An Analysis of Modified Emden-Type Equation $\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$: Exact Explicit Analytical Solution, Lagrangian, Hamiltonian for Arbitrary Values of α and β

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ABSTRACT

The modified Emden-type is being investigated by mathematicians as well as physicists for about a century. However, there exist no exact explicit solution of this equation, $\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$ for arbitrary values of α and β. In this work, the exact analytical explicit solution of modified Emden-type (MEE) equation is derived for arbitrary values of α and β. The Lagrangian and Hamiltonian of MEE are also worked out. The solution is also utilized to find exact explicit analytical solution of Force-free Duffing oscillator-type equation. And exact explicit analytical solution of two-dimensional Lotka-Volterra System is also worked out.

1. INTRODUCTION

The modified Emden-type equation [MEE] or the modified Painleve-Ince equation is

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0 \tag{1.1}$$

where the dot represents derivative with respect to time and α, β are arbitrary parameters. The equation has been investigated by a lot of physicists as well as mathematicians [1-3]. Painleve, himself obtained solutions of (1.1) for $\alpha^2 = 9\beta$ and $\alpha^2 = -\beta$. This equation is extremely important because it arises in many mathematical problems like univalent functions defined by differential equation of second order [4]. This equation also arises in modelling of fusion of pellets [5]. The MEE is a special case of one dimensional analogue [6, 7] of gauze boson theory introduced by Yang and Mills. In the study of equilibrium configuration of spherical gas clouds [8, 9], this equation is also encountered.

As regards solution of MEE *i.e.*, Equation (1.1), some progress in recent past has been done by Chandrasekar *et al.* [10]. These authors have made good contributions by finding Lagrangian, Hamiltonian and Invariant of (1.1) for distinct cases: 1) $\alpha^2 = 8\beta$, 2) $\alpha^2 > 8\beta$ and 3) $\alpha^2 < 8\beta$. However the solution, they have found, is unsatisfactory.

In this paper, exact analytical explicit single solution of (1.1) will be shown valid for arbitrary values of α and β. Also single Lagrangian as well as Hamilton valid for all α and β will be worked out.

2. ANALYTICAL SOLUTION OF MODIFIED EMDEN-TYPE EQUATION (MEE)

To solve (1.1), *i.e.*, modified Emden-type equation we first write

$$x = x_1^m \quad m \text{ to be determined later.} \tag{2.1}$$

Therefore

$$\dot{x} = m x_1^{m-1} \dot{x}_1 \quad \text{dot represents time derivative} \tag{2.2}$$

and

$$\ddot{x} = m(m-1) x_1^{m-2} \dot{x}_1^2 + m x_1^{m-1} \ddot{x}_1 \tag{2.3}$$

Using (2.1), (2.2) and (2.3), (1.1) can be written as

$$m x_1^{m-1} \ddot{x}_1 + m(m-1) x_1^{m-2} \dot{x}_1^2 + \alpha m x_1^{2m-1} \dot{x}_1 + \beta x_1^{3m} = 0 \tag{2.4}$$

After simplification (2.4) gives

$$\ddot{x}_1 + \alpha x_1^m \dot{x}_1 + (m-1)\frac{\dot{x}_1^2}{x_1} + \frac{\beta}{m} x_1^{2m+1} = 0 \tag{2.5}$$

Equation (2.5) is autonomous, so let

$$\dot{x}_1 = u(x_1) \tag{2.6}$$

So that $\ddot{x}_1 = u'(x_1) u(x_1)$ prime denotes derivative w.r. to x_1 (2.7)

Using (2.6) and (2.7) one gets from (2.5)

$$u'(x_1) + \alpha x_1^m + (m-1)\frac{u(x_1)}{x_1} + \frac{\beta}{m}\frac{x_1^{2m+1}}{u(x_1)} = 0 \tag{2.8}$$

To solve (2.8) we use an ansatz for u as

$$u = A x_1^K \quad A \text{ and } K \text{ are constants and } K \neq 1 \tag{2.9}$$

Equation (2.8) now reduces to with ansatz for u given by (2.9):

$$AK x_1^{K-1} + \alpha x_1^m + A(m-1) x_1^{K-1} + \frac{\beta}{mA} x_1^{2m+1-K} = 0 \tag{2.10}$$

Now let

$$2m + 1 - K = K - 1 \tag{2.11}$$

So that

$$m = K - 1 \tag{2.12}$$

with help of (2.11) and (2.12), Equation (2.10) reduces to

$$AK x_1^{K-1} + \alpha x_1^{K-1} + A(m-1) x_1^{K-1} + \frac{\beta}{mA} x_1^{K-1} = 0 \tag{2.13}$$

Cancelling x_1^{K-1} in (2.13) assuming $x_1^{K-1} \neq 0$, we find

$$AK + \alpha + A(m-1) + \frac{\beta}{mA} = 0 \tag{2.14}$$

Simplification of (2.14) gives

$$Am^2 + (AK + \alpha - A)m + \frac{\beta}{A} = 0 \tag{2.15}$$

Equation (2.15) is an algebraic quadratic in m whose solution is

$$m = \frac{(A - \alpha - AK) \pm \sqrt{(A - \alpha - AK)^2 - 4\beta}}{2A} \tag{2.16}$$

Now comparing (2.12) and (2.16) we can write

$$K - 1 = \frac{(A - \alpha - AK) \pm \sqrt{(A - \alpha - AK)^2 - 4\beta}}{2A}$$

i.e.,

$$2A(K-1) - A + \alpha + AK = \pm\sqrt{(A - \alpha - AK)^2 - 4\beta}$$

i.e.,

$$3AK - 3A + \alpha = \pm\sqrt{(A - \alpha - AK)^2 - 4\beta} \tag{2.17}$$

Squarring and simplifying (2.17) we find

$$2A^2K^2 + KA(\alpha - 4A) + (2A^2 - A\alpha + \beta) = 0 \tag{2.18}$$

Equation (2.18) is a quadratic in K whose solution is given by:

$$K = \frac{(4A - \alpha) \pm \sqrt{\alpha^2 - 8\beta}}{4A} \tag{2.19}$$

Next, from (2.12), using (2.19) we find

$$m = K - 1 = \frac{-\alpha \pm \sqrt{\alpha^2 - 8\beta}}{4A} \tag{2.20}$$

Now, from (2.6) and (2.9), we find

$$\dot{x}_1 = Ax_1^K$$

i.e.,

$$\frac{x_1^{1-K}}{1-K} = At + BB \quad = \text{Constant of integration}$$

i.e.,

$$x_1 = \left[A(1-K)t + B(1-K)\right]^{\frac{1}{1-K}} \tag{2.21}$$

From (2.21) using (2.1) we can write

$$\begin{aligned} x = x_1^m &= \left[A(1-K)t + B(1-K)\right]^{\frac{m}{1-K}} \\ &= \left[A(1-K)t + B(1-K)\right]^{-1} \quad \text{using (2.12)} \end{aligned} \tag{2.22}$$

Now from (2.22), we get our sought result:

$$x = \frac{1}{A(1-K)t + B(1-K)}$$

i.e.,

$$x=\frac{1}{\frac{A\left\{\alpha\mp\sqrt{\alpha^{2}-8\beta}\right\}t}{4A}+\frac{B\left\{\alpha\mp\sqrt{\alpha^{2}-8\beta}\right\}}{4A}}\quad \text{using (2.20)}$$

i.e.,

$$x=\frac{4A}{A\left\{\alpha\mp\sqrt{\alpha^{2}-8\beta}\right\}t+B\left\{\alpha\mp\sqrt{\alpha^{2}-8\beta}\right\}}\tag{2.23}$$

Equation (2.23) gives the exact analytic explicit solution of Emden-type Equation (1.1) for arbitrary values of α and β. In (2.23) A and B are two arbitrary constants.

3. LAGRANGIAN AND HAMILTONIAN OF MEE

In this section we will find the Lagrangian and Hamiltonian using a method due to Vujanovic and Jones [11]. These authors have shown after a long calculation that the Lagrangian of the equation:

$$\ddot{x}+\frac{(n-2)}{(n-1)}R'(x)\dot{x}+\frac{1}{(1-n)}R(x)R'(x)=0\tag{3.1}$$

The dot and prime represent derivative with respect to t and x respectively.
Is given by

$$L=\left\{\dot{x}+R(x)\right\}^{n}\tag{3.2}$$

Now let

$$R(x)=\frac{A_0}{2}x^{2}\quad A_0 \text{ is a constant}\tag{3.3}$$

So that

$$R'(x)=A_0x\tag{3.4}$$

Using (3.3) and (3.4) Equation (3.1) assumes the form

$$\ddot{x}+\frac{(n-2)}{(n-1)}A_0x\dot{x}+\frac{1}{(1-n)}\frac{A_0^{2}}{2}x^{3}=0\quad (n\neq 1,\ 2)\tag{3.5}$$

Next, let

$$\frac{(n-2)}{(n-1)}A_0=\alpha\tag{3.6}$$

and

$$\frac{1}{(1-n)}\frac{A_0^{2}}{2}=\beta\tag{3.7}$$

Then (3.5) reduces to

$$\ddot{x}+\alpha x\dot{x}+\beta x^{3}=0\tag{3.8}$$

Equation (3.8) is just modified Emden-type Equation (1.1).
Now (3.6) and (3.7) gives

$$\frac{2(n-2)^2}{(1-n)}=\frac{\alpha^2}{\beta}$$

i.e.,

$$2\left(n^2-4n+4\right)=\frac{\alpha^2}{\beta}-n\frac{\alpha^2}{\beta}$$

Therefore

$$2n^2-n\left(8-\frac{\alpha^2}{\beta}\right)+8-\frac{\alpha^2}{\beta}=0 \tag{3.9}$$

Equation (3.9) is an algebraic quadric in n whose solution is:

$$n=\frac{8-\frac{\alpha^2}{\beta}\pm\sqrt{\left(8-\frac{\alpha^2}{\beta}\right)^2-8\left(8-\frac{\alpha^2}{\beta}\right)}}{4} \tag{3.10}$$

and also from (3.6)

$$\frac{\alpha(n-1)}{(n-2)}=A_0,\ \text{where } n \text{ is given by (3.10)} \tag{3.11}$$

It, therefore turns out from (3.1), 3.2) and (3.8) that Lagrangian of (3.8) or (1.1) *i.e.*, modified Emden-type equation is

$$L=\left[\dot{x}+\frac{A_0}{2}x^2\right]^n \tag{3.12}$$

where A_0 and n are given by (3.11) and (3.10) respectively. This Lagrangian is valid for arbitrary values of α and β.

4. SOME SPECIAL CASES

Cariena and others [12] has considered MEE (1.1) with $\alpha = 3K$ and $\beta = K^2$ and has shown that for this choice the Lagrangian is

$$L_1=\frac{1}{\dot{x}+Kx^2} \tag{4.1}$$

For $\alpha = 3K$ and $\beta = K^2$ another Lagrangian of (1.1) is

$$L_2=\left(2\dot{x}+Kx^2\right)^{\frac{1}{2}} \tag{4.2}$$

In this section we will show that both the above Lagrangians (4.1) and (4.2) follow from our derived general expression for Lagrangian of MEE (3.12)

For, if we take $\alpha = 3K$ and $\beta = K^2$, we find from (3.10)

$$n=\frac{-1\pm\sqrt{1+8}}{4}=\frac{-1\pm3}{4}=-1 \text{ or } +\frac{1}{2} \tag{4.3}$$

and from (3.11)

$$\left.\begin{aligned}&\text{when } n=-1,\ A_0=2K\\&\text{when } n=\frac{1}{2},\ A_0=K\end{aligned}\right\} \tag{4.4}$$

Then for $n = -1$ and $A_0 = 2K$ we get from (3.12)

$$L = \left[\dot{x} + Kx^2\right]^{-1} = \frac{1}{\dot{x} + Kx^2} \tag{4.5}$$

And for $n = 1/2$ and $A_0 = K$ one finds from (3.12)

$$L = \left[\dot{x} + \frac{K}{2}x^2\right]^{1/2} = \frac{\left[2\dot{x} + Kx^2\right]^{1/2}}{\sqrt{2}} \tag{4.6}$$

Rejecting the factor $\sqrt{2}$ in the denominator of R. H. S of (4.6) we get

$$L = \left[2\dot{x} + Kx^2\right]^{1/2} \tag{4.7}$$

Thus it follows from above that our general expression for Lagrangtian of (1.1) correctly produce Lagrangian of (1.1) for $\alpha = 3K$ and $\beta = K^2$, derived earlier by other authors.

The above analysis and examples clearly indicate that the Lagrangian of modified Emden-type Equation (1.1) can be derived for arbitrary values of α and β from Equation (3.12), which is the general uniform expression for Lagrangian of (1.1).

5. HAMILTONIAN OF MODIFIED EMDEN-TYPE EQUATION

The calculation of Hamiltonian of (1.1) is straight forward.

From (3.12)

$$\frac{\partial L}{\partial \dot{x}} = p = n\left[\dot{x} + \frac{A_0}{2}x^2\right]^{n-1} \tag{5.1}$$

Therefore

$$\dot{x} = \left(\frac{p}{n}\right)^{\frac{1}{n-1}} - \frac{A_0}{2}x^2 \tag{5.2}$$

and

$$\left(\frac{p}{n}\right)^{\frac{1}{n-1}} = \dot{x} + \frac{A_0}{2}x^2 \tag{5.3}$$

Hence $H = p\dot{x} - L$

i.e.,

$$H = p\left[\left(\frac{p}{n}\right)^{\frac{1}{n-1}} - \frac{A_0}{2}x^2\right] - \left[\dot{x} + \frac{A_0}{2}x^2\right]^n \quad \text{using (5.2) and (3.12)}$$

i.e.,

$$H = p\left[\left(\frac{p}{n}\right)^{\frac{1}{n-1}} - \frac{A_0}{2}x^2\right] - \left(\frac{p}{n}\right)^{\frac{n}{n-1}} \quad \text{using (5.3)} \tag{5.4}$$

Equation (5.4) is therefore, the expression for Hamiltonian of modified Emden-type Equation (1.1) for arbitrary α, β, where n and A_0 are given by Equations (3.10) and (3.11) respectively.

6. SOME APPLICATIONS

The modified Emden-type Equation (1.1) is closely connected with Duffing oscillator type equation and Lotka-Volterra equation.

1) Force-free Duffing oscillator type equation

The force-free Duffing oscillator equation is:

$$\omega'' + (\alpha\omega + \lambda)\omega' + \beta\omega^3 + \frac{\alpha\gamma}{3}\omega^2 + \frac{2\gamma^2\omega}{9} = 0 \quad (6.1)$$

where $\omega' = \frac{\mathrm{d}\omega}{\mathrm{d}\tau}$.

The specific cases of (6.1) for $\alpha = 0$ and $\beta = 0$ have been studied by many authors [13-15]. However their results are for specific cases only.

The solution of (6.1) for arbitrary α, β and γ can be obtained in a straight forward manner from our results of earlier sections in the following way:

An invertible point transformation [10]

$$\omega = x\mathrm{e}^{-\frac{\gamma}{3}\tau} \text{ and } t = -\frac{3}{\gamma}\mathrm{e}^{-\frac{\gamma}{3}\tau},\ \gamma = \text{arbitrary parameter} \quad (6.2)$$

converts (6.1) into Equation (1.1) *i.e.*, into MEE.

Therefore solution of (6.1) follows directly from (2.23) and (6.2):

$$\omega = \frac{4A\mathrm{e}^{-\frac{\gamma}{3}\tau}}{-\frac{3A}{\gamma}\left\{\alpha \pm \sqrt{\alpha^2 - 8\beta}\right\}\mathrm{e}^{-\frac{\gamma}{3}\tau} + B\left\{\alpha \pm \sqrt{\alpha^2 - 8\beta}\right\}} \quad (6.3)$$

Equation (6.3) is the general solution of force-free Dutting oscillator Equation (6.1) for arbitrary α, β and γ. A and B are two arbitrary constants.

2) Lotka-Volterra Equation in Two Dimension

Two dimensional Lotka-Volterra equation is of the form

$$\dot{x} = x(a_1 + a_2x + a_3y),\ \dot{y} = y(b_1 + b_2x + b_3y) \quad (6.4)$$

where $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ etc are real parameters. This equation has been studied for a long time and its solution is important in mathematical biology [16].

For $b_1 = a_1$ and $b_3 = -a_3$ Equation (6.4) is of the form

$$\dot{x} = x(a_1 + a_2x + a_3y),\ \dot{y} = y(a_1 + b_2x - a_3y) \quad (6.5)$$

Equation (6.5) can be written by eliminating y and $\dot{y}$ as

$$\ddot{x} - \left\{(3a_2 + b_2)x + 3a_1\right\}\dot{x} + a_2(a_2 + b_2)x^3 + a_1(3a_2 + b_2)x^2 + 2a_1^2x = 0 \quad (6.6)$$

Equation (6.6) is of the form (6.1) with

$$\alpha = -(3a_2 + b_2),\ \beta = a_2(a_2 + b_2) \text{ and } \gamma = -3a_1$$

For these choices of α, β and γ solution of (6.6) follows directly from (6.3):

$$x = \frac{4A\mathrm{e}^{a_1t}}{\frac{A}{a_1}\left\{-(3a_2 + b_2) \pm \sqrt{(3a_2 + b_2)^2 - 8a_2(a_2 + b_2)}\right\}\mathrm{e}^{a_1t} + B\left\{-(3a_2 + b_2) \pm \sqrt{(3a_2 + b_2)^2 - 8a_2(a_2 + b_2)}\right\}} \quad (6.7)$$

From (6.7) we can find easily analytic expression for y in the following way:

From (6.4), $a_1 + a_2x + a_3y = \frac{\dot{x}}{x}$

Therefore

$$y=\frac{1}{a_3}\left[\frac{\dot{x}}{x}-a_2x-a_1\right] \tag{6.8}$$

Now for simplicity we write (6.7) as

$$x=\frac{4Ae^{a_1t}}{\frac{A}{a_1}Ze^{a_1t}+BZ} \tag{6.9}$$

where

$$Z=-\left(3a_2+b_2\right)\pm\sqrt{\left(3a_2+b_2\right)^2-8a_2\left(a_2+b_2\right)} \tag{6.10}$$

Therefore

$$\dot{x}=\frac{4Aa_1e^{a_1t}}{\frac{A}{a_1}Ze^{a_1t}+BZ}-\frac{4Ae^{a_1t}\frac{A}{a_1}Za_1e^{a_1t}}{\left(\frac{A}{a_1}Ze^{a_1t}+BZ\right)^2} \tag{6.11}$$

Hence

$$\frac{\dot{x}}{x}=a_1-\frac{AZe^{a_1t}}{\frac{A}{a_1}Ze^{a_1t}+BZ} \tag{6.12}$$

Therefore from (6.8) to (6.12)

$$\begin{aligned} y&=\frac{1}{a_3}\left[-\frac{AZe^{a_1t}}{\frac{A}{a_1}Ze^{a_1t}+BZ}-\frac{4a_2Ae^{a_1t}}{\frac{A}{a_1}Ze^{a_1t}+BZ}\right]\\ &=\frac{1}{a_3}\left[\frac{A_0Ze^{a_1t}}{BZ-\frac{A_0}{a_1}Ze^{a_1t}}+\frac{4a_2A_0e^{a_1t}}{BZ-\frac{A_0}{a_1}Ze^{a_1t}}\right],\ A_0=-A \end{aligned} \tag{6.13}$$

Equation (6.7) and (6.13) give explicit analytical solution of Lotka-Volterra Equation (6.5). It is to be mentioned here that these solutions of Lotka-Volterra equation are perfectly new addition to literature.

7. CONCLUSION

In the above work, the initially proposed things, *i.e.*, analytic explicit solution of Emden-type Equation (1.1) for arbitrary α, β and the Lagrangian and Hamiltonian of Emden-type equation are all worked out. Further, as applications of the results, the Force-free Duffing oscillator and Lotka-Volterra (two dimensional) equation is also analytically solved. All the solutions are explicit and can be directly applied. It turns out that all the results obtained in this paper are new addition to literature.

CONFLICTS OF INTEREST

The author declares no conflicts of interest regarding the publication of this paper.

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