Journal of Modern Physics, 2026, 17(11), 1510-1529
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

Electromagnetic Form Factor of η Meson as a Bound System in 3 + 1 Dimensional QCD

Tero Kurai

Independent Researcher, Tokyo, Japan
Email: teruokurai357@gmail.com

How to cite this paper: Kurai, T. (2026) Electromagnetic Form Factor of η Meson as a Bound System in 3 + 1 Dimensional QCD. *Journal of Modern Physics*, **17**, 1510-1529.
https://doi.org/10.4236/jmp.2026.1711063

Received: October 9, 2026
Accepted: November 20, 2026
Published: November 23, 2026

Copyright © 2026 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/
Open Access

Abstract

We investigate electromagnetic form factor (EMFF) of η meson by using three dimension Fourier Transform of wave functions corresponding to eigen functions of η meson mass. In this process, we impose the physical pole, from which pure physics pole terms generate the main EMFF that corresponds to the charged pion EMFF and we obtain a part of the EMFF whose behavior is close to the so-called TPE EMFF by subtracting pure physics pole terms from our wave functions.

Keywords

Electromagnetic Form Factor, η Meson, Virtual Pion Mass

1. Introduction

Recently, lattice QCD calculation has been surprisingly developed and many results are shown up in Hadron physics. For instance, several results have been published for η meson electromagnetic form factors (EMFF) [1] [2] ([2] also includes η' meson). These calculations involve a specific result that shows EMFF contributed by two photon exchange (TPE) besides EMFF contributed by normal one photon exchange (OPE). For TPE calculations, there are two considerations; one is direct loop calculation (lattice QCD calculation is in this Group) and the other one is based on dispersion theory, however, we can say that there is still no decisive way for evaluating TPE EMFF, yet. While for one photon exchange, there are no published results obtained by nonperturbative methods, either. This means we do not have a full one photon exchange wave function. Thus, we cannot say that it is necessary for TPE to describe each meson's EMFF. In this regard, it may be worthwhile to attempt something else. We consider finding which part of EMFF evaluated by already existing wave functions of η mesons corresponds to their

TPE EMFF as one of them. If we can find this part and how much close to their TPE EMFF, we may use this approach to describe each meson's EMFF more profoundly even these wave functions are not full wave functions. For this consideration, we attempt to find which part of the EMFF evaluated by our wave functions of η mesons obtained in ref. [3] corresponds to their TPE EMFF.

2. Formulation

Before going to argue η meson electromagnetic form factor (eta EMFF), we revisit our equations obtained in ref. [3]. The following equations are slightly revised as shown in erratum [4].

$$P_0\chi_2(r) = -2\frac{\partial}{\partial r}\chi_3(r) - \frac{2}{r}\chi_3(r) - \frac{g^2L_1}{2}r\chi_3(r) + \frac{g^2L_1}{2P_0}\chi_2(r) \tag{1}$$

$$P_0\chi_3(r) = 2\frac{\partial}{\partial r}\chi_2(r) - \frac{g^2L_1}{2}r\chi_2(r) - \frac{g^2L_1}{2P_0}\chi_3(r) \tag{2}$$

Note that we denote $|L_1|$ by L_1 for convenience.

Thus, equations for $\chi_3(r)$ and $\chi_2(r)$ become as follows.

$$0 = \frac{\partial^2\chi_3}{\partial r^2} + \frac{1}{r}\frac{\partial\chi_3}{\partial r} - \frac{1}{r^2}\chi_3 + \frac{1}{4}\left[\left(P_0 - \frac{\frac{g^2L_1}{2}}{P_0}\right)\left(P_0 + \frac{\frac{g^2L_1}{2}}{P_0}\right)\right]\chi_3 - \frac{1}{4}\left(\frac{g^2L_1}{2}\right)^2 r^2\chi_3 \tag{3}$$

$$0 = \frac{\partial^2\chi_2}{\partial r^2} + \frac{1}{r}\frac{\partial\chi_2}{\partial r} + \left[-\frac{g^2L_1}{2} + \frac{1}{4}\left(P_0 - \frac{\frac{g^2L_1}{2}}{P_0}\right)\left(P_0 + \frac{\frac{g^2L_1}{2}}{P_0}\right)\right]\chi_2 - \frac{1}{4}\left(\frac{g^2L_1}{2}\right)^2 r^2\chi_2 \tag{4}$$

For Equation (1) and Equation (2), kinetic terms are twice as large as usual. Now we consider the normal kinetic terms case. Then Equation (1) and Equation (2) become as follows.

$$P_0\chi_2(r) = -\frac{\partial}{\partial r}\chi_3(r) - \frac{1}{r}\chi_3(r) - \frac{g^2L_1}{2}r\chi_3(r) + \frac{g^2L_1}{2P_0}\chi_2(r) \tag{5}$$

$$P_0\chi_3(r) = \frac{\partial}{\partial r}\chi_2(r) - \frac{g^2L_1}{2}r\chi_2(r) - \frac{g^2L_1}{2P_0}\chi_3(r) \tag{6}$$

In this case, equations for $\chi_3(r)$ and $\chi_2(r)$ become as follows.

$$0 = \frac{\partial^2\chi_3}{\partial r^2} + \frac{1}{r}\frac{\partial\chi_3}{\partial r} - \frac{1}{r^2}\chi_3 + \left[\left(P_0 - \frac{\frac{g^2L_1}{2}}{P_0}\right)\left(P_0 + \frac{\frac{g^2L_1}{2}}{P_0}\right)\right]\chi_3 - \left(\frac{g^2L_1}{2}\right)^2 r^2\chi_3 \tag{7}$$

$$0 = \frac{\partial^2\chi_2}{\partial r^2} + \frac{1}{r}\frac{\partial\chi_2}{\partial r} + \left[-2\left(\frac{g^2L_1}{2}\right) + \left(P_0 - \frac{\frac{g^2L_1}{2}}{P_0}\right)\left(P_0 + \frac{\frac{g^2L_1}{2}}{P_0}\right)\right]\chi_2 - \left(\frac{g^2L_1}{2}\right)^2 r^2\chi_2 \tag{8}$$

To solve Equation (7) and Equation (8), using the same argument in Appendix C in ref. [3], that is, taking $\chi_i = r^{-\frac{1}{2}} W_i$, $t = r^2$, and $W_i = t^{-\frac{1}{4}} u_i$ (i denotes 3 and 2), Equation (7) and Equation (8) become as

$$0 = \frac{\partial^2 u_3}{\partial t^2} + \left[-\frac{1}{4}\left(\frac{g^2 L_1}{2}\right)^2 + \frac{\frac{1}{4}\left(P_0 - \frac{\frac{g^2 L_1 \delta(0)}{2}}{P_0}\right)\left(P_0 + \frac{\frac{g^2 L_1 \delta(0)}{2}}{P_0}\right)}{t} \right] u_3 \tag{9}$$

$$0 = \frac{\partial^2 u_2}{\partial t^2} + \left[-\frac{1}{4}\left(\frac{g^2 L_1}{2}\right)^2 + \frac{-\frac{1}{2}\left(\frac{g^2 L_1}{2}\right) + \frac{1}{4}\left(P_0 - \frac{\frac{g^2 L_1 \delta(0)}{2}}{P_0}\right)\left(P_0 + \frac{\frac{g^2 L_1 \delta(0)}{2}}{P_0}\right)}{t} + \frac{\frac{1}{4}}{t^2} \right] u_2 \tag{10}$$

Then taking $\bar{t} = \alpha t$ and $\alpha = \frac{g^2 L_1}{2}$, Equation (9) and Equation (10) become the standard form of the Whittaker differential equation such as

$$0 = \frac{\partial^2 u_3}{\partial \bar{t}^2} + \left[-\frac{1}{4} + \frac{\left(P_0 - \frac{\frac{g^2 L_1 \delta(0)}{2}}{P_0}\right)\left(P_0 + \frac{\frac{g^2 L_1 \delta(0)}{2}}{P_0}\right)}{2g^2 L_1} \right] u_3 \tag{11}$$

$$0 = \frac{\partial^2 u_2}{\partial \bar{t}^2} + \left[-\frac{1}{4} + \frac{-\frac{1}{2} + \left(P_0 - \frac{\frac{g^2 L_1 \delta(0)}{2}}{P_0}\right)\left(P_0 + \frac{\frac{g^2 L_1 \delta(0)}{2}}{P_0}\right)}{2g^2 L_1} + \frac{\frac{1}{4}}{\bar{t}^2} \right] u_2 \tag{12}$$

Comparing Equation (11) and Equation (12) to twice large kinetic terms case that are shown in Appendix C of ref. [3], we notice that the difference is only in

the following two parts.

(a) Denominator of mass terms: $2g^2L_1$ for kinetic × 1 case, while $4g^2L_1$ for kinetic × 2 case.

(b) α value: $\alpha=\frac{g^2L_1}{2}$ for kinetic × 1 case, while $\alpha=\frac{g^2L_1}{4}$ for kinetic × 2 case.

Thus, solutions of $\chi_3(r)$ for kinetic × 1 case become:

$$\chi_3(r)=\text{const}\frac{1}{r}M_{\kappa,\mu}\left(\frac{1}{2}g^2L_1^{(2)}r^2\right) \tag{13}$$

There are two types of solutions, one corresponds to $\mu=-\frac{1}{2}$ and the other one corresponds to $\mu=+\frac{1}{2}$. The important point is that these two types of solutions are totally different. Thus, we have to deal with these two types differently as follows.

For $\mu=-\frac{1}{2}$ case, we denote $\frac{g^2L_1\delta(0)}{2}$ as $\frac{1}{2}g^2L_1^{(1)}\delta_1(0)$ and $\mu=+\frac{1}{2}$ case, we denote $\frac{g^2L_1\delta(0)}{2}$ as $\frac{1}{2}g^2L_1^{(2)}\delta_2(0)$ and further $\frac{1}{2}\frac{g^2L_1^{(2)}}{\gamma}\left(\gamma\delta_2(0)\right)$. The reason for introducing coefficient γ is to discuss for kinetic × 1 case and for kinetic × 2 case on the same footing.

For $\mu=-\frac{1}{2}$ case, the possible solution is only one, as mentioned in ref. [3] because the corresponding confluent hyper geometric series is represented as

$${}_1F_1\left(\mu-\kappa+\frac{1}{2},2\mu+1;z\right)$$

Definition of confluent hyper geometric series ${}_1F_1\left(\bar{\alpha},\bar{\beta};z\right)$ is

$${}_1F_1\left(\bar{\alpha},\bar{\beta};z\right)=1+\sum_{n=1}^{\infty}\frac{\bar{\alpha}\left(\bar{\alpha}+1\right)\cdots\left(\bar{\alpha}+n-1\right)}{\bar{\beta}\left(\bar{\beta}+1\right)\cdots\left(\bar{\beta}+n-1\right)}\frac{z^n}{n!}$$

Note that the first term 1 corresponds to $n=0$

Thus, the denominator part becomes 0 for $\mu=-\frac{1}{2}$ case, that means n can take only the first term ($n=0$), *i.e.* ${}_1F_1\left(\bar{\alpha},\bar{\beta};z\right)=1$. Therefore this corresponds to pion as mentioned in ref. [3].

For $\mu=+\frac{1}{2}$ case, meson mass spectra except pion mass can be described as follows.

For kinetic terms × 2 case

$$\frac{\left(P_0-\frac{\frac{g^2L_1^{(2)}}{\gamma}\left(\gamma\delta_2(0)\right)}{2}{P_0}\right)\left(P_0+\frac{\frac{g^2L_1^{(2)}}{\gamma}\left(\gamma\delta_2(0)\right)}{2}{P_0}\right)}{4g^2L_1^{(2)}}=\kappa \tag{14}$$

where $\kappa = n+1$ (n is a positive integer including 0).

Equation (14) can be rewritten as

$$\frac{\left(P_0 - \dfrac{\dfrac{g^2 L_1^{(2)}}{\gamma^2}\left(\gamma\delta_2(0)\right)}{\dfrac{2}{\dfrac{P_0}{\gamma}}}\right)\left(P_0 + \dfrac{\dfrac{g^2 L_1^{(2)}}{\gamma^2}\left(\gamma\delta_2(0)\right)}{\dfrac{2}{\dfrac{P_0}{\gamma}}}\right)}{4g^2 L_1^{(2)}} = \kappa \tag{15}$$

Equation (15) becomes

$$\frac{\left(\dfrac{P_0^2}{\gamma} - \dfrac{\dfrac{g^2 L_1^{(2)}}{\gamma}\left(\gamma\delta_2(0)\right)}{2\gamma}\right)\left(\dfrac{P_0^2}{\gamma} + \dfrac{\dfrac{g^2 L_1^{(2)}}{\gamma}\left(\gamma\delta_2(0)\right)}{2\gamma}\right)}{4g^2 L_1^{(2)}\dfrac{P_0^2}{\gamma}} = n+1 \tag{16}$$

Choosing $\gamma\delta_2(0) = 1$, Equation (16) becomes

$$\left(\frac{P_0^2}{\gamma}\right)^2 - 4(n+1)g^2 L_1^{(2)}\left(\frac{P_0^2}{\gamma}\right) - \left(\frac{1}{2\gamma}\right)^2\left(\frac{g^2 L_1^{(2)}}{\gamma}\right)^2 = 0 \tag{17}$$

Equation (17) gives the following form for $\dfrac{P_0^2}{\gamma}$.

$$\frac{P_0^2}{\gamma} = \left[2(n+1)\gamma + \sqrt{\left(2(n+1)\gamma\right)^2 + \left(\frac{1}{2\gamma}\right)^2}\right]\frac{g^2 L_1^{(2)}}{\gamma} \tag{18}$$

$$\begin{aligned} P_0^2 &= \left[2(n+1)\gamma + \sqrt{\left(2(n+1)\gamma\right)^2 + \left(\frac{1}{2\gamma}\right)^2}\right] g^2 L_1^{(2)} \\ &= \left[2(n+1)\gamma + \sqrt{\left(2(n+1)\gamma\right)^2 + \left(\frac{1}{2\gamma}\right)^2}\right]\overline{\gamma}\left(\frac{g^2 L_1^{(2)}}{\overline{\gamma}}\right) \end{aligned} \tag{19}$$

For $\overline{\gamma}$ value, it is determined in later, but this shows $\overline{\gamma}$ value does not affect mass spectra.

For kinetic × 1 case, from Equation (11),

$$\frac{\left(P_0 - \dfrac{g^2 L_1^{(2)}\delta_2(0)}{\dfrac{2}{P_0}}\right)\left(P_0 + \dfrac{g^2 L_1^{(2)}\delta_2(0)}{\dfrac{2}{P_0}}\right)}{2g^2 L_1^{(2)}} = k = n+1 \tag{20}$$

$$\left(P_0^2\right)^2 - 2(n+1)g^2 L_1^{(2)} P_0^2 - \left(\frac{g^2 L_1^{(2)}\delta_2(0)}{2}\right)^2 = 0 \tag{21}$$

$$P_0^2 = \left(n+1+\sqrt{(n+1)^2+\left(\frac{\delta_2(0)}{2}\right)^2} \right) g^2 L_1^{(2)} \tag{22}$$

Recalling the corresponding Equation (22) for kinetic × 2 case before introducing γ, that is,

$$P_0^2 = \left(2(n+1)+\sqrt{(2(n+1))^2+\left(\frac{\delta_2(0)}{2}\right)^2} \right) g^2 L_1^{(2)} \tag{23}$$

Then, comparing Equation (22) and Equation (23) to Equation (19) under the condition $\gamma\delta_2(0)=1$, γ value are $\gamma=1$ for kinetic × 2 case and $\gamma=\frac{1}{2}$ for kinetic × 1 case, respectively.

To obtain meson mass spectra, we have to choose a basis meson. In our case, we choose η meson as the basis meson because it is the lightest and has a well-determined mass value.

This means we set $g^2 L_1^{(2)} = (m_\eta)^2 = (0.548)^2 (\mathrm{GeV}^2)$

Then other meson masses except the pion can be obtained as

$$\begin{aligned} M^2 &= \frac{\left[2(n+1)+\sqrt{(2(n+1))^2+\left(\frac{1}{2}\right)^2} \right]}{4+\sqrt{16+\left(\frac{1}{2}\right)^2}} g^2 L_1^{(2)} \\ &= \frac{\left[2(n+1)+\sqrt{(2(n+1))^2+\left(\frac{1}{2}\right)^2} \right]}{4+\sqrt{16+\left(\frac{1}{2}\right)^2}} (0.548)^2 (\mathrm{GeV}^2) \end{aligned} \tag{24}$$

Corresponding mass for each meson are following. $m_{f^0}(n=0)$ m_ρ ($n=2$ case), m_ϖ ($n=3$), $m_{\eta'}$ ($n=4$), m_{a^0} ($n=5$), m_ϕ ($n=6$). Note that γ value for kinetic terms × 2 case is 1.

An interesting point is that this mass spectra formula gives a value close to the pion mass also. Equation (24) gives

$$M^2 = \frac{\frac{1}{2}}{4+\sqrt{16+\left(\frac{1}{2}\right)^2}} (0.548)^2 \tag{25}$$

$$M = 0.13674(\mathrm{GeV})$$

This value is close to the charged pion mass, however, this is only a virtual one as mentioned before.

For kinetic terms × 1 case, recalling the difference shown in (a), in the mass spectra case, we only need to change $4g^2 L_1^{(2)}$ to $2g^2 L_1^{(2)}$ in Equation (15). Then Equation (23) changes to

$$P_0^2 = \left[(n+1) + \sqrt{(n+1)^2 + \left(\frac{1}{2\gamma}\right)^2} \right] g^2 L_1^{(2)} \tag{26}$$

Recalling $\gamma = \frac{1}{2}$ for kinetic × 1 case and again taking η meson mass as a basis, we obtain the mass formula for kinetic terms × 1 case corresponding to Equation (24) as

$$M^2 = \frac{\left[(n+1) + \sqrt{(n+1)^2 + 1} \right]}{2 + \sqrt{4+1}} g^2 L_1^{(2)} = \frac{\left[(n+1) + \sqrt{(n+1)^2 + 1} \right]}{2 + \sqrt{4+1}} (0.548)^2 \tag{27}$$

The important point is that when we divide both numerator and denominator of the last line of Equation (24) by factor 2, this becomes almost the same form as Equation (27). This means mass spectra except pion is independent of choice of working in kinetic terms × 2 case or in kinetic terms × 1 case as shown Table 1.

Table 1. Comparison of mass spectra between kinetic × 1 and kinetic × 2.

k	n	Mass (GeV) (kinetic × 2)	Mass (GeV) (kinetic × 1)	Mass (GeV) (experiment)	mesons
0	−1	0.13674	0.188271	0.13957	$\pi^{\pm}$
1	0	0.389707	0.413701	0.400 - 0.550	$f^0(500)$
2	1	0.548	0.548	0.547862	η
3	2	0.670438	0.6609532	0.77511	ρ
4	3	0.773867	0.758857	0.78265	ω
5	4	0.865053	0.846133	0.95778	η'
6	5	0.947528	0.925512	0.980	$f^0(980)$
7	6	1.023388	0.998703	1.019461	ϕ

Note: Mass value are adopted from Particle data Group (2022).

Note that the pion mass in this table is only the virtual one because we consider $\mu = +\frac{1}{2}$ case. Real pion appears in the case of $\mu = -\frac{1}{2}$ as mentioned before. Even then, we want to insist that kinetic × 2 case only gives a close value of the pion mass. Thus, we think considering kinetic × 2 case has some meaning.

However, the corresponding eigenfunction (wave function) is different from that shown in difference point (b). Explicit forms of $\chi_3(r)$ and $\chi_2(r)$ in kinetic × 2 case are shown in ref. [3] as

$$\begin{aligned} \chi_3(r) &= \text{const}\, r \exp\left(-\frac{1}{8} g^2 L_1^{(2)} r^2\right) {}_1F_1\left(1-\kappa, 2; \frac{1}{4} g^2 L_1^{(2)} r^2\right) \\ &= \text{const}\, \frac{1}{r} M_{\kappa, \frac{1}{2}}\left(\frac{1}{4} g^2 L_1^{(2)} r^2\right) \\ \chi_2(r) &= \text{const} \exp\left(-\frac{1}{8} g^2 L_1^{(2)} r^2\right) {}_1F_1\left(\frac{1}{2}-\kappa, 1; \frac{1}{4} g^2 L_1^{(2)} r^2\right) \end{aligned} \tag{28}$$

$$= \text{const}\frac{1}{r}M_{\bar{\kappa},0}\left(\frac{1}{4}g^2L_1^{(2)}r^2\right) \tag{29}$$

Note that we denote L_1 in ref. [3] to $L_1^{(2)}$ here.

Recalling the difference point (b), $\chi_3(r)$ and $\chi_2(r)$ in kinetic × 1 case are represented as

$$\begin{aligned}\chi_3(r) &= \text{const}\, r\exp\left(-\frac{1}{4}g^2L_1^{(2)}r^2\right){}_1F_1\left(1-\kappa,2;\frac{1}{2}g^2L_1^{(2)}r^2\right)\\ &= \text{const}\frac{1}{r}M_{\kappa,\frac{1}{2}}\left(\frac{1}{2}g^2L_1^{(2)}r^2\right)\end{aligned} \tag{30}$$

$$\begin{aligned}\chi_2(r) &= \text{const}\exp\left(-\frac{1}{4}g^2L_1^{(2)}r^2\right){}_1F_1\left(\frac{1}{2}-\kappa,1;\frac{1}{2}g^2L_1^{(2)}r^2\right)\\ &= \text{const}\frac{1}{r}M_{\bar{\kappa},0}\left(\frac{1}{2}g^2L_1^{(2)}r^2\right)\end{aligned} \tag{31}$$

where $\kappa = \dfrac{\left(P_0 - \dfrac{g^2L_1^{(2)}}{2P_0}\right)\left(P_0 + \dfrac{g^2L_1^{(2)}}{2P_0}\right)}{2g^2L_1}$ for χ_3, and

$\bar{\kappa} = -\dfrac{1}{2} + \dfrac{\left(P_0 - \dfrac{g^2L_1^{(2)}}{2P_0}\right)\left(P_0 + \dfrac{g^2L_1^{(2)}}{2P_0}\right)}{2g^2L_1}$ for χ_2, respectively.

An important point is that each meson's form factor should be independent of the choice of kinetic terms. Comparing Equation (28) and Equation (30) and also comparing Equation (29) and Equation (31), we notice that

$\dfrac{1}{8}g^2L_1^{(2)} = \dfrac{1}{4}\left(\dfrac{g^2L_1^{(2)}}{2}\right) = \dfrac{1}{2}\left(\dfrac{\frac{g^2L_1^{(2)}}{2}}{2}\right)$ and $\dfrac{1}{4}g^2L_1^{(2)} = \dfrac{1}{2}\left(\dfrac{g^2L_1^{(2)}}{2}\right)$. Thus, when we take $\bar{\gamma} = 2$ for $\dfrac{g^2L_1^{(2)}}{\bar{\gamma}}$ for kinetic × 2 and $\bar{\gamma} = 1$ for $\dfrac{g^2L_1^{(2)}}{\bar{\gamma}}$, Equation (28) and Equation (30) and also Equation (29) and Equation (31) become the same representations. Note that the ratio of $\bar{\gamma}$ of kinetic × 2 case to that of kinetic × 1 case is exactly the same as when we consider mass spectra (in this case, notation is γ) and this $\bar{\gamma}$ does not affect to $g^2L_1^{(2)}$ as shown in Equation (19). We consider that this corresponds to an energy scale adjustment (rescaling).

Thus, for both kinetic × 2 case and kinetic × 1 case, wave functions become same form under this consideration. Thus, we use the representation of wave functions for kinetic × 1 case for considering electromagnetic form factors (EMFF).

Then, we consider EMFF of η meson for an example.

Recalling definition of κ for χ_3 and $\bar{\kappa}$ for χ_2 and the condition of terminating ${}_1F_1$ gives following equation;

$$\frac{\left(P_0-\frac{\frac{g^2L_1^{(2)}}{2}}{P_0}\right)\left(P_0+\frac{\frac{g^2L_1^{(2)}}{2}}{P_0}\right)}{2g^2L_1}=n+1 \quad (n \text{ is 0 or a positive integer}) \tag{32}$$

We construct EMFF of η meson by following consideration.

In quantum theory, mass and wave functions are related as eigen value and eigen functions. For charged pion, we obtain $\mu=-\frac{1}{2}$, $\chi_3=\text{const}\frac{1}{r}\exp\left(-\frac{g^2L_1^{(1)}}{8}r^2\right)$ for pion mass (eigen value) $\frac{g^2L_1^{(1)}\delta_1(0)}{2}=m_\pi^2$ [3], while for neutral pion, for same mass eigen value, we obtain $\mu=-\frac{1}{2}$, $\chi_0=\text{const}\frac{2(m_1+m_2)}{g^2L_1^{(1)}}\frac{1}{r^2}\exp\left(-\frac{g^2L_1^{(1)}}{8}r^2\right)$ [5]. Note that χ_3 and χ_0 are vector component and scalar component, respectively and that $\frac{g^2L_1^{(1)}}{8}$ corresponds to pre-rescaling case. According to Particle Data Group 2022, decay mode of $\pi^\pm$ is $\pi^\pm\to\mu^\pm+\nu_\mu$ (almost 100%), and that of π^0 is $\pi^0\to 2\gamma$ (almost 99%). This means that charged pion decays as a vector mode, while neutral pion decays as a scalar mode. Thus, each solution is very reasonable because our solutions of charged pion corresponds to the vector component and that of the neutral pion corresponds to the scalar component, respectively. However, for η meson case, the decay mode is mixed, as the ratio of the neutral mode (scalar mode) to the charged mode (vector mode) is about 2.5:1. Although the scalar mode is still larger, eigen functions are not χ_0 but χ_3 and χ_2 as shown before. Because priority is the relation of eigen value and eigen functions in quantum theory, we consider χ_3 and χ_2 for constructing EMFF of η meson instead of χ_0.

For η meson corresponds to $\kappa=2$ case for χ_3 and $\bar{\kappa}=\frac{3}{2}$ case for χ_2, Equation (30) and Equation (31) give the following descriptions as wave functions:

$$\begin{aligned}\chi_3^\eta(r)&=\text{const}\,r\exp\left(\frac{g^2L_1^{(2)}}{4}r^2\right){}_1F_1\left(-1,2;\frac{g^2L_1^{(2)}}{2}r^2\right)\\&=\text{const}\frac{1}{r}M_{2,\frac{1}{2}}\left(\frac{g^2L_1^{(2)}}{2}r^2\right)\end{aligned} \tag{33}$$

$$\begin{aligned}\chi_2^\eta(r)&=\text{const}\exp\left(\frac{g^2L_1^{(2)}}{4}r^2\right){}_1F_1\left(-1,1;\frac{g^2L_1^{(2)}}{2}r^2\right)\\&=\text{const}\frac{1}{r}M_{\frac{3}{2},0}\left(\frac{g^2L_1^{(2)}}{2}r^2\right)\end{aligned} \tag{34}$$

Then, electromagnetic form factor $F(q)$ is evaluated by 3 dimension Fourier

Transform except for the constant factor as

$$F(q)=\int_0^\infty \mathrm{d}r r^2 \int_0^\pi \sin\theta \mathrm{d}\theta \exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r})\left|a\chi_2(r)\pm b\chi_3(r)\right|^2 \tag{35}$$

Thus, electromagnetic form factor $F(q)$ except constant factor is

$$F(q)=\int_0^\infty \mathrm{d}r r^2 \left|a\chi_2(r)\pm b\chi_3(r)\right|^2 \frac{\sin(qr)}{qr} \tag{36}$$

Then, for η meson case, EMFF is contributed as following 3 parts.

$$F_\eta^{(3)}(q)=\frac{1}{q}\int_0^\infty \mathrm{d}r \frac{1}{r}\sin(qr)M_{\frac{3}{2},0}(z)M_{\frac{3}{2},0}(z) \tag{37}$$

$$F_\eta^{(2)}(q)=\frac{1}{q}\int_0^\infty \mathrm{d}r \frac{1}{r}\sin(qr)M_{2,\frac{1}{2}}(z)M_{\frac{3}{2},0}(z) \tag{38}$$

$$F_\eta^{(1)}(q)=\frac{1}{q}\int_0^\infty \mathrm{d}r \frac{1}{r}\sin(qr)M_{2,\frac{1}{2}}(z)M_{2,\frac{1}{2}}(z) \tag{39}$$

where $z=\dfrac{g^2L_1^{(2)}}{2}r^2$

where $F_\eta^{(3)}(q),F_\eta^{(2)}(q),F_\eta^{(1)}(q)$ denote EMFF contributed by $|\chi_2|^2,|\chi_2\chi_3|,|\chi_3|^2$, respectively.

We use following integral formula [6] for integration.

$$\int_0^\infty \mathrm{d}x x^{\lambda-1}\mathrm{e}^{-\gamma x^2}\sin(qx)=q\frac{\mathrm{e}^{-\frac{q^2}{4\beta}}}{2\beta^{\frac{\lambda+1}{2}}}\Gamma\left(\frac{\lambda+1}{2}\right){}_1F_1\left(1-\frac{\lambda}{2},\frac{3}{2};\frac{q^2}{4\beta}\right) \tag{40}$$

where $\beta=\dfrac{g^2L_1^{(2)}}{2}r^2$

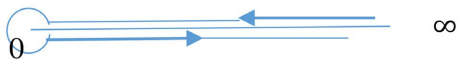
Important point is, for both $\mu=\dfrac{1}{2}$ and $\mu=0$ case, 2μ become 1 and 0, respectively.

When 2μ is integer, $M_{\kappa,\mu}$ and $M_{\kappa,-\mu}$ are not linearly independent each other. Linearly independent solution of $M_{\kappa,-\mu}$ is $W_{\kappa,-\mu}$ and $W_{\kappa,\mu}$ because of $W_{\kappa,-\mu}=W_{\kappa,\mu}$ [7]. We use $\dfrac{1}{r}M_{0,-\frac{1}{2}}(z)$ for charged pion wave function, that is, $\mu=-\dfrac{1}{2}$. As shown in **Table 1**, for kinetic × 2 case, we can obtain close to pion mass for $\kappa=0$ case of which wave function is described as $\dfrac{1}{r}M_{0,\frac{1}{2}}(z)$ for χ_3. Actuary, this is not real pion, however, this suggest that we should use $W_{\kappa,\mu}(z)$ ($\mu=\dfrac{1}{2}$ and 0) instead of $M_{\kappa,\mu}(z)$.

The definition of $W_{\kappa,\mu}(z)$ is as follows [7]

$$W_{\kappa,\mu}(z)=\frac{\Gamma\left(\kappa-\mu+\frac{1}{2}\right)}{2\pi\mathrm{i}}\exp\left(-\frac{1}{2}z\right)z^\kappa\int_{c_A\infty}^{0^+}\mathrm{d}t\mathrm{e}^{-t}(-t)^{\mu-\kappa-\frac{1}{2}}\left(1+\frac{t}{z}\right)^{\mu+\kappa-\frac{1}{2}} \tag{41}$$

where c_A denotes integral counter described as



By definition, each χ_3^2 is represented as

$$\begin{aligned}\chi_3^2 &= \frac{1}{r^2}W_{2,\frac{1}{2}}(z)W_{2,\frac{1}{2}}(z)\\ &= \frac{1}{r^2}\frac{\Gamma(2)\Gamma(2)}{(2\pi\mathrm{i})^2}\mathrm{e}^{-z}z^4\int_{C_A\infty}^{0^+}\mathrm{d}t\mathrm{e}^{-t}\frac{1}{t^2}\left(1+\frac{t}{z}\right)^2\int_{C_A\infty}^{0^+}\mathrm{d}t'\mathrm{e}^{-t'}\frac{1}{t'^2}\left(1+\frac{t'}{z}\right)^2\end{aligned} \tag{42}$$

Integral becomes as

$$I_3 = \int_{C_A\infty}^{0^+}\mathrm{d}t\mathrm{e}^{-t}\left(\frac{1}{t^2}+\frac{1}{t}\left(\frac{2}{z}\right)+\frac{1}{z^2}\right) \tag{43}$$

Using residue theorem I_3 becomes

$$I_3 = 2\pi\mathrm{i}\left(-1+\frac{2}{z}\right) \tag{44}$$

Thus, χ_3^2 is represented as

$$\chi_3^2 = \frac{1}{r^2}\Gamma(2)\Gamma(2)\mathrm{e}^{-z}\left(4z^2-4z^3+z^4\right) \tag{45}$$

Similarly, $\chi_2\chi_3$ and χ_2^2 become

$$\chi_2\chi_3 = \frac{1}{r^2}\Gamma(1)\Gamma(2)\mathrm{e}^{-z}\left(2z^{\frac{3}{2}}-3z^{\frac{5}{2}}+z^{\frac{7}{2}}\right) \tag{46}$$

$$\chi_2^2 = \frac{1}{r^2}\Gamma(1)\Gamma(1)\mathrm{e}^{-z}\left(z-2z^2+z^3\right) \tag{47}$$

Recalling $z=\frac{g^2L_1^{(2)}}{2}r^2$ and using Equation (40), we cannot obtain so called pole form factor which appears in experiment results [8]. To obtain this type of form factor, we have to invoke $\frac{1}{z^2}$ term in Equation (43). In this consideration, we also refer the following facts. For charged pion wave function, E. Ydrefors *et al.* [9] show that EMFF of the charged pion is q^{-4} and less (see Equation (10) and Fig. (2) in ref. [9]) when q^2 is larger and Martín Contreras *et al.* [10] show that EMFF of pion is q^{-4} (Equation (30) in ref. [10]) when q^2 is larger. While, E. Arriora *et al.* [11] show that EMFF of the charged pion becomes slightly larger than experiment results that is slightly larger than q^{-2} when considering extrapolation from time-like region to space-like region using the dispersion relation argument. In addition, K. Quilion *et al.* [12] also show that EMFF of pion is $\frac{\ln q^2}{q^2}$ when Reggeized $\alpha(0)<0$ case using also the dispersion relation argument. These results correspond to the loop calculation approach for the first two cases and the dispersion relation approach for the last two cases, respectively. Obviously, these two approaches obtain different results. In our case, solutions

without physics pole correspond to the first two types q^{-4}, q^{-6}, q^{-8} when q^2 is larger (asymptotic behavior of ${}_1F_1$ is shown on page 67 of ref. [7]) besides Gaussian like behavior ones as shown later (loop calculation approach) and the solution from the pure physics pole corresponds to the last two types (q^{-1} when q^2 is large as shown later) (dispersion relation type). From this consideration, imposing physics pole seems to have actual physics meaning.

Therefore, in this regard, we impose physics pole as

$$I_3 = \int_{C_A\infty}^{0^+} \mathrm{d}t \mathrm{e}^{-t} \left(\frac{1}{t^2} + \frac{1}{t}\left(\frac{2}{z}\right) + \frac{1}{t_{\text{phy}}}\left(\frac{1}{z^2}\right)\right) \tag{48}$$

By defining t_{phy} as $t_{\text{phy}} = \dfrac{t}{\alpha_{\text{phy}}}$, I_3 becomes

$$I_3 = 2\pi \mathrm{i}\left(\frac{\alpha_{\text{phy}}}{z^2} + \frac{2}{z} - 1\right) \tag{49}$$

Thus, χ_3^2 changes to

$$\begin{aligned} \chi_3^2 &= \frac{1}{r^2}\Gamma(2)\Gamma(2)\mathrm{e}^{-z}\left(\left(\alpha_{\text{phy}}\right)^2 + 4\alpha_{phy} z + \left(4-2\alpha_{\text{phy}}\right)z^2 - 4z^3 + z^4\right) \\ &= \frac{1}{r^2}\Gamma(2)\Gamma(2)\left(\left(\chi^{(3)}_{\text{phy pole only}}\right)^2 + \left(\chi^{(3)}_{\text{except phy pole}}\right)^2\right) \end{aligned} \tag{50}$$

We denote $\left(\chi^{(3)}_{\text{phy pole only}}\right)^2 = \mathrm{e}^{-z}\left(\alpha_{\text{phy}}\right)^2$ and

$\left(\chi^{(3)}_{\text{except phy pole}}\right)^2 = \mathrm{e}^{-z}\left(4\alpha_{\text{phy}} z + \left(4-2\alpha_{\text{phy}}\right)z^2 - 4z^3 + z^4\right)$, respectively.

For $\chi_2\chi_3$ case, $\chi_2\chi_3$ changes to following form

$$\chi_2\chi_3 = \frac{1}{r^2}\Gamma(1)\Gamma(2)\mathrm{e}^{-z}\left(\alpha_{\text{phy}} z^{\frac{1}{2}} + \left(2-\alpha_{\text{phy}}\right)z^{\frac{3}{2}} - 3z^{\frac{5}{2}} + z^{\frac{7}{2}}\right) \tag{51}$$

For χ_2^2 case, there is no change at all. Thus, even after imposing physics pole χ_2^2 is described as

$$\chi_2^2 = \frac{1}{r^2}\Gamma(1)\Gamma(1)\mathrm{e}^{-z}\left(z - 2z^2 + z^3\right) \tag{52}$$

Therefore, when we use Equation (50) type notation, $\chi_2\chi_3$ and χ_2^2 are represented as

$$\chi_2\chi_3 = \frac{1}{r^2}\Gamma(1)\Gamma(2)\left(\chi_2\chi_3\right)_{\text{except phy pole}} \tag{53}$$

$$\chi_2{}^2 = \frac{1}{r^2}\Gamma(1)\Gamma(1)\left(\chi^{(2)}_{\text{except phy pole}}\right)^2 \tag{54}$$

Recalling $\Gamma(1) = \Gamma(2) = 1$, we omit Gamma function from now on.

Using Equation (40), $\left(\chi^{(3)}_{\text{phy pole only}}\right)^2$ term gives

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\left(2g^2 L_1^{(2)}\right)^{\frac{1}{2}}} \exp\left(-\frac{q^2}{2g^2 L_1^{(2)}}\right) {}_1F_1\left(1, \frac{3}{2}; \frac{q^2}{2g^2 L_1^{(2)}}\right)$$

This form factor is exactly the same form as our EMFF of the charged pion $\pi^{\pm}$ whose asymptotic behavior is q^{-1} and this corresponds to a normal pole type form factor whose asymptotic behavior is q^{-2}. Recalling the fact that experimental results for η meson show this type of form factor, we can consider that most of the contribution to the form factor comes from the $\left(\chi^{(3)}_{\text{phy pole only}}\right)^2$ part.

Thus, the EMFF of η meson described in Equation (36) can be rewritten as

$$\begin{aligned} F_\eta &= \int_0^\infty \mathrm{d}r r^2 \left|a\chi_2 \pm b\chi_3\right|^2 \frac{\sin(qr)}{qr} = \left|\int_0^\infty \mathrm{d}r r^2 \left(a\chi_2 \pm b\chi_3\right)^2 \frac{\sin(qr)}{qr}\right| \\ &= \left|\int_{r_0}^\infty \mathrm{d}r r^2 b^2 \left(\chi^{(3)}_{\text{phy pole only}}\right)^2\right| \\ &\quad + \left|\int_0^{r_0} \mathrm{d}r r^2 \left(a^2 \left(\chi^{(2)}_{\text{except phy pole}}\right)^2 \pm 2ab\left(\chi_2\chi_3\right)_{\text{except phy pole}} + b^2 \left(\chi^{(3)}_{\text{except phy pole}}\right)^2\right)\right| \end{aligned} \tag{55}$$

Because, as mentioned above, $\left|\int_{r_0}^\infty \mathrm{d}r r^2 b^2 \left(\chi^{(3)}_{\text{phy pole only}}\right)^2 \frac{\sin(qr)}{qr}\right|$ contributes to experiment results, r_0 must be small, very close to 0. Because of $\left|\int_{r_0}^\infty \mathrm{d}r r^2 b^2 \left(\chi^{(3)}_{\text{phy pole only}}\right)^2 \frac{\sin(qr)}{qr}\right| = \left|\int_{r_0}^\infty \mathrm{d}r r^2 \left(-b^2\right)\left(\chi^{(3)}_{\text{phy pole only}}\right)^2 \frac{\sin(qr)}{qr}\right|$, we have to rewrite

$$\left|\int_0^{r_0} \mathrm{d}r r^2 \left(a^2 \left(\chi^{(2)}_{\text{except phy pole}}\right)^2 \pm 2ab\left(\chi_2\chi_3\right)_{\text{except phy pole}} + b^2 \left(\chi^{(3)}_{\text{except phy pole}}\right)^2\right)\frac{\sin(qr)}{qr}\right|$$

as $\left|\int_0^{r_0} \mathrm{d}r r^2 \left(a^2 \left(\chi^{(2)}_{\text{except phy pole}}\right)^2 \pm 2ab\left(\chi_2\chi_3\right)_{\text{except phy pole}} - b^2 \left(\chi^{(3)}_{\text{except phy pole}}\right)^2\right)\frac{\sin(qr)}{qr}\right|$

in the case of $\left|\int_{r_0}^\infty \mathrm{d}r r^2 \left(-b^2\right)\left(\chi^{(3)}_{\text{phy pole only}}\right)^2 \frac{\sin(qr)}{qr}\right|$. In addition, because r_0 is very small, we can describe

$$\left|\int_0^{r_0} \mathrm{d}r r^2 \left(a^2 \left(\chi^{(2)}_{\text{except phy pole}}\right)^2 \pm 2ab\left(\chi_2\chi_3\right)_{\text{except phy pole}} - b^2 \left(\chi^{(3)}_{\text{except phy pole}}\right)^2\right)\frac{\sin(qr)}{qr}\right|$$

as $\left|\int_0^{r_0} \mathrm{d}r\, r^2 \left(\left(\chi^{(2)}_{\text{except phy pole}}\right)^2 - F\left(\chi_2\chi_3\right)_{\text{except phy pole}} - b^2 \left(\chi^{(3)}_{\text{except phy pole}}\right)^2\right)\frac{\sin(qr)}{qr}\right|$

under the condition that F and b must be small compared to 1 (we set $a = 1$ in this case). This is because χ_2 goes to a constant when r approaches 0, while χ_3 goes to 0 when r approaches 0 (recalling $\chi_2 = \frac{1}{r} W_{\frac{3}{2},0}\left(\frac{g^2 L_1^{(2)}}{2} r^2\right)$ and $\chi_3 = \frac{1}{r} W_{2,\frac{1}{2}}\left(\frac{g^2 L_1^{(2)}}{2} r^2\right)$). F and b are used as parameters to obtain a closed form for the EMFF corresponding to the TPE EMFF by Alexandorou [1] because we consider that contribution of

$\left|\int_0^{r_0} \mathrm{d}r r^2 \left(\left(\chi^{(2)}_{\text{except phy pole}}\right)^2 - F\left(\chi_2\chi_3\right)_{\text{except phy pole}} - b^2 \left(\chi^{(3)}_{\text{except phy pole}}\right)^2\right)\frac{\sin(qr)}{qr}\right|$ part corresponds to TPE EMFF by Alexandrou even though our case is one photon exchange case (OPE).

To evaluate $\left|\int_0^{r_0} \mathrm{d}r r^2 \left(\left(\chi^{(2)}_{\text{except phy pole}} \right)^2 - F\left(\chi_2 \chi_3 \right)_{\text{except phy pole}} - b^2 \left(\chi^{(3)}_{\text{except phy pole}} \right)^2 \right) \frac{\sin(qr)}{qr} \right|$ we set r_0 as ∞, while to evaluate $\left|\int_{r_0}^{\infty} \mathrm{d}r r^2 b^2 \left(\chi^{(3)}_{\text{phy pole only}} \right)^2 \frac{\sin(qr)}{qr} \right|$ we set r_0 as 0. Thus, $\left|\int_0^{r_0} \mathrm{d}r r^2 \left(\left(\chi^{(2)}_{\text{except phy pole}} \right)^2 - F\left(\chi_2 \chi_3 \right)_{\text{except phy pole}} - b^2 \left(\chi^{(3)}_{\text{except phy pole}} \right)^2 \right) \frac{\sin(qr)}{qr} \right|$ integral becomes

$$\left|\int_0^{\infty} \mathrm{d}r r^2 \left(\left(\chi^{(2)}_{\text{except phy pole}} \right)^2 - F\left(\chi_2 \chi_3 \right)_{\text{except phy pole}} - b^2 \left(\chi^{(3)}_{\text{except phy pole}} \right)^2 \right) \frac{\sin(qr)}{qr} \right| \quad (56)$$

Recalling $z = \frac{g^2 L_1^{(2)}}{2} r^2$ and using Equation (40), the integral of each exponent of z in Equation (56) becomes as follows;

For χ_3^2 and χ_2^2 case, (note that $g^2 L_1^{(2)} = m_\eta^2$ (m_η is η meson mass).

$$\frac{1}{q}\int_0^{\infty} \mathrm{d}r \frac{1}{r} \sin(qr) \mathrm{e}^{-z} z = \frac{\Gamma\left(\frac{3}{2}\right)}{\left(2m_\eta^2\right)^{\frac{1}{2}}} \mathrm{e}^{-x} {}_1F_1\left(0, \frac{3}{2}; x\right) = \frac{\Gamma\left(\frac{3}{2}\right)}{\left(2m_\eta^2\right)^{\frac{1}{2}}} \mathrm{e}^{-x}$$

$$\frac{1}{q}\int_0^{\infty} \mathrm{d}r \frac{1}{r} \sin(qr) \mathrm{e}^{-z} z^2 = \frac{\Gamma\left(\frac{5}{2}\right)}{\left(2m_\eta^2\right)^{\frac{1}{2}}} \mathrm{e}^{-x} {}_1F_1\left(-1, \frac{3}{2}; x\right) = \frac{\Gamma\left(\frac{5}{2}\right)}{\left(2m_\eta^2\right)^{\frac{1}{2}}} \mathrm{e}^{-x} \left(1 - \frac{2}{3}x\right)$$

$$\frac{1}{q}\int_0^{\infty} \mathrm{d}r \frac{1}{r} \sin(qr) \mathrm{e}^{-z} z^3 = \frac{\Gamma\left(\frac{7}{2}\right)}{\left(2m_\eta^2\right)^{\frac{1}{2}}} \mathrm{e}^{-x} {}_1F_1\left(-2, \frac{3}{2}; x\right)$$

$$= \frac{\Gamma\left(\frac{7}{2}\right)}{\left(2m_\eta^2\right)^{\frac{1}{2}}} \mathrm{e}^{-x} \left(1 - \frac{4}{3}x + \frac{4}{15}x^2\right)$$

$$\frac{1}{q}\int_0^{\infty} \mathrm{d}r \frac{1}{r} \sin(qr) \mathrm{e}^{-z} z^4 = \frac{\Gamma\left(\frac{9}{2}\right)}{\left(2m_\eta^2\right)^{\frac{1}{2}}} \mathrm{e}^{-x} {}_1F_1\left(-3, \frac{3}{2}; x\right)$$

$$= \frac{\Gamma\left(\frac{9}{2}\right)}{\left(2m_\eta^2\right)^{\frac{1}{2}}} \mathrm{e}^{-x} \left(1 - 2x + \frac{4}{5}x^2 - \frac{8}{105}x^3\right)$$

Note that $\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$, $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$, $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\frac{1}{2}\sqrt{\pi}$, $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

For $\chi_2 \chi_3$ case

$$\frac{1}{q}\int_0^\infty \mathrm{d}r \frac{1}{r}\sin(qr)\mathrm{e}^{-z} z^{\frac{1}{2}} = \frac{\Gamma(1)}{\left(2m_\eta^2\right)^{\frac{1}{2}}}\mathrm{e}^{-x}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};x\right)$$

$$\frac{1}{q}\int_0^\infty \mathrm{d}r \frac{1}{r}\sin(qr)\mathrm{e}^{-z} z^{\frac{3}{2}} = \frac{\Gamma(2)}{\left(2m_\eta^2\right)^{\frac{1}{2}}}\mathrm{e}^{-x}\,{}_1F_1\left(-\frac{1}{2},\frac{3}{2};x\right)$$

$$\frac{1}{q}\int_0^\infty \mathrm{d}r \frac{1}{r}\sin(qr)\mathrm{e}^{-z} z^{\frac{5}{2}} = \frac{\Gamma(3)}{\left(2m_\eta^2\right)^{\frac{1}{2}}}\mathrm{e}^{-x}\,{}_1F_1\left(-\frac{3}{2},\frac{3}{2};x\right)$$

$$\frac{1}{q}\int_0^\infty \mathrm{d}r \frac{1}{r}\sin(qr)\mathrm{e}^{-z} z^{\frac{7}{2}} = \frac{\Gamma(4)}{\left(2m_\eta^2\right)^{\frac{1}{2}}}\mathrm{e}^{-x}\,{}_1F_1\left(-\frac{5}{2},\frac{3}{2};x\right)$$

where x denotes $x=\dfrac{q^2}{2m_\eta^2}$, and note that $\Gamma(3)=2$, $\Gamma(4)=6$.

Asymptotic form of ${}_1F_1$ is described as [7]

$${}_1F_1(\alpha',\gamma';z) \sim \frac{\Gamma(\gamma')}{\Gamma(\alpha')}\mathrm{e}^z z^{\alpha'-\gamma'} + \frac{\Gamma(\gamma')}{\Gamma(\gamma'-\alpha')}(-z)^{-\alpha'}$$

where $iz \neq$ negative real value and $|z|$ is sufficiently large.

Note that we omit the summation part for simplicity.

Thus, asymptotic behavior of solutions without physics pole which correspond to $\mathrm{e}^{-x}\,{}_1F_1\left(-\frac{1}{2},\frac{3}{2};x\right)$, $\mathrm{e}^{-x}\,{}_1F_1\left(-\frac{3}{2},\frac{3}{2};x\right)$ and $\mathrm{e}^{-x}\,{}_1F_1\left(-\frac{3}{2},\frac{3}{2};x\right)$ become as $\dfrac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)}q^{-4}$, $\dfrac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(-\frac{3}{2}\right)}q^{-6}$, $\dfrac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(-\frac{5}{2}\right)}q^{-8}$ as mentioned before. Note that

$\Gamma\left(-n+\frac{1}{2}\right)=\dfrac{(-1)^n 2^n}{(2n-1)!!}\sqrt{\pi}$ [7] so that $-\chi_2\chi_3$ is appropriate.

For pure physics pole case, ${}_1F_1\left(1,\frac{3}{2};x\right) \sim \dfrac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}\left(q^2\right)^{-\frac{1}{2}} = \dfrac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)}q^{-1}$.

Important point is description of α_{phy}. For $\chi_2\chi_3$ case, each exponent of x is described as

$$-\chi_2\chi_3 = -2+\left(\frac{14}{3}+\frac{-2\alpha_{\mathrm{phy}}}{3}\right)x+\left(\frac{-2\alpha_{\mathrm{phy}}}{15}-\frac{7}{3}\right)x^2+\left(\frac{-\alpha_{\mathrm{phy}}}{35}+\frac{19}{105}\right)x^3$$
$$+\sum_{n=4}^{\infty}\left(-\alpha_{\mathrm{phy}}\frac{\frac{1}{2}}{n+\frac{1}{2}}+\frac{2-\alpha_{\mathrm{phy}}}{4}\frac{1}{\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}+\frac{\frac{9}{4}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}\right.$$
$$\left.+\frac{\frac{45}{8}}{\left(n-\frac{5}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)}\right)\frac{x^n}{n!}$$

Thus we set $\alpha_{\text{phy}} = \frac{\bar{\alpha}}{b} = -\frac{|\bar{\alpha}|}{b}$ because this setting guarantees that the corresponding pole type part becomes independent of b and the exponent of x larger than 4 terms becomes absolutely positive.

Therefore Equation (55) becomes as

$$\begin{aligned} F_\eta^{(2)} + F_\eta^{(3)} = \frac{1}{\left(2m_\eta^2\right)^{\frac{1}{2}}} \mathrm{e}^{-x} \Bigg[& \left(\frac{7}{8} - \frac{33}{16} b^2 + b|\bar{\alpha}| \right) \sqrt{\pi} - 2F \\ & + \left(\left(-\frac{3}{2} + \frac{41}{8} b^2 + 2b|\bar{\alpha}| \right) \sqrt{\pi} + \left(\frac{14}{3} + 2\frac{\bar{\alpha}}{b} \right) F \right) x \\ & + \left(\left(\frac{1}{2} - \frac{13}{4} b^2 \right) \sqrt{\pi} + \left(-\frac{7}{3} + \frac{2}{15} \frac{|\bar{\alpha}|}{b} \right) F \right) x^2 \\ & + \left(\left(\frac{1}{2} b^2 \right) \sqrt{\pi} + \left(\frac{19}{105} + \frac{1}{35} \frac{|\bar{\alpha}|}{b} \right) F \right) \\ & + F \left(\sum_{n=4}^{\infty} \left(\frac{|\alpha|}{b} \frac{\frac{1}{2}}{\left(n+\frac{1}{2}\right)} + \frac{\frac{1}{4}\frac{|\bar{\alpha}|}{b} + \frac{1}{2}}{\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)} \right. \right. \\ & + \frac{\frac{9}{4}}{\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)} \\ & \left. \left. + \frac{\frac{45}{8}}{\left(n-\frac{5}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{1}{2}\right)\left(n+\frac{1}{2}\right)} \right) \right) \Bigg] \end{aligned} \tag{57}$$

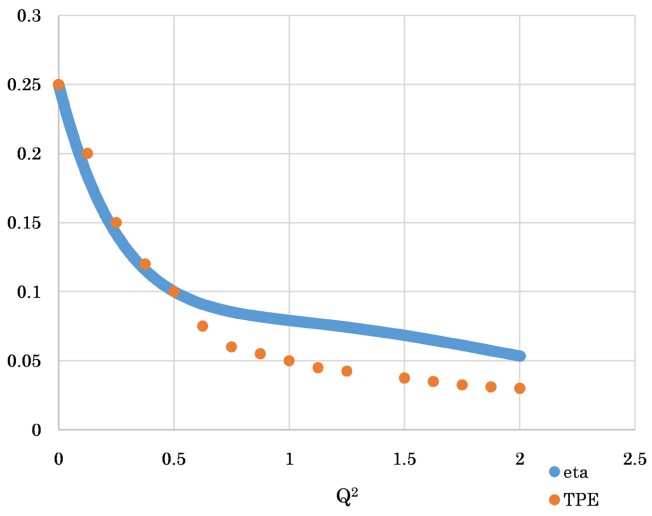


Figure 1. EMFF contributed by except pure physics pole part of our wave functions (blue curve): Image of EMFF contributed by two photon exchange by Alexandrou (red circle; exact form of this EMFF refer to ref. [1].

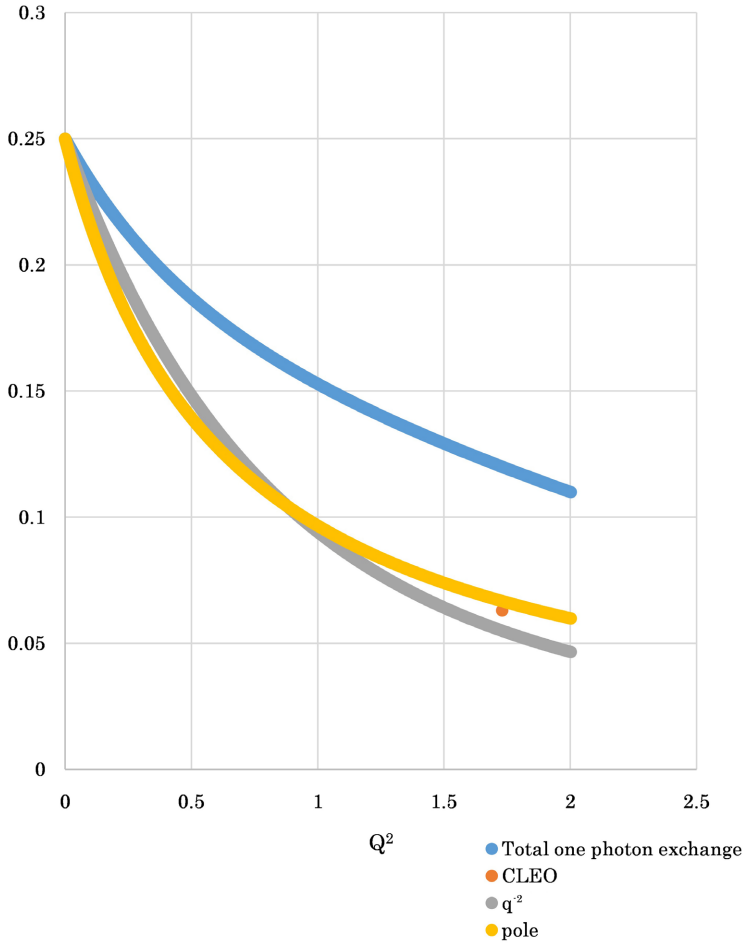


Figure 2. Our EMFF of η mesons that is due to one photon exchange (blue curve); CLEO experiment data (red circle); q^{-2} shows $\mathrm{e}^{-x}{}_1F_1\left(\frac{1}{2},\frac{3}{2};x\right)$ curve of which asymptotic form is $\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\mathrm{e}^x x^{-1}$ $\left(x=\frac{q^2}{2m_\eta^2}\right)$ $\left(-1=\frac{1}{2}-\frac{3}{2}\right)$ [7] (gray curve); pole shows normal pole type EMFF of which form is $\frac{1}{1+\frac{q^2}{\Lambda^2}}$ (Λ^2 is a parameter, in this figure $\Lambda^2=0.63\left(\mathrm{GeV}^2\right)$ (yellow curve).

For pure pole part,

$$F_\eta^{(1)}=\frac{\sqrt{\pi}}{\left(2m_\eta^2\right)^{\frac{1}{2}}}\left|\bar{\alpha}\right|^2\mathrm{e}^{-x}{}_1F_1\left(1,\frac{3}{2};x\right) \tag{58}$$

where $x=\frac{q^2}{2m_\eta^2}$

Because EMFF (electromagnetic form factor) is normalized, we omit $\frac{1}{\left(2m_\eta^2\right)^{\frac{1}{2}}}$

for evaluation. Then using b, F and $|\bar{\alpha}|$ values as $b=0.1$, $F=b^2=0.01$, $|\bar{\alpha}|=3$, $m_\eta=0.548(\mathrm{GeV})$, we obtain the following **Figure 1** for $F_\eta^{(2)}+F_\eta^{(3)}$ and **Figure 2** for $F_\eta^{(1)}$

Figure 1 shows a comparison of our η meson EMFF contributed by the part of our wave functions excluding pure physics pole part to the image of η meson TPE EMFF by Alexandrou [1]. This supports our consideration that η meson EMFF contributed by the parts of our wave functions excluding pure physics pole part should be close to Alexandrou's TPE EMFF even though this is only an image. **Figure 2** shows our η meson EMFF contributed by our full wave functions and CLEO collaboration experiment data. There is only one data point, but this shows a clear discrepancy. However, our EMFF is $\mathrm{e}^{-x}{}_1F_1\left(1,\frac{3}{2},x\right)$ that is exactly the same form as our charged pion EMFF whose asymptotic behavior is q^{-1} [3]. Our charged pion EMFF itself is little bit larger than experiment results of q^{-2}. This asymptotic q^{-2} behavior appears in η meson EMFF case as shown in **Figure 2**. This means that our η meson EMFF is consistent with the results in our formalism.

3. Summary

We obtain a part of the EMFF of η meson whose behavior is close to Alexandrou's TPE EMFF η meson by imposing a physics pole. In addition, pole type EMFF appears for all κ case from $\left|\int_{r_0}^{\infty}\mathrm{d}r r^2 b^2\left(\chi^{(3)}_{\text{phy pole only}}\right)^2\frac{\sin(qr)}{qr}\right|$ term when $r_0=0$ in χ_3^2 term once invoking the physics pole. Thus, for ρ and η' mesons case, our consideration gives $\mathrm{e}^{-x}{}_1F_1\left(1,\frac{3}{2};x\right)$ as EMFF ($x=\frac{q^2}{2m_i^2}$ $i=\rho$ or η'). However, for ρ meson case, ρ meson is not a pseudoscalar meson but a vector meson which has spin 1 and cannot deal with ρ meson as a point-like particle like the pion. Thus, in experimental results, pole type EMFF has not appeared, while experimental results show η' meson (pseudoscalar meson, same as the charged pion and η meson) has pole type EMFF [8] [13]. Thus, we consider that imposing a physics pole is necessary to investigate the EMFF of pseudoscalar mesons. Even ρ meson case, $\left|\int_0^{\infty}\mathrm{d}r r^2\left(\left(\chi^{(2)}_{\text{except phy pole}}\right)^2-F\left(\chi_2\chi_3\right)_{\text{except phy pole}}-b^2\left(\chi^{(3)}_{\text{except phy pole}}\right)^2\right)\frac{\sin(qr)}{qr}\right|$ part of imposing the physics pole method may work for investigating one of the EMFF before including quark charge corresponding to F_1^ρ in Hutauruk *et al.*'s paper [14] because by multiplying by a factor of 0.25, behavior of F_1^ρ seems to be similar to TPE EMFF by Alexandorou [1] up to $Q^2=2\left(\mathrm{GeV}^2\right)$ and following property of ρ meson. Note that ρ meson corresponds to $\kappa=3$ for χ_3 and $\bar{\kappa}=\frac{5}{2}$ for χ_2 so that largest exponent of z for χ_3^2 gives ${}_1F_1\left(-5,\frac{3}{2},x\right)$,

that is, largest exponent of x term is $-x^5$ and largest exponent of z for χ_2^2 gives ${}_1F_1\left(-4,\frac{3}{2},x\right)$, that is, largest exponent of x term is $+x^4$. Also, the largest exponent of z of $\chi_2\chi_3$ gives ${}_1F_1\left(-\frac{9}{2},\frac{3}{2},x\right)$ and smallest exponent of z of $\chi_2\chi_3$ gives ${}_1F_1\left(-\frac{1}{2},\frac{3}{2},x\right)$ so that sign of asymptotic behavior of each $\chi_2\chi_3$ term is $-,+,-,+,-$, while the sign of each exponent of z of $\chi_2\chi_3$ is $+,-,+,-,+$. Thus, all terms of $\chi_2\chi_3$ of which exponent of x is larger than 6 become negative so that we need $-\chi_2\chi_3$ which is exactly the same sign as η meson case. Therefore, we can use the same argument as that for η meson case. However, we can safely say that our approach may work only for pseudoscalar mesons basically. Note that, for η' meson case, that is, $\kappa=5$ for χ_3 and $\bar{\kappa}=\frac{9}{2}$ for χ_2, we also notice that we can use the same argument as that for η meson case by checking the sign of χ_2^2, χ_3^2 and $\chi_2\chi_3$.

Therefore, imposing the physics pole method shown in η meson case may work for describing the pseudoscalar meson's EMFF more profoundly.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Alexandrou, C., Bcchio, S., Burri, S., Hadjiyiannkou, K., *et al.* (2023) The $\eta\to\gamma^*\gamma^*$ Transition form Factor and the Hadronic Light-by-Light η-Pole Contribution to the Muon $g-2$ from Lattice QCD. *Physical Review D*, **108**, Article 054509.

[2] Gerardin, A., Verpland, W.E.A., Wang, G., Fodor, Z., Guenther, J.N., Lellouch, L., Szabo, K.K. and Varnhorst, L. (2025) Lattice Calculation of π^0,η and η' Transition form Factors and Hadronic Light-by-Light Contribution to Muon $g-2$. *Physical Review D*, **111**, Article 054511.

[3] Kurai, T. (2018) Light Meson Mass Spectra and Pion Electromagnetic Form Factor as a Bound System in 3 + 1 Dimensional QCD. *Results in Physics*, **10**, 865-881. https://doi.org/10.1016/j.rinp.2018.07.034

[4] Kurai, T. (2022) Erratum: Light Meson Mass Spectra and Pion Electromagnetic Form Factor as a Bound System in 3+1 Dimensional QCD [Results Phys. 10 (2018) 865-881]. *Results in Physics*, **43**, Article 106109. https://doi.org/10.1016/j.rinp.2022.106109

[5] Kurai, T. (2021) Light Meson Mass Spectra with Massive Quarks. *Journal of Modern Physics*, **12**, 1545-1572. https://doi.org/10.4236/jmp.2021.1211093

[6] Gradshteyn, I.S. and Ryuzhik, M. (1980) Table of Integrals, Series, and Products (Corrected and Enlarged Edition). Academic Press.

[7] Moriguti, S., Udagawa, K. and Hitotsumatu, S. (1975) Mathematical Formula III (Special Functions). Iwanami.

[8] Gronberg, J., Bloom, K.A. and CLEO Collaboration (1998) Measurements of the Me-

son-Photon Transition Formfactors of Pseud-Scalar Light Meson at Large Momentum Transfer. *Physical Review D*, **57**, 1-23.

[9] Ydrefors, E., de Paula, W., Alvarenga Nogueira, J.H., Frederico, T. and Salmè, G. (2021) Pion Electromagnetic Form Factor with Minkowskian Dynamics. *Physics Letters B*, **820**, Article 136494. https://doi.org/10.1016/j.physletb.2021.136494

[10] Martín Contreras, M.A., Capossoli, E.F., Li, D., Vega, A. and Boschi-Filho, H. (2022) Pion Form Factor from an Ads Deformed Background. *Nuclear Physics B*, **977**, Article 115726. https://doi.org/10.1016/j.nuclphysb.2022.115726

[11] Arriora, E.R. and Sanchez-Puertas, P. (2024) Phase of the Electromagnetic Form Factor of the Pion. *Physical Review D*, **110**, Article 054003. https://doi.org/10.1103/physrevd.110.054003

[12] Quirion, K., Fernández-Ramírez, C., Mathieu, V., Montaña, G., Perry, R. ., Pilloni, A., *et al.* (2025) Nonperturbative Aspects of the Electromagnetic Pion Form Factor at High Energies. *Physical Review D*, **111**, Article 054034. https://doi.org/10.1103/physrevd.111.054034

[13] Acciarri, M., Adriani, O., L3 Collaboration, *et al.* (1989) Measurement of η' (958) Formation in Two Photon Collision at LEP-1. *Physics Letters B*, **418**, 399-410.

[14] Hutauruk, P.T., Mart, T. and Tsushima, K. (2025) Medium Effects on the Electromagnetic form Factors of the ρ Meson. *Physical Review D*, **112**, Article 114030. https://doi.org/10.1103/95yl-nwds