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Equilibrium and Stability in the Cusp Catastrophe Framework

Omeed Adwal Ali

Department of Civil Engineering, College of Engineering, Al-Qalam University, Kirkuk, Iraq
Email: umed.ali@alqalam.edu.iq

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Abstract

Background: This paper explores equilibrium configurations and stability characteristics within the framework of the cusp catastrophe. **Methodology:** The study employs discriminant analysis of the governing cubic equilibrium equation to classify the number and nature of steady states as functions of control parameters. Stability conditions are derived using derivative-based criteria, and the bifurcation set is constructed to partition the parameter space into regions of distinct dynamical behavior. **Methods:** Equilibria are obtained analytically from the cubic equation, their stability assessed through local linearization, and the cusp boundary identified via vanishing discriminant conditions. Geometric visualization of the equilibrium surface is used to interpret mechanisms of sudden transitions. **Results:** The analysis reveals that the system admits either one or three equilibria, depending on parameter values, with bistability arising within the cusp region. Stable equilibria coexist with unstable separators, and transitions occur along fold lines where stability is lost. **Discussion:** These findings highlight the geometric role of the cusp catastrophe in explaining abrupt qualitative changes in nonlinear systems. The bifurcation structure provides a rigorous framework for understanding how small parameter variations can induce large-scale state shifts. **Conclusion:** The cusp catastrophe serves as a minimal yet powerful model for nonlinear systems exhibiting multi-stability and discontinuous change, offering valuable insight into the mechanisms driving sudden transitions across diverse domains.

Subject Areas

Dynamical System

Keywords

Cusp Catastrophe, Stability Analysis, Bifurcation Set, Multi-Stability, Sudden Transitions

1. Introduction

Many nonlinear systems encountered in science and engineering possess multiple equilibrium states whose stability depends on external parameters. When parameter changes lead to abrupt transitions between these states, classical linear analysis is often insufficient to describe the observed behavior. Catastrophe theory offers an alternative approach by focusing on the qualitative structure of equilibrium surfaces and their singularities.

The cusp catastrophe provides a minimal model for describing systems with bistability and sudden switching behavior. It arises when a single state variable interacts with two independent control parameters, producing a characteristic cusp-shaped bifurcation diagram. This diagram divides the parameter space into regions corresponding to different equilibrium configurations [1]-[4].

Within the cusp region, the system exhibits three equilibrium states, two of which are stable and one unstable. As parameters vary, the system may jump abruptly between stable states, giving rise to hysteresis phenomena. Such behavior has been observed in a variety of applications, including phase transitions, ecological systems, and decision dynamics [5]-[8].

This paper examines the equilibrium structure and stability properties of the cusp catastrophe framework. By analyzing the stationary equation and the corresponding stability conditions, we provide a systematic description of the parameter regions associated with different dynamical behaviors.

Research Objectives:

- To investigate the concept of equilibrium points within the cusp catastrophe framework and analyze their mathematical characteristics.
- To examine the stability behavior of equilibrium states under varying control parameters in cusp catastrophe systems.
- To identify bifurcation conditions that lead to sudden transitions between stable and unstable equilibrium states.
- To explore the relationship between catastrophe manifolds and system stability in nonlinear dynamical models.
- To develop analytical criteria for determining stability regions and critical thresholds in cusp catastrophe models.
- To demonstrate the applicability of cusp catastrophe theory in explaining abrupt structural changes in complex dynamic systems.

2. Methodology

The methodology of this study is based on equilibrium analysis and stability theory within the framework of catastrophe theory. The cusp catastrophe model is formulated using a potential function that defines the energy landscape of the system [6]-[9].

Equilibrium states are obtained by solving the first-order stationary condition of the potential function. Analytical methods are then used to determine how the number of equilibrium states depends on the control parameters.

The bifurcation set is identified by analyzing the degeneracy conditions of the equilibrium equation. This analysis determines the boundary separating regions with one equilibrium from those with three equilibria [10]-[15].

Local stability of the equilibria is evaluated using second-derivative criteria derived from the potential function. Furthermore, qualitative analysis of the system dynamics is used to explain transitions between stable states and the emergence of hysteresis [16]-[20].

The methodology provides a systematic framework for studying nonlinear systems exhibiting sudden transitions and multi-stability.

3. Mathematical Model

The paper is based on the canonical cusp catastrophe potential function, which is typically written in the form:

$$V(x) = \frac{x^4}{4} + \frac{\boldsymbol{a}x^2}{2} + \boldsymbol{b}x$$

where:

- x is the state variable.
- $\boldsymbol{a}$ and $\boldsymbol{b}$ are control parameters.

The equilibrium points are obtained from the stationary condition:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = x^3 + \boldsymbol{a}x + \boldsymbol{b} = 0$$

This cubic equation represents the equilibrium equation of the cusp catastrophe model [1]-[5].

4. Results

The equilibrium analysis demonstrates that the cusp catastrophe framework provides a minimal mathematical model capable of producing multi-stability. The equilibrium equation yields parameter-dependent solutions whose stability is determined by second-order conditions.

The results indicate that:

- The parameter space is divided into regions with one or three equilibrium states.
- Within the cusp region, two stable equilibria coexist and are separated by an unstable state.
- Stability loss occurs when parameters approach the bifurcation set.

The analysis also shows that transitions between stable equilibria occur when the system crosses fold boundaries of the cusp surface. These transitions produce discontinuous changes in the state variable despite continuous variations in parameters.

5. Discussion

The equilibrium and stability analysis carried out in this work demonstrates that the cusp catastrophe framework offers an effective mathematical approach for studying nonlinear systems with multiple steady states. The existence of a bifurcation set dividing the parameter space into distinct regions is a key feature that determines the qualitative behavior of the system.

Within the cusp region, the coexistence of two stable equilibria introduces the possibility of sudden transitions between system states. Such transitions occur when parameter changes push the system beyond the stability limits of one equilibrium branch. As a result, the system rapidly moves toward another stable configuration [17]-[22].

The presence of hysteresis further illustrates the path-dependent nature of system behavior in the cusp catastrophe model. Once the system transitions to a different equilibrium state, reversing the parameter change does not necessarily restore the original state until another critical boundary is crossed.

These findings confirm that the cusp catastrophe framework provides valuable insight into the mechanisms governing multistability and abrupt changes in nonlinear dynamical systems.

6. Conclusions and Future Work

This study examined equilibrium configurations and stability properties within the cusp catastrophe framework. The analysis showed that the model provides a minimal mathematical structure capable of generating multi-stability and abrupt transitions. The bifurcation set divides the control parameter space into regions with different equilibrium characteristics, explaining the emergence of hysteresis and sudden changes in system behavior.

The results highlight the usefulness of the cusp catastrophe framework in analyzing nonlinear systems where classical linear models fail to capture complex transitions. The theoretical insights obtained from this analysis can serve as a foundation for modeling various phenomena involving multistable dynamics.

Future work may include incorporating time-dependent dynamics into the model to study transient behavior and attractor transitions. Moreover, applying cusp catastrophe theory to practical problems in physics, biology, and economics could further demonstrate its value as a tool for understanding nonlinear stability and critical transitions.

Conflicts of Interest

The author declares no conflicts of interest.

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