\le\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|x-y|}v_{\gamma,a}^2(y)\mathrm{d}y\left|v_{\gamma,a}\frac{\partial v_{\gamma,a}}{\partial x_j}\right|$$

$$\le C\mathrm{e}^{-\frac{4}{3}|x|}+\varepsilon_{\gamma,a}^4\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2$$

and

$$a\varepsilon_{\gamma,a}\left|\frac{\partial K\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right)}{\partial x_j}\right|\left|v_{\gamma,a}^3\frac{\partial v_{\gamma,a}}{\partial x_j}\right|\le\left|\frac{a^2}{4}\nabla K(x)\right|^2|v_{\gamma,a}|^6+\left|\varepsilon_{\gamma,a}\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2$$

$$\le C\mathrm{e}^{-4|x|}+\varepsilon_{\gamma,a}^2\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2,$$

uniformly for $|x|>R$ as $a\nearrow a^*$. Moreover, from (3.16) we get

$$\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\ln\left(1+|x-y|^{-1}\right)v_{\gamma,a}^2(y)\mathrm{d}y\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2\le C\varepsilon_{\gamma,a}^2\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2 \text{ in } \mathbb{R}^2.$$

Noting that

$$\left|\nabla\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2+\varepsilon_{\gamma,a}^2\left(|\varepsilon_{\gamma,a}x+x_{\gamma,a}|-A\right)^2\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2$$

$$+\frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\ln\left(1+|x-y|\right)v_{\gamma,a}^2(y)\mathrm{d}y\left|\frac{\partial v_{\gamma,a}}{\partial x_j}\right|^2\ge 0 \text{ in } \mathbb{R}^2,$$

we then derive from (3.4) and (3.26) that

$$\left(-\frac{1}{2}\Delta+\frac{5}{6}-3a^*K\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right)v_{\gamma,a}^2\right)|\nabla v_{\gamma,a}|^2\le C\mathrm{e}^{-|x|} \text{ uniformly for } |x|\ge R \text{ as } a\nearrow a^*.$$

Combining this with (3.22) further implies that

$$\left(-\Delta+\frac{6}{5}\right)|\nabla v_{\gamma,a}|^2\le C\mathrm{e}^{-|x|} \text{ uniformly for } |x|\ge R \text{ as } a\nearrow a^*. \tag{3.28}$$

Applying the De Giorgi-Nash-Moser theory, we then deduce from (3.28) that there exists a constant $C>0$, independent of $\gamma>0$ and $a\in(0,a^*)$, such that

$$\max_{B_1(\xi)}|\nabla v_{\gamma,a}(x)|^2\le C\left(\int_{B_2(\xi)}|\nabla v_{\gamma,a}|^2\mathrm{d}x+\left\|\mathrm{e}^{-|x|}\right\|_{L^2(B_2(\xi))}\right),$$

where $\xi\in\mathbb{R}^2$ is arbitrary. Together with (3.6), it yields that

$\left|\nabla v_{\gamma,a}(x)\right| \in L^\infty(\mathbb{R}^2)$ and $\lim_{|x|\to\infty}\left|\nabla v_{\gamma,a}(x)\right| = 0$ uniformly for $\gamma > 0$ as $a \nearrow a^*$.

Using the comparison principle to (3.28), we further obtain that

$$\left|\nabla v_{\gamma,a}(x)\right| \le C\mathrm{e}^{-\frac{|x|}{2}} \text{ uniformly for } |x| \ge R \text{ as } a \nearrow a^*.$$

□

Now we prove the refined limiting behavior of positive minimizers of $e(\gamma,a)$ in $\mathcal{H}\cap L^\infty(\mathbb{R}^2)$ for $\gamma>0$ as $a \nearrow a^*$.

Proof of Theorem 1.2. In view of the above Lemma, we need to prove

$$\frac{\left|x_{\gamma,a}\right| - A}{\left(a^*-a\right)^{\frac{1}{2}}} \to C_0 \text{ as } a \nearrow a^*, \tag{3.29}$$

$$v_{\gamma,a}(x) \to \frac{Q(|x|)}{\sqrt{a^*}} \text{ in } \mathcal{X}\cap L^\infty(\mathbb{R}^2) \text{ as } a \nearrow a^* \tag{3.30}$$

and

$$\varepsilon_{\gamma,a} = \left[\frac{8\pi\left(a^*-a\right)}{\gamma a^*}\right]^{\frac{1}{2}}\left(1+o(1)\right), \tag{3.31}$$

where $x_{\gamma,a}$ is the unique maximum point of the positive minimizer $u_{\gamma,a}$ and $v_{\gamma,a}$ is defined by (3.3).

Now we prove (3.29). Due to (1.4) (1.12) and (3.3), that

$$\begin{aligned} e(\gamma,a) &= E_{\gamma,a}\left(u_{\gamma,a}\right) \\ &\ge \frac{\gamma}{8\pi}\ln\varepsilon_{\gamma,a} + \frac{\gamma}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\, v_{\gamma,a}^2(x)v_{\gamma,a}^2(y)\,\mathrm{d}x\mathrm{d}y \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left(\left|\varepsilon_{\gamma,a}x + x_{\gamma,a}\right| - A\right)^2 v_{\gamma,a}^2(x)\,\mathrm{d}x. \end{aligned} \tag{3.32}$$

We claim that $\left\{\frac{\left|x_{\gamma,a}\right|-A}{\varepsilon_{\gamma,a}}\right\} \subset \mathbb{R}$ is uniformly bounded as $a \nearrow a^*$. Otherwise, there exists a sequence of $\{a\}$, denoted by $\{a_k\}$, such that $\left|\frac{\left|x_{\gamma,a_k}\right|-A}{\varepsilon_{\gamma,a_k}}\right| \to \infty$ as $k\to\infty$. We can get for any constant $C>0$ that

$$\lim_{k\to\infty}\varepsilon_{\gamma,a_k}^{-2}\int_{\mathbb{R}^2}\left(\left|\varepsilon_{\gamma,a_k}x + x_{\gamma,a_k}\right| - A\right)^2 v_{\gamma,a_k}^2(x)\,\mathrm{d}x \ge C. \tag{3.33}$$

Indeed, applying the mean value theorem yields

$$\begin{aligned} &\lim_{k\to\infty}\varepsilon_{\gamma,a_k}^{-2}\int_{\mathbb{R}^2}\left(\left|\varepsilon_{\gamma,a_k}x + x_{\gamma,a_k}\right| - A\right)^2 v_{\gamma,a_k}^2(x)\,\mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^2}\left(\frac{\left|\varepsilon_{\gamma,a_k}x + x_{\gamma,a_k}\right| - \left|x_{\gamma,a_k}\right|}{\varepsilon_{\gamma,a_k}} + \frac{\left|x_{\gamma,a_k}\right| - A}{\varepsilon_{\gamma,a_k}}\right)^2 v_{\gamma,a_k}^2(x)\,\mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^2}\left(\frac{\left(\varepsilon_{\gamma,a_k}\xi + x_{\gamma,a_k}\right)x}{\left|\varepsilon_{\gamma,a_k}\xi + x_{\gamma,a_k}\right|} + \frac{\left|x_{\gamma,a_k}\right| - A}{\varepsilon_{\gamma,a_k}}\right)^2 v_{\gamma,a_k}^2(x)\,\mathrm{d}x \end{aligned}$$

where $\xi \in \mathbb{R}^2$. Together with $\left|\frac{\left|x_{\gamma,a_k}\right|-A}{\varepsilon_{\gamma,a_k}}\right| \to \infty$ as $k \to \infty$, (3.33) can be proved. This estimate and (3.32) imply that

$$e(\gamma,a_k) \ge \frac{\gamma}{8\pi}\ln\varepsilon_{\gamma,a_k} + \frac{\gamma}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a_k}^2(x)v_{\gamma,a_k}^2(y)\mathrm{d}x\mathrm{d}y + C\varepsilon_{\gamma,a_k}^2$$

holds for any $C>0$. Contradict with Lemma 3.1 we have $e(\gamma,a_k)\to+\infty$ as $C\to+\infty$ for large k. So, the above claim is proved. Then, up to subsequence we obtain that there exists a constant $C_0>0$ independent of $\gamma>0$ and $0<a<a^*$ such that

$$\frac{\left|x_{\gamma,a}\right|-A}{\left(a^*-a\right)^{\frac{1}{2}}} \to C_0 \ \text{ as } a \nearrow a^*.$$

It further implies that for any $\gamma>0$, $\left|x_{\gamma,a}\right| \to A$ as $a \nearrow a^*$.

Then we prove the uniqueness of the global maximum point $x_{\gamma,a}$ of $u_{\gamma,a}$ as $a \nearrow a^*$. Denote

$$\begin{aligned} K_{\gamma,a}(x) :=& -\varepsilon_{\gamma,a}^2\left(\left|\varepsilon_{\gamma,a}x+x_{\gamma,a}\right|-A\right)^2 v_{\gamma,a} - \frac{\gamma\varepsilon_{\gamma,a}^2}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(y)\mathrm{d}y v_{\gamma,a} \\ &+ \lambda_{\gamma,a}\varepsilon_{\gamma,a}^2 v_{\gamma,a} + aK\left(\varepsilon_{\gamma,a}x+x_{\gamma,a}\right)v_{\gamma,a}^3 - \frac{\gamma}{2\pi}\varepsilon_{\gamma,a}^2\ln\varepsilon_{\gamma,a}v_{\gamma,a} \ \text{ in } \mathbb{R}^2 \end{aligned}$$

in (3.14), so we get

$$-\Delta v_{\gamma,a} = K_{\gamma,a}(x) \ \text{ in } \mathbb{R}^2. \tag{3.34}$$

We have already obtained (3.15), it means that $\{v_{\gamma,a}\}$ is bounded uniformly in $L^q(\mathbb{R}^2)$ for $q\in[2,\infty)$ and $K_{\gamma,a}(x)$ is bounded uniformly in $L^q_{loc}(\mathbb{R}^2)$. Applying the L^p-estimate (cf. [21], Theorem 9.11) to (3.34) yields the uniform boundedness of $\{v_{\gamma,a}\}$ in $W^{2,q}_{\mathrm{loc}}(\mathbb{R}^2)$ as $a \nearrow a^*$. By the Sobolev embedding theorem, this in turn implies that $\{v_{\gamma,a}\}$ is uniformly bounded in $C^{1,\mu}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\mu\in(0,1)$ as $a \nearrow a^*$.

A key observation from ([15], Proposition 2.3) is that $\varepsilon_{\gamma,a}^2\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(y)\mathrm{d}y$ lies in $C^{2,\mu}_{\mathrm{loc}}(\mathbb{R}^2)$ as $a \nearrow a^*$. Consequently, $\{K_{\gamma,a}\}$ remains uniformly bounded in $C^{\mu}(B_r(0))$ for sufficiently large $r>0$ in this limit. With the Schauder estimate (cf. [21], Theorem 6.2) applied to (3.34), we conclude that $\{v_{\gamma,a}\}$ is uniformly bounded in $C^{2,\mu}(B_r(0))$ as $a \nearrow a^*$. Hence, one can extract a subsequence such that $v_{\gamma,a}$ converges in $C^2(B_r(0))$ to some limit $\tilde{v}_0 \in C^2(B_r(0))$ as $a \nearrow a^*$. In view of (3.6), this limit coincides with v_0, leading to

$$v_{\gamma,a}(x) \to v_0(x) = \frac{Q(|x|)}{\sqrt{a^*}} \ \text{ in } C^2(B_r(0)) \text{ as } a \nearrow a^*, \tag{3.35}$$

for any sufficiently large $r>0$.

Because the origin 0 is the unique global maximum point of $Q(x)$, (3.35) implies that, as $a\nearrow a^*$, all local maxima of $v_{\gamma,a}$ must approach 0 and stay inside $B_\delta(0)$ for some small $\delta>0$. The fact that $Q''(0)<0$ guarantees $Q''(t)<0$ for $t\in[0,\delta)$. An argument adapted from ([22], Lemma 4.2) then shows that, as $a\nearrow a^*$, each $v_{\gamma,a}$ admits exactly one maximum point, which is located at 0. Consequently, the global maximum $x_{\gamma,a}$ of $u_{\gamma,a}$ is unique as $a\nearrow a^*$. The proof of the limit and uniqueness of $x_{\gamma,a}$ is now accomplished.

The proof of (3.30) relies on L^p estimates for elliptic equations and Sobolev compact embedding. Local estimates and compactness yield a convergent subsequence, while uniqueness allows us to extend the convergence to the whole space and eliminate the subsequence dependence, leading to strong convergence. A detailed proof of this conclusion can be found in [20].

Next, we prove (3.31) as follows. We begin by deriving the refined estimate of $\varepsilon_{\gamma,a}$ as $a\nearrow a^*$. Taking $\tau=\left[\frac{\gamma a^*}{8\pi(a^*-a)}\right]^{\frac{1}{2}}$ in (2.10) leads to the following upper bound for $e(\gamma,a)$ as $a\nearrow a^*$:

$$\begin{aligned}e(\gamma,a)\le{}&\frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y\\&+\frac{\gamma}{16\pi}\left[\ln\frac{8\pi(a^*-a)}{\gamma a^*}+1\right]+\frac{A^2}{2}+o(1)\ \text{ as } a\nearrow a^*.\end{aligned}\tag{3.36}$$

We now give the lower estimate of $e(\gamma,a)$ as $a\nearrow a^*$. From [20] we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(x)v_{\gamma,a}^2(y)\mathrm{d}x\mathrm{d}y\\&\to\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|\frac{Q^2(x)}{a^*}\frac{Q^2(y)}{a^*}\mathrm{d}x\mathrm{d}y\ \text{ as } a\nearrow a^*.\end{aligned}\tag{3.37}$$

Then combining (1.4), (1.12), (1.13), (3.30) with (3.37) that

$$\begin{aligned}e(\gamma,a)&=E_{\gamma,a}(u_{\gamma,a})\\&=\frac{\varepsilon_{\gamma,a}^{-2}}{2}\int_{\mathbb{R}^2}|\nabla v_{\gamma,a}|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^2}\left(|\varepsilon_{\gamma,a}x+x_{\gamma,a}|-A\right)^2v_{\gamma,a}^2(x)\mathrm{d}x\\&\quad+\frac{\gamma}{8\pi}\ln\varepsilon_{\gamma,a}+\frac{\gamma}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|v_{\gamma,a}^2(x)v_{\gamma,a}^2(y)\mathrm{d}x\mathrm{d}y\\&\quad-\frac{a\varepsilon_{\gamma,a}^{-2}}{4}\int_{\mathbb{R}^2}K(\varepsilon_{\gamma,a}x+x_{\gamma,a})v_{\gamma,a}^4\mathrm{d}x\\&\ge\frac{\gamma}{8\pi}\ln\varepsilon_{\gamma,a}+\frac{a^*-a}{2a^*}(1+o(1))\varepsilon_{\gamma,a}^{-2}\\&\quad+\frac{\gamma}{8\pi(a^*)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|Q^2(x)Q^2(y)\mathrm{d}x\mathrm{d}y+o(1)\\&\quad+\frac{\gamma}{16\pi}\left[\ln\frac{8\pi(a^*-a)}{\gamma a^*}+1\right]+o(1)\ \text{ as } a\nearrow a^*,\end{aligned}\tag{3.38}$$

where equality in the above inequality is attained at

$$\varepsilon_{\gamma,a} = \left(\frac{8\pi\left(a^* - a\right)}{\gamma a^*} \right)^{\frac{1}{2}} \left(1 + o(1)\right) \text{ as } a \nearrow a^*. \tag{3.39}$$

Combining (3.36) with (3.32), we conclude that as $a \nearrow a^*$,

$$e(\gamma, a) \approx \frac{\gamma}{16\pi} \ln \frac{8\pi\left(a^* - a\right)}{\gamma a^*} + C + o(1),$$

and $\varepsilon_{\gamma,a} > 0$ satisfies (3.39). Moreover, we derive from (3.30) and (3.39) that

$$\lim_{a \nearrow a^*} \left(\frac{8\pi\left(a^* - a\right)}{\gamma a^*} \right)^{\frac{1}{2}} u_{\gamma,a} \left(\left(\frac{8\pi\left(a^* - a\right)}{\gamma a^*} \right)^{\frac{1}{2}} x + x_{\gamma,a} \right) = \frac{Q(x)}{\sqrt{a^*}} \text{ in } \mathcal{X} \cap L^\infty\left(\mathbb{R}^2\right).$$

This completes the proof of Theorem 1.2. □

Conflicts of Interest

The author declares no conflicts of interest.

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