

# The Single-Rotation Quantum Search Algorithm: Amplitude Amplification with One Oracle Call

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## Abstract

This paper presents the Single-Rotation Quantum Search Algorithm, which achieves amplitude-amplification with exactly one oracle call and one rotation operation. We prove that for unstructured search over  $N$  items, a single rotation by  $\pi/4$  radians is both necessary and sufficient to amplify the marked state probability from  $1/N$  to  $1/2$ . This represents the theoretical minimum: no algorithm can succeed with fewer than one oracle call, and no additional rotations are needed beyond the single  $\pi/4$  rotation. We provide explicit closed forms for the rotation operator with numerical values, demonstrating complete physical realizability. The algorithm is implemented by one oracle call, one diffusion call, and  $\log(N)$  reflections. For  $N = 2^n$  search spaces, the algorithm achieves success probability exactly  $1/2$  with one iteration, compared to  $O(\sqrt{N})$  iterations for Grover's algorithm.

## Keywords

Quantum Computing, Grover's Algorithm, Oracle, Amplitude Amplification, Quantum Circuits, Unitary Matrix, Single-Rotation Quantum Search

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## 1. Introduction

Quantum amplification enables quantum computers to find marked items in unstructured databases quadratically faster than classical computers [1]. Standard Grover's algorithm requires  $O(\sqrt{N})$  iterations [1], while the Exponential Speedup Algorithm reduces this to  $O(\log N)$  iterations [2]. A natural question arises: What is the absolute minimum number of iterations needed? This paper answers this question definitively: exactly one iteration suffices.

**Main Result:** For unstructured search over  $N$  items with one marked item, a

single rotation by  $\pi/4$  radians amplifies the marked state probability from  $1/N$  (initial uniform superposition) to exactly  $1/2$  (measurement threshold), achieving optimal amplitude-amplification with minimal iterations.

### 1.1. Prior Work

#### Standard Grover's Algorithm [1]:

- Iterations:  $K \approx (\pi/4)\sqrt{N}$
- Fixed operator  $G$  applied repeatedly
- Arithmetic series amplitude growth:  $r(k) \approx 2k + 1$

#### Exponential Speedup Algorithm [2]:

- Iterations:  $K = n/2 = (1/2) \log_2 N$  (for  $\beta = 2$ )
- Iteration-dependent operators  $U_0, U_1, \dots, U_{K-1}$
- Geometric series amplitude growth:  $r(k) = \beta^k$

**Single-Rotation Quantum Search:** The Single-Rotation Quantum Search circuit was first constructed by the author in 2024 [3] for a special case where only one qubit in the marked state is 1. This paper works for general cases and has many new results than that of [3].

#### This Work (Single-Rotation Algorithm):

- Iterations:  $K = 1$ ,
- Single rotation:  $\Delta\theta = \pi/4$ ,
- Direct amplification:  $\theta_0 \approx 0 \rightarrow \theta_1 = \pi/4$ .

**Relationship to Exponential Speedup Algorithm:** The Single-Rotation Algorithm is the limiting case of the Exponential Speedup Algorithm [2] as  $\beta \rightarrow \sqrt{N}$ :

- As  $\beta$  increases, fewer iterations are needed,
- In the limit  $\beta \rightarrow \sqrt{N}$ , exactly one iteration suffices,
- The single rotation  $\Delta\theta = \pi/4$  directly achieves the target angle.

### 1.2. Main Contributions

This paper establishes:

- 1) **Theoretical Minimum:** One oracle call plus one  $\pi/4$  rotation is necessary and sufficient for amplitude amplification (Section 3).
- 2) **Explicit Numerical Matrices:** Complete  $N \times N$  unitary matrix construction with closed-form numerical values for the  $\pi/4$  rotation (Section 5).
- 3) **Physical Realizability:** The rotation operator is unitary and can be implemented with  $\log(N)$  steps (Section 6).
- 4) **Implementation:** The  $\pi/4$  rotation is implemented by one oracle call, one diffusion call and  $\log(N)$  reflections (Sections 7 and 8).
- 5) **Lower Bounds:** The standard lower bounds for unstructured quantum search still apply (Section 9).

This work provides a possible quantum search algorithm—a single  $\pi/4$  rotation.

### 1.3. Notation

Throughout this paper:

- $n$  Number of qubits,
- $N = 2^n$  Search space size,
- $K$  Iteration count,
- $|\alpha\rangle$  the single marked state,
- $|\beta\rangle$  uniform superposition of  $N - 1$  unmarked states,
- $a_1$  Amplitude of marked state (assumed to be state  $|\alpha\rangle$ ),
- $a_0$  Amplitude of each unmarked state,
- $\theta$  Angle in geometric representation, relative to  $|\beta\rangle$ ,
- $\Delta\theta$  Rotation angle implemented by  $U$ .

Section 1 provides motivation and context. Section 2 compares rotation vs. inversion-about-average. Section 3 proves that one rotation by  $\pi/4$  is necessary and sufficient. Section 4 analyzes the variations. Section 5 constructs explicit  $N \times N$  unitary matrices with numerical values. Section 6 proves physical realizability. Section 7 introduces implementation strategy. Section 8 implements the Single-Iteration algorithm. Section 9 discusses time complexity. Section 10 discusses implicit assumptions used in the proposed algorithm. Section 11 concludes.

## 2. Grover's Algorithm: Rotation vs. "Inversion about Average"

Grover's algorithm [1] is commonly described in two ways, which might suggest two different implementation strategies. This section clarifies that only one is physically realizable.

This section provides a pedagogical comparison between two equivalent descriptions of Grover's algorithm: the rotation viewpoint and the "inversion about average" interpretation commonly found in textbooks. It is important to emphasize at the outset that both descriptions correspond to the same underlying unitary operator, the Grover diffusion operator, which has a well-defined quantum circuit implementation using standard gates.

The "inversion about average" description, introduced by Grover [1] and popularized in quantum computing textbooks [4], employs arithmetic operations on amplitude values to illustrate how the algorithm amplifies the marked state. This description involves intermediate steps where amplitudes are manipulated algebraically, for example, computing average amplitudes, reflecting about this average, and updating state coefficients. Some of these intermediate calculations may appear to violate normalization constraints when viewed in isolation, leading to potential confusion about whether the operations correspond to valid quantum states.

**Clarification on normalization:** The non-normalized intermediate arithmetic that appears in "inversion about average" explanations is a calculational tool for understanding amplitude evolution, not a claim about physical state normalization. At each step of the actual quantum algorithm, the quantum state remains properly normalized with total probability equal to one. The intermediate arithmetic serves as a pedagogical bridge for readers more comfortable with classical

reasoning about averages and reflections, but it should not be interpreted as suggesting that Grover's algorithm violates quantum mechanics or requires non-unitary operations.

The rotation viewpoint, by contrast, describes the same algorithm as a sequence of rotations in a two-dimensional subspace of the Hilbert space, spanned by the marked state  $|\alpha\rangle$  and the unmarked superposition  $|\beta\rangle$ . This geometric description makes the unitary nature of the algorithm manifest: rotations in Hilbert space are inherently unitary transformations that preserve state normalization. Both perspectives, "inversion about average" (algebraic) and rotation (geometric), describe identical amplitude evolution and lead to the same quantum circuit implementation.

The purpose of this section is not to argue that the "inversion about average" description is incorrect or that it implies non-unitarity. Rather, we aim to demonstrate that the rotation viewpoint provides certain analytical advantages, particularly for understanding the single-rotation algorithm presented in this paper. The rotation framework naturally generalizes to variable rotation angles (as in the exponential speedup algorithm) and makes the connection to optimal amplitude-amplification more transparent. Both descriptions remain valid and useful depending on the pedagogical or analytical context.

With this clarification established, we now proceed to compare the two descriptions in detail, showing their equivalence while highlighting why the rotation perspective proves advantageous for the developments in subsequent sections.

### 2.1. Description 1: Rotation in Hilbert Space

The diffusion operator acts as a reflection in the  $N$ -dimensional Hilbert space, which combined with the oracle creates a rotation in the 2D subspace  $\{|\alpha\rangle, |\beta\rangle\}$  [4]. The superposition over all  $2^n$  computational states is given in Equation (1):

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle \quad (1)$$

Note:

$$|\psi\rangle = H^{\otimes n} |0\rangle^{\otimes n} \quad (2)$$

Throughout this paper, let the marked state be  $|1\rangle$  for convenience of discussion whenever necessary, assumed without loss of generality [2]. The quantum state can be visualized as a vector in a 2-dimensional subspace spanned by the marked state in Equation (3):

$$|\alpha\rangle = |1\rangle \quad (3)$$

and uniform superposition of unmarked states in Equation (4)

$$|\beta\rangle = \frac{1}{\sqrt{N-1}} \sum_{x=0, x \neq 1}^{N-1} |x\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq 1} |x\rangle \quad (4)$$

Mathematical form of the diffusion operator is:

$$D = 2|\psi\rangle\langle\psi| - I \quad (5)$$

where  $|\psi\rangle$  is the uniform superposition. Physical realizability requires that only unitary operators can be implemented in quantum hardware.

## 2.2. Description 2: Inversion About Average

The diffusion operator is often pedagogically described [4]-[6] as a process:

- 1) **Invert** the amplitude of the marked state (oracle)
- 2) **Compute mean** amplitude:  $\langle a \rangle = (1/N)\sum_i a_i$
- 3) **Shift coordinate system** to center at mean:  $a_i \rightarrow a_i - \langle a \rangle$
- 4) **Reflect** all amplitudes about zero:  $a_i - \langle a \rangle \rightarrow -(a_i - \langle a \rangle)$
- 5) **Shift back**:  $-(a_i - \langle a \rangle) \rightarrow 2\langle a \rangle - a_i$

**Net effect:** Each amplitude transforms as  $a_i \rightarrow \langle a \rangle - a_i$  (reflection about the mean). Inversion About Average is Not physically realizable step-by-step (see Section 2.3).

## 2.3. The Intermediate States Are Unphysical

**THEOREM 2.1** (Non-Unitarity of Intermediate Steps): The “shift” and “reflect” operations in the “inversion about average” description do not preserve quantum state normalization and therefore cannot represent physical quantum states.

**Proof by Example:** Consider  $N = 8$ , let the marked state be  $|1\rangle$ , starting from uniform superposition in Equation (1).

**Initial state:**

- All amplitudes:  $a = 1/\sqrt{8} = 0.3536$
- All probabilities:  $a^2 = 1/8 = 0.125$
- **Sum of probabilities: 1.0** ✓ (normalized)

**After oracle** (invert state 1):

- State 1 amplitude:  $-0.3536$
- Other amplitudes:  $+0.3536$
- All probabilities:  $0.125$
- **Sum of probabilities: 1.0** ✓ (normalized-oracle is unitary)

**After shifting** (center at mean =  $0.2652$ ):

- State 1:  $-0.3536 - 0.2652 = -0.6187 \rightarrow$  probability =  $0.3828$
- Other states:  $+0.3536 - 0.2652 = +0.0884 \rightarrow$  probability =  $0.0078$  each
- **Sum of probabilities:  $0.3828 + 7(0.0078) = 0.4375$**  (NOT normalized!)

**After reflecting:**

- State 1:  $+0.6187 \rightarrow$  probability =  $0.3828$
- Other states:  $-0.0884 \rightarrow$  probability =  $0.0078$  each
- **Sum of probabilities:  $0.4375$**  (still not normalized!)

**After shifting back (add mean):**

- State 1:  $+0.6187 + 0.2652 = 0.8839 \rightarrow$  probability =  $0.7813$
- Other states:  $-0.0884 + 0.2652 = 0.1768 \rightarrow$  probability =  $0.0313$  each
- **Sum of probabilities:  $0.7813 + 7(0.0313) = 1.0$**  ✓ (normalized again)

**Conclusion:** The intermediate states after “shifting” and “reflecting” have  $\sum |a_i|^2 = 0.4375 \neq 1$ , violating the fundamental requirement for quantum states.

These are unphysical mathematical intermediates, not realizable quantum states. **Table 1** lists sums of probabilities.  $\square$

**Table 1.** Normalization throughout inversion about average process.

Step	Sum of Probabilities	Physical State?
Initial superposition	1.0	✓ YES
After oracle	1.0	✓ YES
After shifting	0.4375	✗ NO
After reflecting	0.4375	✗ NO
After shifting back	1.0	✓ YES

## 2.4. Only Rotation Is Physically Realizable

**THEOREM 2.2** (Single Implementation Path): Both Grover’s algorithm and the Single-Rotation Algorithm (to be proposed in this paper) have exactly one physically realizable implementation: rotation in Hilbert space.

**Proof: For Grover’s algorithm:**

- Description 1 (rotation): ✓ Unitary operator, preserves normalization
- Description 2 (inversion about average): ✗ Intermediate steps are unphysical (Theorem 2.1)
- **Conclusion:** Only rotation is implementable

**For Single-Rotation Algorithm:**

- Single  $\pi/4$  rotation: ✓ Unitary operator, preserves normalization throughout
- **Conclusion:** Rotation is the only option

Therefore, both algorithms have exactly one implementation pathway: unitary rotation in the  $\{|a\rangle, |\beta\rangle\}$  subspace.  $\square$

## 2.5. Implementation Requirements Are Identical

**COROLLARY 2.3** (Equal Implementation Complexity): Grover’s algorithm and the Single-Rotation Algorithm have identical implementation requirements.

**Both require:**

- 1) **Oracle access:** To identify the marked state  $|a\rangle$ 
  - Grover: Used in every iteration ( $K \approx \sqrt{N}$  times)
  - Single-Rotation: Used once

2) **Rotation in  $\{|a\rangle, |\beta\rangle\}$  subspace:**

- Grover: Small angle  $\Delta\theta$  [2], repeated  $K$  times
- Single-Rotation: Large angle  $\Delta\theta = \pi/4$ , once

No implementation advantage exists for either algorithm in terms of the fundamental operations required.  $\square$

## 2.6. Refuting the “Easier Implementation” Argument

**THEOREM 2.4** (Exponential Time Complexity): The “inversion about average”

has an exponential Time Complexity.

**Proof.** The time complexities are:

- 1) **Invert** the amplitude of the marked state (oracle): NA
- 2) **Compute mean**-amplitude:  $\langle a \rangle = (1/N) \sum_i a_i$ ,  $\mathcal{O}(N)$
- 3) **Shift coordinate system** to center at mean:  $a_i \rightarrow a_i - \langle a \rangle$ ,  $\mathcal{O}(N)$
- 4) **Reflect** all amplitudes about zero:  $a_i - \langle a \rangle \rightarrow -(a_i - \langle a \rangle)$ ,  $\mathcal{O}(N)$
- 5) **Shift back**:  $-(a_i - \langle a \rangle) \rightarrow 2\langle a \rangle - a_i$ ,  $\mathcal{O}(N)$

Therefore,  $T = \mathcal{O}(N) = \mathcal{O}(2^n)$ .  $\square$

**Common Misconception:** “Grover’s algorithm is easier to implement because the ‘inversion about average’ provides a simple step-by-step recipe: compute mean, shift, reflect, shift back. The Single-Rotation Algorithm requires identifying a direction in Hilbert space by oracle,  $|a\rangle$ , ‘constructing’ of  $|\beta\rangle$ , and rotating  $|\beta\rangle$  toward  $|a\rangle$ , which is more complex.”

**Refutation:** This argument is **invalid** because:

1) **The “inversion about average” is not implementable step-by-step** (Theorem 2.1)

- The intermediate states are unphysical
- Only the overall unitary operator D can be implemented
- Exponential complexity

2) **Both algorithms require the same fundamental operation:** rotation in the  $\{|a\rangle, |\beta\rangle\}$  subspace (Theorem 2.2), geometrically,

- Both need oracle to identify  $|a\rangle$
- Both need to “construct”  $|\beta\rangle$
- Both need to rotate toward  $|a\rangle$  in this 2D subspace

3) **The Single-Rotation Algorithm is simpler:**

- Grover:  $K$  iterations of rotation
- Single-Rotation: 1 iteration of rotation
- Both use the same per-iteration mechanics

**Conclusion:** Neither algorithm has an “ease of implementation” advantage. The Single-Rotation Algorithm is strictly simpler, requiring fewer iterations of the same fundamental operation.  $\square$

## 2.7. Summary

**Key Results:**

- **Theorem 2.1:** The “inversion about average” intermediate steps are unphysical
- **Theorem 2.2:** Both algorithms have only one implementation: rotation in Hilbert space
- **Corollary 2.3:** Implementation requirements are identical
- **Theorem 2.4:** The “inversion about average” is exponential
- **Refutation:** No “easier implementation” advantage exists for Grover’s algorithm

**The playing field is level:** Both algorithms require unitary rotation in a 2D subspace. The Single-Rotation Algorithm simply requires fewer iterations of the

same operation. This establishes that the comparison between algorithms reduces purely to iteration count and total complexity, with no advantage in fundamental implementation difficulty for either approach.

### 3. The Single-Rotation Algorithm

This section presents the complete Single Rotation algorithm or Single Iteration algorithm.

#### 3.1. Algorithm Specification

**ALGORITHM 1** (Single-Rotation Quantum Search):

**Input:**

- Search space size  $N = 2^n$
- Oracle  $O$  that marks the target state

**Output:**

- Marked state with probability  $1/2$

**Steps:**

- 1) **Initialize:** Create uniform superposition in Equation (1)
- 2) **Oracle:** oracle,  $O = I - 2|\alpha\rangle\langle\alpha|$ , identifies marked state,  $|\alpha\rangle$
- 3) **Apply Single Rotation:** Rotate by  $\pi/4$  from  $|\beta\rangle$  to  $|\alpha\rangle$  in  $\{|\alpha\rangle, |\beta\rangle\}$  plane

$$|\psi_1\rangle = U(\pi/4)O|\psi_0\rangle \quad (6)$$

where  $U(\pi/4)$  is the rotation operator (defined in Section 6)

- 4) **Measure:** Measure in computational basis
- 5) **Verify:** Check if measured state is marked
- 6) **Repeat if necessary:** If measurement fails, repeat from Step 1. Expected attempts:  $\leq 2$ .

#### 3.2. Geometric Framework

Following [4], we work in the 2-dimensional subspace spanned by  $\{|\alpha\rangle, |\beta\rangle\}$ . The marked state is  $|\alpha\rangle$ . The unmarked superposition is given in Equation (4). Any state at iteration  $k$  can be written in Equation (7):

$$\begin{aligned} |\psi_k\rangle &= a_1(k)|1\rangle + a_0(k)\sqrt{N-1}\frac{1}{\sqrt{N-1}}\sum_{x=0, x \neq 1}^{N-1} |x\rangle \\ &= a_1(k)|\alpha\rangle + a_0(k)\sqrt{N-1}|\beta\rangle \end{aligned} \quad (7)$$

Equation (7) can be written in Equation (8):

$$|\psi_k\rangle = \sin(\theta_k)|\alpha\rangle + \cos(\theta_k)|\beta\rangle \quad (8)$$

where  $\theta_k$  is the angle from the  $|\beta\rangle$  axis at iteration  $k$  and the rotation is from  $|\beta\rangle$  axis to  $|\alpha\rangle$  axis. Amplitude of the marked state is given in Equation (9):

$$\sin(\theta_k) = a_1(k) \quad (9)$$

The amplitude of unmarked superposition is given in Equation (10):

$$\cos(\theta_k) = a_0(k) \cdot \sqrt{N-1} \quad (10)$$

Initial angle ( $k = 0$ ), from the initial superposition in Equation (1), is given in Equation (11):

$$\theta_0 = \arcsin\left(1/\sqrt{N}\right) \tag{11}$$

Target angle ( $k = 1$ ) is given in Equation (12):

$$\theta_1 = \pi/4 \tag{12}$$

Required rotation is given in Equation (13):

$$\Delta\theta = \theta_1 - \theta_0 = \pi/4 - \arcsin\left(1/\sqrt{N}\right) \tag{13}$$

### 3.3. Key Theorem: Single Rotation Suffices

**THEOREM 3.1** (Single Rotation Optimality): For unstructured search over  $N$  items with one marked item, a single rotation by angle  $\Delta\theta = \pi/4 - \arcsin(1/\sqrt{N})$  is both necessary and sufficient to achieve success probability  $P(\text{marked}) = 1/2$ .

**Proof: Part 1 (Sufficiency):** A single rotation by  $\Delta\theta$  achieves the threshold.

Starting from  $\theta_0 = \arcsin(1/\sqrt{N})$ , after rotation by  $\Delta\theta$ :

$$\theta_1 = \theta_0 + \Delta\theta = \arcsin\left(1/\sqrt{N}\right) + \left[\pi/4 - \arcsin\left(1/\sqrt{N}\right)\right] = \pi/4$$

Amplitude of marked state:

$$a_1(1) = \sin(\theta_1) = \sin(\pi/4) = 1/\sqrt{2}$$

Success probability:

$$P_1 = |a_1(1)|^2 = 1/2 \quad \checkmark$$

**Part 2 (Necessity):** No rotation smaller than  $\Delta\theta$  can achieve  $P \geq 1/2$ .

For  $P(\text{marked}) \geq 1/2$ , we need  $|a_1|^2 \geq 1/2$ , which requires:

$$|\sin(\theta)| \geq 1/\sqrt{2}$$

This is satisfied when  $\theta \geq \pi/4$  (for  $0 \leq \theta \leq \pi/2$ ).

Since  $\theta_0 < \pi/4$  (for all  $N > 1$ ), we must rotate to at least  $\theta = \pi/4$ , requiring:

$$\Delta\theta \geq \pi/4 - \theta_0$$

Therefore,  $\Delta\theta = \pi/4 - \arcsin(1/\sqrt{N})$  is the minimum rotation needed.  $\square$

### 3.4. Large $N$ Approximation

**THEOREM 3.2** (Large  $N$  Limit): For large  $N$ , the required rotation angle approaches  $\pi/4$ :

$$\Delta\theta = \theta_1 - \theta_0 = \pi/4 \tag{14}$$

**Proof:** For large  $N$ :

$$\theta_0 = \arcsin\left(1/\sqrt{N}\right) \approx 1/\sqrt{N} \rightarrow 0$$

Therefore:

$$\Delta\theta = \pi/4 - \arcsin\left(1/\sqrt{N}\right) \approx \pi/4 - 0 = \pi/4 \quad \square$$

**Example 3.1:**  $\Delta\theta$  vs.  $N$  is given in **Table 2**.

**Table 2.**  $\Delta\theta$  vs.  $N$ .

$n$	$N$	$\theta_0$ (radians)	$\theta_0$ (degrees)	$\Delta\theta$ (radians)	$\Delta\theta$ (degrees)
3	8	0.3614	20.7°	0.4250	24.3°
10	1024	0.0313	1.8°	0.7540	43.2°
20	1,048,576	0.0010	0.056°	0.7844	44.9°
128	$2^{128}$	$5.4 \times 10^{-20}$	$3.1 \times 10^{-18}^\circ$	$\pi/4$	45°

### 3.5. Practical Simplification

**COROLLARY 3.3** ( $\pi/4$  Rotation for any  $N$ ): For any  $N$ , the single rotation can be approximated as exactly  $\pi/4$ .

**Proof:** From Theorem 3.1,  $\Delta\theta = \pi/4 - \arcsin(1/\sqrt{N})$  is the minimum rotation needed.  $\Delta\theta = \pi/4 \geq \pi/4 - \arcsin(1/\sqrt{N})$  is larger than required minimum rotation.  $\square$

**Practical Implication:** For all realistic applications, we can implement the rotation as exactly  $U(\pi/4)$  without computing the arcsin correction term in Equation (13).

### 3.6. Construction of $|\beta\rangle$

This subsection establishes that  $|\beta\rangle$  need not be explicitly constructed, avoiding potential exponential overhead.

**THEOREM 3.4** (Implicit  $|\beta\rangle$  Representation): The state  $|\beta\rangle$  in Equation (4) can be represented as:

$$|\beta\rangle = \frac{1}{\sqrt{N-1}} \cdot (\sqrt{N} \cdot |\psi_0\rangle - |\alpha\rangle) \quad (15)$$

where  $|\psi_0\rangle$  is the initial uniform superposition in Equation (1) and  $|\alpha\rangle$  is the marked state.

**Proof:** The initial uniform superposition in Equation (1) can be decomposed:

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} |\alpha\rangle + \frac{1}{\sqrt{N}} \sqrt{N-1} \frac{1}{\sqrt{N-1}} \sum_{x=0, x \neq \alpha}^{N-1} |x\rangle$$

By definition of Equation (4):

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} |\alpha\rangle + \frac{1}{\sqrt{N}} \sqrt{N-1} |\beta\rangle$$

Solving for  $|\beta\rangle$  will give Equation (15)  $\square$

## 4. Exploration of Various Rotation Angles

While this paper focuses on the  $\pi/4$  rotation that achieves the standard threshold probability  $P = 1/2$ , it is instructive to consider what happens with different rotation angles, particularly the  $\pi/2$  rotation that achieves maximum success proba-

bility.

**THEOREM 4.1** ( $\pi/2$  Rotation for Maximum Success): For large  $N$ , a single rotation by  $\Delta\theta = \pi/2$  achieves success probability  $P = 1$  (certainty).

**Proof:** For large  $N$ , the initial angle:

$$\theta_0 = \arcsin(1/\sqrt{N}) \approx 1/\sqrt{N} \rightarrow 0$$

After rotation by  $\Delta\theta = \pi/2$ :

$$\theta_1 = \theta_0 + \Delta\theta \approx 0 + \pi/2 = \pi/2$$

Amplitude of marked state:

$$a_1(1) = \sin(\theta_1) = \sin(\pi/2) = 1$$

Success probability:

$$P_1 = |a_1(1)|^2 = 1$$

Therefore, a single  $\pi/2$  rotation achieves certainty (100% success probability) for large  $N$ .  $\square$

**REMARK 4.1** The  $\pi/2$  rotation is not a good choice, see Section 6.

**REMARK 4.2:** This paper emphasizes the  $\pi/4$  rotation for several reasons:

1) **Standard threshold:** The probability  $P = 1/2$  is the standard threshold used in amplitude amplification literature [7].

2) **Comparison consistency:** Allows direct comparison with Grover's algorithm and other quantum search algorithms, which typically aim for  $P \geq 1/2$ .

3) **Theoretical significance:** Proves that the standard threshold can be achieved with exactly one oracle call, establishing the fundamental limit.

4) **Generalizability:** The framework extends naturally to other angles, including  $\pi/2$ .

**REMARK 4.3** (Precise  $\pi/2$  Rotation Angle): The exact rotation angle for  $P = 1$  is:

$$\Delta\theta = \pi/2 - \arcsin(1/\sqrt{N})$$

## 5. Explicit Unitary Matrix Construction

This section provides the complete  $N \times N$  unitary matrix for  $U(\pi/4)$  with explicit numerical values. This section assumes Equation (3),  $|\alpha\rangle = |1\rangle$ .

### 5.1. Rotation Operator Definition

Following [2], the rotation operator for angle  $\theta$  is:

$$U = I + (\cos\theta - 1)|\alpha\rangle\langle\alpha| + \sin\theta|\alpha\rangle\langle\beta| - \sin\theta|\beta\rangle\langle\alpha| + (\cos\theta - 1)|\beta\rangle\langle\beta| \quad (16)$$

For  $\theta = \pi/4$ , we have numerical values:

$$c = \cos(\pi/4) = 1/\sqrt{2} \approx 0.7071 \quad (17)$$

$$s = \sin(\pi/4) = 1/\sqrt{2} \approx 0.7071 \quad (18)$$

The matrix elements of the operator  $U$  are defined in Equation (19):

$$U_{ij} = \langle i|U|j \rangle \tag{19}$$

The matrix elements of  $U(\theta)$ , for  $|\alpha\rangle = |1\rangle$ , are:

$$U_{ij} = \delta_{ij} + (\cos \theta - 1) \left( \delta_{i1} \delta_{j1} + \frac{(1 - \delta_{i1})(1 - \delta_{j1})}{N - 1} \right) + \frac{\sin \theta}{\sqrt{N - 1}} (\delta_{i1} (1 - \delta_{j1}) - (1 - \delta_{i1}) \delta_{j1}) \tag{20}$$

### 5.2. Matrix Elements Formula

**THEOREM 5.1** (Matrix Elements for  $\pi/4$  Rotation): The matrix elements of  $U(\pi/4)$ , for  $|\alpha\rangle = |1\rangle$ , are:

$$U_{ij} = \delta_{ij} + \left( \frac{1}{\sqrt{2}} - 1 \right) \left( \delta_{i1} \delta_{j1} + \frac{(1 - \delta_{i1})(1 - \delta_{j1})}{N - 1} \right) + \frac{1}{\sqrt{2}\sqrt{N - 1}} (\delta_{i1} (1 - \delta_{j1}) - (1 - \delta_{i1}) \delta_{j1}) \tag{21}$$

**Proof:** Substituting  $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$  into Equation (20) will prove (21) [2]. □

**Example 5.1 Case  $N = 2$  ( $n = 1$ ):**

$$U(\pi/4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{22}$$

**Example 5.2 Case  $N = 4$  ( $n = 2$ ):**

$$U(\pi/4) = \begin{bmatrix} 0.9036 & -0.4082 & -0.0964 & -0.0964 \\ 0.4082 & 0.7071 & 0.4082 & 0.4082 \\ -0.0964 & 0.4082 & 0.9036 & -0.0964 \\ -0.0964 & 0.4082 & -0.0964 & 0.9036 \end{bmatrix}$$

**Example 5.3 Case  $N = 8$  ( $n = 3$ ):**

$$U(\pi/4) = \begin{bmatrix} 0.9515 & -0.3780 & -0.0485 & -0.0485 & -0.0485 & -0.0485 & -0.0485 & -0.0485 \\ 0.3780 & 0.7071 & 0.3780 & 0.3780 & 0.3780 & 0.3780 & 0.3780 & 0.3780 \\ -0.0485 & 0.3780 & 0.9515 & -0.0485 & -0.0485 & -0.0485 & -0.0485 & -0.0485 \\ -0.0485 & 0.3780 & -0.0485 & 0.9515 & -0.0485 & -0.0485 & -0.0485 & -0.0485 \\ -0.0485 & 0.3780 & -0.0485 & -0.0485 & 0.9515 & -0.0485 & -0.0485 & -0.0485 \\ -0.0485 & 0.3780 & -0.0485 & -0.0485 & -0.0485 & 0.9515 & -0.0485 & -0.0485 \\ -0.0485 & 0.3780 & -0.0485 & -0.0485 & -0.0485 & -0.0485 & 0.9515 & -0.0485 \\ -0.0485 & 0.3780 & -0.0485 & -0.0485 & -0.0485 & -0.0485 & -0.0485 & 0.9515 \end{bmatrix}$$

### 5.3. General Pattern

**THEOREM 5.2** (Matrix Structure): For arbitrary  $N = 2^n$ , the matrix  $U(\pi/4)$ , for  $|\alpha\rangle = |1\rangle$ , has the structure:

Row  $i = 1$  (marked state row):

$$U_{1j} = \frac{1}{\sqrt{2}} \delta_{1j} + \frac{1}{\sqrt{2}\sqrt{N-1}} (1 - \delta_{1j}) \quad (23)$$

Column  $j = 1$  (marked state column):

$$U_{i1} = \frac{1}{\sqrt{2}} \delta_{i1} + \frac{1}{\sqrt{2}\sqrt{N-1}} (1 - \delta_{i1}) \quad (24)$$

Unmarked block ( $i \neq 1, j \neq 1$ ):

$$U_{ij} = \delta_{ij} + \left( \frac{1}{\sqrt{2}} - 1 \right) \left( \frac{1}{N-1} \right) \quad i \neq 1, j \neq 1 \quad (25)$$

**Proof:** Direct substitution of  $\theta = \pi/4$  into Equation (21) will do [2].  $\square$

## 6. Single-Rotation Operators

Shende, Bullock, Markov [8] shows that arbitrary state preparation can be exponential  $O(2^n)$ . The Single-Rotation Algorithm is physically realizable with polynomial complexity (next section). This section addresses the critical  $|\beta\rangle$  construction problem and provides explicit formulas for implementation. A key insight is that  $|\beta\rangle$  can be represented implicitly as a linear combination of  $|\psi_0\rangle$  and  $|\alpha\rangle$ , see Equation (15), avoiding exponential construction overhead and enabling polynomial-time implementation. For large  $N$ ,  $|\beta\rangle$  can be approximated by  $|\psi_0\rangle$ . The single-rotation operator is not unique, as discussed in Section 4.

### 6.1. Unitarity Proof

**THEOREM 6.1** (Unitarity of  $U(\pi/4)$ ): The operator  $U(\pi/4)$  defined in Equation (16) is unitary for all  $N = 2^n$ .

**Proof 1:** From [2], the general rotation operator  $U(\theta)$  is unitary for any angle  $\theta$ . Since  $U(\pi/4)$  is a special case with  $\theta = \pi/4$ , it inherits unitarity.  $\square$

### 6.2. Implicit Construction of $|\beta\rangle$

The state  $|\beta\rangle$  serves two distinct purposes:

#### Geometric Role:

- Forms orthonormal basis  $\{|\alpha\rangle, |\beta\rangle\}$  for the 2D amplification subspace.
- Clarifies the rotation mechanism (from  $\theta_0$  to  $\theta_0 + \pi/4$ ).
- Explains amplitude growth through  $\sin(\theta)$  increase.
- Connects to Grover's geometric interpretation [4].

#### Computational Role:

- Need NOT be explicitly constructed as a quantum state.
- Represented implicitly via Theorem 3.4.
- All operations involving  $|\beta\rangle$  rewritten using  $|\psi_0\rangle$  and  $|\alpha\rangle$  in Equation (15).
- Avoids exponential construction overhead.
- Approximated by  $|\psi_0\rangle$  for large  $N$ .

The implicit representation in Equation (15) is the key to polynomial-time implementation.

**THEOREM 6.2** (Projector Expansion): The projector  $|\beta\rangle\langle\beta|$  can be expanded as:

$$|\beta\rangle\langle\beta| = \frac{1}{\sqrt{N-1}} \cdot (N \cdot |\psi_0\rangle\langle\psi_0| - \sqrt{N} \cdot |\psi_0\rangle\langle\alpha| - \sqrt{N} \cdot |\alpha\rangle\langle\psi_0| + |\alpha\rangle\langle\alpha|) \quad (26)$$

**Proof:** From Equation (15), direct expansion of  $|\beta\rangle\langle\beta|$ :

$$|\beta\rangle\langle\beta| = \frac{1}{\sqrt{N-1}} (\sqrt{N} \cdot |\psi_0\rangle - |\alpha\rangle) \cdot \frac{1}{\sqrt{N-1}} (\sqrt{N} \cdot \langle\psi_0| - \langle\alpha|). \quad \square$$

**THEOREM 6.3** (Off-Diagonal Terms): The operators  $|\alpha\rangle\langle\beta|$  and  $|\beta\rangle\langle\alpha|$  can be expanded as:

$$|\alpha\rangle\langle\beta| = \frac{1}{\sqrt{N-1}} \cdot (\sqrt{N} \cdot |\alpha\rangle\langle\psi_0| - |\alpha\rangle\langle\alpha|) \quad (27)$$

$$|\beta\rangle\langle\alpha| = \frac{1}{\sqrt{N-1}} \cdot (\sqrt{N} \cdot |\psi_0\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|) \quad (28)$$

**Proof:** Direct substitutions of Equation (15) into  $|\alpha\rangle\langle\beta|$  and  $|\beta\rangle\langle\alpha|$  complete proof.  $\square$

### 6.3. General Rotation Formula

**THEOREM 6.4** (Rotation Operator): The complete rotation operator  $U(\pi/4)$  can be expanded using only  $|\psi_0\rangle$  and  $|\alpha\rangle$ :

$$U(\theta) = I + (\cos\theta - 1)|\alpha\rangle\langle\alpha| + \sin\theta|\alpha\rangle\langle\beta| - \sin\theta|\beta\rangle\langle\alpha| + (\cos\theta - 1)|\beta\rangle\langle\beta| \quad (29)$$

where

$|\beta\rangle\langle\beta|$  is given in Equation (26),

$|\alpha\rangle\langle\beta|$  is given in Equation (27), and

$|\beta\rangle\langle\alpha|$  is given in Equation (28).

**Proof.** Substituting Theorem 6.2 and 6.3 into the standard rotation operator formula in Equation (16) will prove the theorem.  $\square$

**THEOREM 6.5** (Rotation Operator): The complete rotation operator  $U(\pi/4)$  can be expanded using only  $|\psi_0\rangle$  and  $|\alpha\rangle$ :

$$\begin{aligned} U(\theta) = & I + (c-1)|\alpha\rangle\langle\alpha| + s \cdot \frac{1}{\sqrt{N-1}} (\sqrt{N} \cdot |\alpha\rangle\langle\psi_0| - |\alpha\rangle\langle\alpha|) \\ & - s \cdot \frac{1}{\sqrt{N-1}} (\sqrt{N} \cdot |\psi_0\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|) \\ & + (c-1) \cdot \frac{1}{N-1} (N \cdot |\psi_0\rangle\langle\psi_0| - \sqrt{N} \cdot |\psi_0\rangle\langle\alpha| - \sqrt{N} \cdot |\alpha\rangle\langle\psi_0| + |\alpha\rangle\langle\alpha|) \end{aligned} \quad (30)$$

where  $c = \cos(\theta)$  and  $s = \sin(\theta)$ .

**Proof.** Expanding  $|\beta\rangle\langle\beta|$ ,  $|\alpha\rangle\langle\beta|$  and  $|\beta\rangle\langle\alpha|$  in Theorem 6.4 will prove the theorem.  $\square$

### 6.4. Rotation Operator $\pi/4$

**THEOREM 6.6** (Rotation Operator for  $\pi/4$ ): The complete rotation operator  $U(\pi/4)$  can be expanded using only  $|\psi_0\rangle$  and  $|\alpha\rangle$ :

$$\begin{aligned}
U(\pi/4) = & I + (1/\sqrt{2} - 1) \cdot N/(N-1) \cdot |\alpha\rangle\langle\alpha| \\
& + (1/\sqrt{2} - 1) \cdot N/(N-1) \cdot |\psi_0\rangle\langle\psi_0| \\
& + \left[ -(1/\sqrt{2} - 1) \cdot \sqrt{N}/(N-1) - (1/\sqrt{2}) \cdot \sqrt{N}/\sqrt{N-1} \right] \cdot |\psi_0\rangle\langle\alpha| \\
& + \left[ -(1/\sqrt{2} - 1) \cdot \sqrt{N}/(N-1) + (1/\sqrt{2}) \cdot \sqrt{N}/\sqrt{N-1} \right] \cdot |\alpha\rangle\langle\psi_0|
\end{aligned} \tag{31}$$

**Proof.** Using  $\pi/4$ , organizing Theorem 6.5 by  $I$ ,  $|\psi_0\rangle\langle\psi_0|$ ,  $|\psi_0\rangle\langle\alpha|$ ,  $|\alpha\rangle\langle\psi_0|$ , and  $|\alpha\rangle\langle\alpha|$  will prove this theorem.  $\square$

**THEOREM 6.7** (Rotation Operator for  $\pi/4$ ): The complete rotation operator  $U(\pi/4)$  can be expanded using only  $|\psi_0\rangle$  and  $|\alpha\rangle$ :

$$U(\pi/4) = I + A \cdot |\alpha\rangle\langle\alpha| + A \cdot |\psi_0\rangle\langle\psi_0| + B \cdot |\psi_0\rangle\langle\alpha| + C \cdot |\alpha\rangle\langle\psi_0| \tag{32}$$

where:

- $A = (1/\sqrt{2} - 1) \cdot N/(N-1)$
- $B = -\sqrt{N} \cdot \left[ (1/\sqrt{2} - 1)/(N-1) + (1/\sqrt{2})/\sqrt{N-1} \right]$
- $C = -\sqrt{N} \cdot \left[ (1/\sqrt{2} - 1)/(N-1) - (1/\sqrt{2})/\sqrt{N-1} \right]$

**Proof.** Organizing Theorem 6.6 by  $A$ ,  $B$  and  $C$  will prove this theorem.  $\square$

Every term in these expansions involves only Identity  $I$ ,  $|\psi_0\rangle\langle\psi_0|$ ,  $|\psi_0\rangle\langle\alpha|$ ,  $|\alpha\rangle\langle\psi_0|$ ,  $|\alpha\rangle\langle\alpha|$ .

## 6.5. Rotation Operators for $\pi/2$

This subsection provides the explicit formula for the rotation operator of  $\pi/2$ . Theorem 6.4 and 6.5 can be applied to any angle.

**THEOREM 6.8** ( $\pi/2$  Rotation Case): For  $\theta = \pi/2$ ,

$$\begin{aligned}
U\left(\frac{\pi}{2}\right) = & I - \frac{N}{N-1} \cdot (|\alpha\rangle\langle\alpha| + |\psi_0\rangle\langle\psi_0|) \\
& + \frac{\sqrt{N}}{N-1} \cdot [1 - \sqrt{N-1}] \cdot |\psi_0\rangle\langle\alpha| \\
& + \frac{\sqrt{N}}{N-1} \cdot [1 + \sqrt{N-1}] \cdot |\alpha\rangle\langle\psi_0|
\end{aligned} \tag{33}$$

**Proof:** we have  $\cos(\pi/2) = 0$  and  $\sin(\pi/2) = 1$ , Equation (31) gives:

$$\begin{aligned}
U(\pi/2) = & I + (-1) \cdot N/(N-1) \cdot |\alpha\rangle\langle\alpha| \\
& + (-1) \cdot N/(N-1) \cdot |\psi_0\rangle\langle\psi_0| \\
& + \left[ -(-1) \cdot \sqrt{N}/(N-1) - (1) \cdot \sqrt{N}/\sqrt{N-1} \right] \cdot |\psi_0\rangle\langle\alpha| \\
& + \left[ -(-1) \cdot \sqrt{N}/(N-1) + (1) \cdot \sqrt{N}/\sqrt{N-1} \right] \cdot |\alpha\rangle\langle\psi_0|
\end{aligned}$$

Simplifying:

$$\begin{aligned}
U(\pi/2) = & I - N/(N-1) \cdot |\alpha\rangle\langle\alpha| \\
& - N/(N-1) \cdot |\psi_0\rangle\langle\psi_0| \\
& + \left[ \sqrt{N}/(N-1) - \sqrt{N}/\sqrt{N-1} \right] \cdot |\psi_0\rangle\langle\alpha| \\
& + \left[ \sqrt{N}/(N-1) + \sqrt{N}/\sqrt{N-1} \right] \cdot |\alpha\rangle\langle\psi_0|
\end{aligned}$$

For the  $|\psi_0\rangle\langle\alpha|$  coefficient:

$$\sqrt{N}/(N-1) - \sqrt{N}/\sqrt{N-1} = \sqrt{N}/(N-1) \cdot [1 - \sqrt{N-1}]$$

For the  $|\alpha\rangle\langle\psi_0|$  coefficient:

$$\sqrt{N}/(N-1) + \sqrt{N}/\sqrt{N-1} = \sqrt{N}/(N-1) \cdot [1 + \sqrt{N-1}]$$

Final form is Equation (32).  $\square$

**THEOREM 6.9** ( $\pi/2$  Rotation Case): For  $\theta = \pi/2$ ,

$$U(\pi/2) = I + A \cdot |\alpha\rangle\langle\alpha| + A \cdot |\psi_0\rangle\langle\psi_0| + B \cdot |\psi_0\rangle\langle\alpha| + C \cdot |\alpha\rangle\langle\psi_0|$$

where:

- $A = -N/(N-1)$
- $B = \sqrt{N}/(N-1) \cdot [1 - \sqrt{N-1}] = \sqrt{N}/(N-1) - N/\sqrt{N-1}$
- $C = \sqrt{N}/(N-1) \cdot [1 + \sqrt{N-1}] = \sqrt{N}/(N-1) + N/\sqrt{N-1}$

### 6.6. Selection of Angles

For  $\pi/2$ , the off-diagonal coefficients B and C grow as  $\pm\sqrt{N}$ , while for  $\pi/4$  they remain  $O(1)$ . This suggests the  $\pi/2$  rotation involves more significant amplitude transfer between  $|\alpha\rangle$  and  $|\psi_0\rangle$  components, thus, the  $\pi/2$  rotation is not a good choice.

**THEOREM 6.10** ( $\pi/4$  Rotation Coefficient Scaling): For the  $\pi/4$  rotation operator  $U(\pi/4)$  with large  $N$ , all coefficients scale as  $O(1)$ :

$$U(\pi/4) = I + A \cdot |\alpha\rangle\langle\alpha| + A \cdot |\psi_0\rangle\langle\psi_0| + B \cdot |\psi_0\rangle\langle\alpha| + C \cdot |\alpha\rangle\langle\psi_0|$$

where:

$$\begin{aligned} A &= O(1) \\ B &= O(1) \\ C &= O(1) \end{aligned}$$

**Proof:** For  $\theta = \pi/4$ , we have  $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$ . The coefficients are:

$$A = (1/\sqrt{2} - 1) \cdot N/(N-1)$$

For large  $N$ ,  $N/(N-1) \rightarrow 1$ , therefore:

$$A = (1/\sqrt{2} - 1) \cdot 1 = 1/\sqrt{2} - 1$$

Next,

$$B = -\sqrt{N} \cdot [(1/\sqrt{2} - 1)/(N-1) + (1/\sqrt{2})/\sqrt{N-1}]$$

For large  $N$ :

$$\begin{aligned} -\sqrt{N} \cdot (1/\sqrt{2} - 1)/(N-1) &\approx -(1/\sqrt{2} - 1) \cdot \sqrt{N}/N = -(1/\sqrt{2} - 1)/\sqrt{N} \rightarrow 0 \\ -\sqrt{N} \cdot (1/\sqrt{2})/\sqrt{N-1} &\approx -(1/\sqrt{2}) \cdot \sqrt{N}/\sqrt{N} = -1/\sqrt{2} \end{aligned}$$

Therefore:

$$B \rightarrow -1/\sqrt{2}$$

Next

$$C = -\sqrt{N} \cdot \left[ \frac{(1/\sqrt{2} - 1)}{(N-1)} + \frac{(1/\sqrt{2})}{\sqrt{N-1}} \right]$$

For large  $N$ :

$$\begin{aligned} &-\sqrt{N} \cdot \frac{(1/\sqrt{2} - 1)}{(N-1)} \rightarrow 0 \\ &+\sqrt{N} \cdot \frac{(1/\sqrt{2})}{\sqrt{N-1}} \approx +\frac{(1/\sqrt{2}) \cdot \sqrt{N}}{\sqrt{N}} = +1/\sqrt{2} \end{aligned}$$

Therefore:

$$C \rightarrow +1/\sqrt{2}$$

All three coefficients are bounded constants for large  $N$ .  $\square$

**THEOREM 6.11** ( $\pi/2$  Rotation Coefficient Scaling): For the  $\pi/2$  rotation operator  $U(\pi/2)$  with large  $N$ , the diagonal coefficients scale as  $O(1)$  while the off-diagonal coefficients scale as  $O(\sqrt{N})$ :

$$U(\pi/2) = I + A \cdot |\alpha\rangle\langle\alpha| + A \cdot |\psi_0\rangle\langle\psi_0| + B \cdot |\psi_0\rangle\langle\alpha| + C \cdot |\alpha\rangle\langle\psi_0|$$

where:

$$A = O(1)$$

$$B = O(\sqrt{N})$$

$$C = O(\sqrt{N})$$

**Proof:** For  $\theta = \pi/2$ , we have  $\cos(\pi/2) = 0$  and  $\sin(\pi/2) = 1$ . The coefficients are

$$A = (0-1) \cdot N/(N-1) = -N/(N-1)$$

For large  $N$ ,  $N/(N-1) \rightarrow 1$ , therefore:

$$A \rightarrow -1$$

Next,

$$B = \sqrt{N}/(N-1) \cdot [1 - \sqrt{N-1}] = \sqrt{N}/(N-1) - N/\sqrt{N-1}$$

For large  $N$ :

$$\begin{aligned} \sqrt{N}/(N-1) &\approx \sqrt{N}/N = 1/\sqrt{N} \rightarrow 0 \\ -N/\sqrt{N-1} &\approx -N/\sqrt{N} = -\sqrt{N} \end{aligned}$$

Therefore:

$$B \rightarrow -\sqrt{N}$$

Next,

$$\begin{aligned} C &= \sqrt{N}/(N-1) \cdot [1 + \sqrt{N-1}] \\ &= \sqrt{N}/(N-1) + \sqrt{N} \cdot \sqrt{N-1}/(N-1) \\ &= \sqrt{N}/(N-1) + N/\sqrt{N-1} \\ \sqrt{N}/(N-1) &\approx \sqrt{N}/N = 1/\sqrt{N} \rightarrow 0 \\ +N/\sqrt{N-1} &\approx +N/\sqrt{N} = +\sqrt{N} \end{aligned}$$

Therefore:

$$C \rightarrow +\sqrt{N}$$

The off-diagonal coefficients,  $B$  and  $C$ , grow asymptotically as  $\sqrt{N}$ , while the diagonal coefficient,  $A$ , remains bounded.  $\square$

For large-scale quantum search problems (large  $N$ ), the  $\pi/4$  rotation is preferable to the  $\pi/2$  rotation despite the  $\pi/2$  rotation achieving higher success probability. Reasons are: 1) Bounded coefficients; 2) Numerical stability: For  $N = 2^n$  with large  $n$ , the  $\pi/2$  coefficients become extremely large; 3. Potential error sensitivity: the extremely large coefficients in  $\pi/2$  rotation may introduce:

- **Numerical precision issues** in classical parameter computation.
- **Potential error amplification** if quantum gate errors scale with parameter magnitude.
- **Implementation challenges** on quantum hardware with finite control precision.

**Conclusion:** For large  $N$ , the  $\pi/4$  rotation is better due to being bound by  $O(1)$  coefficients, numerical stability, and robustness to potential implementation imperfections.

Both  $\pi/4$  and  $\pi/2$  rotations have the same gate complexity structure.

### 6.7. Simplified Formula (Large $N$ )

**THEOREM 6.12 (Universal Simplified Form for Practical Applications):** For all practical quantum search applications with  $n \geq 7$  ( $N \geq 128$ ), the  $\pi/4$  rotation operator with less than 5% error is:

$$U(\pi/4) = I + (1/\sqrt{2} - 1) \cdot |\alpha\rangle\langle\alpha| + (1/\sqrt{2} - 1) \cdot |\psi_0\rangle\langle\psi_0| - (1/\sqrt{2}) \cdot |\psi_0\rangle\langle\alpha| + (1/\sqrt{2}) \cdot |\alpha\rangle\langle\psi_0| \tag{34}$$

**Proof.** From **Table 3**, for  $n \geq 7$  or  $N \geq 128$ , less than 5% error can be achieved.

**Table 3.** Exact vs. simplified coefficients.

$n$	$A$ (exact)	$A$ (simplified)	$B$ (exact)	$B$ (simplified)	$C$ (exact)	$C$ (simplified)	Max Error
1	-0.5858	-0.2929	-0.5858	-0.7071	+1.4142	+0.7071	100%
2	-0.3905	-0.2929	-0.6213	-0.7071	+1.0116	+0.7071	43%
3	-0.3347	-0.2929	-0.6377	-0.7071	+0.8741	+0.7071	24%
4	-0.3125	-0.2929	-0.6601	-0.7071	+0.7946	+0.7071	15%
5	-0.3023	-0.2929	-0.6764	-0.7071	+0.7567	+0.7071	8%
6	-0.2975	-0.2929	-0.6885	-0.7071	+0.7342	+0.7071	5%
<b>7</b>	<b>-0.2952</b>	<b>-0.2929</b>	<b>-0.6844</b>	<b>-0.7071</b>	<b>+0.7363</b>	<b>+0.7071</b>	<b>4%</b>
8	-0.2940	-0.2929	-0.6956	-0.7071	+0.7217	+0.7071	2%
128	-0.2929	-0.2929	-0.7071	-0.7071	+0.7071	+0.7071	<0.001%

## 7. Implementation Strategy

This section introduces the basic strategy; the next section implements the strategy.

The basic strategy is  $\log(N)$  reflections with doubling reflection angle in each subsequent step. The state can be written as:

$$|\psi_k\rangle = \sin(\theta_k)|\alpha\rangle + \cos(\theta_k)|\beta\rangle \tag{35}$$

where  $\theta_k$  is the angle from the  $|\beta\rangle$  axis (horizontal axis) at iteration,  $k$ , and the rotation is from  $|\beta\rangle$  axis to  $|\alpha\rangle$  axis (vertical axis). The amplitude of the marked state is  $\sin(\theta_k)$ . The state evolution can be written in Equation (36):

$$|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, \dots, |\psi_K\rangle \tag{36}$$

**Step 0 The initial state.** The initial state is

$$|\psi_0\rangle = \frac{1}{\sqrt{N}}|\alpha\rangle + \frac{1}{\sqrt{N}}\sqrt{N-1}|\beta\rangle \tag{37}$$

From Equation (35) and (37), for large  $N$ ,

$$\sin(\theta_0) = \theta_0 = \frac{1}{\sqrt{N}} \tag{38}$$

The angle between the initial uniform superposition and  $|\beta\rangle$  axis (horizontal axis) is  $\theta_0$ .

**Step 1. The first reflection Axis.**

The first reflection axis is obtained by the Grover model, *i.e.*, applying the oracle operator and diffusion operator. The oracle operator,  $O$ , is,

$$O = I - 2|\alpha\rangle\langle\alpha| \tag{39}$$

The diffusion operator is

$$D = 2|\psi\rangle\langle\psi| - I \tag{40}$$

where  $|\psi\rangle$  is the uniform superposition. Apply oracle:

$$O|\psi_0\rangle = (I - 2|\alpha\rangle\langle\alpha|)|\psi_0\rangle = |\psi_0\rangle - \frac{2}{\sqrt{N}}|\alpha\rangle \tag{41}$$

The diffusion operator has identities:

$$D|\psi_0\rangle = |\psi_0\rangle \tag{42}$$

$$D|\alpha\rangle = (2|\psi_0\rangle\langle\psi_0| - I)|\alpha\rangle = \frac{2}{\sqrt{N}}|\psi_0\rangle - |\alpha\rangle \tag{43}$$

Applying both oracle and diffusion,

$$|\psi_1\rangle = DO|\psi_0\rangle = D\left(|\psi_0\rangle - \frac{2}{\sqrt{N}}|\alpha\rangle\right) = \left(1 - \frac{4}{N}\right)|\psi_0\rangle + \frac{2}{\sqrt{N}}|\alpha\rangle \tag{44}$$

As an approximation, we can drop  $O(1/N)$  term,

$$|\psi_1\rangle = DO|\psi_0\rangle \approx |\psi_0\rangle + \frac{2}{\sqrt{N}}|\alpha\rangle \tag{45}$$

Equation (44) or (45) indicated that the rotation angle in this step is  $2/\sqrt{N}$ . This is the first reflection axis. After this step,  $|\psi_1\rangle$  is  $2\theta_0$  above  $|\psi_0\rangle$  and  $3\theta_0$  above the  $|\beta\rangle$  axis.

**Step 2. The first reflection.**

Now,  $|\psi_0\rangle$  is reflected across  $|\psi_1\rangle$  to generate  $|\psi_2\rangle$ . The angle between  $|\psi_0\rangle$  and  $|\psi_1\rangle$  is  $2\theta_0$ . After the reflection,  $|\psi_2\rangle$  is  $4\theta_0$  above  $|\psi_0\rangle$  and  $(4 + 1)\theta_0$  above the  $|\beta\rangle$  axis.

**Step 3. The next reflection.**

Now,  $|\psi_0\rangle$  is reflected across  $|\psi_2\rangle$  to generate  $|\psi_3\rangle$ . The angle between  $|\psi_0\rangle$  and  $|\psi_2\rangle$  is  $4\theta_0$ . After the reflection,  $|\psi_3\rangle$  is  $8\theta_0$  above  $|\psi_0\rangle$  and  $(8 + 1)\theta_0$  above the  $|\beta\rangle$  axis.

Rotation angles double in each step, this is a typical exponential growth, resulting  $O(\log N)$  time complexity.

## 8. Implementation for $\pi/4$ Rotations

The operator  $R(\pi/4)$  can be implemented in  $\log(N)$  iterations. After oracle call and diffusion call,  $\log(N)$  reflections will produce a geometric series for rotation angles.

### 8.1. Cartan-Dieudonné Theorem in Mathematics

As an example of Cartan-Dieudonné theorem in Mathematics, consider the usual 2-D  $x$ - $y$  coordinate system, let:

- $B = (1, 0) \rightarrow$  pointing along the  $x$ -axis
- $\alpha = (\cos\theta, \sin\theta) \rightarrow$  rotated by angle  $\theta$

Furthermore, let  $\theta = 30^\circ$ , so:

- $\alpha = (\cos 30^\circ, \sin 30^\circ)$
- $\beta = (1, 0)$

First reflection: Reflect across the line perpendicular to  $\alpha$  (this corresponds to  $R_\alpha$ ). If you start at  $\beta = (1, 0)$ , after reflecting across  $\alpha$ , you land on the other side of  $\alpha$ , symmetric with respect to  $\alpha$  ( $240^\circ$ ):

- $\alpha = (\cos 30^\circ, \sin 30^\circ)$
- $\beta = (\cos 240^\circ, \sin 240^\circ)$

Second reflection: Now reflect across the line perpendicular to  $\beta$ . Since  $\beta = (1, 0)$ , this is just Reflection across the  $y$ -axis, so  $(x, y) \rightarrow (-x, y)$ :

- $\alpha = (\cos 30^\circ, \sin 30^\circ)$
- $\beta = (-\cos 240^\circ, \sin 240^\circ)$

Note,  $\cos(2 \times 30^\circ) = -\cos 240^\circ$ ,  $\sin(2 \times 30^\circ) = \sin(240^\circ)$ , we have:

- $\alpha = (\cos 30^\circ, \sin 30^\circ)$
- $\beta = (\cos(2 \times 30^\circ), \sin(2 \times 30^\circ))$

This demonstrates “Two flips = one rotation”:

$$R_\beta R_\alpha = \text{rotation by } 2\theta$$

### 8.2. Efficient Implementation for Reflections

The rotation  $R(2\theta)$  can be implemented using two reflections of  $\theta$ . The basic ideal

is the key identity [7]:

$$R_\psi = I - 2|\psi\rangle\langle\psi| = U(I - 2|0\rangle\langle 0|)U^\dagger \quad (46)$$

Where  $U$  prepares the state:  $U|0\rangle = |\psi\rangle$ . Instead of implementing:

$$|\psi\rangle\langle\psi|$$

Which looks exponential, we do:

- 1) Uncompute  $|\psi\rangle$  to  $|0\rangle$ :  $|\psi\rangle = U|0\rangle$
- 2) Apply a simple phase flip on  $|0\rangle$ :  $|0\rangle \rightarrow -|0\rangle$
- 3) Compute back:  $U|0\rangle = |\psi\rangle$

So:

$$R_\psi = U(\text{phase flip})U^\dagger$$

Three steps for reflections are:

Step 1: Apply  $U^\dagger$

- cost ( $U^\dagger$ )

Step 2: Flip phase of  $|0\rangle$

This is  $|0^n\rangle \rightarrow -|0^n\rangle$ , implemented by:

- multi-controlled Z gate
- cost:  $O(n)$

Step 3: Apply  $U$

- cost ( $U$ )

### 8.3. Rotation from $|\beta\rangle$ to $|\alpha\rangle$ by $\theta$ : From Two Reflections to One Reflection

We will use **Cartan-Dieudonné theorem** in Mathematics, which essentially states that any rotation can be “factored” into a pair of reflections. The core rule is the **Double Angle Rule**: the angle of the resulting rotation is exactly twice the angle between the two lines (or planes) of reflections.

However, the direct application of **Cartan-Dieudonné theorem** with one rotation will not work because the cost is exponential, this problem can be resolved by multiple rotations.

**Construction of rotations for  $\theta$** : To specifically rotate a vector, we can follow these two-step reflections:

1) **First Reflection**: Choose the first line (or plane) to pass directly through the vector  $|\beta\rangle$ . When you reflect  $|\beta\rangle$  across a line that it already lies on, nothing happens—the vector stays exactly where it is.

2) **Second Reflection**: Choose the second line to be the **angle bisector** between  $|\beta\rangle$  and  $|\alpha\rangle$ . When you reflect the vector  $|\beta\rangle$  (which is still at position  $|\beta\rangle$ ) across this bisector, it will “jump” across the line and land perfectly on  $|\alpha\rangle$ .

If we denote the reflection operation as  $R_\beta, R_\alpha$ , the combined operation is:

$$R_\alpha R_\beta |\beta\rangle = |\alpha\rangle \quad (47)$$

Because the angle between  $|\beta\rangle$  and the bisector is half of the total angle between  $|\beta\rangle$  and  $|\alpha\rangle$ , the composition results in a total rotation of  $\theta$ . To find the angle bi-

sector, one must normalize them first:

$$|v\rangle = \frac{1}{\sqrt{2}} \left( \frac{|\beta\rangle}{|\beta|} + \frac{|\alpha\rangle}{|\alpha|} \right) \quad (48)$$

Standard reflection uses:

$$R_v = I - 2|v\rangle\langle v| \quad (49)$$

New rotation operator is

$$R = R_v R_\beta \quad (50)$$

In general, the state preparation,  $|v\rangle$ , in Equation (41) is NOT poly ( $n$ ). In this section, we have shown that when you reflect  $|\beta\rangle$  across a line that it already lies on, nothing happens—the vector stays exactly where it is. Now by a proper choice, two reflections above are reduced to one reflection.

### 8.4. Geometric Series for Rotation Angles

When rotating from  $|\beta\rangle$  to  $|\alpha\rangle$  by  $2\theta$  within 2-D plane,  $\{|\alpha\rangle, |\beta\rangle\}$ , a single reflection around  $\theta$ -line is enough. If geometric series for Rotation Angles is:

$$\frac{2^0}{\sqrt{N}} \frac{2^1}{\sqrt{N}} \frac{2^2}{\sqrt{N}} \cdots \frac{2^K}{\sqrt{N}} = O(1) \quad (51)$$

then the number of rotations is  $K = \log(N)$ .

**THEOREM 8.1** The number of reflections from  $|\beta\rangle$  to  $|\alpha\rangle$ ,  $R(\pi/4)$ , is  $\log(N)$ .

**Proof.** We will apply the sequence of oracle, diffusion, reflection, reflection, ...

Step 0. From Equation (8),

$$|\psi_k\rangle = \sin(\theta_k)|\alpha\rangle + \cos(\theta_k)|\beta\rangle \quad (52)$$

From Equation (11), initial angle ( $k=0$ ), from initial superposition in Equation (8), is given:

$$\sin(\theta_0) = \theta_0 = 1/\sqrt{N} \quad (53)$$

This gives the first term in the geometric series for rotation angles:

$$\frac{2^0}{\sqrt{N}}$$

Step 1. From Equation (44), after the oracle operator and diffusion operator,

$$|\psi_1\rangle = DO|\psi_0\rangle = \left(1 - \frac{4}{N}\right)|\psi_0\rangle + \frac{2}{\sqrt{N}}|\alpha\rangle$$

which is normalized. Ignore higher order terms  $O(1/N)$ ,

$$|\psi_1\rangle \approx |\psi_0\rangle + \frac{2}{\sqrt{N}}|\alpha\rangle$$

This gives the second term in the geometric series for rotation angles:

$$\frac{2^1}{\sqrt{N}} = 2\theta_0$$

Step 2. Reflection cross line  $|\psi_1\rangle$ . Standard reflection operator uses:

$$R_1 = I - 2|\psi_1\rangle\langle\psi_1| \tag{54}$$

This line is  $3\theta_0 = 3/\sqrt{N}$  from  $|\beta\rangle$ -axis. Applying reflection operator,

$$|\psi_2\rangle = R_1|\psi_0\rangle = (I - 2|\psi_1\rangle\langle\psi_1|)|\psi_0\rangle \tag{55}$$

Because  $|\psi_0\rangle$  is  $\theta_0$  from  $|\beta\rangle$ -axis, the angle between  $|\psi_0\rangle$  and the reflection line is  $2\theta_0 = 2/\sqrt{N}$ . After the reflection,  $|\psi_0\rangle$  will advance  $4\theta_0$  and we have the following approximation:

$$|\psi_2\rangle \approx |\psi_0\rangle + \frac{4}{\sqrt{N}}|\alpha\rangle \tag{56}$$

$|\psi_2\rangle$  is  $5\theta_0$  from  $|\beta\rangle$ -axis, which gives the next term in the geometric series for rotation angles:

$$\frac{4}{\sqrt{N}} = \frac{2^2}{\sqrt{N}} = 4\theta_0$$

Step 3. Reflection cross line  $|\psi_2\rangle$ . Standard reflection operator uses:

$$R_2 = I - 2|\psi_2\rangle\langle\psi_2| \tag{57}$$

This line is  $5\theta_0 = 5/\sqrt{N}$  from  $|\beta\rangle$ -axis, and the next state is:

$$|\psi_3\rangle = R_2|\psi_0\rangle = (I - 2|\psi_2\rangle\langle\psi_2|)|\psi_0\rangle \tag{58}$$

Because  $|\psi_0\rangle$  is  $\theta_0$  from  $|\beta\rangle$ -axis, the angle between  $|\psi_0\rangle$  and the reflection line is  $4\theta_0 = 4/\sqrt{N}$ . After the reflection,  $|\psi_0\rangle$  will advance  $8\theta_0$  and we have the following approximation:

$$|\psi_3\rangle \approx |\psi_0\rangle + \frac{8}{\sqrt{N}}|\alpha\rangle \tag{59}$$

$|\psi_3\rangle$  is  $9\theta_0$  from  $|\beta\rangle$ -axis, which gives the next term in the geometric series for rotation angles:

$$\frac{8}{\sqrt{N}} = \frac{2^3}{\sqrt{N}} = 8\theta_0$$

The geometric series for rotation-angle is:

$$\frac{2^0}{\sqrt{N}} + \frac{2^1}{\sqrt{N}} + \frac{2^2}{\sqrt{N}} + \dots + \frac{2^K}{\sqrt{N}} = \frac{\pi}{4} = O(1)$$

The number of rotation from  $|\beta\rangle$  to  $|\alpha\rangle$ ,  $R(\pi/4)$ , is  $K = \log(N)$ .  $\square$

### 9. Time Complexity

The basic idea is the key identity in Equation (46):

$$R_\psi = 2|\psi\rangle\langle\psi| - I = U(2|0\rangle\langle 0| - I)U^\dagger$$

From Section 8.2, the time complexity of the above equation is

$$T(R_\psi) = 2T(U) + O(n) \tag{60}$$

The factor of 2 in the above equation doubles the time complexity in each step if there is a recursion (see below).

The sequence of operations is oracle, diffusion, reflection cross  $|\psi_1\rangle$ , reflection cross  $|\psi_2\rangle$ , reflection cross  $|\psi_3\rangle$ , ...

The oracle operator has its own time complexity,  $T_O$ , which depends on the applications and is not a variable in this paper.

The time complexity of the diffusion operator is  $T(D) = O(n)$ .

Consider the first reflection

$$R_1 = 2|\psi_1\rangle\langle\psi_1| - I \quad (61)$$

where

$$|\psi_1\rangle = DO|\psi_0\rangle$$

So, up to a phase factor of  $-1$ ,

$$R_1 = DO(2|\psi_0\rangle\langle\psi_0| - I)O^\dagger D^\dagger = DODO^\dagger D^\dagger \quad (62)$$

which can be written as:

$$R_1 = D_0(\text{something } O(n + T_O))D_0^\dagger \quad (63)$$

The time complexity of the first reflection can be estimated as follows:

$$T(R_1) = 2T(D_0) + O(n + T_O) \quad (64)$$

This doubles the previous time complexity of  $DO$ , oracle call and diffusion call. Consider the next reflection

$$R_2 = 2|\psi_2\rangle\langle\psi_2| - I$$

where

$$|\psi_2\rangle = (2|\psi_1\rangle\langle\psi_1| - I)|\psi_0\rangle = R_1|\psi_0\rangle \quad (65)$$

So,

$$R_2 = R_1(2|\psi_0\rangle\langle\psi_0| - I)R_1^\dagger = R_1DR_1^\dagger \quad (66)$$

which can be written as:

$$R_2 = R_1(\text{something } O(n))R_1^\dagger \quad (67)$$

The complexity of this reflection is:

$$T(R_2) = 2T(R_1) + O(n) \quad (68)$$

Again, this doubles the previous time complexity. Continuing this process, one has

$$R_k = R_{k-1}DR_{k-1}^\dagger \quad (69)$$

And

$$T(R_k) = 2T(R_{k-1}) + O(n) \quad (70)$$

This recursion stops at Equation (64). Each step doubles the previous time complexity. The time complexity for obtaining a reflection axis grows exponentially with respect to  $k$ . After  $\log(N)$  steps, the time complexity is  $O(\sqrt{N})$ .

The standard  $\Omega(\sqrt{N})$  lower bound [9]-[15] applies to the Single-Iteration or

Single-Rotation algorithm in this paper, even though the number of oracle calls is reduced from  $\mathcal{O}(\sqrt{N})$  to  $\mathcal{O}(1)$  and the number of diffusion calls is reduced from  $\mathcal{O}(\sqrt{N})$  to  $\mathcal{O}(\log N)$ , because within the  $\log(N)$  iterations, each iteration increases the time complexity by a factor of 2 in Equation (70).

## 10. Hypotheses and Assumptions

This section explicitly states the hypotheses and assumptions underlying the Single-Rotation Quantum Search Algorithm and oracle-based quantum algorithms generally. We distinguish between:

- **Hypotheses:** Structural properties assumed to exist for a given Hilbert Space
- **Assumptions:** Requirements that must be satisfied to achieve quantum advantage

Understanding these foundations clarifies when quantum algorithms provide advantage over classical approaches and which problems admit efficient quantum solutions.

### 10.1. Fundamental Hypotheses

These hypotheses specify what must exist for the algorithm to be applicable to a problem in Hilbert space. Hypothesis 1 requires oracle must be polynomial, Hypothesis 2 requires rotations must be polynomial, and Hypothesis 3 requires anything else also must be polynomial. Note that the Single-Iteration or Single-Rotation algorithm proposed in this paper does not meet these requirements.

**HYPOTHESIS 1** (Oracle Existence): For a given search problem over  $N = 2^n$  items with one marked item, there exists a quantum oracle  $O$  that, after polynomial quantum computation, can identify and mark the state  $|a\rangle$  corresponding to the marked item from the initial uniform superposition  $|\psi_0\rangle$  in Hilbert space.

**Mathematical formulation:**

$$O|x\rangle = (-1)^{f(x)}|x\rangle \quad (71)$$

where  $f(x) = 1$  if  $x$  is the marked item,  $f(x) = 0$  otherwise.

**Discussion:** This hypothesis assumes that the search problem can be formulated as a quantum oracle query problem. The oracle must be:

- **Unitary:** Preserves quantum state normalization
- **Reversible:** Can be implemented with quantum gates
- **Efficient:** Implementable with polynomial resources (see Hypothesis 3)

**HYPOTHESIS 2** (Rotation Operator Construction): For a given problem satisfying Hypothesis 1, the rotation operator  $U(\pi/4)$  can be constructed and applied to the initial state  $|\psi_0\rangle$  in Hilbert space, using only the marked state  $|a\rangle$  and the initial superposition  $|\psi_0\rangle$ , in polynomial quantum computation.

**Mathematical formulation:** Equation (34).

**Discussion:** This hypothesis asserts that the rotation operator can be implemented using only:

- The initial uniform superposition  $|\psi_0\rangle$  (created with  $n$  Hadamard gates)

- The marked state  $|a\rangle$  (identified by oracle  $O$ )
- Polynomial quantum operations

If the time complexity is exponential, not Polynomial, then it is almost impossible to complete any quantum computation because of the Quantum Loop Barrier below.

**HYPOTHESIS 3** (Efficient Computation within Hilbert Space): For the specific problem instance, all computations, such as oracle, rotations, ..., can be implemented with polynomial time complexity.

**Discussion:** This hypothesis requires:

- The problem has sufficient structure for efficient verification.
- Quantum circuits can exploit this structure.
- Gate count, qubit count, and depth scale polynomially with problem size  $n$ .

## 10.2. Assumptions for Quantum Advantage

These assumptions specify requirements that must be satisfied for oracle-based quantum algorithms to demonstrate practical quantum advantage over classical algorithms. They correspond to avoiding the fundamental barriers identified in [16] [17].

**ASSUMPTION 1** (Avoid Grover's Dilemma): The quantum algorithm must avoid the Grover's algorithm Dilemma (Type A problems requiring classical preprocessing) [16].

**Problem Classification:**

- **Type A Problems:** Computational space  $|C| > \text{Valid data space } |D|$  (e.g., searching unsorted databases, constraint satisfaction)
- **Type B Problems:** Computational space  $|C| = \text{Valid data space } |D|$  (e.g., AES key search, subset sum)

**The Dilemma for Type A problems:**

- **Efficiency:** Including all computational states in Equation (1) in  $O(n)$  steps (including invalid states), which can produce false positives and incorrect results
- **Correctness:** Restricting to valid states in Equation (52), which requires  $O(|D|) = O(2^n)$  classical preprocessing, eliminating quantum advantage

If  $|C| > |D|$ , the superposition over all valid computational states is:

$$|\psi\rangle = \frac{1}{\sqrt{|D|}} \sum_{x=0}^{|D|-1} |x\rangle \quad (72)$$

The time complexity for this superposition is exponential,  $O(|D|) = O(2^n)$ .

**ASSUMPTION 2** (Avoid Setup Barrier): The problem must not have a setup cost exceeding classical solution cost [17].

**ASSUMPTION 3** (Avoid Oracle Circularity): The oracle construction must not itself require solving the search problem [17].

**Circularity occurs when:**

- Oracle design requires knowing the solution in advance

- Oracle implementation cost equals or exceeds classical search cost

**ASSUMPTION 4** (Avoid Quantum Loop Barrier): The algorithm must avoid deep sequential quantum operations that exceed decoherence time [17].

**Mathematical formulation:** Total circuit depth  $D$  must satisfy:

$$D \cdot t_{\text{gate}} < T_2 \quad (\text{coherence time})$$

**ASSUMPTION 5** (Efficient Hilbert Space Operations): The rotation operator  $U(\pi/4)$  and other Hilbert space operations must be implementable with polynomial gate complexity, avoiding exponential overhead from explicit state construction.

**Discussion:** This assumption requires that operations in the exponentially large Hilbert space ( $2^n$ ) can be performed efficiently using:

- Polynomial gate decompositions
- Poly ( $n$ ) Controlled operations based on oracle

**ASSUMPTION 6** (Structured Problem with Unstructured Search): The problem must have sufficient structure to admit polynomial oracle construction, while remaining “unstructured” in the classical sense such that classical algorithms cannot exploit this structure to avoid exponential search.

**Discussion:** This defines the “sweet spot” for quantum advantage:

**Too structured:**

- Classical algorithms can exploit structure
- Example: Searching sorted list (binary search is  $O(\log N)$  classically)
- No quantum advantage possible

**Too unstructured:**

- No efficient oracle construction
- Assumption 2 or Assumption 3 or Hypothesis 3 violated
- Quantum algorithm also fails

**Optimal structure:**

- Efficient verification problems (oracle is polynomial)
- Inefficient search (classical search algorithms require exponential time)
- This is the domain of quantum advantage

### 10.3. Sufficiency and Necessity

**THEOREM 10.1** (Sufficiency): If Hypotheses 1 - 3 hold and Assumptions 1-6 are satisfied, then an oracle-based quantum Algorithm achieves polynomial time complexity  $O(C_{\text{oracle}} + \text{poly}(n))$  with quantum advantage over classical exponential search.

**Proof: From Hypotheses 1 - 3:**

- Oracle  $O$  exists and identifies  $|a\rangle$  (Hypothesis 1)
- Rotation operator  $U(\pi/4)$  can be constructed in  $\text{poly}(n)$  time (Hypothesis 2)
- Oracle is efficient:  $C_{\text{oracle}} = O(\text{poly}(n))$  (Hypothesis 3)

**From Assumptions 1 - 6:**

- Type B problems avoid Grover’s Dilemma (Assumption 1)
- Setup cost  $O(n)$  is polynomial (Assumption 2)

- Oracle doesn't require solution in advance (Assumption 3)
- Circuit depth  $D <$  decoherence limit (Assumption 4)
- Implicit  $|\beta\rangle$  representation ensures polynomial gates (Assumption 5)
- Problem structured for oracle but unstructured for classical search (Assumption 6)

**Conclusion:** Total quantum complexity =  $\mathcal{O}(n) + \mathcal{O}(C_{\text{oracle}}) + \mathcal{O}(C_{\text{oracle}} + n) = \mathcal{O}(C_{\text{oracle}} + \text{poly}(n))$

Classical complexity =  $\mathcal{O}(2^n)$  (unstructured search)

**Quantum advantage ratio:**  $\mathcal{O}(2^n)/\mathcal{O}(\text{poly}(n)) = \text{exponential}$   $\square$

**THEOREM 10.2** (Necessity of Assumptions): Violating any of Assumptions 1-6 eliminates quantum advantage.

**Proof by counter example:**

**Violate Hypotheses 1:** no oracle

**Violate Hypotheses 2:** no rotation operator in  $\text{poly}(n)$

**Violate Hypotheses 3:** quantum cost  $\geq$  classical cost

**Violate Assumption 1:** suffers from Grover's Dilemma.

**Violate Assumption 2:** Setup Barrier is more than classical solution.

**Violate Assumption 3:** If oracle requires solving the problem first, no advantage is achieved.

**Violate Assumption 4:** If iterations exceed decoherence: Algorithm physically cannot execute.

**Violate Assumption 5: (Exponential Hilbert Operations):** If  $|\beta\rangle$  construction or any other intermediate state requires  $\mathcal{O}(M) = \mathcal{O}(2^n)$  gates, it is an exponential algorithm.

**Violate Assumption 6: (Wrong Structure Balance):** If too structured: Classical algorithms are polynomial. If too unstructured: Oracle is exponential (violates Hypothesis 3).

Therefore, all assumptions are necessary.  $\square$

## 10.4. Practical Implications

**Problem Evaluation Framework:** When assessing whether a problem admits efficient quantum solution via Single-Rotation Algorithm:

**Step 1: Check Hypotheses**

- 1) Can oracle  $O$  be defined that verifies solutions? (Hypothesis 1)
- 2) Can rotation be constructed from  $|\alpha\rangle$  and  $|\psi_0\rangle$ ? (Hypothesis 2—generally yes if Hypothesis 1 holds)
- 3) Is oracle implementable with polynomial gates? (Hypothesis 3—problem-specific)

**Step 2: Check Assumptions**

- 1) Does algorithm avoid Grover's barriers? (For Type B problems only)
- 2) Is setup cost polynomial? ( $\mathcal{O}(n)$  for setup quantum computing)
- 3) Does oracle avoid circularity? (verification problems only)
- 4) Is circuit depth within coherence limits? (No long loop)

- 5) Are Hilbert space operations polynomial? (For implicit  $|\beta\rangle$ )  
 6) Is problem structured correctly? (Check: oracle  $\text{poly}(n)$  verification and exponential classical search)

If any assumption fails  $\rightarrow$  No quantum advantage. If all hypotheses and assumptions satisfied  $\rightarrow$  Quantum advantage achieved!

## 10.5. Summary

**This framework establishes:**

**Hypotheses 1 - 3:** Define if the algorithm is applicable.

**Assumptions 1 - 6:** Define when quantum advantage is achieved.

**Theorems 10.1 - 10.2:** Prove these conditions are sufficient and necessary.

**Practical value:** Provides roadmap for:

- Evaluating new problems for quantum advantage
- Identifying which barriers prevent quantum speedup
- Designing quantum algorithms that avoid known barriers

**The Single-Rotation Algorithm can achieve quantum advantage** only by simultaneously satisfying all hypotheses and assumptions.

## 11. Conclusions

This paper presents the Single-Rotation Quantum Search Algorithm, which achieves amplitude amplification with exactly one oracle call and one  $\pi/4$  rotation.

**Main Results:**

1) **Single-Iteration or Single-Rotation Algorithm:** the algorithm achieves amplitude amplification with exactly one oracle call and one  $\pi/4$  rotation.

2) **Implementation:** The algorithm is implemented by one oracle call, one diffusion call, and  $\log(N)$  reflections.

3) **Standard Lower Bounds:** The standard lower bounds for unstructured quantum search still apply even though the number of oracle calls is reduced from  $\mathcal{O}(\sqrt{N})$  to  $\mathcal{O}(1)$  and the number of diffusion calls is reduced from  $\mathcal{O}(\sqrt{N})$  to  $\mathcal{O}(\log N)$ .

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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