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Some Revised Aspects of Type-2 Fuzzy Sets Use

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Abstract

In this work we revise some less investigated aspects of a type-2 *fuzzy logic system* (*FLS*), which can handle rule uncertainties. The specificity of this type-2 *FLS* usually comprises the operations of fuzzification, inference, and output processing. We focus on "inference", which in our opinion should be based on an operation of fuzzy implication, rather than on commonly used t-norm one. This paper also investigates some original properties of fuzzy grades under the operations of join $\sqcup$, meet $\sqcap$ and implication $\longrightarrow$, which are defined by using the extension principal L. A. Zadeh, and shows that convex fuzzy grades form normal convex fuzzy grades, which in their turn form a distributive lattice under $\sqcup$ and $\sqcap$. In this article we present much more effective implementation of such operations. We also explore an important feature of *IF-THEN* rules for type-2 *fuzzy sets*. This study introduces the notion of *semantic similarity* measure of these rules and their importance for logical inference in type-2 *FLS*. All theoretical concepts presented in this work are thoroughly illustrated by relevant examples.

Keywords

Type-2 Fuzzy Set, Type-2 Fuzzy Logic System, Fuzzy Implication, Fuzzy Conditional Inference, IF-THEN Rules, Semantic Similarity

1. Introduction

It should be mentioned that the concept of *fuzzy sets* (*FS*) *of type 2* has been defined by L. A. Zadeh [1]-[3] as an extension of ordinary *FS*. The further investigation of *FS* of *type 2* was done in [4] and [5]. The *FS of type 2* can be characterized by a fuzzy membership function the grade (or *fuzzy grade*), of which is a fuzzy set in the unit interval [0, 1], rather than a point in [0, 1]. Moreover, the algebraic properties of fuzzy grades under the operations $\sqcup$ and $\sqcap$ are slightly different from canonical ones. In this study we revisit operations, algebraic properties, and rule semantics in type-2 systems. We propose normalization of physical values and their mapping to a discrete universe. This article provides formulas for constructing membership functions using singletons and linguistic terms. We prove that fuzzy grades defined via triangular secondary MFs are convex. Note that in [6]-[8] the concept of *type-2 fuzzy logic system* (*FLS*) was introduced and thoroughly investigated. This paper studies several underexplored aspects of such *type-2 FLS*. In particular, this article argues that *inference in type-2 FLS* should rely on *fuzzy implication*, not t-norms, because implication better reflects semantic reasoning.

2. Fuzzy Logic Features

2.1. Fuzzy Logic Type-1

Let us consider the set of some kind of *physical values* $p \in \left[P^{min}, P^{max}\right]$.

To *normalize* these values of p we utilize the following expression

$$p^{norm} = \frac{p - P^{min}}{P^{max} - P^{min}} \tag{2.1}$$

Hence, in our research we always consider to operate on a *universe of discourse* U, *i.e.* on a *set of integers* rather, than on *original physical scale* of $p \in \left[P^{min}, P^{max}\right]$. For this purpose, we use the following mapping

$$\gamma : P \to U \mid u = Ent\left[CardU \cdot p^{norm}\right] \tag{2.2}$$

Let P be correspondent to p^{norm} from (2.1) *type-1 FS* and let the relevant *membership function* (*MF*) of $u \in U$ in P be $\mu_p(u)$, which is a crisp number in $[0,1]$.

Thus, the fuzzy set is presented as

$$P = \int_U \frac{\mu_p(u)}{u} \tag{2.3}$$

On the other hand, to determine the estimates of the *MF* in terms of singletons from (2.3) in the form $\mu_p(u_i)/u_i \mid \forall i \in [1, CardU]$, given (2.1) and (2.2) we propose the following procedure. $\forall i \in [1, CardU]$

$$\mu_p(u_i) = \left\{1 - \frac{1}{CardU - 1} \cdot \left| i - Ent\left[(CardU) \cdot p^{norm}\right]\right|\right\}^{\alpha_i} \tag{2.4}$$

where α_i is a weight coefficient of the *i*-th input cluster [9] for $\forall i \in [1, CardU]$.

In more details from (2.4) we have

$$P = \frac{\mu_p(u_1)}{u_1} + \frac{\mu_p(u_2)}{u_2} + \cdots + \frac{\mu_p(u_n)}{u_n} = \sum_i \frac{\mu_p(u_i)}{u_i}, u_i \in U \tag{2.5}$$

In *type-1* fuzzy set P is represented as a *triplet* of the form

$$P = \left\{p, U, \tilde{P} >\right\}, p \in T(u), u_i \in U, i = \overline{1, CardU} \tag{2.6}$$

where $\tilde{P}$ is normal *FS* with correspondent *MF* $\mu_p : U \to [0,1]$. $T(u)$ could be the *term* from **Table 1**.

For each fuzzy set P one could define a *linguistic term* from the following set

$$T(u)=\{low,\cdots,highest\} \tag{2.7}$$

The following **Table 1** depicts linguistic terms from (2.7).

Table 1. Linguistic terms.

Value of variable	
$T(k)$	$k \in U$
low	*1*
higher than low	*2*
middle	*3*
higher than middle	*4*
high	*5*
highest	*6*

To determine the estimates of the *MF* in terms of singletons from (2.5) in the form $\mu_p(u_i)/u_i \mid \forall i \in [1, CardU]$, for each *k*-th term from **Table 1**. we use the following procedure.

$$\mu_p(u_i)=\left\{1-\frac{1}{CardU-1}\cdot|i-k|\right\}, \forall i \in [1, CardU] \tag{2.8}$$

Note, that MF from (2.8) has *triangular* form, which is defined on entire *universe of discourse* U.

2.2. Fuzzy Logic Type-2

A *type-2* fuzzy set in X is $\tilde{P}$ and the *membership grade* of $x \in X$ in $\tilde{P}$ is $\mu_{\tilde{P}}(x)$, which is a *type-1 FS* in $[0,1]$. The elements of the domain of $\mu_{\tilde{P}}(x)$ are called *primary memberships* of x in $\tilde{P}$ and the memberships of the primary memberships in $\mu_{\tilde{P}}(x)$ are called *secondary memberships* of x in $\tilde{P}$; the latter defines the possibilities for the primary membership, e.g., the *type-2 FS*, $\tilde{P} \in X$ has *fuzzy grades* (*FS* in $J_X \subseteq [0,1]$), $\mu_{\tilde{P}}(x)$ that can be denoted for each $x \in X$ as $\mu_{\tilde{P}}: X \to [0,1]^{J_X}$

$$\mu_{\tilde{P}}(x)=\frac{f(v_1)}{v_1}+\frac{f(v_2)}{v_2}+\cdots+\frac{f(v_n)}{v_n}=\sum_i \frac{f(v_i)}{v_i}, v_i \in J_X \tag{2.9}$$

And to define $v_i \in J_X$ in (2.7) we propose the following expression

$$v_i=\left\{\frac{i}{CardJ_X-1}\right\}, v_i \in J_X, \forall i \in [1, CardJ_X] \tag{2.10}$$

Thus, we presume that *membership grade* $\mu_{\tilde{P}}(x)$ is normal, therefore $\exists! l^* \in [1, CardJ_X] \mid f(v_{l^*})=\max_{\forall i \in [1, CardJ_X]}\{f(v_i)\}==1$ and the *MF* of fuzzy grades $f(v_i)$ we define as

$$f(v_i)=\left\{1-\frac{1}{CardJ_X-1}\cdot|i-l^*|\right\}, \forall i \in [1, CardJ_X] \tag{2.11}$$

Example 1.

Suppose the $CardJ_X = 6$, then from (2.10) we obtain $J_X = \{0, 0.2, 0.4, 0.6, 0.8, 1\} \subseteq [0,1]$. And also suppose the $X = \{1,2,3,4,5,6\}$ is the set of numbers u from (2.2), which is correspondent to the set of normalized values p^{norm} from (2.1) and that P is *FS* of *type-2* of *Normalized Variable* (Figure 1).

$$P = \textit{Normalized Variable} = low/1 + middle/3 + high/5$$

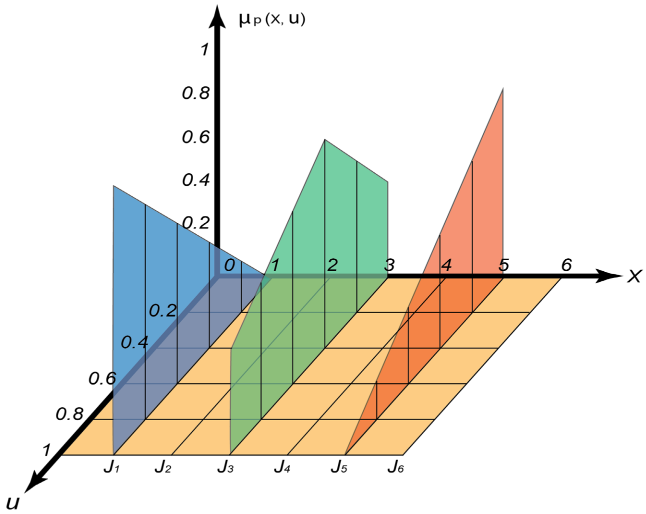

Figure 1. Type-2 MF of "Normalized Variable" from Ex. 1.

From (2.9) and (2.11) by using *terms* from Table 1. we find the following *membership grades* ($\mu_{\tilde{P}}(1), \mu_{\tilde{P}}(3)$ and $\mu_{\tilde{P}}(5)$ are shown on Figure 1).

$\mu_{\tilde{P}}(1)$ = ***low*** = **1/0** + 0.8/0.2 + 0.6/0.4 + 0.4/0.6 + 0.2/0.8 + 0/1

$\mu_{\tilde{P}}(3)$ = ***middle*** = 0.6/0 + 0.8/0.2 + **1/0.4** + 0.8/0.6 + 0.6/0.8 + 0.4/1

$\mu_{\tilde{P}}(5)$ = ***high*** = 0/0 + 0.2/0.2 + 0.4/0.4 + 0.6/0.6 + 0.8/0.8 + **1/1**

Note, that in *Ex. 1* $J_1 = J_3 = J_5 = J_X$, whereas $J_2 = J_4 = J_6 = \varnothing$.

2.3. Operations of FS Type-2

Let us deliberate some important feature of fuzzy grades, which could be utilize later in this study.

Lemma 1.

If fuzzy grade $\mu_P(x) = \sum_i \frac{f(v_i)}{v_i}, v_i \in J_X$

from (2.9) is defined as (2.10), then it is a ***convex*** for any integers $i, k \in [1, CardJ_X]$, $\exists j \in [1, CardJ_X] \mid i \le j \le k$, *i.e.* $f(v_j) \ge \min\{f(v_i), f(v_k)\} \mid i \le j \le k$.

Proof.

Let us consider $f(v_{n-1}), f(v_n), f(v_{n+1})$ from (2.11) $\forall n \in [1, Card\,J_X]$. We have the following cases

1) $n+1 \le l^*$;

2) $n \le l^*, n+1 > l^*$;

3) $n-1 \le l^*, n > l^*$;

4) $n-1 > l^*$.

For case 1. from (2.11)

$$i = n+1 \Rightarrow \Delta_{n+1} = \left|i - l^*\right| = l^* - n - 1;$$

$$i = n \Rightarrow \Delta_n = \left|i - l^*\right| = l^* - n;$$

$$i = n-1 \Rightarrow \Delta_{n-1} = \left|i - l^*\right| = l^* - n + 1.$$

From where we have

$$\Delta_{n-1} > \Delta_n > \Delta_{n+1} \Rightarrow f\left(v_{n-1}\right) < f\left(v_n\right) < f\left(v_{n+1}\right)$$

$$\Rightarrow f\left(v_n\right) > \min\left\{f\left(v_{n-1}\right), f\left(v_{n+1}\right)\right\}$$

For case 2. from (2.11)

$$i = n \Rightarrow \Delta_n = \left|i - l^*\right| = l^* - n;$$

$$i = n-1 \Rightarrow \Delta_{n-1} = \left|i - l^*\right| = l^* - n + 1;$$

$$i = n+1 \Rightarrow \Delta_{n+1} = \left|i - l^*\right| = n + 1 - l^*.$$

Given that $\Delta_{n+1} - \Delta_n = 2 \cdot l^* - 2 \cdot n - 1 \mid n \le l^* \Rightarrow \Delta_{n+1} - \Delta_n < 0$ we have

$$\Delta_{n-1} > \Delta_n > \Delta_{n+1} \Rightarrow f\left(v_{n-1}\right) < f\left(v_n\right) < f\left(v_{n+1}\right)$$

$$\Rightarrow f\left(v_n\right) > \min\left\{f\left(v_{n-1}\right), f\left(v_{n+1}\right)\right\}$$

For case 3. from (2.11)

$$i = n-1 \Rightarrow \Delta_{n-1} = \left|i - l^*\right| = l^* - n + 1;$$

$$i = n \Rightarrow \Delta_n = \left|i - l^*\right| = n - l^*;$$

$$i = n+1 \Rightarrow \Delta_{n+1} = \left|i - l^*\right| = n + 1 - l^*.$$

Given that $\Delta_n - \Delta_{n-1} = 2 \cdot n - 2 \cdot l^* - 1 \mid n > l^* \Rightarrow \Delta_n - \Delta_{n-1} < 0$ we have

$$\Delta_{n-1} > \Delta_n > \Delta_{n+1} \Rightarrow f\left(v_{n-1}\right) < f\left(v_n\right) < f\left(v_{n+1}\right)$$

$$\Rightarrow f\left(v_n\right) > \min\left\{f\left(v_{n-1}\right), f\left(v_{n+1}\right)\right\}$$

For case 4. from (2.11)

$$i = n-1 \Rightarrow \Delta_{n-1} = \left|i - l^*\right| = n - 1 - l^*;$$

$$i = n \Rightarrow \Delta_n = \left|i - l^*\right| = n - l^*;$$

$$i = n+1 \Rightarrow \Delta_{n+1} = \left|i - l^*\right| = n + 1 - l^*.$$

From where we have

$$\Delta_{n+1} > \Delta_n > \Delta_{n-1} \Rightarrow f\left(v_{n+1}\right) < f\left(v_n\right) < f\left(v_{n-1}\right)$$

$$\Rightarrow f\left(v_n\right) > \min\left\{f\left(v_{n-1}\right), f\left(v_{n+1}\right)\right\}. \blacksquare$$

Let $\mu_P\left(x\right)$ and $\mu_Q\left(x\right)$ be two *fuzzy grades* that is, *FSs* in $J_X \subseteq \left[0,1\right]$ of *FSs* of *type 2*, P and Q, respectively, represented as

$$\mu_P(x) = \frac{f(v_1)}{v_1} + \frac{f(v_2)}{v_2} + \cdots + \frac{f(v_n)}{v_n} = \sum_i \frac{f(v_i)}{v_i}, v_i \in J_X, \tag{2.12}$$

$$\mu_Q(x) = \frac{g(w_1)}{w_1} + \frac{g(w_2)}{w_2} + \cdots + \frac{g(w_n)}{w_n} = \sum_j \frac{g(w_j)}{w_j}, w_j \in J_X, \tag{2.13}$$

where the functions f and g are *MF*s of fuzzy grades (*FS*s in $J_X \subseteq [0,1]$) $\mu_P(x)$ and $\mu_Q(x)$, respectively, and the values $f(v_i)$ and $g(w_j)$ in [0, 1] denote the grades for v_i and w_j in J_X, respectively. Thus, the operations for *FS* of *type 2* are expressed by the following.

Union.

$$\begin{aligned} P \cup Q \Leftrightarrow \mu_{P\cup Q}(x) &= \mu_p(x) \sqcup \mu_Q(x) = \sum_i \frac{f(v_i)}{v_i} \sqcup \sum_j \frac{g(w_j)}{w_j} \\ &= \sum_{i,j} \big(f(v_i) \wedge g(w_j)\big) \big/ \big(v_i \vee w_j\big), \end{aligned} \tag{2.14}$$

where $\wedge$ represent ***min*** and $\vee$ denote ***max***, whereas $\sqcup$ is *fuzzy grades* operation ***join***.

Intersection.

$$\begin{aligned} P \cap Q \Leftrightarrow \mu_{P\cap Q}(x) &= \mu_p(x) \sqcap \mu_Q(x) = \sum_i \frac{f(v_i)}{v_i} \sqcap \sum_j \frac{g(w_j)}{w_j} \\ &= \sum_{i,j} \big(f(v_i) \wedge g(w_j)\big) \big/ \big(v_i \wedge w_j\big), \end{aligned} \tag{2.15}$$

where $\wedge$ represent ***min***, whereas $\sqcap$ is *fuzzy grades* operation ***meet***.

Remark.

In general, if $\mu_P(x)$ and $\mu_Q(x)$ are two *fuzzy grades* that is, *FS*s in $J_X \subseteq [0,1]$ of *FS* of *type 2*, P and Q in X respectively, denoted as (2.12) and (2.13), and $\divideontimes$ is a binary operation defined in X, then the operation $\divideontimes$ can be extended to *FS* P and Q by the definition relation (extension principle).

$$\begin{aligned} P \divideontimes Q \Leftrightarrow \mu_{P\divideontimes Q}(x) &= \mu_p(x) \divideontimes \mu_Q(x) = \sum_i \frac{f(v_i)}{u_i} \divideontimes \sum_j \frac{g(w_j)}{w_j} \\ &= \sum_{i,j} \big(f(v_i) \wedge g(w_j)\big) \big/ \big(v_i \divideontimes w_j\big) \end{aligned} \tag{2.16}$$

Example 2.

Let $J_X = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ and let fuzzy grades $\mu_P(x)$ and $\mu_Q(x)$ be given as

$\mu_P(x)$ = *0.8/0* + ***1/0.2*** + 0.8/0.4 + 0.6/0.6 + 0.4/0.8 + 0.2/1

$\mu_Q(x)$ = 0.2/0 + 0.4/0.2 + 0.6/0.4 + 0.8/0.6 + ***1/0.8*** + 0.8/1

Then for ***Join*** operation from (2.14) we have

$\mu_p(x) \sqcup \mu_Q(x)$ = (0.8/0 + ***1/0.2*** + 0.8/0.4 + 0.6/0.6 + 0.4/0.8 + 0.2/1) $\sqcup$ (0.2/0 + 0.4/0.2 + 0.6/0.4 + 0.8/0.6 + ***1/0.8*** + 0.8/1)

$\text{MATRIX}\left(\mu_p(x) \sqcup \mu_Q(x)\right) =$

(0.8∧0.2)/(0∨0)	(0.8∧0.4)/(0∨0.2)	(0.8∧0.6)/(0∨0.4)	(0.8∧0.8)/(0∨0.6)	(0.8∧1)/(0∨0.8)	(0.8∧0.8)/(0∨1)
(1∧0.2)/(0.2∨0)	(1∧0.4)/(0.2∨0.2)	(1∧0.6)/(0.2∨0.4)	(1∧0.8)/(0.2∨0.6)	**(1∧1)/(0.2∨0.8)**	(1∧0.8)/(0.2∨1)
(0.8∧0.2)/(0.4∨0)	(0.8∧0.4)/(0.4∨0.2)	(0.8∧0.6)/(0.4∨0.4)	(0.8∧0.8)/(0.4∨0.6)	(0.8∧1)/(0.4∨0.8)	(0.8∧0.8)/(0.4∨1)
(0.6∧0.2)/(0.6∨0)	(0.6∧0.4)/(0.6∨0.2)	(0.6∧0.6)/(0.6∨0.4)	(0.6∧0.8)/(0.6∨0.6)	(0.6∧1)/(0.6∨0.8)	(0.6∧0.8)/(0.6∨1)
(0.4∧0.2)/(0.8∨0)	(0.4∧0.4)/(0.8∨0.2)	(0.4∧0.6)/(0.8∨0.4)	(0.4∧0.8)/(0.8∨0.6)	(0.4∧1)/(0.8∨0.8)	(0.4∧0.8)/(0.8∨1)
(0.2∧0.2)/(1∨0)	(0.2∧0.4)/(1∨0.2)	(0.2∧0.6)/(1∨0.4)	(0.2∧0.8)/(1∨0.6)	(0.2∧1)/(1∨0.8)	(0.2∧0.8)/(1∨1)

=

0.2/0	***0.4/0.2***	***0.6/0.4***	***0.8/0.6***	0.8/0.8	***0.8/1***
0.2/0.2	***0.4/0.2***	***0.6/0.4***	***0.8/0.6***	**1/0.8**	***0.8/1***
0.2/0.4	0.4/0.4	***0.6/0.4***	***0.8/0.6***	0.8/0.8	***0.8/1***
0.2/0.6	0.4/0.6	0.6/0.6	0.6/0.6	0.6/0.8	0.6/1
0.2/0.8	0.4/0.8	0.4/0.8	0.4/0.8	0.4/0.8	0.4/1
0.2/1	0.2/1	0.2/1	0.2/1	0.2/1	0.2/1

=0.2/0 + (0.2∨0.4∨0.4)/0.2 + (0.2∨0.4∨0.6∨0.6∨0.6)/0.4 + (0.2∨0.4∨0.6∨0.6∨0.8∨0.8∨0.8)/0.6 + (0.2∨0.4∨0.4∨0.4∨0.4∨0.6∨0.8∨1∨0.8)/0.8 + (0.2∨0.2∨0.2∨0.2∨0.2∨0.2∨0.4∨0.6∨0.8∨0.8∨0.8)/1 = 0.2/0 + 0.4/0.2 + 0.6/0.4 + 0.8/0.6 + *1/0.8* + 0.8/1

For ***Meet*** operation from (2.15)

$\mu_p(x) \sqcap \mu_Q(x)$ = (0.8/0 + ***1/0.2*** + 0.8/0.4 + 0.6/0.6 + 0.4/0.8 + 0.2/1) ⊓ (0.2/0 + 0.4/0.2 + 0.6/0.4 + 0.8/0.6 + ***1/0.8*** + 0.8/1)

$\text{MATRIX}\left(\mu_p(x) \sqcap \mu_Q(x)\right) =$

(0.8∧0.2)/(0∧0)	(0.8∧0.4)/(0∧0.2)	(0.8∧0.6)/(0∧0.4)	(0.8∧0.8)/(0∧0.6)	(0.8∧1)/(0∧0.8)	(0.8∧0.8)/(0∧1)
(1∧0.2)/(0.2∧0)	(1∧0.4)/(0.2∧0.2)	(1∧0.6)/(0.2∧0.4)	(1∧0.8)/(0.2∧0.6)	**(1∧1)/(0.2∧0.8)**	(1∧0.8)/(0.2∧1)
(0.8∧0.2)/(0.4∧0)	(0.8∧0.4)/(0.4∧0.2)	(0.8∧0.6)/(0.4∧0.4)	(0.8∧0.8)/(0.4∧0.6)	(0.8∧1)/(0.4∧0.8)	(0.8∧0.8)/(0.4∧1)
(0.6∧0.2)/(0.6∧0)	(0.6∧0.4)/(0.6∧0.2)	(0.6∧0.6)/(0.6∧0.4)	(0.6∧0.8)/(0.6∧0.6)	(0.6∧1)/(0.6∧0.8)	(0.6∧0.8)/(0.6∧1)
(0.4∧0.2)/(0.8∧0)	(0.4∧0.4)/(0.8∧0.2)	(0.4∧0.6)/(0.8∧0.4)	(0.4∧0.8)/(0.8∧0.6)	(0.4∧1)/(0.8∧0.8)	(0.4∧0.8)/(0.8∧1)
(0.2∧0.2)/(1∧0)	(0.2∧0.4)/(1∧0.2)	(0.2∧0.6)/(1∧0.4)	(0.2∧0.8)/(1∧0.6)	(0.2∧1)/(1∧0.8)	(0.2∧0.8)/(1∧1)

=

0.2/0	0.4/0	0.6/0	***0.8/0***	***0.8/0***	***0.8/0***
0.2/0	0.4/0.2	0.6/0.2	0.8/0.2	***1/0.2***	0.8/0.2
0.2/0	0.4/0.2	0.6/0.4	***0.8/0.4***	***0.8/0.4***	***0.8/0.4***
0.2/0	0.4/0.2	0.6/0.4	***0.6/0.6***	***0.6/0.6***	***0.6/0.6***
0.2/0	0.4/0.2	0.4/0.4	0.4/0.6	***0.4/0.8***	***0.4/0.8***
0.2/0	0.2/0.2	0.2/0.4	0.2/0.6	0.2/0.8	***0.2/1***

=(0.2∨0.2∨0.2∨0.2∨0.2∨0.2∨0.4∨0.6∨0.8∨0.8∨0.8)/0 + (0.2∨0.4∨0.4∨0.4∨0.4∨0.6∨0.8∨1∨0.8)/0.2 + (0.2∨0.4∨0.6∨0.6∨0.8∨0.8∨0.8)/0.4 + (0.2∨0.4∨0.6∨0.6∨0.6)/0.6 + (0.2∨0.4∨0.4)/0.8 + 0.2/1 = 0.8/0 + *1/0.2* + 0.8/0.4 + 0.6/0.6 + 0.4/0.8 + 0.2/1

Note. For both ***Join*** and ***Meet*** operations the number N_1 of *calculations* needed was $N_1 = 3 \times CardJ_X^2$, In our case for $CardJ_X = 6$ we obtain $N_1 = 108$,

2.4. Fast Operations of FS Type-2

Now we introduce significantly more effective way to operate with type-2 *FS*.

Lemma 2.

If fuzzy grades

$$\mu_P(x) = \sum_i \frac{f(v_i)}{v_i}, v_i \in J_X$$

and

$$\mu_Q(x) = \sum_j \frac{g(w_j)}{w_j}, w_j \in J_X$$

are both normal, *i.e.*

$$\begin{cases} \exists! k^* \in [1, CardJ_X] \mid f(v_{k^*}) = \max_{\forall i \in [1, CardJ]} \{f(v_i)\} == 1; \\ \exists! l^* \in [1, CardJ] \mid g(w_{l^*}) = \max_{\forall j \in [1, CardJ]} \{g(w_j)\} == 1 \end{cases} \tag{2.17}$$

then the following features are taking place

$$v_{k^*}, w_{l^*} \in J \mid v_{k^*} \le w_{l^*} \Rightarrow \mu_p(x) \sqcap \mu_Q(x) = \mu_p(x);$$

$$v_{k^*}, w_{l^*} \in J \mid v_{k^*} > w_{l^*} \Rightarrow \mu_p(x) \sqcap \mu_Q(x) = \mu_Q(x);$$

$$v_{k^*}, w_{l^*} \in J \mid v_{k^*} \le w_{l^*} \Rightarrow \mu_p(x) \sqcup \mu_Q(x) = \mu_Q(x);$$

$$v_{k^*}, w_{l^*} \in J \mid v_{k^*} > w_{l^*} \Rightarrow \mu_p(x) \sqcup \mu_Q(x) = \mu_p(x)$$

Proof.

1)

$$v_{k^*}, w_{l^*} \in J \mid v_{k^*} \le w_{l^*} \Rightarrow b^* = v_{k^*} \wedge w_{l^*} = v_{k^*}$$

$$l(b^*) = l(v_{k^*}) = f(v_{k^*}) = 1$$

$$\mu_p(x) \sqcap \mu_Q(x) = f(v_{k^*}) + \sum_{v_i \wedge w_j \ne b^*} f(v_i) \wedge g(w_j)$$

$$\mu_p(x) \sqcap \mu_Q(x) = 1 + \sum_{v_i \wedge w_j \ne b^*} f(v_i) \wedge g(w_j) = \mu_p(x)$$

2)

$$v_{k^*}, w_{l^*} \in J \mid v_{k^*} > w_{l^*} \Rightarrow b^* = v_{k^*} \wedge w_{l^*} = w_{l^*}$$

$$l(b^*) = l(w_{l^*}) = g(w_{l^*}) = 1$$

$$\mu_p(x) \sqcap \mu_Q(x) = g(w_{l^*}) + \sum_{v_i \wedge w_j \ne b^*} f(v_i) \wedge g(w_j)$$

$$\mu_p(x) \sqcap \mu_Q(x) = 1 + \sum_{v_i \wedge w_j \ne b^*} f(v_i) \wedge g(w_j) = \mu_Q(x)$$

3)

$$v_{k^*}, w_{l^*} \in J \mid v_{k^*} \le w_{l^*} \Rightarrow b^* = v_{k^*} \vee w_{l^*} = w_{l^*}$$

$$l\left(b^*\right)=l\left(w_{l^*}\right)=g\left(w_{l^*}\right)=1$$

$$\mu_p(x)\sqcup\mu_Q(x)=g\left(w_{l^*}\right)+\sum_{v_i\vee w_j\neq b^*}f(v_i)\wedge g\left(w_j\right)$$

$$\mu_p(x)\sqcup\mu_Q(x)=1+\sum_{v_i\vee w_j\neq b^*}f(v_i)\wedge g\left(w_j\right)=\mu_Q(x)$$

4) $$v_{k^*},w_{l^*}\in J\,|\,v_{k^*}>w_{l^*}\Rightarrow b^*=v_{k^*}\vee w_{l^*}=v_{k^*}$$

$$l\left(b^*\right)=l\left(w_{l^*}\right)=f\left(v_{k^*}\right)=1$$

$$\mu_p(x)\sqcup\mu_Q(x)=f\left(v_{k^*}\right)+\sum_{v_i\vee w_j\neq b^*}f(v_i)\wedge g\left(w_j\right)$$

$$\mu_p(x)\sqcup\mu_Q(x)=1+\sum_{v_i\vee w_j\neq b^*}f(v_i)\wedge g\left(w_j\right)=\mu_p(x).\ \blacksquare$$

Corollary 1.

From the conditions (2.17) of **Lemma 2** we can formulate the following *fast* and *simple* implementations of ***Meet*** and ***Join*** operations for *type-2 FS*

$$\mu_p(x)\sqcap\mu_Q(x)=\begin{cases}\mu_p(x),\text{if } k^*\leq l^*,\\ \mu_Q(x),\text{othervise}\end{cases}$$

$$\mu_p(x)\sqcup\mu_Q(x)=\begin{cases}\mu_p(x),\text{if } k^*\geq l^*,\\ \mu_Q(x),\text{othervise}\end{cases}$$

Note. For both ***Join*** and ***Meet*** operations the number N_2 of *calculations* needed was $N_2=\left(2\times CardJ_X\right)+1$, In our case for $CardJ_X=6$ we obtain $N_2=13$, Therefore, this algorithm is more than *8.3 times* faster than the original one.

3. Type-2 Fuzzy Logic System (FLS)

3.1. Type-2 FLS: Overview

In this chapter we revisit and reconsider some important features of *type-2 FLS*, in which the *antecedent* or *consequent* membership functions are *type-2* FS.

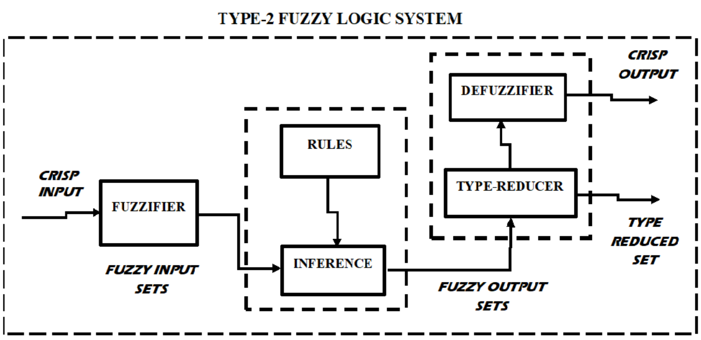

Figure 2. The structure of a type-2 FLS.

Figure 2 shows the structure of a *type-2 FLS* [6]. It is very similar to the structure of a *type-1 FLS*. The fuzzifier maps the crisp input into a *FS*. This fuzzy set can, in general, be a *type-2* set. The inference engine combines rules and gives a mapping from input *type-2 FS* to output *type-2 FS*. To do this one needs to find *unions* and *intersections* of *type-2 FS* sets, as well as *compositions* of *type-2* relations. In the *type-2* case, the output of the *inference* engine is a *type-2* set. Usually, in *type-2 FLS* extended defuzzification is utilized, which gives a *type-1* fuzzy set. Since this operation transforms *type-2* output sets to the *FLS* to a *type-1* set, this operation is called *type reduction* and the type-reduced set so obtained a *type-reduced* set. In [6]-[8] it was mentioned that in order to get a *crisp output* from a *type-2* FLS, the *type-reduced* set has to be *de-fuzzified*.

Hence, in order to develop a *type-2* FLS, one needs to be able to:

1) perform the set theoretic operations of union, intersection, and complement on *type-2* sets;

2) know properties (e.g., commutativity, associativity, identity laws) of membership grades of *type-2* sets;

3) deal with *type-2* fuzzy relations and their compositions;

4) perform type reduction and defuzzification to obtain a set-valued or crisp output from the *FLS* [7] [8].

We will concentrate our attention on 1., 2. and 3.

3.2. Type-2 FLS: Inference

Let us consider a type-2 *FLS* having *K* inputs, $x_1 \in X_1, x_2 \in X_2, \cdots, x_K \in X_K$, and one output $y \in Y$. We assume that this *FLS* has *M* rules where the *m*-th rule has the form

$$R^m : \text{IF } x_1 \text{ is } \widetilde{P_1^m} \text{ and } x_2 \text{ is } \widetilde{P_2^m} \text{ and } \cdots \text{ and } x_K \text{ is } \widetilde{P_K^m} \text{ THEN } y \text{ is } \tilde{Q}^m \tag{3.1}$$

This rule represents a *type-2* fuzzy relation between the input space $X_1 \times X_2 \times \cdots \times X_K$ and the output space Y of the *FLS*. We denote the membership function of this *type-2* relation as,

$\forall m \in [1, M] \,|\, \mu_{\widetilde{P_1^m} \times \widetilde{P_2^m} \times \cdots \times \widetilde{P_K^m} \to \tilde{Q}^m}(x, y)$, where $\widetilde{P_1^m} \times \widetilde{P_2^m} \times \cdots \times \widetilde{P_K^m}$ denotes the *Cartesian* product of $P_1^m, P_2^m, \cdots, P_K^m$ and $x = \{x_1, x_2, \cdots, x_K\}$. When an input x' is applied, the *composition* of the fuzzy set $\widetilde{P'}$ to which x' have its place and the rule R^m is found by using the extended *sup-star composition* [6]-[8]

$$\mu_{\widetilde{P'} \circ \widetilde{P_1^m} \times \widetilde{P_2^m} \times \cdots \times \widetilde{P_K^m} \to \tilde{Q}^m}(y) = \coprod_{x \in \widetilde{P'}} \left[\mu_{\widetilde{P'}}(x) \sqcap \mu_{\widetilde{P_1^m} \times \widetilde{P_2^m} \times \cdots \times \widetilde{P_K^m} \to \tilde{Q}^m}(x, y) \right] \tag{3.2}$$

Let us take a look at the implication membership function from (3.2) for each ***m***-th rule and denote it in the following form

$$\mu_{\widetilde{P_1^m} \times \widetilde{P_2^m} \times \cdots \times \widetilde{P_K^m} \to \tilde{Q}^m}(x, y) = \left[\prod_{\forall i \in [1, K]} \mu_{\widetilde{P_i^m}}(x) \right] \to \mu_{\widetilde{Q^m}}(y) \tag{3.3}$$

We have to underline the fact that the implication operation in (3.3) is an essential element of a decision-making mechanism, known as *Fuzzy Conditional Inference Rule* (*FCIR*) of the (3.1) form.

Thus, we use the following type of implication [10]

$$P \to Q = \begin{cases} (1-p)\cdot q, & p > q, \\ 1, & p \le q \end{cases} \tag{3.4}$$

For practical purposes, described down below, we will use *Fuzzy Conditional Rule* (*FCR*) [11] of the following type

$$(P \to Q) \cap (\neg P \to \neg Q) = \begin{cases} (1-q)\cdot p, & p < q, \\ 1, & p = q, \\ (1-p)\cdot q, & p > q. \end{cases} \tag{3.5}$$

From (3.3) the correspondent *fuzzy binary relationship* would look like that $\forall m \in [1, M]$

$$\begin{aligned} \mu_{\widetilde{P_1^m} \times \widetilde{P_2^m} \times \cdots \times \widetilde{P_K^m} \to \tilde{Q}^m}(x,y) = & \left\{ \left[\prod_{\forall i \in [1,K]} \mu_{\widetilde{P_i^m}}(x) \right] \to \mu_{\widetilde{Q^m}}(y) \right\} \\ & \wedge \left\{ \left(1 - \left[\prod_{\forall i \in [1,K]} \mu_{\widetilde{P_i^m}}(x) \right] \right) \to \left(1 - \mu_{\widetilde{Q^m}}(y) \right) \right\} \end{aligned} \tag{3.6}$$

Given (3.5) from (3.6) we are getting $\forall m \in [1, M]$

$$\begin{aligned} & \mu_{\widetilde{P_1^m} \times \widetilde{P_2^m} \times \cdots \times \widetilde{P_K^m} \to \tilde{Q}^m}(x,y) \\ & = \begin{cases} \left(1 - \left[\prod_{\forall i \in [1,K]} \mu_{\widetilde{P_i^m}}(x) \right] \right) \cdot \mu_{\widetilde{Q^m}}(y), & \left[\prod_{\forall i \in [1,K]} \mu_{\widetilde{P_i^m}}(x) \right] < \mu_{\widetilde{Q^m}}(y), \\ 1, & \left[\prod_{\forall i \in [1,K]} \mu_{\widetilde{P_i^m}}(x) \right] = \mu_{\widetilde{Q^m}}(y), \\ \left(1 - \mu_{\widetilde{Q^m}}(y) \right) \cdot \left[\prod_{\forall i \in [1,K]} \mu_{\widetilde{P_i^m}}(x) \right], & \mu_{\widetilde{Q^m}}(y) < \left[\prod_{\forall i \in [1,K]} \mu_{\widetilde{P_i^m}}(x) \right]. \end{cases} \end{aligned} \tag{3.7}$$

To further **investigate** the structure and possible implementation of $\left[\prod_{\forall i \in [1,K]} \mu_{\widetilde{P_i^m}}(x) \right]$ in (3.7) we introduce the following.

Definition 1.

The set of values $J_X = \{v_i\} = \{v_1, v_2, \cdots, v_N\} \subseteq [0,1]$ | is a *lattice* ($J_X, \le$), *i.e.* $v_1 < v_2 < \cdots < v_N$ and $\forall a, b \in J_X \mid a \le b \Rightarrow a \wedge b = a; a \vee b = b$.

Definition 2.

For a *lattice* $J_X \subseteq [0,1]$ and *N* various *normal* fuzzy grades in J_X, *i.e.*

$$\forall i \in [1, CardJ_X], \forall j \in [1, N] \mid \mu_{P_j}(x) = \sum_i \frac{f_j(v_{ij})}{v_{ij}} \Rightarrow$$

$\exists! k_j^* \in [1, CardJ_X] \mid f_j\left(v_{k_j^*}\right) = \max_{\forall i \in [1, CardJ]} \{ f_j(v_{ij}) \} == 1$, we define

$$v^{upper} = \max_{\forall j \in [1,N]} \left\{ v_{k_j^*} \right\} \tag{3.8}$$

$$v^{lower} = \min_{\forall j \in [1,N]} \left\{ v_{k_j^*} \right\} \tag{3.9}$$

Therefore, the relevant indexes from (2.8) are

$$i^{upper} = \left(CardJ_X\right) \cdot v^{upper} \tag{3.10}$$

$$i^{lower} = \left(CardJ_X\right) \cdot v^{lower} \tag{3.11}$$

From (3.8) and (3.10) we define the *upper* fuzzy grade for $\forall i \in \left[1, CardJ_X\right]$

$$\begin{aligned} \mu_P^{upper}(x) &= \frac{f\left(v^{upper}\right)}{v^{upper}} + \sum_{i \neq i^{upper}} \frac{f\left(v_i\right)}{v_i} \\ &= \frac{1}{v^{upper}} + \sum_{i \neq i^{upper}} \frac{f\left(v_i\right)}{v_i} \end{aligned} \tag{3.12}$$

And from (3.9) and (3.11) we define its *lower* counterpart for $\forall i \in \left[1, CardJ_X\right]$

$$\begin{aligned} \mu_P^{lower}(x) &= \frac{f\left(v^{lower}\right)}{v^{lower}} + \sum_{i \neq i^{lower}} \frac{f\left(v_i\right)}{v_i} \\ &= \frac{1}{v^{lower}} + \sum_{i \neq i^{lower}} \frac{f\left(v_i\right)}{v_i}. \end{aligned} \tag{3.13}$$

Based on *Definition1* and *Definition2* we shall prove the following.

Lemma 3.

If $J_X \subseteq [0,1]$ is a *lattice*, then for *N* various *normal* fuzzy grades in J_X , *i.e.*

$$\mu_{P_j}(x) = \sum_i f_j\left(v_i\right)/v_i \mid \forall i \in \left[1, CardJ_X\right], \forall j \in 1, N$$

the *following* is taking place

$$\begin{aligned} \prod_{j=1,N} \mu_{P_j}(x) &= \prod_{j=1,N} \sum_{i=CardJ_X} \frac{f_j\left(v_i\right)}{v_i} = \mu_P^{lower}(x) \\ \coprod_{j=1,N} \mu_{P_j}(x) &= \coprod_{j=1,N} \sum_{i=1,CardJ_X} \frac{f_j\left(v_i\right)}{v_i} = \mu_P^{higher}(x) \end{aligned} \tag{L3.1}$$

Proof.

From *Definition 2* we have the set $V \subseteq J_X = \left\{v_{k^*}\right\} = \left\{v_{k_1^*}, v_{k_2^*}, \cdots, v_{k_M^*}\right\} \mid V \subseteq \left[v^{lower}, v^{upper}\right]$.

Since $v^{lower}, v^{upper} \in J_X \mid v^{lower} < v^{upper}$, from (3.12) and (3.13) we obtain the following

$$\prod_{j=1,N} \mu_{P_j}(x) = \mu_P^{lower}(x) \sqcap \mu_P^{upper}(x) = \mu_P^{lower}(x),$$

$$\coprod_{j=1,N} \mu_{P_j}(x) = \mu_P^{lower}(x) \sqcup \mu_P^{upper}(x) = \mu_P^{upper}(x). \blacksquare$$

3.3. Type-2 FLS: Implication

It was mentioned in [6]-[8] that "the *distinction* between type-1 and type-2 is associated with the *nature* of the *MF*s, which is *not important* while forming 'IF-THEN' rules". In opposite, from our point of view, this aspect needs additional attention. To further elaborate on *type-2 FLS inference* specificity and on correspondent "IF-THEN" rules formulation, we have to mention that in contrast with *type-1* case, these rules are describing not a knowledge about input/output system behavior, but rather between our *semantic interpretation*/ *understanding* of a nature of things under consideration. For this particular reason we are formulating the following.

Proposition 1.

Let $\mu_P(x)$ and $\mu_Q(x)$ be two *fuzzy grades* in $J_X \subseteq [0,1]$ of *FS* of *type 2*, P and Q, represented as (2.12) and (2.13) correspondingly. Therefore, by applying an extension Zadeh principle [2.16], for *type-2 FS implication* operator we obtain the following

$$\begin{aligned} P \to Q \Leftrightarrow \mu_{P\to Q}(x) &= \mu_p(x) \to \mu_Q(x) \\ &= \sum_i \frac{f(u_i)}{u_i} \to \sum_j \frac{g(w_j)}{w_j} \\ &= \sum_{i,j} \left(f(u_i) \wedge g(w_j)\right) / \left(u_i \to w_j\right), \end{aligned} \tag{3.14}$$

where

$$(u \to w) = \begin{cases} (1-w)\cdot u, & u < w, \\ 1, & u = w, \\ (1-u)\cdot w, & u > w. \end{cases} \tag{3.15}$$

Let us represent the implication operation on *type-2 FS* from (3.14) and (3.15), as

$$\begin{aligned} \mu_p(x) \to \mu_Q(x) = R(u,w) &= \sum_{i,j} \left(f(u_i) \wedge g(w_j)\right) / \left(u_i \to w_j\right) \\ &= \sum_{u_i < w_j} \frac{f(u_i) \wedge g(w_j)}{(1-w_j)\cdot u_i} + \sum_{u_i = w_j} \frac{f(u_i) \wedge g(w_j)}{1} \\ &\quad + \sum_{u_i > w_j} \frac{f(u_i) \wedge g(w_j)}{(1-u_i)\cdot w_j} \end{aligned} \tag{3.16}$$

We have to mark the fact that the *implication* operator (3.15). utilized in (3.14), indicates a degree of *closeness* between $\forall u, w \in J_X$ values. That *closeness,* by the very nature of type-2 *FS linguistic scales* in use, might be interpreted as a degree of *semantic similarity*.

3.4. Type-2 FLS "IF-THEN" Rules Semantic Dissimilarity

Let us denote a fuzzy binary relationship $R(u,w)$ from (3.16) like a sum of three subsets, *i.e.* for $\forall i, j \in [1, CardJ_X]$ we have

$$R(u,w) = R_{u_i < w_j}(u_i, w_j) + R_{u_i = w_j}(u_i, w_j) + R_{u_i > w_j}(u_i, w_j) \tag{3.17}$$

Based on that let us first consider two subsets of $R(u,w)$ from (3.17) $R_{u_i < w_j}(u_i, w_j)$ and $R_{u_i > w_j}(u_i, w_j)$ for $\forall i, j \in [1, CardJ_X]$, in which

$$\left(u\to w\right)=\begin{cases}\left(1-w\right)\cdot u, u<w,\\ \left(1-u\right)\cdot w, u>w.\end{cases}$$

In fact, these subsets, when $R_{u_i\neq w_j}\left(u_i,w_j\right)$ for $\forall i,j\in\left[1,CardJ_X\right]$ indicate the degree of *semantic dissimilarity* between $\forall u,w\in J_X$ values. In connection with that we have to investigate the following feature of values within *single* interval $a\in\left[0,1\right]$. For this reason, let us formulate the following.

Lemma 4.

If $f\left(a\right)=a\cdot\left(1-a\right),\forall a\in\left[0,1\right]$, then the $\max_{\forall a\in[0,1]}\left[f\left(a\right)\right]=0.25$

Proof.

We have $f\left(a\right)=a\cdot\left(1-a\right)=a-a^2$. To get the value of a, which corresponds to $\max_{\forall a\in[0,1]}\left[a\cdot\left(1-a\right)\right]$ we use some basic principle of an optimization theory by taking *first derivative* of $f\left(a\right)$ and equalize it to zero, *i.e.* $f'\left(a\right)=0$. Since $f'\left(a\right)=1-2\cdot a=0$, then we obtain $a=0.5$. Therefore, we are getting the following $\max_{\forall a\in[0,1]}\left[f\left(a\right)\right]=0.25$. ■

Corollary 2.

From the result of *Lemma 4* and from (3.15) we obtain the following

$\forall a,b\in\left[0,1\right]$, $f\left(a,b\right)=\begin{cases}a\cdot\left(1-b\right), a<b\\ \left(1-a\right)\cdot b, a>b\end{cases}$. And since

$a\neq b\Rightarrow a\cdot\left(1-b\right)<\max_{\forall a\in[0,1]}\left[a\cdot\left(1-a\right)\right]$ and $\left(1-a\right)\cdot b<\max_{\forall a\in[0,1]}\left[a\cdot\left(1-a\right)\right]$ we have $a\cdot\left(1-b\right)<0.25$ and $\left(1-a\right)\cdot b<0.25$.

Example 4.

Assume the $CardJ_X=6$, then again from (2.10) we have $J_X=\{0,0.2,0.4,0.6,0.8,1\}\subseteq\left[0,1\right]$. Given the fact that $\forall u,w\in J_X$, then the *closest value* for both $\left(1-w\right)\cdot u$, $u<w$ and $\left(1-u\right)\cdot w$, $u>w$ expressions to *max possible* value of 0.25 is 0.16 ($u=0.4$; $w=0.6$) and ($u=0.6$; $w=0.4$).

3.5. Type-2 FLS: "IF-THEN" Rules Semantic Similarity

Now let us consider the issue of *semantic similarity* between $\forall u,w\in J_X$ values, *i.e.* the subset $R_{u_i=w_j}\left(u_i,w_j\right)$ for $\forall i,j\in\left[1,CardJ_X\right]$ from (3.17), in which $\left(u\to w\right)=1$, $u=w$.

From (3.16) and (3.17) the subset $R_{u_i=w_j}\left(u_i,w_j\right)$ for $\forall i,j\in\left[1,CardJ_X\right]$ is in fact, the *major diagonal* of a matrix $R\left(u,w\right)$. And it looks like that $\forall i,j\in\left[1,CardJ_X\right]$,

$$R_{u_i=w_j}\left(u_i,w_j\right)=\sum_{u_i=w_j}\frac{f\left(u_i\right)\wedge g\left(w_j\right)}{1}\quad(3.18)$$

Lemma 5.

If *MFs* $f\left(u\right)$ and $g\left(w\right)$ of *fuzzy grades*

$$\mu_P\left(x\right)=\sum_i\frac{f\left(u_i\right)}{u_i},u_i\in J_X\text{ and }\mu_Q\left(x\right)=\sum_j\frac{f\left(w_j\right)}{w_j},w_j\in J_X\quad(L5.1)$$

are both normal (2.17), *i.e.*

$$\begin{cases} \exists ! k^* \in [1, CardJ_X] \mid f(u_{k^*}) = \max_{\forall i \in [1, CardJ_X]} \{ f(u_i) \} == 1; \\ \exists ! l^* \in [1, CardJ_X] \mid g(w_{l^*}) = \max_{\forall j \in [1, CardJ_X]} \{ g(w_j) \} == 1 \end{cases} \quad \text{(L5.2)}$$

and if *MF* $h(v) \mid \forall v \in J_X$, is an intersection of *MFs* $f(u)$ and $g(w)$, $h(v) = f(u) \wedge g(w)$ from (3.18), and it is presented as

$$h(v) = \sum_{\forall m \in [1, CardJ_X]} h(v_m) = \sum_{u_i \to w_j = 1} f(u_i) \wedge g(w_j), \quad \text{(L5.3)}$$

then it is *normal and unimodal* iff

$\forall i, j \in [1, CardJ_X] \mid u_i \to w_j = 1 \Rightarrow f(u_i) == g(w_j)$.

Proof.

By definition of a logical conjunction operation, we have the following $\forall a, b \in [0,1]$, $F(a,b) = a \wedge b = 1 \Rightarrow a = b = 1$. Therefore, from *normality* of *MFs* $f(u)$ and $g(w)$ from (2.17)

$\exists ! k^*, l^* \in [1, CardJ_X] \Rightarrow \exists ! m^* \mid h(v_{m^*}) = f(u_{k^*}) \wedge g(w_{l^*}) = 1$. In other words, we obtain the case when

$m^* = k^* = l^* \mid h(v) = f(u) = g(w) \Rightarrow h(v)$ is *normal* and *unimodal*. ■

Example 5.

1) The case when *fuzzy grades* for both input and output are ***"low"***

$\mu_P(x)$ = ***low*** = ***1/0*** + 0.8/.2 + 0.6/.4 + 0.4/.6 + 0.2/.8 + 0/1

$\mu_Q(x)$ = ***low*** = ***1/0*** + 0.8/.2 + 0.6/.4 + 0.4/.6 + 0.2/.8 + 0/1

$R(u,w) = (\mu_P(x) \to \mu_Q(x)) =$

1/1	0.8/0	0.6/0	0.4/0	0.2/0	0/0
0.8/0	***0.8/1***	0.6/0.12	0.4/08	0.2/04	0/0
0.6/0	0.6/0.12	***0.6/1***	0.4/0.16	0.2/08	0/0
0.4/0	0.4/08	0.4/0.16	***0.4/1***	0.2/0.12	0/0
0.2/0	0.2/04	0.2/08	0.2/0.12	***0.2/1***	0/0
0/0	0/0	0/0	0/0	0/0	***0/1***

The *major diagonal* of a matrix $R(u,w)$ is $\forall i, j \in [1, CardJ_X]$

$R_{u_i = w_j}(u_i, w_j)$ = ***1/1*** + 0.8/1 + 0.6/1 + 0.4/1 + 0.2/1 + 0/1

2) The case when *fuzzy grades* for both input and output are ***"high"***

$\mu_P(x)$ = ***high*** = 0/0 + 0.2/0.2 + 0.4/0.4 + 0.6/0.6 +0.8/0.8 + ***1/1***

$\mu_Q(x)$ = ***high*** = 0/0 +0.2/0.2 +0.4/0.4 +0.6/0.6 +0.8/0.8 + ***1/1***

$R(u,w) = (\mu_P(x) \to \mu_Q(x)) =$

0/1	0/0	0/0	0/0	0/0	0/0
0/0	***0.2/1***	0.2/0.12	0.2/08	0.2/04	0.2/0
0/0	0.2/0.12	***0.4/1***	0.4/0.16	0.4/08	0.4/0
0/0	0.2/08	0.4/0.16	***0.6/1***	0.6/0.12	0.6/0
0/0	0.2/04	0.4/08	0.6/0.12	***0.8/1***	0.8/0
0/0	0.2/0	0.4/0	0.6/0	0.8/0	***1/1***

The *major diagonal* of a matrix $R(u,w)$ is $\forall i,j \in [1, CardJ_X]$

$R_{u_i=w_j}(u_i,w_j)$ = 0/1 + 0.2/1 + 0.4/1 + 0.6/1 + 0.8/1 + ***1/1***

3) The case when *fuzzy grades* for both input and output are ***"middle"***

$\mu_P(x)$ = ***middle*** = 0.6/0 + 0.8/0.2 + ***1/.4*** +0.8/0.6 +0.6/0.8 +0.4/1

$\mu_Q(x)$ = ***middle*** = *0.6/0 +0.8/0.2 +* ***1/.4*** +0.8/0.6 +0.6/0.8 +0.4/1

$R(u,w) = (\mu_P(x) \to \mu_Q(x)) =$

0.6/1	0.6/0	0.6/0	0.6/0	0.6/0	0.4/0
0.6/0	***0.8/1***	0.8/0.12	0.8/08	0.6/04	0.4/0
0.6/0	0.8/0.12	***1/1***	0.8/0.16	0.6/08	0.4/0
0.6/0	0.8/08	0.8/0.16	***0.8/1***	0.6/0.12	0.4/0
0.6/0	0.6/04	0.6/08	0.6/0.12	***0.6/1***	0.4/0
0.4/0	0.4/0	0.4/0	0.4/0	0.4/0	***0.4/1***

The *major diagonal* of a matrix $R(u,w)$ is $\forall i,j \in [1, CardJ_X]$

$R_{u_i=w_j}(u_i,w_j)$ = 0.6/1 + 0.8/1 + ***1/1*** + 0.8/1 + 0.6/1 + 0.4/1

Corollary 3.

If *MFs* $f(u)$ and $g(w)$ of *fuzzy grades* of an input $\mu_P(x)$ and an output $\mu_Q(x)$ correspondingly, satisfy conditions (*L5.1*) - (*L5.3*) from *Lemma 5*, but $\forall i,j \in [1, CardJ_X] \mid u_i \to w_j = 1 \Rightarrow f(u_i) \neq g(w_j)$, then MF $h(v)$ from (L5.3) is *subnormal*.

Proof.

From (L5.1) and (L5.2) and $f(u_i) \neq g(w_j), \mid \forall i,j \in [1, CardJ_X] \mid u_i \to w_j = 1$ $\Rightarrow k^* \neq l^*$, therefore $\forall m \in [1, CardJ_X]$, $h(v_m) = f(u_i) \wedge g(w_j) < f(u_{k^*}) == 1$ or $h(v_m) = f(u_i) \wedge g(w_j) < g(w_{l^*}) == 1$. Hence $\exists! m^* \mid h(v_{m^*}) == \max_{\forall m \in [1, CardJ_X]} \{h(v_m)\} < 1$. ■

Definition 3.

If $\mu_P(x)$ and $\mu_Q(x)$ are two *fuzzy grades* in $J_X \subseteq [0,1]$ of *FS* of *type 2*, P and Q, respectively, represented as

$$\mu_P(x) = \sum_i \frac{f(v_i)}{v_i} \text{ and } \mu_Q(x) = \sum_i \frac{g(v_i)}{v_i}, v_i \in J_X, \tag{3.19}$$

then we call *FS* of *type 2* Q from (3.19) *Semantically Inversive* in relation to *FS* of *type 2* P if $g(v_i) = 1 - f(v_i)$, $v_i \in J_X$. In other words we have the following

$$\mu_Q(x) = \sum_i \frac{1 - f(v_i)}{v_i}, v_i \in J_X \tag{3.20}$$

Lemma 6.

If $\forall a \in [0,1] \mid f(a) = a \wedge (1-a) = \min(a, 1-a)$, then $f(a) \leq 0.5$

Proof.

From *algebraic* definition of a min of two numbers we obtain the following

$$f(a)=\min(a,1-a)=\frac{a+(1-a)-|a-(1-a)|}{2}.$$

Let present $f(a)$ in the following way $f(a)=\begin{cases} f_1(a), & a\geq(1-a) \\ f_2(a), & a<(1-a) \end{cases}$.

Thus, we have

$$f_1(a)=\frac{1-a+(1-a)}{2}=\frac{2-2a}{2}=1-a.$$

And

$$f_2(a)=\frac{1-1+2a}{2}=\frac{2a}{2}=a.$$

Finally

$$f(a)=\begin{cases} 1-a, & a\geq(1-a) \\ a, & a<(1-a) \end{cases}=\begin{cases} 1-a, & 2a\geq 1 \\ a, & 2a<1 \end{cases}$$

$$=\begin{cases} 1-a, & a\geq 0.5 \\ a, & a<0.5 \end{cases}=\begin{cases} 1-a, & 1-a\leq 0.5 \\ a, & a<0.5 \end{cases}$$

$$\Rightarrow f(a)\leq 0.5. \blacksquare$$

Corollary 4.

Hence, from *Lemma 6*, if *MF* $h(v)\,|\,\forall v\in J_X$ is an *intersection* of *MFs* $f(u)$ and $g(w)$, $h(v)=f(u)\wedge g(w)$ from (3.19) and (3.2), and if it is presented as (L5.3), then it is *subnormal* and

$$\max_{\forall m\in[1,CardJ_X]}\{h(v_m)\}\leq 0.5. \quad (3.21)$$

Example 6.

Here is a case of *Semantically Inversive* fuzzy grades ***"low"*** and ***"high"*** in use.

4) The case when input *fuzzy grade* is ***"low"*** and output one is ***"high"***

$\mu_P(x)$ = ***low*** = ***1/0*** + 0.8/0.2 + 0.6/0.4 + 0.4/0.6 + 0.2/0.8 + 0/1

$\mu_Q(x)$ = ***high*** = 0/0 + 0.2/0.2 + 0.4/0.4 + 0.6/0.6 + 0.8/0.8 + ***1/1***

$$R(u,w)=(\mu_P(x)\rightarrow\mu_Q(x))=$$

0/1	0.2/0	0.4/0	0.6/0	0.8/0	1/0
0/0	**0.2/1**	0.4/0.12	0.6/08	0.8/04	0.8/0
0/0	0.2/0.12	**0.4/1**	0.6/0.16	0.6/08	0.6/0
0/0	0.2/08	0.4/0.16	**0.4/1**	0.4/0.12	0.4/0
0/0	0.2/04	0.2/08	0.2/0.12	**0.2/1**	0.2/0
0/0	0/0	0/0	0/0	0/0	**0/1**

The *major diagonal* of a matrix $R(u,w)$ is $\forall i,j\in[1,CardJ_X]$

$R_{u_i=w_j}\left(u_i, w_j\right)$ = 0/1 + 0.2/1 + ***0.4/1 + 0.4/1*** + 0.2/1 + 0/1

3.6. Type-2 FLS: Improved Sup-Star Composition

Let, as usual $\mu_P(x)$ and $\mu_Q(x)$ be two *fuzzy grades* in $J_X \subseteq [0,1]$ of *FS* of *type 2*, P and Q, represented as a system input (2.12) and an output (2.13) correspondingly. And let an input/output *fuzzy relationship matrix* be defined as $R(u,w) = \mu_p(x) \to \mu_Q(x)$ from (3.16). Hence, we use a *Fuzzy Conditional Inference Rule* (*FCIR*), formulated by means of "common sense" as a following conditional clause:

$$R = \text{"IF } (p \text{ is } P)\text{, THEN } (q \text{ is } Q)\text{"} \tag{3.22}$$

In other words, we use fuzzy conditional inference of the following type [11]:

Ant 1: If Input is P, then Output is Q
Ant2: Input is P'
-- (3.23)
Cons: Output is Q'

where $P, P' \subseteq U$ and $Q, Q' \subseteq W$. Note, that $\mu_{P'}(x)$ and $\mu_{Q'}(x)$ are two *fuzzy grades* in $J_X \subseteq [0,1]$ of *FS* of *type 2*, P' and Q' from (3.23), represented as a system *current input* of type

$$\begin{gathered}\mu_{P'}(x) = \sum_i \frac{f'(v_i)}{v_i}, v_i \in J_X \\ \mu_{Q'}(x) = \sum_i \frac{g'(w_i)}{w_i}, w_i \in J_X.\end{gathered} \tag{3.24}$$

Given a *unary* relationship $R(u) = P'$ one can obtain the consequence $R(w)$ by *Sup-Star Composition* to $R(u)$ and $R(u,w)$ of type (3.16):

$$\begin{aligned}R(w) &= R(u) \circ R(u,w) = \sum_i \frac{f'(u_i)}{u_i} \circ \sum_{i,j} \left(f(u_i) \wedge g(w_j)\right) \big/ \left(u_i \to w_j\right) \\ &= \sum_{i,j} \coprod_i \left[\frac{f'(u_i)}{u_i} \sqcap \frac{\left(f(u_i) \wedge g(w_j)\right)}{(u_i \to w_j)}, \forall i,j \in [1, CardJ_X]\right.\end{aligned} \tag{3.25}$$

The value of an output fuzzy set Q' could be consider as

$$Q' = R(w) \tag{3.26}$$

Lemma 7.

Assume that $\mu_P(x), \mu_{P'}(x)$ and $\mu_Q(x)$ are *fuzzy grades* in $J_X \subseteq [0,1]$ of *FS* of *type 2*; P and Q represent a system input (2.12) and an output (2.13), correspondingly and P' represents a system *current input* (3.24). If an input/output *fuzzy relationship matrix* is defined as $R(u,w) = \mu_p(x) \to \mu_Q(x)$ from (3.16), then *Sup-Star Composition* (3.25) is reduced to the following $R(w) = R(u) \circ R(u,w) = R(u) \sqcap R_{u_i=w_j}\left(u_i, w_j\right)$ for $\forall i,j \in [1, CardJ_X]$. Where $R_{u_i=w_j}\left(u_i, w_j\right)$ for $\forall i,j \in [1, CardJ_X]$ from (3.18) is the *major diagonal* of a matrix $R(u,w)$.

Proof.

Since $R(u,w)$ from (3.17) consists of three subsets $R_{u_i<w_j}(u_i,w_j)$, $R_{u_i=w_j}(u_i,w_j)$ and $R_{u_i>w_j}(u_i,w_j)$, then from (3.25) we obtain

$$\begin{aligned} R(w) = & \sum_{u_i<w_j} \coprod_i \left[\frac{f'(u_i)}{u_i} \sqcap \frac{\left(f(u_i)\wedge g(w_j)\right)}{(1-w_j)\cdot u_i} \right. \\ & + \sum_{u_i=w_j} \coprod_i \left[\frac{f'(u_i)}{u_i} \sqcap \frac{\left(f(u_i)\wedge g(w_j)\right)}{1} \right. \\ & + \sum_{u_i>w_j} \coprod_i \left[\frac{f'(u_i)}{u_i} \sqcap \frac{\left(f(u_i)\wedge g(w_j)\right)}{(1-u_i)\cdot w_j}, \forall i,j \in \left[1, CardJ_X\right] \right. \end{aligned} \tag{3.27}$$

From (3.27) for *meet* operator $\sqcap$ we consider the following three cases for $\forall i,j \in \left[1, CardJ_X\right]$

$$\begin{cases} \dfrac{f'(u_i)}{u_i} \sqcap \dfrac{\left(f(u_i)\wedge g(w_j)\right)}{(1-w_j)\cdot u_i}, u_i < w_j; \\ \dfrac{f'(u_i)}{u_i} \sqcap \dfrac{\left(f(u_i)\wedge g(w_j)\right)}{1}, u_i = w_j; \\ \dfrac{f'(u_i)}{u_i} \sqcap \dfrac{\left(f(u_i)\wedge g(w_j)\right)}{(1-u_i)\cdot w_j}, u_i > w_j;. \end{cases} \tag{3.28}$$

Note, that for all (3.28) cases the *meet* operation is realized for involved *MF*s like $h(u_i,w_j) = f^{'}(u_i) \wedge f(u_i) \wedge g(w_j)$ for $\forall i,j \in \left[1, CardJ_X\right]$, whereas for values $\forall u_i, w_j \in J_X$ we have the following

$$\begin{cases} v_{ij} = u_i \wedge (1-w_j)\cdot u_i, & u_i < w_j; \\ v_{ij} = u_i \wedge 1, & u_i = w_j; \\ v_{ij} = u_i \wedge (1-u_i)\cdot w_j, & u_i > w_j \end{cases} \tag{3.29}$$

From (3.29), given the result of *Corollary 2*, $(1-w_j)\cdot u_i < 0.25 \mid u_i < w_j$ and $(1-u_i)\cdot w_j < 0.25 \mid u_i > w_j$. Hence, $v_{ij} < 0.25 \Rightarrow v_{ij} \notin J_X \mid u_i \neq w_j$, whereas $v_{ij} = u_i \Rightarrow v_{ij} \in J_X \mid u_i = w_j$ and therefore we can rewrite (3.27) as follows $\forall i,j \in \left[1, CardJ_X\right]$

$$R(w) = \sum_{u_i=w_j} \coprod_i \left[\frac{f'(u_i)}{u_i} \sqcap \frac{\left(f(u_i)\wedge g(w_j)\right)}{1} = R(u) \sqcap R_{u_i=w_j}(u_i,w_j) \right. \quad \blacksquare$$

Example 7.

Let P and Q represent a system input (2.12) and an output (2.13).

Correspondingly, with the following *fuzzy grades*

$\mu_P(x)$ = ***in between low and mid*** = 0.8/0 + ***1/0.2*** + 0.8/0.4 + 0.6/0.6 + 0.4/0.8 + 0.2/1

$\mu_Q(x)$ = ***in between low and mid*** = 0.8/0 + ***1/0.2*** + 0.8/0.4 + 0.6/0.6 + 0.4/0.8 + 0.2/1

An input/output *fuzzy relationship matrix* is

$R(u,w) = (\mu_P(x) \to \mu_Q(x)) =$

0.8/1	0.8/0	0.8/0	0.6/0	0.4/0	0.2/0
0.8/0	***1/1***	0.8/0.12	0.6/0.08	0.4/0.04	0.2/0
0.8/0	0.8/0.12	***0.8/1***	0.6/0.16	0.4/0.08	0.2/0
0.6/0	0.6/0.08	0.6/0.16	***0.6/1***	0.4/0.12	0.2/0
0.4/0	0.4/0.04	0.4/0.08	0.4/0.12	***0.4/1***	0.2/0
0.2/0	0.2/0	0.2/0	0.2/0	0.2/0	***0.2/1***

The *major diagonal* of a matrix $R(u,w)$ is $\forall i,j \in [1, CardJ_X]$

$R_{u_i=w_j}(u_i,w_j)$ = 0.8/1 + ***1/1*** + 0.8/1 + 0.6/1 + 0.4/1 + 0.2/1

And let a system *current input* P' from (3.24) to be

$\mu_{P'}(x)$ = ***middle*** = 0.6/0 + 0.8/0.2 + ***1/0.4*** + 0.8/0.6 + 0.6/0.8 + 0.4/1

The proposed version of *Sup-Star Composition* would be

$R(w) = R(u) \sqcap R_{u_i=w_j}(u_i,w_j) = \mu_{P'}(x) \sqcap R_{u_i=w_j}(u_i,w_j) =$

(0.6∧0.8)/(0∧1)	(0.6∧1)/(0∧1)	(0.6∧0.8)/(0∧1)	(0.6∧0.6)/(0∧1)	(0.6∧0.4)/(0∧1)	(0.6∧0.2)/(0∧1)
(0.8∧0.8)/(0.2∧1)	(0.8∧1)/(0.2∧1)	(0.8∧0.8)/(0.2∧1)	(0.8∧0.6)/(0.2∧1)	(0.8∧0.4)/(0.2∧1)	(0.8∧0.2)/(0.2∧1)
(1∧0.8)/(0.4∧1)	***(1∧1)/(0.4∧1)***	(1∧0.8)/(0.4∧1)	(1∧0.6)/(0.4∧1)	(1∧0.4)/(0.4∧1)	(1∧0.2)/(0.4∧1)
(0.8∧0.8)/(0.6∧1)	(0.8∧1)/(0.6∧1)	(0.8∧0.8)/(0.6∧1)	(0.8∧0.6)/(0.6∧1)	(0.8∧0.4)/(0.6∧1)	(0.8∧0.2)/(0.6∧1)
(0.6∧0.8)/(0.8∧1)	(0.6∧1)/(0.8∧1)	(0.6∧0.8)/(0.8∧1)	(0.6∧0.6)/(0.8∧1)	(0.6∧0.4)/(0.8∧1)	(0.6∧0.2)/(0.8∧1)
(0.4∧0.8)/(1∧1)	(0.4∧1)/(1∧1)	(0.4∧0.8)/(1∧1)	(0.4∧0.6)/(1∧1)	(0.4∧0.4)/(1∧1)	(0.4∧0.2)/(1∧1)

=

0.6/0	0.6/0	0.6/0	0.6/0	0.4/0	0.2/0
0.8/0.2	0.8/0.2	0.8/0.2	0.6/0.2	0.4/0.2	0.2/0.2
0.8/0.4	***1/0.4***	0.8/0.4	0.6/0.4	0.4/0.4	0.2/0.4
0.8/0.6	0.8/0.6	0.8/0.6	0.6/0.6	0.4/0.6	0.2/0.6
0.6/0.8	0.6/0.8	0.6/0.8	0.6/0.8	0.4/0.8	0.2/0.8
0.4/1	0.4/1	0.4/1	0.4/1	0.4/1	0.2/1

Thus, a system *current output*

Q' = 0.6/0 + 0.8/0.2 + ***1/0.4*** + 0.8/0.6 + 0.6/0.8 + 0.4/1 $\Rightarrow \mu_{Q'}(x)$ = ***middle***

Example 8.

Here is a case of *Semantically Inversive* fuzzy grades for input ***"high"*** and output ***"low"***.

$\mu_P(x)$ = ***high*** = 0/0 + 0.2/0.2 + 0.4/0.4 + 0.6/0.6 + 0.8/0.8 + ***1/1***

$\mu_Q(x)$ = ***low*** = ***1/0*** + 0.8/.2 + 0.6/0.4 +0.4/0.6 +0.2/0.8 +0/1

$R(u,w)=\left(\mu_P(x)\to\mu_Q(x)\right)=$

0/1	0/0	0/0	0/0	0/0	0/0
0.2/0	***0.2/1***	0.2/0.12	0.2/08	0.2/04	0/0
0.4/0	0.4/0.12	***0.4/1***	0.4/0.16	0.2/08	0/0
0.6/0	0.6/08	0.6/0.16	***0.4/1***	0.2/0.12	0/0
0.8/0	0.8/04	0.6/08	0.4/0.12	***0.2/1***	0/0
1/0	0.8/0	0.6/0	0.4/0	0.2/0	***0/1***

The *major diagonal* of a matrix $R(u,w)$ is $\forall i,j\in\left[1,CardJ_X\right]$

$R_{u_i=w_j}\left(u_i,w_j\right)$ = 0/1 + 0.2/1 + ***0.4/1 + 0.4/1***+ 0.2/1 + 0/1

And let a system *current input* P' from (3.24) to be

$\mu_{P'}(x)$ = ***middle*** = 0.6/0 + 0.8/0.2 + ***1/0.4*** + 0.8/0.6 + 0.6/0.8 + 0.4/1

Again, the proposed version of *Sup-Star Composition* would be

$R(w)=R(u)\sqcap R_{u_i=w_j}\left(u_i,w_j\right)=\mu_{P'}(x)\sqcap R_{u_i=w_j}\left(u_i,w_j\right)=$

$(0.6\wedge0)/(0\wedge1)$	$(0.6\wedge0.2)/(0\wedge1)$	$(0.6\wedge0.4)/(0\wedge1)$	$(0.6\wedge0.4)/(0\wedge1)$	$(0.6\wedge0.2)/(0\wedge1)$	$(0.6\wedge0)/(0\wedge1)$
$(0.8\wedge0)/(0.2\wedge1)$	$(0.8\wedge0.2)/(0.2\wedge1)$	$(0.8\wedge0.4)/(0.2\wedge1)$	$(0.8\wedge0.4)/(0.2\wedge1)$	$(0.8\wedge0.2)/(0.2\wedge1)$	$(0.8\wedge0)/(0.2\wedge1)$
$(1\wedge0)/(0.4\wedge1)$	$(1\wedge0.2)/(0.4\wedge1)$	$(1\wedge0.4)/(0.4\wedge1)$	$(1\wedge0.4)/(0.4\wedge1)$	$(1\wedge0.2)/(0.4\wedge1)$	$(1\wedge0)/(0.4\wedge1)$
$(0.8\wedge0)/(0.6\wedge1)$	$(0.8\wedge0.2)/(0.6\wedge1)$	$(0.8\wedge0.4)/(0.6\wedge1)$	$(0.8\wedge0.4)/(0.6\wedge1)$	$(0.8\wedge0.2)/(0.6\wedge1)$	$(0.8\wedge0)/(0.6\wedge1)$
$(0.6\wedge0)/(0.8\wedge1)$	$(0.6\wedge0.2)/(0.8\wedge1)$	$(0.6\wedge0.4)/(0.8\wedge1)$	$(0.6\wedge0.4)/(0.8\wedge1)$	$(0.6\wedge0.2)/(0.8\wedge1)$	$(0.6\wedge0)/(0.8\wedge1)$
$(0.4\wedge0)/(1\wedge1)$	$(0.4\wedge0.2)/(1\wedge1)$	$(0.4\wedge0.4)/(1\wedge1)$	$(0.4\wedge0.4)/(1\wedge1)$	$(0.4\wedge0.2)/(1\wedge1)$	$(0.4\wedge0)/(1\wedge1)$

=

0/0	0.2/0	0.4/0	0.4/0	0.2/0	0/0
0/0.2	0.2/0.2	0.4/0.2	0.4/0.2	0.2/0.2	0/0.2
0/0.4	0.2/0.4	0.4/0.4	0.4/0.4	0.2/0.4	0/0.4
0/0.6	0.2/0.6	0.4/0.6	0.4/0.6	0.2/0.6	0/0.6
0/0.8	0.2/0.8	0.4/0.8	0.4/0.8	0.2/0.8	0/0.8
0/1	0.2/1	0.4/1	0.4/1	0.2/1	0/1

Hence, a system *current output*

Q' = 0.4/0 + 0.4/0.2 + 0.4/0.4 + 0.4/0.6 + 0.4/0.8 + 0.4/1 $\Rightarrow \mu_{Q'}(x)$ = ***un-known!!!***

Thus, *Semantically Inversive* fuzzy grades *shouldn't be used* for *fuzzy relationship matrix* build. Using them leads to *fruitless* results of *type-2 FLS inference*.

3.7. Type-2 FLS: Streamlined Inference

3.7.1. Knowledge Base

Remind that we still consider a type-2 *FLS* with *K* inputs, $x_1\in X_1,x_2\in X_2,\cdots,x_K\in X_K$, and one output $y\in Y$. Presumably this *FLS* has *M* rules, where the *m*-th rule has the form (3.1). Each rule represents a *type-2* fuzzy relation between the input space $X_1 \times X_2 \times \cdots \times X_K$ and the output space Y of the *FLS*. And again, when an input x' is applied, the *composition* of the fuzzy set $\widetilde{X'}$, to which x' takes place and the rule R^m is found by using the extended *sup-star composition* (3.2). And also, the *implication MF* from (3.2) for each *m*-th rule is presented in in (3.3) form.

Given that in (3.13) we define the *lower* fuzzy grade $\mu_{\widetilde{P^m}}^{lower}(x)$, then from (L3.1) of *Lemma 3* we obtain for *K* inputs and *m* rules of *FLS* the following

$$\prod_{j=1,K} \mu_{\widetilde{P_j^m}}(x) = \prod_{j=1,K} \sum_{i=1,CardJ_X} \frac{f_j^m(u_i)}{u_i} = \mu_{\widetilde{P^m}}^{lower}(x), \forall m \in [1, M] \tag{3.30}$$

From (3.7) given (3.30) we are getting for $\forall m \in [1, M]$

$$\begin{aligned} &\mu_{\widetilde{P_1^m} \times \widetilde{P_2^m} \times \cdots \times \widetilde{P_K^m} \to \tilde{Q}^m}(x, y) \\ &= \prod_{j=1,K} \mu_{\widetilde{P_j^m}}(x) \to \mu_{\widetilde{Q^m}}(y) = \mu_{\widetilde{P^m}}^{lower}(x) \to \mu_{\widetilde{Q^m}}(y) \end{aligned} \tag{3.31}$$

Given that

$$\mu_{\widetilde{P^m}}^{lower}(x) = \sum_{i=1,CardJ_X} f^m\left(u_i^{lower}\right) / u_i^{lower}$$

$$\mu_{\widetilde{Q^m}}(y) = \sum_{j=1,CardJ_X} g^m\left(w_j\right) / w_j$$

We obtain $\forall m \in [1, M]$

$$\begin{aligned} \mu_{\widetilde{P^m}}^{lower}(x) \to \mu_{\widetilde{Q^m}}(y) &= \sum_{i,j} \left(f^m\left(u_i^{lower}\right) \wedge g^m\left(w_j\right) \right) / \left(u_i^{lower} \to w_j \right) \\ &= \sum_{u_i^{lower} < w_j} \frac{f^m\left(u_i^{lower}\right) \wedge g^m\left(w_j\right)}{\left(1 - w_j\right) \cdot u_i^{lower}} \\ &\quad + \sum_{u_i^{lower} = w_j} \frac{f^m\left(u_i^{lower}\right) \wedge g^m\left(w_j\right)}{1} \\ &\quad + \sum_{u_i^{lower} > w_j} \frac{f^m\left(u_i^{lower}\right) \wedge g^m\left(w_j\right)}{\left(1 - u_i^{lower}\right) \cdot w_j}. \end{aligned} \tag{3.32}$$

From (3.17) and **(**3.32**)** for each ***m*** rule

$$\begin{aligned} R^m(u, w) &= \mu_{\widetilde{P^m}}^{lower}(x) \\ &\to \mu_{\widetilde{Q^m}}(y) = R^m_{u_i^{lower} < w_j}\left(u_i^{lower}, w_j\right) + R^m_{u_i^{lower} = w_j}\left(u_i^{lower}, w_j\right) \\ &\quad + R^m_{u_i^{lower} > w_j}\left(u_i^{lower}, w_j\right) \end{aligned} \tag{3.33}$$

Now from (3.32) for each *m* rule and $\forall i, j \in [1, CardJ_X]$ we introduce

$$h^m\left(v_{ij}\right) = f^m\left(u_i^{lower}\right) \wedge g^m\left(w_j\right) \tag{3.34}$$

From (3.33) and (3.34) we obtain for each *m* rule

$$R^m(u, w) = \sum_{\forall i, j \in [1, CardJ_X]} h^m\left(v_{ij}\right) / v_{ij} \tag{3.35}$$

Hence, for $\forall i, j \in [1, CardJ_X]$, whereas for values $u_i^{lower}, w_j \in J_X$ from (3.35) we have the following

$$\begin{cases} v_{ij} = \left[\left(1 - w_j\right) \cdot u_i^{lower} \right], & u_i^{lower} < w_j; \\ v_{ij} = 1, & u_i^{lower} = w_j; \\ v_{ij} = \left[\left(1 - u_i^{lower}\right) \cdot w_j \right], & u_i^{lower} > w_j. \end{cases} \tag{3.36}$$

From (3.36) given the result of *Corollary 2*

$$\begin{cases} v_{ij} \le 0.25, & u_i^{lower} < w_j; \\ v_{ij} = 1, & u_i^{lower} = w_j; \\ v_{ij} \le 0.25, & u_i^{lower} > w_j. \end{cases} \tag{3.37}$$

Given (3.37) and from the results of *Lemma 7* we obtain for each *m* rule its fuzzy relationship matrix *major diagonals*

$$R^m(u,w) = \sum_{\forall k \in [1, CardJ_X]} h^m(v_k)/1 \tag{3.38}$$

The *knowledge base* for *m* rules of *FLS* has to be represented as a *superposition* of all single rules (3.38).

$$\begin{aligned} R^{sup}(u,w) &= \coprod_{\forall m \in [1,M]} R^m(u,w) = \coprod_{\forall m \in [1,M]} \sum_{\forall k \in [1, CardJ_X]} h^m(v_k)/1 \\ &= \bigvee_{\forall m \in [1,M]} \sum_{\forall k \in [1, CardJ_X]} h^m(v_k)/1 = \sum_{\forall k \in [1, CardJ_X]} h^{sup}(v_k)/1 \end{aligned} \tag{3.39}$$

Note. The *usability* of *FS* $R^{sup}(u,w)$ for it use in sup-max composition depends on the question of whether *FS*s for each *m*-th rule $h^m(v_k)$ are normal, unimodal and more or less semantically similar.

3.7.2. Decision Making

When an input x' is applied with *K* inputs, $x_1 \in X'_1, x_2 \in X'_2, \cdots, x_K \in X'_K$. The input space $X'_1 \times X'_2 \times \cdots \times X'_K$ is presented with the *fuzzy grade*

$$\mu_{\tilde{P}'}(x) = \mu_{\tilde{P}'_1 \times \tilde{P}'_2 \times \cdots \times \tilde{P}'_K}(x) = \prod_{j=1,K} \mu_{\tilde{P}'_j}(x) = \mu_{\tilde{P}'}^{lower}(x) \tag{3.40}$$

Given (3.39) and (3.40), the correspondent output space Y' will be obtained by using streamlined *sup-star composition* as the following *fuzzy grade*

$$\begin{aligned} \mu_{\tilde{Q}'}(y) &= \mu_{\tilde{P}'}^{lower}(x) \sqcap R^{sup}(u,w) \\ &= \sum_{\forall i \in [1, CardJ_X]} f'\left(u_i^{lower}\right)/u_i^{lower} \sqcap \sum_{\forall k \in [1, CardJ_X]} h^{sup}(v_k)/1 \end{aligned}$$

4. Conclusions

The paper argues that *inference in type-2 FLS should rely on fuzzy implication*, not t-norms, because implication better reflects semantic reasoning. The work revisits operations, algebraic properties, and rule semantics in type-2 systems. It defines *normalization* of physical values and mapping to a discrete universe. It provides formulas for constructing membership functions using singletons and linguistic terms. It proves that fuzzy grades defined via triangular secondary MFs are *convex*. The paper revises several underexplored aspects of *type-2 fuzzy sets* and *type-2 fuzzy logic systems* (*FLS*), proposing:

- a shift from *t-norm-based inference* to *fuzzy implication-based inference*. The paper argues that *t-norm-based inference* is semantically weak for type-2 rules.
- *dramatically* faster method for *join, meet*, operations on type-2 fuzzy grades (Speedup: *8.3× faster*), by providing full matrix-based definitions using Zadeh's extension principle and by showing that these operations form a *distributive lattice* for *convex* fuzzy grades.
- a new framework for analyzing *semantic similarity* and *dissimilarity* of type-2 IF-THEN rules,
- an improved formulation of *sup-star composition* for inference. It uses a *conditional implication operator* and a more symmetric *Fuzzy Conditional Rule* (*FCR*).

The study demonstrates that *standard sup-star composition* is computationally heavy. The paper proves that, under the proposed implication operator, *only the diagonal of the implication matrix matters*. This *greatly* reduces computational cost in inference. Numerous examples illustrating the theory.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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