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# Physical and Mathematical Inconsistency of T. Damour's Criticisms of the Janus Cosmological Model

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## Abstract

Since 2019, two articles by T. Damour—published on his personal webpage hosted by the IHES site and addressing the alleged physical and mathematical inconsistency of the Janus cosmological model—have posed a major obstacle to the scientific community's consideration of this model, despite several points of qualitative agreement with astrophysical and cosmological observations. As these criticisms were not published in peer-reviewed journals, the controversy could not unfold within the standard editorial framework where critiques and responses are subject to review by referees. We first outline the independent origins of the Janus cosmological model, its theoretical foundations, its main results, and its qualitative observational consequences. We revisit the simultaneous satisfaction of the constraints arising from the two Bianchi identities—established in an explicit solution published as early as 2019 but overlooked in T. Damour's subsequent critiques. We then re-examine, point by point, the objections he raised. In particular, although a variational derivation of the coupled Janus field equations has already been published, we present a particularly direct illustration of their variational origin. In 2002, Damour and Kogan introduced an action describing two interacting dynamical metrics within the framework of non-linear massive bigravity. We show that by choosing gravitational coupling constants equal in magnitude but opposite in sign, the characteristic structure of the Janus model's coupled field equations emerges naturally from this construction. Regarding the contradiction concerning the nature of gravitational interactions raised by T. Damour in 2019—and maintained in his 2022 article—we demonstrate that his reasoning requires the same four-velocity field of a test particle to simultaneously satisfy the geodesic equations associated with both metrics. This prescription is not among the assumptions defining the Janus model: positive-mass test particles follow the geodesics of the first metric, whereas negative-mass test

particles follow those of the second. In the stationary regime, spherically symmetric exterior metrics thus allow for the direct recovery of mutual attraction between masses of the same sign and mutual repulsion between masses of opposite signs.

## Keywords

Bimetric Gravity, Janus Cosmological Model, Negative Mass, Positive and Negative Masses, Gravitational Coupling Constants, Opposite Gravitational Couplings, Damour-Kogan Bigravity, Hossenfelder Bimetric Theory, Runaway Problem, Interacting Metrics, Bianchi Identities, FLRW Cosmology, Accelerated Expansion, Tolman-Oppenheimer-Volkoff Equations, Newtonian Limit.

## 1. Introduction

The standard cosmological model is based on the field equation introduced in 1915 by A. Einstein [1]. It assumes that the universe is a four-dimensional manifold associated with a Lorentzian metric $g_{\mu\nu}$, wherein masses follow non-null geodesics derived from that metric, while light follows null geodesics. In its most general form, this equation includes a constant Λ—known as the cosmological constant—whose role was long overlooked, as there was no clear observational data to which it could be linked.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \chi T_{\mu\nu} \tag{1}$$

By adding the assumption of homogeneity and isotropy—where $k$ is the curvature index, $(u,\theta,\varphi)$ represents the dimensionless forms of the spatial coordinates under spherical symmetry, and $a(t)$ is a scale factor—the metric takes the FLRW form:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a\left(\frac{\mathrm{d}u^2}{1-ku^2} + u^2\mathrm{d}\theta^2 + u^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{2}$$

When introduced into the field equation, this metric leads to three possible solutions, depending on the value of the curvature index $k \in \{-1,0,+1\}$

$\rho > 10^{-29}\ \mathrm{gr/cm^3} \rightarrow k = +1$ Positive curvature, elliptic model

$\rho < 10^{-29}\ \mathrm{gr/cm^3} \rightarrow k = -1$ Negative curvature, hyperbolic model

$\rho = 10^{-29}\ \mathrm{gr/cm^3} \rightarrow k = 0$ Zero curvature, parabolic model

In all scenarios, the expansion proceeded with a deceleration:

$$\frac{\mathrm{d}^2a}{\mathrm{d}t^2} < 0 \tag{3}$$

One problem remained—and still remains unresolved: this solution confirmed the observational discovery of the universe's expansion. Consequently, tracing

back in time implied the existence of a hyper-dense, extremely hot state in which the universe's content consisted of a mixture of antimatter in equilibrium with radiation; high-energy photons were constantly creating pairs of matter and antimatter particles, which would almost immediately annihilate each other, regenerating those same high-energy photons. As the wavelength of these photons increased with the scale factor, the persistence of such primordial radiation would manifest as the observation of centimeter-wavelength photons forming an isotropic, homogeneous background—a phenomenon indeed observed as early as the 1960s. However, since these photons could no longer offset the annihilations, the result should have been the complete disappearance of those matter and antimatter particles. Yet:

- One particle of matter in a billion survived.
- No trace of its antimatter counterpart is found.

All attempts to detect this primordial antimatter have ended in failure. Thus, in this model, one begins by losing half the universe's content—no small matter. To date, no theoretical justification has been proposed to account for this state of affairs. The situation has reached the point where there is no debate comparing competing theories, nor any international conference where this issue is addressed.

In the 1970s, the shape of galactic rotation curves—exhibiting a plateau in the outer regions—could no longer be reconciled with the gravitational field generated solely by visible matter. Specialists were thus led to propose that this field might be attributed to a vast halo of a new, unknown component termed "dark matter." Despite decades of experimental research, no particle constituting dark matter has yet been conclusively identified.

Twenty years later, measurements taken from distant sources led specialists to acknowledge that the universe was undergoing accelerated expansion. The cosmological constant was simply reintroduced into the field equation, on the grounds that its mere presence accounted for this phenomenon—though the nature of this new component, termed "dark energy," remained unexplained. In any case, within this model, the expansion and the acceleration are exponential.

The DESI results from the first three years, announced in March 2025 [2], reinforce the indication that dark energy might evolve over time. DESI indicates that the preference for dynamic dark energy strengthens when BAO, CMB, and Type Ia supernova data are combined. Notably, extended analyses show that this indication is robust at low redshift ($z < 0.3$). This remains a statistical indication, not a definitively established discovery.

Once the map of the cosmic microwave background radiation was produced, its extreme uniformity became apparent. Consequently, the standard model was supplemented by a phase known as inflation, during which—in a timespan ranging from $10^{-36}$ to $10^{-30}$ seconds—the universe is thought to have expanded by a factor exceeding $10^{26}$ under the influence of a field generated by particles dubbed "inflatons." To date, there is no model for such particles, nor any mechanism ex-

plaining their conversion into the known and measured constituents of the universe.

Nevertheless, in-depth analysis of CMB fluctuations has given rise to a new discipline based on the use of a six-parameter model—namely, baryon density, cold dark matter density, the acoustic angular scale, the optical depth of reionization, and the amplitude and spectral index of primordial perturbations.

This involved adjusting the value of one of the parameters—the Hubble constant—to a very specific figure: $H_0 = 67\ \mathrm{km\cdot s^{-1}\cdot Mpc^{-1}} \pm 0.5\ \mathrm{km\cdot s^{-1}}$. For a long time, this value could be considered to fall within the margin of uncertainty associated with the observational data. However, improved precision leading to a value of $H_0 = 73\ \mathrm{Mpc^{-1}} \pm 0.8\ \mathrm{km\cdot s^{-1}}$ has given rise to what has been termed the "Hubble tension". See Figure 1.

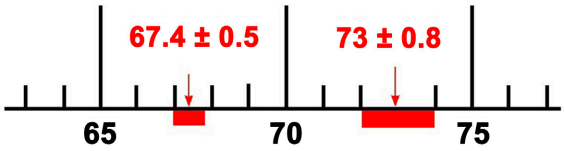



Figure 1. The Hubble tension. The two ranges do not overlap [3].

All these factors justify considering a shift in the cosmological paradigm. This is what the Janus Cosmological Model proposes; it essentially involves introducing particles with negative mass and energy into the model.

## 2. Negative Masses? Impossible, Because of the Runaway Effect!

Such an introduction was proposed as early as 1975 by Hermann Bondi [4]. General relativity, through its Newtonian approximation, immediately reveals the nature of the forces acting on test particles, leading to an immediate observation: in a field created by a given source, there exists only a single family of geodesics, which are followed by any test particle—whether it possesses positive or negative mass. The simplest approach is to focus on non-null geodesics associated with a field created by a mass $M$. If this mass is positive, the resulting metric is:

$$\mathrm{d}s^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-\frac{2GM}{c^2 r}} - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) \tag{4}$$

The geodesics corresponding to positive or negative test masses then have the form as shown on Figure 2.

Let us now imagine that this field is created by a negative mass. The external metric gives geodesics paths as shown on Figure 3.

This leads to the force configuration shown below, which immediately gives rise to a paradox known as "runaway": when two masses of equal absolute value but opposite signs are placed near each other, the negative mass flees, pursued by the positive mass. Both acquire kinetic energy (negative for the negative mass), and

their absolute values increase indefinitely. Furthermore, the action-reaction principle is violated. See the schema of Figure 4.

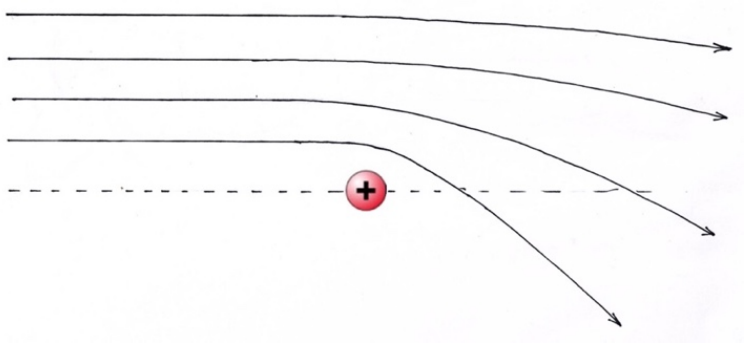


Figure 2. Geodesics in a field created by a positive mass in general relativity.

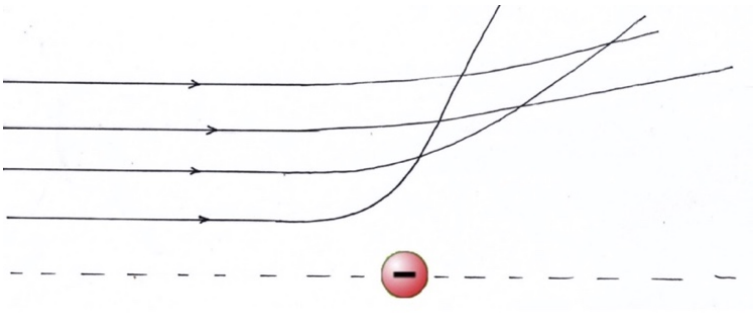


Figure 3. Geodesics in a field created by a negative mass, in general relativity.

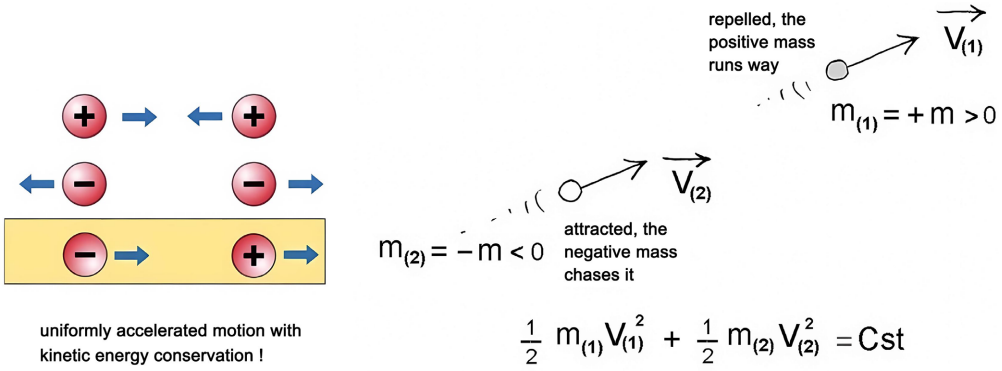



Figure 4. Runaway phenomenon.

Hence the rejection by the entire scientific community of the idea of introducing negative masses into the cosmological model.

## 3. In Search of a Different Geometric Context

As early as 1974, in the now-defunct journal Il Nuovo Cimento, we published [5] what can be considered an initial sketch of the geometric framework of the Janus model.

The article proposed a rudimentary field equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \chi\left[T_{\mu\nu} - A\left(T_{\mu\nu}\right)\right] \tag{5}$$

A evoking antipodality.

The central geometric idea is to view a closed spacetime as the double covering of a projective space. The involution that swaps the two sheets of this covering is the antipodal involution. We will show how this can induce a $PT$ symmetry.

To make this construction immediately visualizable, let us begin with its two-dimensional analogue.

Let us consider a compact two-dimensional spacetime, comprising one spatial dimension and one temporal dimension. We represent it as a sphere, $S^2$. The meridians represent the lines of time, while the parallels represent space. Its extent is zero at the first pole—representing the Big Bang—increases to a maximum value at the equator, and then decreases to the second pole, representing the Big Crunch. See Figure 5.

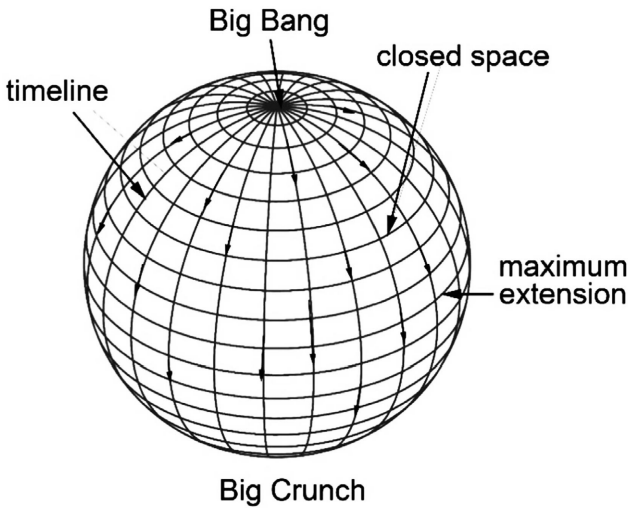



Figure 5. A two-dimensional spacetime.

Let us now consider the antipodal involution.

$$A: X \mapsto -X \tag{6}$$

It maps every point $M$ on the sphere to its antipode $M'$. Identifying all pairs of antipodal points defines the real projective plane:

$$\mathbb{RP}^2 = S^2 / \left(X \sim -X\right) \tag{7}$$

Its canonical projection

$$\pi : S^2 \to \mathbb{RP}^2 \tag{8}$$

is a two-sheeted covering. For any point $q \in \mathbb{RP}^2$

$$\pi^{-1}\left(q\right) = \left\{X, -X\right\} \tag{9}$$

The projective plane $\mathbb{RP}^2$ cannot be embedded in $\mathbb{R}^3$ without self-intersections. It can, however, be immersed, and Boy's surface is one of its classical repre-

sentations.

Figure 6 provides a visual representation of this construction. On the left are two antipodal points. In the center and on the right is an immersion of the projective quotient. In this particular representation, the neighborhood of the equator takes the form of a Möbius strip with three half-twists. See Figure 6.

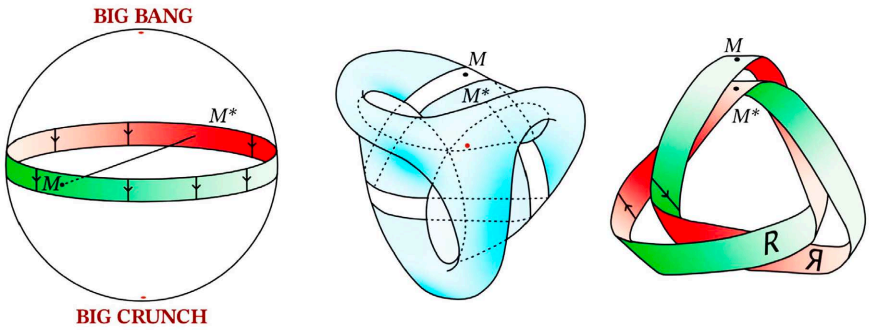



Figure 6. Identification of antipodal points on $S^2$ and representation of the projective quotient by a Boy immersion [6].

This construction has an interesting consequence when the meridians are interpreted as timelines. The two points $X$ and $-X$, which correspond to the same point of the projective quotient, belong to regions of the covering in which the time orientations are opposite.

This is illustrated in Figure 7: antipodal identification brings into correspondence portions of spacetime endowed with antiparallel arrows of time. See Figure 7.

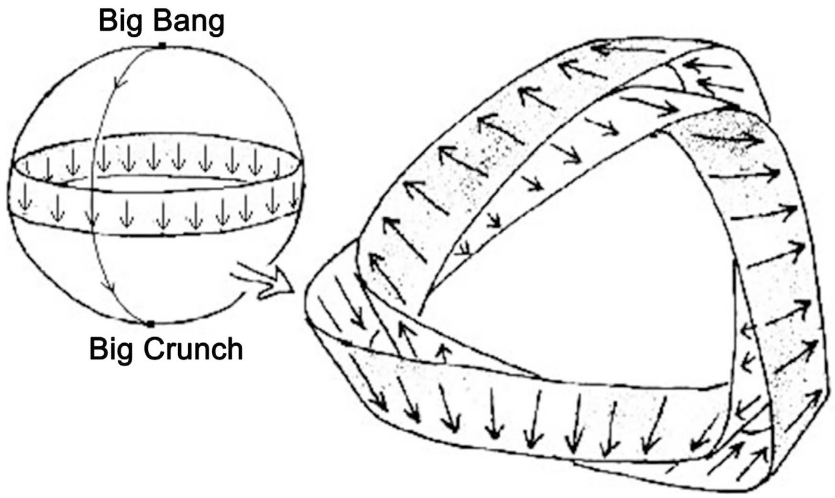



Figure 7. Opposite time orientations induced by the two-sheeted covering of a spherical 2D spacetime [6].

Let us now examine this property in the neighborhood of the meridians.

On a sphere, all meridian curves issuing from one pole converge at the diametrically opposite pole. These two poles are antipodal and therefore correspond to

the same point of the quotient $\mathbb{RP}^2$. In the Boy immersion, they consequently appear joined at a single point.

Figure 8 shows the meridian lines and the neighborhood of each of them. Whereas the neighborhood of the equator is represented in this immersion by a Möbius strip with three half-twists, the neighborhood of a meridian curve has the structure of a Möbius strip with one half-twist. See Figure 8.

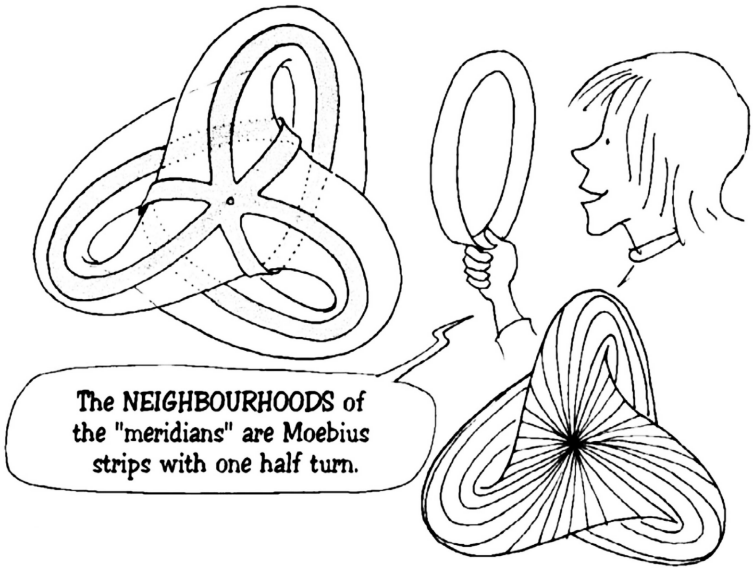



Figure 8. Meridian lines of Boy's surface and their neighborhoods [6].

This representation also makes it possible to visualize separately the two families of curves derived from the meridians and parallels of the initial sphere.

As shown in Figure 9. Figure 9(a) shows the meridians, Figure 9(b) the parallels, and Figure 9(c) their superposition.

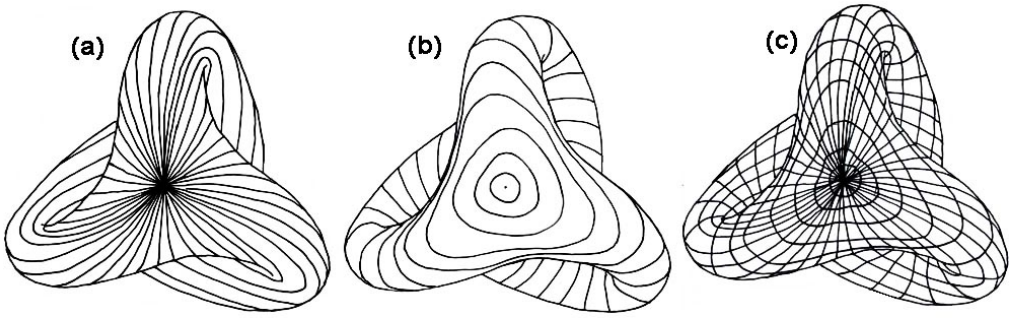



Figure 9. Meridian and parallel curves on Boy's surface. (a) Meridians; (b) parallels; (c) superposition of the two families [6].

We can now recover on the meridians the phenomenon already observed in Figure 7 in the neighborhood of the equator.

Figure 10 shows how the two determinations of the covering related by the antipodal involution carry opposite time orientations. See Figure 10.

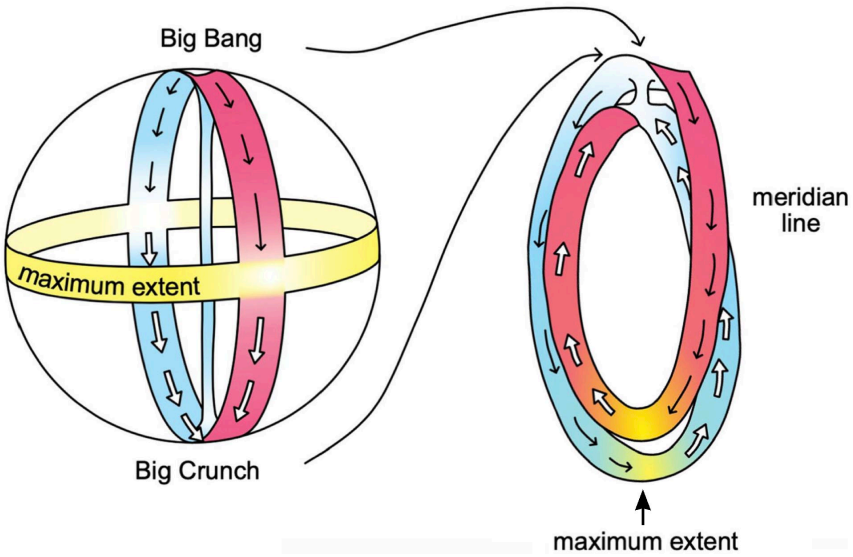



Figure 10. Opposite time orientations on the two determinations of the covering in the neighborhood of a meridian of Boy's surface.

This two-dimensional construction thus geometrically exhibits a reversal of time orientation, which we denote by $T$.

The non-orientability of the projective quotient likewise entails a reversal of spatial orientation. Figure 11 gives an elementary representation of this effect: two objects carried by the two determinations of the covering appear as enantiomorphs. This spatial reversal corresponds to the parity operation $P$. See Figure 11.

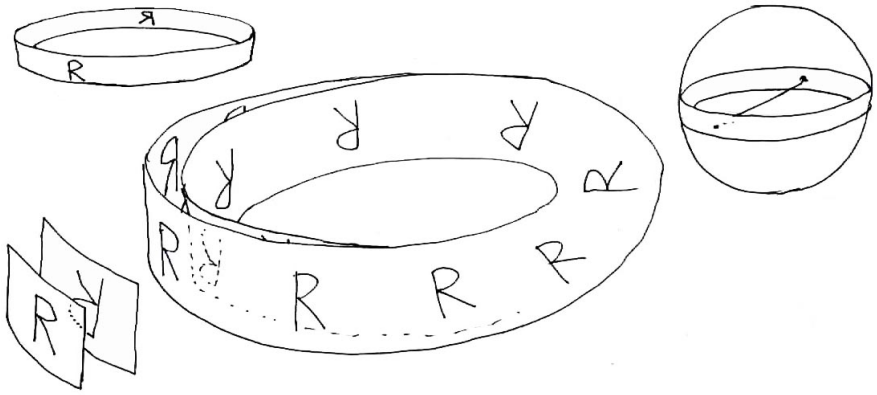


Figure 11. Enantiomorphism induced by the covering of a non-orientable structure [6].

We can now make the dimensional transition that constitutes the essential element of this construction.

Consider a four-dimensional spacetime, with three spatial dimensions and one time dimension. We represent it by a sphere immersed in a five-dimensional space.

We write

$$X = \left(x^0, x^1, x^2, x^3, x^4\right) \tag{10}$$

With

$$\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 = R^2 \tag{11}$$

The four intrinsic coordinates are locally identified with

$$\left(x^0, x^1, x^2, x^3\right) = (t, x, y, z) \tag{12}$$

the fifth coordinate being required to represent the sphere $S^4$ in the ambient space.

We then consider the four-dimensional real projective space:

$$\mathbb{RP}^4 = S^4 / (X \sim -X) \tag{13}$$

whose $S^4$ canonical two-sheeted covering is given by:

$$S^4 \xrightarrow{2:1} \mathbb{RP}^4 \tag{14}$$

The antipodal involution is

$$A: X \mapsto -X \tag{15}$$

On the four intrinsic coordinates, it therefore acts as

$$(t, x, y, z) \mapsto (-t, -x, -y, -) \tag{16}$$

Let us define separately the time reversal

$$T: (t, x, y, z) \mapsto (-t, x, y, z) \tag{17}$$

and the spatial inversion

$$P: (t, x, y, z) \mapsto (t, -x, -y, -z) \tag{18}$$

Thus, on the intrinsic spacetime coordinates,

$$A = PT \tag{19}$$

The antipodal transformation that exchanges the two determinations of the covering $S^4 \to \mathbb{RP}^4$ therefore induces a $PT$ transformation.

This direct passage from two to four dimensions is not arbitrary. One might be tempted to introduce an intermediate step involving a three-dimensional real projective space $\mathbb{RP}^3$. However, real projective spaces satisfy

$$\mathbb{RP}^4 \text{ orientable} \Leftrightarrow \text{odd} \tag{20}$$

Thus, the even-dimensional cases $\mathbb{RP}^2$ and $\mathbb{RP}^4$ considered here are non-orientable, whereas the three-dimensional real projective space $\mathbb{RP}^3$ is orientable. The property sought here would therefore disappear at this intermediate step.

Let us now return to the two-dimensional model.

In Figure 10, the two polar singularities correspond to the Big Bang and the Big Crunch. One may consider removing these two singularities by connecting their neighborhoods. The preceding projective construction is then replaced by a topology whose classical two-dimensional model is the two-sheeted covering of the Klein bottle by the torus:

$$T^2 \xrightarrow{2:1} K^2 \tag{21}$$

Figure 12 illustrates this construction.

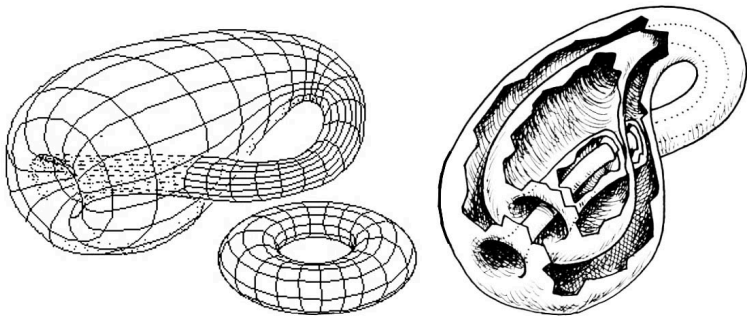


Figure 12. Two-sheeted covering of the Klein bottle by the torus [6].

An analogous generalization may be sought in four dimensions in order to replace the initial and final singularities by a regular structure. In four dimensions, however, the quotient manifold under consideration and the free involution generating its covering must be defined explicitly. We shall return to this point.

This construction is conceptually related to that envisaged by Andrei Sakharov, who considered two conjugate universes related by a CPT symmetry, without introducing between them the gravitational interaction considered here.

Figure 13 gives a didactic representation of this cosmology, first with a singularity and then in a form in which the singularity disappears.

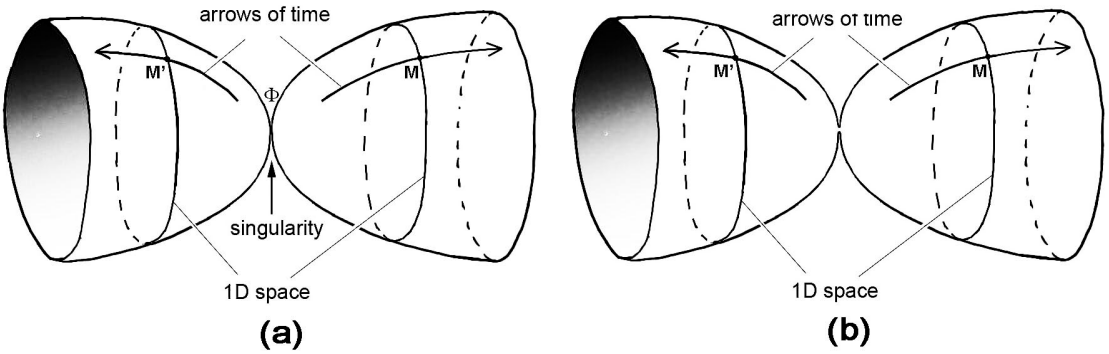



Figure 13. Didactic representation of Andrei Sakharov's cosmology: (a) with a singularity; (b) without a singularity.

It remains to examine how the C symmetry, *i.e.* charge conjugation, could be incorporated into this geometrical construction.

In [7], drawing on Souriau's theory of dynamical groups and on the Janus group, an extension of the full Poincaré group [8], we proposed introducing a fibration involving $p$ compact dimensions.

To each spacetime event $x$ is thus associated a compact fiber

$$F = T^p = \left(S^1\right)^p \tag{22}$$

described by $p$ angular coordinates

$$\theta_1, \theta_2, \cdots, \theta_p \tag{23}$$

The motion of a material point then has two components: its motion in four-dimensional spacetime and its motion along the compact internal dimensions.

The continuous translations

$$\theta_i \mapsto \theta_i + \varepsilon_i \tag{24}$$

constitute symmetries $U(1)$. When the dynamics is invariant under these translations, Noether's theorem associates conserved quantities with them

$$Q_i = \frac{\partial L}{\partial \dot{\theta}_i} \tag{25}$$

which may play the role of charges or quantum numbers.

These continuous translations must then be distinguished from a discrete transformation of the fiber.

An involution

$$C_F : F \to F \qquad C_F^2 = id \tag{26}$$

may be chosen so as to induce

$$Q_i \mapsto -Q_i \tag{27}$$

Such an involution provides a geometrical candidate for representing charge conjugation $C$.

Figure 14 shows the didactic picture originally proposed to illustrate this idea: motion in a compact dimension is added to the trajectory in spacetime, with the two orientations associated with the two signs of the charge. See Figure 14.

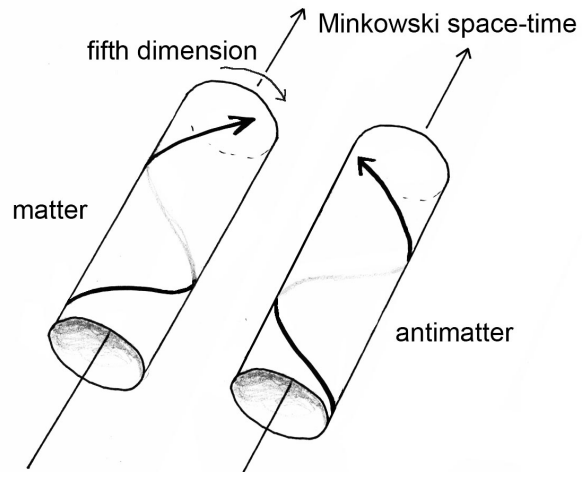



Figure 14. Outline of a geometrical interpretation of matter-antimatter symmetry based on a compact internal dimension [9].

The proposed construction thus has two distinct components

$$PT : \left( X \mapsto -X \right) \tag{28}$$

in spacetime geometry, and

$$C:(Q\mapsto -Q) \tag{29}$$

in the internal dimensions.

It therefore suggests the mapping

$$\boxed{(X,Q)\mapsto(-X,-Q)} \tag{30}$$

combining the transformations $PT$ and $C$.

## 4. Result of a Heuristic Exploration

Pending the development of a more sophisticated system of field equations, a heuristic exploration is attempted using the following force scheme, which satisfies the action-reaction principle, as shown in Figure 15:

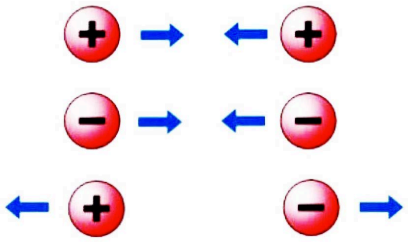


Figure 15. The heuristic hypothesis concerning the force field.

The initial trials in 1994 [5], involving the interaction of sets of positive and negative point masses, produced a percolation phenomenon that did not appear to be readily linkable to observations. See Figure 16.

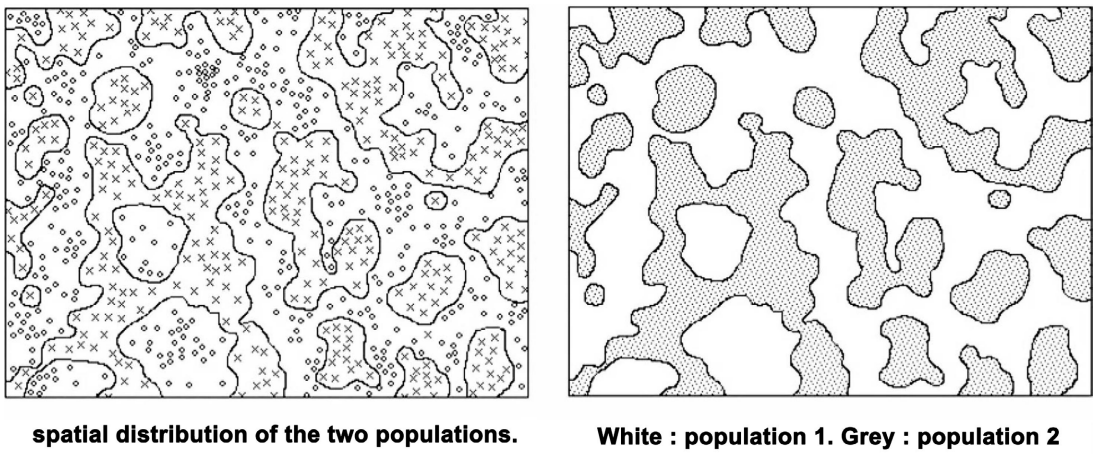



Figure 16. Full symmetry $(|\bar{\rho}|=\rho)$: percolation phenomenon [5].

However, by choosing to introduce significant asymmetry $(|\bar{\rho}|>\rho)$, this simulation immediately results in the formation of a very large-scale structure, in which the negative mass forms a regular network of spheroidal clusters, while the positive mass is confined to the remaining space, taking the form of contiguous bubbles. See Figure 17.

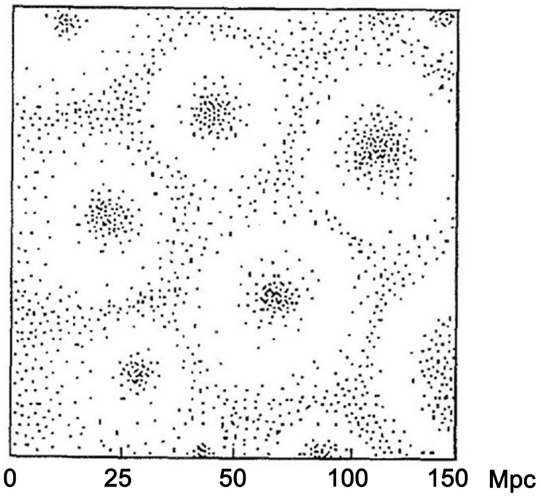



Figure 17. Formation of the very large structure [10].

The explanation is immediately apparent. Since the Jeans timescales of the two populations differ significantly, it is the one with the shorter timescale—the negative population—that gives rise to an organized structure first.

$$\bar{t}_J = \frac{1}{\sqrt{8\pi G\bar{\rho}}} \ll t_J = \frac{1}{\sqrt{8\pi G\rho}} \tag{31}$$

Even before considering the geometric and mathematical context, the approach appears promising. The two subsystems mutually confer stability upon one another. The contiguous spheres of matter entrap the negative-mass clusters, preventing them from meeting and merging; these clusters, in turn, serve as anchor points for this porous structure. See Figure 18.

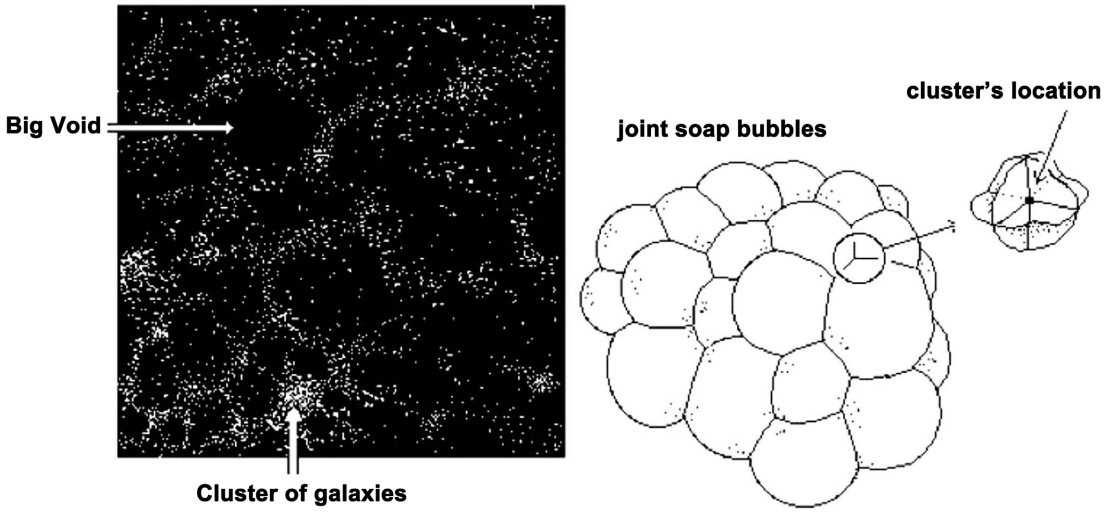



Figure 18. Joint soap bubbles VLS structure.

This suggests a novel scenario for the formation of stars and galaxies. The scientific community is currently faced with the appearance of massive, fully formed

spiral galaxies at an extremely early stage—phenomena that standard models cannot explain. The primary obstacle lies in the time required to dissipate the heat generated as matter compresses into spheroidal structures. Here, however, the situation is quite different. The positive mass is not left to its own devices but undergoes intense compression between sheets; this geometry is optimal for maximizing energy dissipation through radiation. While 3D simulations could quantify this phenomenon, we lack the resources to carry out such a project. We believe that, in this context, massive spiral galaxies could form within the first hundred million years. See Figure 19.

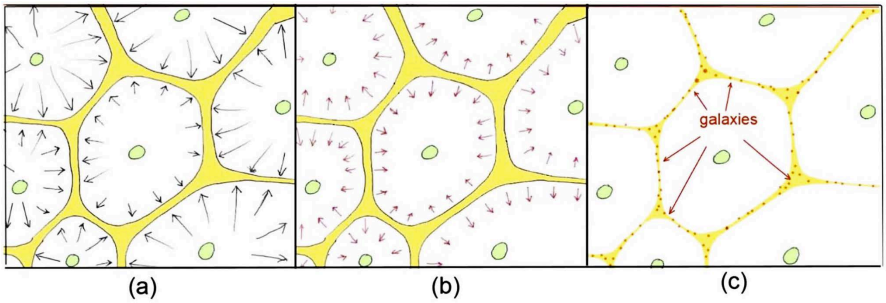



Figure 19. Rapid galaxy formation [9].

In Figure 12(a), the matter (in yellow), subjected to the back-compression exerted by negative-mass clusters (in green), undergoes violent heating. In Figure 12(b), this energy is rapidly dissipated through radiation. In Figure 12(c), having become unstable, the positive mass gives rise to all the galaxies as we observe them today.

There is no subsequent galaxy formation; all galaxies are born within the first hundred million years. As early as 1992, 2D simulations—with initial conditions describing a rotating galaxy confined by its negative-mass environment—showed the very rapid emergence (within a few rotations) of a barred spiral structure. See Figure 20.

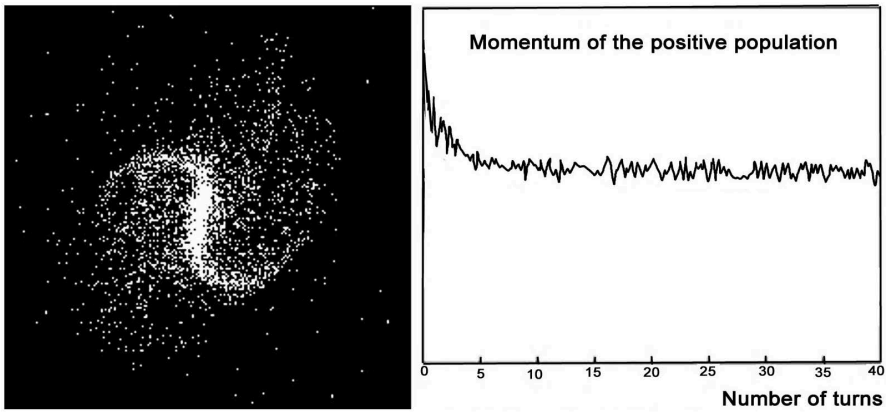



Figure 20. Barred spiral and the evolution of its angular momentum as a function of the number of rotations [9].

The initial conditions governing these simulations were based on solving a system of two Vlasov equations coupled by the gravitational field, yielding these density profiles [5]. See Figure 21 and Figure 22.

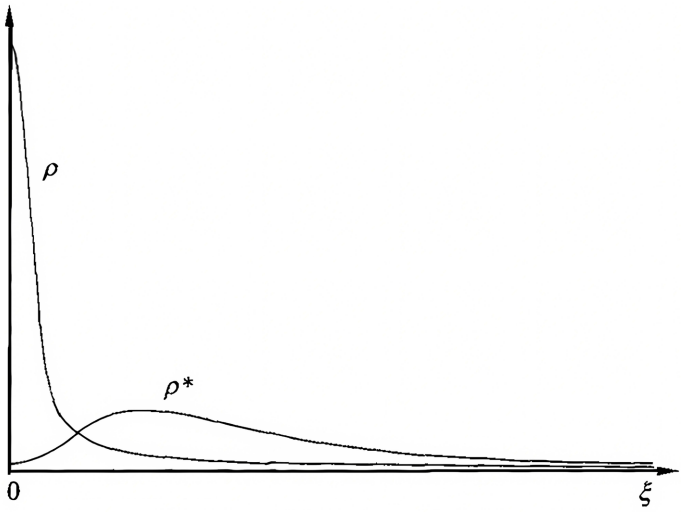



Figure 21. Elliptic solution of two coupled Vlasov equations. Densities of positive masses $\rho$ and negative masses $\rho^*$ (*i.e.*, of negative mass) [5].

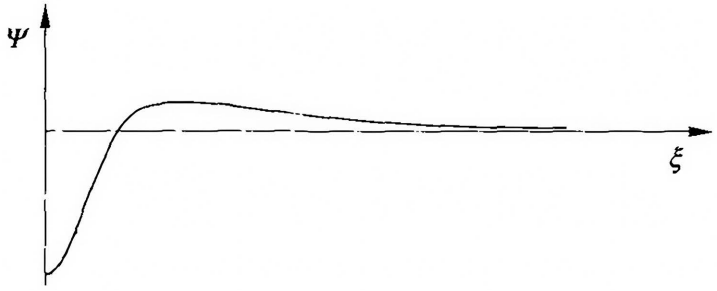



Figure 22. The corresponding gravitational potential [5].

Thirty-four years ago, these simulations thus provided the answer to the question that the entire community of specialists continues to ask: why does this spiral structure form, and why does it persist?

*- The answer is that it is the way in which collisionless systems exchange energy and angular momentum with their environment, through the agency of density waves. Thus, this spiral structure is destined to persist as long as the galaxy contains gas—that is, virtually indefinitely.*

Despite the very positive result obtained using this simple heuristic hypothesis regarding the direction of the forces, all specialized journals refused to review the submitted article, replying, "Sorry, we don't publish speculative works." We can only hope that today, thirty-four years later, a team with sufficient resources will take up this research again.

In this regard, one might wonder whether there is a fairly simple reason why other teams have not attempted—even on an exploratory basis—this type of com-

putational experiment. Simulation programs—evolving from 2D to 3D—have undergone considerable development over recent decades. Thanks to tree-coding, it has become possible to handle increasingly large numbers of point masses and to incorporate both multiple stellar populations and gas. Yet, is there any program that asks the user to "specify the sign of this component's mass"? This seems unlikely, given that the scientific community operates on the basis of an irrevocable decision to exclude negative masses from cosmology and astrophysics.

However, a cosmological model cannot be based on mere heuristic speculations, however fruitful and encouraging they may be. Since a dynamic based on a single field equation immediately ran up against the "runaway" phenomenon, a bimetric approach was considered.

## 5. First Bimetric Description by T. Damour and I. Kogan [11]

In 2002, a 40-page article presenting the first bimetric approach was published in Physical Review D, the highest-ranked journal in the field. The authors initially envisioned the interaction of *n* branes, each equipped with its own metric, before narrowing the scope of their project to two branes: "Left" and "Right." Their aim was not to introduce negative masses. From the outset, the authors referred to the concept known as "massive gravity." Back in 1939, Fierz and Pauli [12] had added a mass term to the tensor field, thereby assigning a non-zero mass to the graviton. This gives rise to a characteristic wavelength equivalent to the Compton wavelength.

$$\lambda_g = \frac{\hbar}{m_g} \tag{32}$$

By linking this to the Hubble constant and associating it with a modification of gravity at large distances, this yields an estimate of the order of magnitude of this hypothetical graviton mass:

$$m_g c^2 = \hbar H_0 \sim 10^{-33}\,\text{eV} \tag{33}$$

In 2011, C. de Rham, G. Gabadadze, *et al.* [13] linked this graviton mass to the recently discovered phenomenon of accelerated expansion, leading them to a graviton mass value of the same order of magnitude.

In 1972, Boulware and Deser [14] investigated the extension of Fierz-Pauli theory to the non-linear regime. They demonstrated that such an extension generically gives rise to a sixth degree of freedom—absent from the linear theory—associated with a "ghost" instability. In this context, any cosmological model based on a modification of gravitational dynamics must therefore satisfy a crucial requirement:

- *Is this model ghost-free*?

This question lies at the heart of the concerns of the authors, T. Damour and I. Kogan [11]. Not only do their gravitons possess mass—distributed according to a specific spectrum—but they also posit a hypothetical gap separating light gravitons from heavy ones. See **Figure 23**.

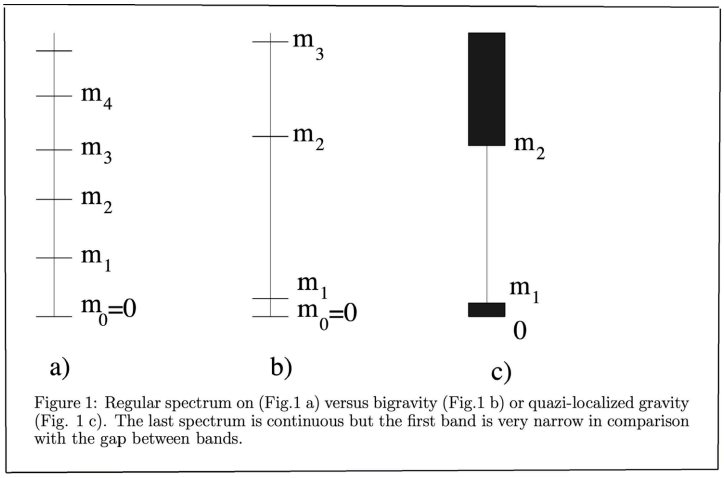



Figure 1: Regular spectrum on (Fig.1 a) versus bigravity (Fig.1 b) or quazi-localized gravity (Fig. 1 c). The last spectrum is continuous but the first band is very narrow in comparison with the gap between bands.

**Figure 23.** The hypothetical gap between light gravitons and heavy gravitons [11].

The fact that this article has become a major reference in the bimetric literature has contributed to the frequent association of bigravity with *massive gravity*—often referred to as "*massive bigravity*"—yet this association is not a mathematical necessity. The bimetric nature of a theory does not, in itself, imply the existence of a massive graviton. The Janus Cosmological Model is precisely not based on such an assumption. Consequently, the specific problem of the Boulware-Deser ghost, which arises in certain non-linear extensions of massive gravity, cannot be applied to it without first demonstrating that the mechanism generating this degree of freedom is indeed present in the Janus system.

Therefore, the question of whether the Janus cosmological model is free from the Boulware-Deser ghost is irrelevant in this context, since the Janus model is not constructed as a non-linear theory of massive gravitons. This specific instability, which is a central concern in massive gravity, is thus not the one governing its construction.

The same applies to a second bimetric model, introduced between 2006 and 2008 by S. Hossenfelder [15] [16]. Here, the focus is different, aiming to construct an antigravitational interaction capable of having cosmological consequences. Indeed, the 2006 paper explicitly frames the approach within the context of "antigravity." Hossenfelder demonstrates that the kinetic energy of the new fields remains positive, while the fields themselves can contribute negatively to the source term in Einstein's equations.

Damour-Kogan, Hossenfelder, and the Janus model thus propose constructions involving two gravitational sectors and coupled field equations, albeit with different physical motivations. The question of interest here is the construction of a Lagrangian that allows such a system to be obtained variationally. To this end, we will adopt the construction proposed by Damour and Kogan.

## 6. Construction of the System of Coupled Field Equations by T. Damour and I. Kogan [11]

The authors consider two sectors, "left: $L$" and "right: $R$". The proposed action is

then:

$$S=\int\left(M_R^2R^R-L^R-\Lambda^R\right)\sqrt{\left|g^R\right|}\mathrm{d}^4x+\int\left(M_L^2R^L-L^L-\Lambda^L\right)\sqrt{\left|g^L\right|}\mathrm{d}^4x \\ -\mu^4\int\mathrm{d}^4x\left(g_Lg_R\right)^{\frac{1}{4}}V\left(g_L,g_R\right). \tag{34}$$

The constants $M_L$ and $M_R$ are "Planck masses." The inverses of their squares play the role of the Einstein-Hilbert gravitational coupling constants. What emerges from this formalism is the fact that the authors assign them the same sign.

In this action we find two elementary volumes $\sqrt{\left|g_R\right|}\mathrm{d}^4x$ and $\sqrt{\left|g_L\right|}\mathrm{d}^4x$, and elementary hypervolume $\mathrm{d}^4x\left(g_Lg_R\right)^{\frac{1}{4}}$ involved in the interaction between the two populations. $\mu$ is an interaction constant and $V\left(g_L,g_R\right)$ is an interaction potential.

Let us forget the two cosmological constants. $\Lambda^R$ and $\Lambda^L$.

When performing the variational calculation, the terms $R^R\sqrt{\left|g_R\right|}$ and $R^L\sqrt{\left|g_L\right|}$ yield the "geometric tensors" $G_{\mu\nu}^R$ and $G_{\mu\nu}^L$—that is, the left-hand sides of the two field equations.

The two Lagrangian densities of the two materials $L^R\sqrt{\left|g^R\right|}$ et $L^L\sqrt{\left|g_L\right|}$ give source tensors $T_{\mu\nu}^R$ and $T_{\mu\nu}^L$

The variations on the interaction term $-\mu^4\int\mathrm{d}^4x\left(g_Lg_R\right)^{\frac{1}{4}}V\left(g_L,g_R\right)$.

Gives interaction tensors $K_{\mu\nu}^{L/R}$ and $K_{\mu\nu}^{R/L}$

- The notation $K_{\mu\nu}^{L/R}$ represents the contribution of matter *L* to the field acting on matter *R*.
- The notation $K_{\mu\nu}^{R/L}$ represents the contribution of matter *R* to the field acting on matter *L*.

Hence the system of two field equations coupled by their interaction terms:

$$G_{\mu\nu}^R=R_{\mu\nu}^R-\frac{1}{2}R^Rg_{\mu\nu}^R=\frac{1}{M_R^2}\left[T_{\mu\nu}^R+t_{\mu\nu}^{L/R}\right] \tag{35a}$$

$$G_{\mu\nu}^L=R_{\mu\nu}^L-\frac{1}{2}R^Lg_{\mu\nu}^L=\frac{1}{M_L^2}\left[T_{\mu\nu}^L+t_{\mu\nu}^{R/L}\right] \tag{35b}$$

Which can be written by explicitly showing the gravitational coupling constants $\chi_R$ and $\chi_L$:

$$G_{\mu\nu}^R=R_{\mu\nu}^R-\frac{1}{2}R^Rg_{\mu\nu}^R=\chi_R\left[T_{\mu\nu}^R+t_{\mu\nu}^{L/R}\right] \tag{36a}$$

$$G_{\mu\nu}^L=R_{\mu\nu}^L-\frac{1}{2}R^Lg_{\mu\nu}^L=\chi_L\left[T_{\mu\nu}^L+t_{\mu\nu}^{R/L}\right] \tag{36b}$$

Noting that the Damour-Kogan approach posits that:

$$\chi_R\chi_L>0 \tag{37}$$

Based on the two metrics, we have covariant derivative operators. $\nabla_R^\nu$ and $\nabla_L^\nu$

As we have, by construction:

$$\nabla_R^\nu G_{\mu\nu}^R=\nabla_L^\nu G_{\mu\nu}^L=0 \tag{38}$$

The mathematical consistency of the system requires:

$$\nabla_R^{\nu}\left(T_{\mu\nu}^{R}+t_{\mu\nu}^{L/R}\right)=0 \tag{39a}$$

$$\nabla_L^{\nu}\left(T_{\mu\nu}^{L}+t_{\mu\nu}^{R/L}\right)=0 \tag{39b}$$

After reviewing various forms of interaction potentials, this 40-page article leads to no results that can be compared with observational data.

## 7. S. Hossenfelder's System of Coupled Field Equations [16]

S. Hossenfelder's approach also relies on an action.:

$$\begin{aligned} S=&\int \mathrm{d}^4x\left[\sqrt{-g}\left({}^{(g)}R/8\pi G+\mathcal{L}\left(\psi\right)\right)+\sqrt{-h}P_{\underline{h}}\left(\underline{\mathcal{L}}\left(\underline{\phi}\right)\right)\right]\\ &+\int \mathrm{d}^4x\left[\left({}^{(h)}R/8\pi G+\underline{\mathcal{L}}\left(\underline{\phi}\right)\right)+\sqrt{-\underline{g}}P_g\left(\mathcal{L}\left(\psi\right)\right)\right]\end{aligned} \tag{40}$$

We have added the brackets, which are missing from the article. Its way of managing the interaction between its two sectors $g$ and $h$ is different in that two terms appear representing:

- One, the action of the field $\phi$, originating from the population $h$, on the elements of the sector $g$ a term weighted by the elementary hypervolume $\sqrt{-h}\,\mathrm{d}^4x$
- The other, the action of the field $\psi$, originating from the population $g$, on the elements of the sector $h$ a term weighted by the elementary hypervolume $\sqrt{-g}\,\mathrm{d}^4x$.

It is this approach that leads to the system of coupled field equations:

$${}^{(g)}R_{\kappa\nu}-\frac{1}{2}g_{\kappa\nu}\,{}^{(g)}R=T_{\kappa\nu}-\underline{V}\sqrt{\frac{\underline{h}}{g}}a_{\nu}^{\underline{\nu}}a_{\kappa}^{\underline{\kappa}}\underline{T}_{\underline{\nu}\underline{\kappa}} \tag{41a}$$

$${}^{(\underline{h})}R_{\underline{\nu}\underline{\kappa}}-\frac{1}{2}h_{\underline{\nu}\underline{\kappa}}\,{}^{(h)}R=\underline{T}_{\underline{\nu}\underline{\kappa}}-W\sqrt{\frac{g}{\underline{h}}}a_{\underline{\kappa}}^{\kappa}a_{\underline{\nu}}^{\nu}T_{\kappa\nu} \tag{41b}$$

It is the similarity in construction that also gives rise to the square root of the ratio of the determinants of the two metrics—and its inverse—in the Janus cosmological model, subject to a change of notation: the metrics are then $g_{\mu\nu}$ and $\overline{g}_{\mu\nu}$.

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\chi\left(T_{\mu\nu}+\sqrt{\frac{\overline{g}}{g}}\hat{\overline{T}}_{\mu\nu}\right) \tag{42a}$$

$$\overline{R}_{\mu\nu}-\frac{1}{2}\overline{R}\overline{g}_{\mu\nu}=-\chi\left(\overline{T}_{\mu\nu}+\sqrt{\frac{g}{\overline{g}}}\hat{T}_{\mu\nu}\right) \tag{42b}$$

A strong similarity is thus observed between the three systems of coupled field equations. But how do they differ? In the Janus model, the gravitational coupling constants are equal and opposite. In Hossenfelder's model, they are equal. In the Damour-Kogan model, they have the same sign.

## 8. Standardization of Notation

The shift to a bimetric system sheds new light on this issue. What about the ap-

proaches proposed by Damour and Hossenfelder? Since the components of their coupled field equation systems satisfy the mathematical constraints arising from the shared form of the Bianchi identities, the choice of free parameters allows for a wide range of possible models. Let us express both systems in a common form by changing the notation:

$$g_{\mu\nu}^{L} = g_{\mu\nu} \to g_{\mu\nu} \tag{43}$$

$$g_{\mu\nu}^{R} = h_{\mu\nu} \to \overline{g}_{\mu\nu} \tag{44}$$

$$R_{\mu\nu}\left(g^{L}\right) = {}^{(g)}R_{\mu\nu} \to R_{\mu\nu} \tag{45}$$

$$R_{\mu\nu}\left(g^{R}\right) = {}^{(h)}R_{\mu\nu} \to \overline{R}_{\mu\nu} \tag{46}$$

$$R\left(g^{L}\right) = {}^{(g)}R \to R \tag{47}$$

$$R\left(g^{R}\right) = {}^{(h)}R \to \overline{R} \tag{48}$$

$$T_{\mu\nu}^{L} = T_{\mu\nu} \to T_{\mu\nu} \tag{49}$$

$$T_{\mu\nu}^{R} = \underline{T}_{\mu\nu} \to \overline{T}_{\mu\nu} \tag{50}$$

$$t_{\mu\nu}^{L} = -\underline{V}\sqrt{\frac{\underline{h}}{g}}\, a_{\nu}^{\underline{\nu}}\, a_{\kappa}^{\underline{\kappa}}\, \underline{T}_{\underline{\nu}\,\underline{\kappa}} \to \overline{K}_{\mu\nu} \tag{51}$$

$$t_{\mu\nu}^{R} = -W\sqrt{\frac{g}{\underline{h}}}\, a_{\underline{\mu}}^{\mu}\, a_{\underline{\nu}}^{\nu}\, T_{\mu\nu} \to K_{\mu\nu} \tag{52}$$

In summary, in both models we have two types of matter associated with masses $m$ and $\overline{m}$, which follow geodesics derived from metrics $g_{\mu\nu}$ and $\overline{g}_{\mu\nu}$. From these, Ricci tensors $R_{\mu\nu}$ and $\overline{R}_{\mu\nu}$ and corresponding Ricci scalars $R$ and $\overline{R}$. are constructed. The source tensors associated with these two types of matter are $T_{\mu\nu}$ and $\overline{T}_{\mu\nu}$.

- $\overline{K}_{\mu\nu}$ is a tensor—referred as interaction tensor (denoted by $t_{\mu\nu}^{L}$ by Damour and Kogan) that accounts for the action of the matter field $\overline{m}$ on the geodesics followed by the masses $m$.

- $K_{\mu\nu}$ is a tensor—referred to as the interaction tensor (denoted by $t_{\mu\nu}^{R}$ by Damour and Kogan)—that accounts for the action of the matter field $m$ on the geodesics followed by the masses $\overline{m}$.

Under these conditions, the two systems—those of Damour and Hossenfelder—can be written in the form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} + \overline{K}_{\mu\nu} \tag{53a}$$

$$\overline{R}_{\mu\nu} - \frac{1}{2}\overline{R}\overline{g}_{\mu\nu} = \overline{T}_{\mu\nu} + K_{\mu\nu} \tag{53b}$$

In both approaches, the authors assume the conservation of both types of matter—that is to say:

$$\nabla^{\nu}T_{\mu\nu} = \overline{\nabla}^{\nu}\overline{T}_{\mu\nu} = 0 \tag{54}$$

$\nabla^\nu$ and $\bar{\nabla}^\nu$ being the two covariant derivative operators constructed from the metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ this immediately leads to the two constraints that determine the form of the interaction tensors:

$$\nabla^\nu \bar{K}_{\mu\nu} = \bar{\nabla}^\nu K_{\mu\nu} = 0 \tag{55}$$

Following the two cited articles, the discussions depend on the intended objectives. In [11], there is no mention of negative masses at any point. In that article, the observer is assumed to reside on one of the two branes, which therefore corresponds to matter—to standard mass. The authors note (in Section 4.3) that bigravity exactly satisfies the equivalence principle and that this model is compatible with local observational data. Regarding what is termed a "bi-cosmological configuration," they suggest that the action of this second type of matter could account for the recently discovered phenomenon of accelerating cosmic expansion. In Section VII of [11], which addresses potential observables, the author writes—we quote:

- *Since the matter contents of both types of fields is identical the source terms in the second members of the field equations cancel identically, no matter how large their values are.*

The equations of the Janus cosmological model also derive from an action (EPJC 2024 [9]). For reasons to be explained later, we will show that the Janus model equations can be derived from the Damour-Kogan action, subject to a slight modification.

## 9. Thanks to a Slight Modification, the Janus Model Becomes a Variant of [11]

The gravitational coupling constants $\chi_R$ and $\chi_L$ are already present in the action of [11].

$$\begin{aligned} S = &\int \left(R^R - \chi_R L^R - \Lambda^R\right)\sqrt{|g^R|}\,\mathrm{d}^4x + \int \left(R^L - \chi_L L^L - \Lambda^L\right)\sqrt{|g^L|}\,\mathrm{d}^4x \\ &- \mu^4 \int \mathrm{d}^4x \left(g_L g_R\right)^{\frac{1}{4}} V\left(g_L, g_R\right). \end{aligned} \tag{56}$$

The system of equations being:

$$R^R_{\mu\nu} - \frac{1}{2}R^R g^R_{\mu\nu} = \chi_R\left[T^R_{\mu\nu} + t^{L/R}_{\mu\nu}\right] \tag{57a}$$

$$R^L_{\mu\nu} - \frac{1}{2}R^L g^L_{\mu\nu} = \chi_L\left[T^L_{\mu\nu} + t^{R/L}_{\mu\nu}\right] \tag{57b}$$

Given that at Damour and Kogan $\chi_R\chi_L > 0$

Or, using our notation, by posing $\chi_R = \chi$ and $\chi_L = \bar{\chi}$:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} = \bar{\chi}\left[\bar{T}_{\mu\nu} + K_{\mu\nu}\right] \tag{57c}$$

As we are opting to preserve both types of materials:

$$\nabla^\nu T_{\mu\nu} = \bar{\nabla}^\nu \bar{T}_{\mu\nu} = 0 \tag{58}$$

Which will necessitate:

$$\nabla^{\nu}\overline{K}_{\mu\nu} = \overline{\nabla}^{\nu}K_{\mu\nu} = 0 \tag{59}$$

If this condition is met, one is then free to choose the signs of the source terms, except for the one that would be identified with standard, self-attractive matter.

This is where the benefit of the Janus model's heuristic approach lies; at the outset, this approach involved assigning a priori force directions corresponding to Figure 9. We therefore chose, heuristically, to adopt a system in which the two types of masses repel each other. We will adjust the signs of the terms in the various tensors and the Einstein constants to align with these force directions. Let us consider the Newtonian approximation regime. Under these conditions:

$$T_{00} = \rho > 0 \left(i \neq 0; j \neq 0\right) \rightarrow T_{ij} \cong 0 \tag{60}$$

$$\overline{T}_{00} = \overline{\rho} < 0 \left(i \neq 0; j \neq 0\right) \rightarrow \overline{T}_{ij} \cong 0 \tag{61}$$

For the second type of matter to be self-attractive as well, the coupling constants must have opposite signs. And we shall choose them to be equal and opposite:

$$\boxed{\overline{\chi} = -\chi} \tag{62}$$

We shall assume that the Newtonian approximation has the same effect on the interaction tensors and, that is to say:

$$K_{00} > 0 \left(i \neq 0; j \neq 0\right) \rightarrow K_{ij} \cong 0 \tag{63}$$

$$\overline{K}_{00} < 0 \left(i \neq 0; j \neq 0\right) \rightarrow \overline{K}_{ij} \cong 0 \tag{64}$$

Under these conditions, the heuristic laws hold true. The principles of equivalence and action-reaction are satisfied. We recall the benefits gained from the heuristic approach. A similar procedure could be carried out starting from Sabine Hossenfelder's model and also adding Einstein constants of opposite signs $\left(\overline{\chi} = -\chi\right)$; however, we shall limit ourselves to the derivation presented in [9].

It now remains for us to revisit the construction of solutions for the Janus cosmological model and the issue of their satisfying the Bianchi conditions.

## 10. Unsteady-State Solution for Isotropic and Homogeneous Media

By assigning the following forms to the interaction tensors:

$$K_{\mu\nu} = T_{\mu\nu};\ \ \overline{K}_{\mu\nu} = \overline{T}_{\mu\nu} \tag{65}$$

We obtain the trivial unsteady solution:

$$g_{\mu\nu} = \overline{g}_{\mu\nu} \tag{66}$$

With:

$$\left|\overline{\rho}\right| = \rho \tag{67}$$

In other words, zero density and a zero gravitational field. This recovers the Dirac-Milne linear expansion model [17].

Still seeking a time-dependent solution under conditions of isotropy and homogeneity, we looked for a solution [18] such that:

$$K_{\mu\nu} = \phi T_{\mu\nu};\ \ \overline{K}_{\mu\nu} = \varphi \overline{T}_{\mu\nu} \tag{68}$$

where $\varphi$ and $\phi$ are functions of time. Symmetry gives the metrics the following FLRW forms:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2\left(\frac{\mathrm{d}u^2}{1-ku^2} + u^2\mathrm{d}\Omega^2\right) \tag{69}$$

$$\mathrm{d}\overline{s}^2 = \overline{c}^2\mathrm{d}t^2 - \overline{a}^2\left(\frac{\mathrm{d}u^2}{1-\overline{k}u^2} + u^2\mathrm{d}\Omega^2\right) \tag{70}$$

By introducing these FLRW metrics into our bi-metric system, we obtain a system of four second-order differential equations yielding the scale factors $a$ and $\overline{a}$. Requiring the system to be consistent will provide the equivalent of the Bianchi condition for this type of solution. The system of field equations becomes:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \chi\left[T_{\mu\nu} + \varphi\overline{T}_{\mu\nu}\right] \tag{71a}$$

$$\overline{R}_{\mu\nu} - \frac{1}{2}\overline{R}\overline{g}_{\mu\nu} = -\chi\left[\overline{T}_{\mu\nu} + \phi T_{\mu\nu}\right] \tag{71b}$$

With:

$$\chi = -\frac{8\pi G}{c^2} \tag{72}$$

By considering the two scenarios using the equation of state:

$$p = \frac{\beta\rho c^2}{3} \qquad \overline{p} = \frac{\beta\overline{\rho}\overline{c}^2}{3} \tag{73}$$

With $\beta = 0$ for the matter-dominated era and $\beta = 1$ for the radiation-dominated era, compatibility is ensured (**Appendix 1**) by the equation:

$$\boxed{\rho c^2 a^{(3+\beta)} + \overline{\rho}\overline{c}^2\overline{a}^{(3+\beta)} = E = Cst} \tag{74}$$

Which translates to a generalized conservation of energy. The system then gives (**Appendix 1**), for the matter-dominated era:

$$\dot{a} = c\sqrt{-k - \frac{8\pi GE}{3a}} \tag{75}$$

$$\ddot{a} = -\frac{4\pi GE}{3a^2} \tag{76}$$

$$\dot{\overline{a}} = \overline{c}\sqrt{-\overline{k} + \frac{8\pi GE}{3\overline{a}}} \tag{77}$$

$$\ddot{\overline{a}} = \frac{4\pi GE}{3\overline{a}^2} \tag{78}$$

Observation yields $\ddot{a} > 0$, hence $E < 0$. Negative mass dominates and drives galactic dynamics. This provides an alternative interpretation to that of dark energy. It also implies $k = -1$.

The evolution of the negative species occurs for $\ddot{\overline{a}} < 0$ and corresponds to the hyperbolic Friedmann solution. At infinity, it tends toward $\overline{c}t$. Similarly, at in-

finity, *a* tends toward *ct*. The equations also impose $\overline{k} = -1$.

The exact solution to Equation (52a) was constructed by William Bonnor [19] in parametric form. With $x^o = ct$:

$$a(\mu) = \alpha^2 c \,\mathrm{ch}^2 u \tag{79}$$

$$t(\mu) = 2\alpha^2 c\left(1 + \frac{\mathrm{sh}\,2u}{2} + u\right) \tag{80}$$

Which give us the curves of Figure 24.

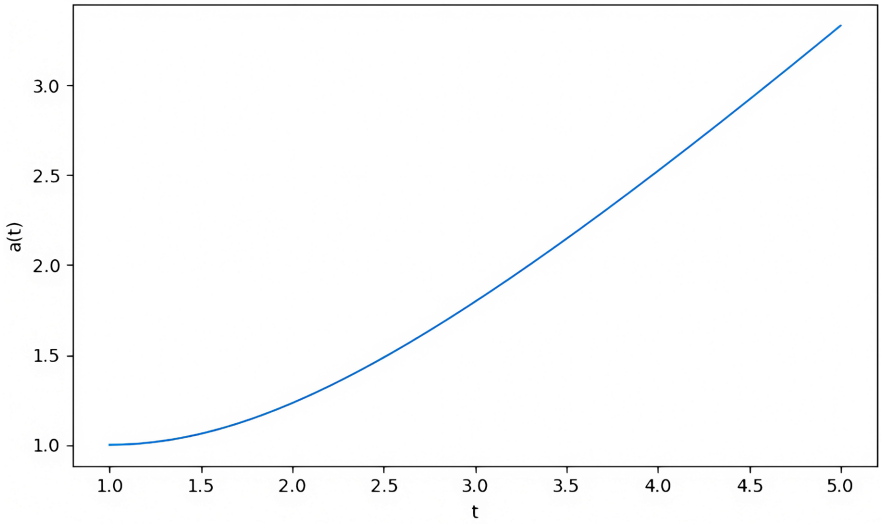



Figure 24. Expansion of the universe.

Profile corresponding to variable acceleration is shown on Figure 25.

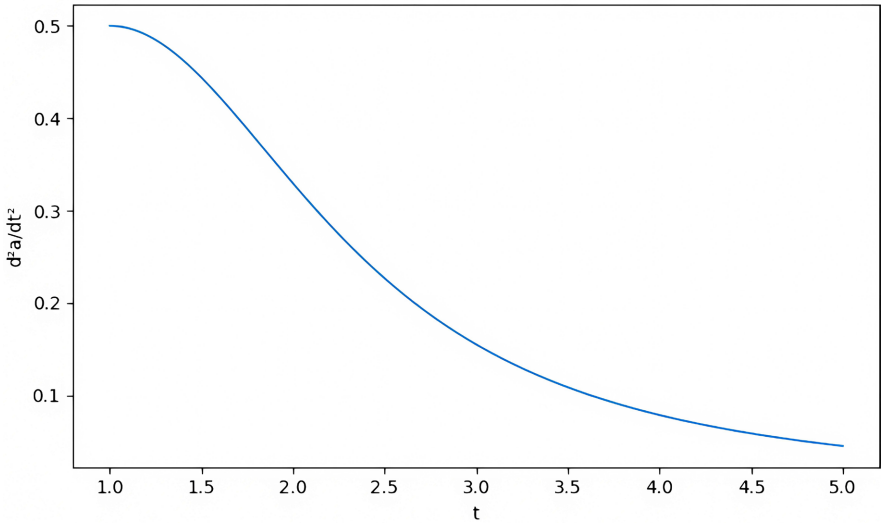



Figure 25. Evolution of acceleration.

This yields an interpretation of the origin of the accelerated expansion as an alternative to the dark energy hypothesis. The model predicts a significant accel-

eration value in the past—a finding seemingly supported by initial observational data from DESI [2], though confirmation from further observational data is still required.

Data regarding the dynamics of the negative species are not amenable to observation. For the radiation-dominated era and positive mass, the choice is made to extend the model with a curvature index of −1.

It is thus evident that the system of field equations is capable of yielding a mathematically consistent non-stationary solution describing isotropic and homogeneous media.

## 11. Spherically Symmetric Stationary Solution

In a vacuum, the satisfaction of the Bianchi conditions is automatic.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{81a}$$

$$\overline{R}_{\mu\nu} - \frac{1}{2} \overline{R} \overline{g}_{\mu\nu} = 0 \tag{81b}$$

In a region of space with non-zero matter density, two cases must be considered.

**First case**: The field is created by a positive mass. The system becomes:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \chi T_{\mu\nu} \tag{82a}$$

$$\overline{R}_{\mu\nu} - \frac{1}{2} \overline{R} \overline{g}_{\mu\nu} = -\chi K_{\mu\nu} \tag{82b}$$

The solution to the first equation is the classical solution—along with its TOV equation—which applies equally to the Newtonian approximation and to very dense objects such as neutron stars. The classical TOV equation is then:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\left(\rho + p/c^2\right)\left(m + 4\pi G p r^3/c^4\right)c^2}{r\left(r - 2m\right)} \tag{83}$$

And the interior metric is:

$$\mathrm{d}s^2 = \left[\frac{3}{2}\sqrt{1 - \frac{R_a^{\,2}}{\hat{R}^2}} - \frac{1}{2}\sqrt{1 - \frac{R^2}{\hat{R}^2}}\right]^2 c^2 \dot{t}^2 - \frac{\dot{R}^2}{1 - \frac{R^2}{\hat{R}^2}} - R^2\left(\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2\right) \tag{84}$$

If one were to consider constructing the interior metric $\overline{g}_{\mu\nu}^{int}$, it would then be necessary to define the form of the interaction tensor $K_{\mu\nu}$. However, this approach would not yield any elements capable of being compared with observational data, since our measuring instruments are unable to detect negative-energy photons. We are therefore not required to provide these elements.

**Second case**: the field is created by a negative mass. The system becomes:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \chi \overline{K}_{\mu\nu} \tag{85a}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} = -\chi\bar{T}_{\mu\nu} \tag{85b}$$

In this instance, we must define the interaction tensor $\bar{K}_{\mu\nu}$, as it is what yields the observables. The two equations will give rise to their own TOV equations—the equivalents of the Bianchi identities in this context. For the solution to be mathematically consistent, compatibility must be ensured.

As early as 2019 [20], it was shown that by opting for:

$$\bar{K}_{\mu}^{\nu} = \begin{pmatrix} \bar{\rho}\bar{c}^2 & 0 & 0 & 0 \\ 0 & \bar{q} & 0 & 0 \\ 0 & 0 & \bar{q} & 0 \\ 0 & 0 & 0 & \bar{q} \end{pmatrix} \tag{86}$$

Namely, the source tensor with the signs of the pressure terms inverted; this led to the TOV equation:

$$\bar{q}' = -\frac{\left(\bar{\rho}\bar{c}^2 + \bar{q}'\right)\left(-\bar{m} + 4\pi G\bar{q}' r^3/c^4\right)}{r\left(r + 2\bar{m}\right)} \tag{87}$$

Compatibility is then ensured—asymptotically, in the Newtonian approximation—with $\bar{q} = \bar{p}$. The two TOV equations then yield the same Euler equation, representing the equilibrium between pressure forces and the force of gravity:

$$\bar{p}' = \frac{\bar{\rho}\bar{c}^2\bar{m}}{r^2} = \frac{G\bar{\rho}\bar{M}}{r^2} \tag{88}$$

The Newtonian approximation means that this type of solution cannot handle cases where the negative-mass object—the source of the field—falls outside that framework. However, numerical simulations modeling the universe's large-scale structure produce vast, spherically symmetric negative-mass conglomerates made of negative-mass antihydrogen and antihelium. The cooling time for such objects exceeds the age of the universe; they do not evolve. Their temperature remains around 3000 K, and they simply radiate in the red and infrared spectra as negative-energy photons that escape detection. Yet, for travelers aboard a spacecraft capable of inverting its mass—thereby utilizing this negative realm as a sort of parallel universe—this is the spectacle that would meet their eyes. This world does not evolve and gives rise to neither stars nor galaxies; life is absent from it.

In conclusion, this Newtonian solution suffices to describe the full range of phenomena, as the negative-mass realm does not harbor superdense objects. From an observational standpoint, the interior metric $g_{\mu\nu}^{int}$ will be used to calculate the luminosity attenuation effect for objects located behind a negative-mass conglomerate—specifically, the one we believe lies at the center of the formation known as the "dipole repeller" [21]. Details regarding the construction of this metric are provided in **Appendix 2**.

In the vicinity of this negative-mass object, the exterior metric is:

$$\mathrm{d}s^2 = \left(1 + \frac{8\pi G\rho r^2)}{3c^2}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{8\pi G\rho r^2}{3c^2}\right)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{89}$$

Which matches the external metric, also linearized.

$$ds^2 = \left(1+\frac{2G|\bar{M}|}{c^2 r}\right)c^2 dt^2 - \left(1-\frac{2G|\bar{M}|}{c^2 r}\right)dr^2 - r^2 d\Omega^2 \quad (90)$$

This result should make it possible to assess the loss of brightness of background sources due to negative lensing, in order to compare them with observational data once the latter become available.

## 12. Re-Examination of Objections Raised against the Consistency of the Janus Model

### 12.1. Chronology of Objections and Responses

On January 4, 2019, based on a paper published in 2014 [22], T. Damour posted an initial highly critical titled "On the Janus cosmological model" [23] on his personal page on the IHES website. He informed J.-P. Petit of this via a registered letter with acknowledgment of receipt [24] sent to his home address.

A reply was sent to him, pointing out that some of his criticisms concerned equations affected by a typographical error in the article on which he based his analysis—specifically, the notation distinguishing interaction tensors from matter tensors had been omitted in the published version. This reply was accompanied by an article titled "Physical and Mathematical Consistency of the Janus Model," published in *Progress in Physics* that same month [20], as well as a proposal for a meeting to examine the calculations in question directly. This proposal received no response.

It should be noted, however, that although T. Damour specifies that his analysis is based on the two articles published in 2014, he does not discuss—nor does he subsequently do so—the non-stationary, homogeneous, and isotropic cosmological solution constructed in one of them [18]. Yet this solution constitutes an explicit case in which both systems of field equations are simultaneously satisfied, including the constraints resulting from the Bianchi identities.

In September 2019, a much more detailed document—explicitly setting out the calculations and, in particular, the analysis of the Tolman-Oppenheimer-Volkoff equations in the Newtonian limit—was sent to T. Damour via registered mail with acknowledgment of receipt [25]. This letter also went unanswered. The document was made public in 2020, in both French and English, as an open letter posted on Jean-Pierre Petit's website, so that the calculations could be independently examined.

In the years that followed, the published document prompted several French and foreign scientists to examine the calculations. In 2022, Marie-France Duval, a former astrophysics lecturer at the University of Marseille, initiated a new approach to T. Damour together with several colleagues. A joint letter, accompanied by an analysis of the calculations in question, was sent to him via registered mail with acknowledgment of receipt on November 31, 2022 [26].

On December 12, 2022, T. Damour informed J.-P. Petit by email that he had

posted a new article [27] on the IHES website—bearing the same title as before, "Physical and mathematical inconsistency of the Janus cosmological model"—which, in his view, contained "serious inconsistencies."

While in this article—as in the 2019 one—he reiterates that his analysis focuses on the two papers published in 2014 K [18] [22], he says nothing about the paper published in "Astrophysics and Space Science" [18], which presents a non-stationary solution—under conditions of homogeneity and isotropy—that satisfies the Bianchi conditions. Everything suggests, therefore, that he has still not consulted it.

However, with a three-year delay, he shows that he has examined the details of the calculations published in January 2019 [20]—the details of which had been sent to him in September 2019 [25] and which demonstrated the possibility of constructing a solution in the Newtonian approximation under steady-state conditions—and writes, see **Figure 26**.

Starting from the Janus-2019 field equations, the analogues of the Tolman-Oppenheimer-Volkoff equations are obtained from Eqs. (1) by setting $T^{-}_{\mu\nu}$ to zero and solving the equations for the two metrics in a stationary, spherically symmetric case. These solutions were written in Eqs. (45), (46) of PDD19, namely (with $' = d/dr$)

$$p'^{+} = -G(\rho^{+} + p^{+}/c^2)\,[M^{+}(r) + 4\pi p^{+} r^3/c^2] \,/\, [r(r - 2GM^{+}(r)/c^2)],$$

$$p'^{+} = -G(\rho^{+} - p^{+}/c^2)\,[M^{+}(r) - 4\pi p^{+} r^3/c^2] \,/\, [r(r + 2GM^{+}(r)/c^2)]. \quad (7)$$

Here $p^{+}(r)$ is the pressure (of ordinary matter), $\rho^{+}(r)$ its density, and $M^{+}(r) = 4\pi \int_0^r dr\, r^2 \rho^{+}(r)$ is the (positive) mass contained within radius r. Note that one passes from the first equation (7) to the second by the substitutions $p^{+} \to -p^{+}$ and $G \to -G$.

It is true that if one formally takes the Newtonian limit $1/c^2 \to 0$ in Eqs. (7), these two equations become compatible, because they both reduce to the single Newtonian structure equation (6).

But it is physically unacceptable to neglect the fact that the Janus-2019 model predicts that the radial pressure variation in an ordinary-matter star must satisfy two mutually incompatible equations. Indeed, if one considers, for example, a neutron star, the additional relativistic terms in (7), namely $\pm p^{+}/c^2$, $\pm 4\pi p^{+} r^3/c^2$, and $\pm 2GM^{+}(r)/c^2$, are numerically very significant (of order 10%) and conceptually very important because they greatly modify the value of the maximum mass of a neutron star. As recalled in Eq. (5), the analogues of Eqs. (7) in Janus-2014 had right-hand sides differing by about 200%. The modification introduced in Janus-2019 (geometrically undefined because it violates covariance) has merely formally reduced the difference between the right-hand sides from 200% to, say, about 20% (for a neutron star). This does not alter the fundamental mathematical inconsistency that the Janus-2019 model predicts that the radial pressure variation in an ordinary-matter star must satisfy two mathematically (and physically) incompatible equations.

CONCLUSION: The field equations of the 2019 version of the "Janus model", Eqs. (1), have no logical derivation from a covariant variational principle and are mathematically and physically contradictory. Moreover, they predict (contrary to the basic idea of the Newtonian Janus model) that a negative central mass must repel the negative masses around it.

**Figure 26.** Excerpt from the article posted on the IHES website by T. Damour on December 12, 2022 [27].

However, in the last line (underlined in red), he writes that the Janus equations predict that negative masses must repel each other. In a letter dated December 14, 2022 [28], we pointed out his error. T. Damour then withdrew the version dated December 12 and replaced it with a new document dated December 28, 2022 [29], bearing the same title. This second version, like the January 2019 document, remains accessible—four years after its initial posting—via T. Damour's personal

page on the IHES website. The differences between the versions dated December 12, 2022 [27] and December 28, 2022 [29], as well as their relationship to the objection raised in the letter dated December 14, 2022 [28], will be examined later.

A further step was taken in 2024 with the publication in European Physical Journal [9] of a derivation of the Janus system from an action. This formulation notably allows for the definition of interaction tensors via variation of the action and enables the issue of compatibility with Bianchi identities to be revisited within this framework. The article also reviews the analysis conducted as early as 2019 [20] regarding the spherically symmetric stationary case and its Newtonian limit.

In 2025, a draft letter calling for a scientific debate on the Janus model to be held at the Academy of Sciences was made public on J.-P. Petit's website to gather co-signatures. This letter, co-signed by 54 French and international scientists, was subsequently sent to Françoise Combes, President of the Academy of Sciences. As of the time of writing, this request has received no response [30].

This chronology is presented neither as an argument in favor of the Janus model nor as a response to the objections raised against it. Its purpose is to identify the various documents to which we will refer and to specify the sequence in which arguments were formulated, responses were communicated, and texts were modified. The question of the model's consistency must be resolved independently of this history, through an examination of the field equations, the Bianchi identities, and the solutions they admit.

The two articles by T. Damour addressing the alleged inconsistency of the Janus cosmological model—published in January 2019 [23] and December 2022 [29] respectively, and still available on his personal IHES webpage—were not published in a peer-reviewed scientific journal but were instead posted online on that personal institutional page. This circumstance obviously implies nothing regarding the validity or invalidity of the arguments contained therein. However, it has significant implications for the conditions under which scientific debate takes place. When a critique explicitly targeting an author's published work appears in a scientific journal, editorial procedures generally allow the author in question to submit a response—such as a "Letter," "Comment," or "Reply," depending on the journal's specific guidelines. The author can then address the objections raised in detail and refer back to published articles where the issues in question were already discussed. This response is, in turn, subject to the journal's editorial process and can thus become part of the scientific literature on the same footing as the critique to which it responds. Posting T. Damour's two critiques on his personal IHES webpage did not provide such a framework for a structured exchange of views. Therefore, we will now examine the objections raised in these two articles one by one and compare them against the equations, solutions, and scientific publications presented or cited in the preceding sections.

### 12.2. First Objection: Compatibility of the Two Bianchi Identities

In his document dated January 4, 2019 [23], he examines the system of equations

he then attributes to the Janus model and concludes that it is physically and mathematically inconsistent.

This question must be distinguished from the issue of whether arbitrarily chosen source tensors could simultaneously satisfy two independent sets of differential constraints. In the model considered here, the interaction tensors are precisely not arbitrary; they are subject to the conservation conditions resulting from the Bianchi identities.

Using the notation employed previously, the field equations are written in the form:

$$G_{\mu\nu} = \chi\left[T_{\mu\nu} + \bar{K}_{\mu\nu}\right] \tag{91a}$$

$$\bar{G}_{\mu\nu} = -\chi\left[\bar{T}_{\mu\nu} + K_{\mu\nu}\right] \tag{91b}$$

Geometric identities:

$$\nabla_\nu G^{\mu\nu} = \bar{\nabla}_\nu \bar{G}^{\mu\nu} = 0 \tag{92}$$

therefore require:

$$\nabla_\nu\left[T_{\mu\nu} + \bar{K}_{\mu\nu}\right] = \bar{\nabla}_\nu\left[\bar{T}_{\mu\nu} + K_{\mu\nu}\right] = 0 \tag{93}$$

The question of compatibility is therefore not an external condition added to the system; it constitutes a constraint on the form of the interaction tensors.

Let us first consider the non-stationary case describing homogeneous and isotropic media. The initial 2014 paper [16] provides precisely such a mathematically consistent solution. Although this paper is explicitly cited by T. Damour in his two documents [23] [29], the solution itself is not examined therein.

For two homogeneous and isotropic FLRW-type metrics and for the equation of state:

$$p = \beta\frac{\rho c^2}{3} \qquad \bar{p} = \beta\frac{\bar{\rho}\bar{c}^2}{3} \tag{94}$$

The two constraints derived from the Bianchi identities lead, as shown in **Appendix 1**, to:

$$\varphi = \left(\frac{\bar{a}}{a}\right)^{3+\beta} \qquad \phi = \left(\frac{a}{\bar{a}}\right)^{3+\beta} \tag{95}$$

and consequently to the generalized conservation law:

$$\rho c^2 a^{3+\beta} + \bar{\rho}\bar{c}^2\bar{a}^{3+\beta} = E = Cst \tag{96}$$

Thus, within this class of solutions, the two Bianchi constraints are satisfied simultaneously. They do not lead to a contradiction but rather determine the relations that the interaction terms must satisfy.

The existence of such a solution obviously does not constitute a proof of the existence of solutions for any arbitrary distribution of matter. It suffices, however, to show that a general and necessary incompatibility between the two Bianchi identities cannot be deduced solely from the bimetric nature of the system.

This point is all the more important given that the homogeneous and isotropic

cosmological solution had already been constructed in the 2014 paper Negative mass hypothesis in cosmology and the nature of dark energy [18]—explicitly cited in the January 2019 document—without that solution being examined therein.

### 12.3. Second Objection: Incompatibility of the Structural Equations

In his January 2019 paper [23], T. Damour then considers the spherically symmetric, stationary case. For the equations he employs, he derives—in the Newtonian limit—two structure equations with opposite signs. This objection was mathematically valid regarding the system as it appeared in the second 2014 paper [22]. And this is where an unfortunate set of circumstances comes into play: when the manuscript for that paper was typeset by the publisher, the hats we had included disappeared. See Figure 27.

In our model, the Universe is an $M_4$ manifold associated not to one single metric, but to two: $g_{\mu\nu}^{(+)}$ and $g_{\mu\nu}^{(-)}$, the former linked to species of positive mass and energy, the latter to species of negative mass and energy. From these metrics, one can build the associated Ricci tensors, $R_{\mu\nu}^{(+)}$ and $R_{\mu\nu}^{(-)}$. A system of two coupled field equations was then proposed:[20]

$$R_{\mu\nu}^{(+)} - \frac{1}{2}R^{(+)}g_{\mu\nu}^{(+)} = \chi\left(T_{\mu\nu}^{(+)} + T_{\mu\nu}^{(-)}\right), \tag{2a}$$

hat missing

$$R_{\mu\nu}^{(-)} - \frac{1}{2}R^{(-)}g_{\mu\nu}^{(+)} = -\chi\left(T_{\mu\nu}^{(+)} + T_{\mu\nu}^{(-)}\right). \tag{2b}$$

where the tensors $T_{\mu\nu}^{(+)}$ and $T_{\mu\nu}^{(+)}$ represent positive and negative energy contents (and positive and negative mass contents as well). Previously, in 1957, Bondi[23] did study the possibility of introducing negative masses into the Einsteinian model with a single metric. As a result, positive masses attracted everything and negative masses repelled everything. It then led to a phenomenon that was called "runaway": when a mass $+m$ met a mass $-m$, the positive mass fled, pursued by the negative mass, both undergoing a continuous acceleration, with conservation of energy, since one of the energies was positive and the other negative.

**Figure 27.** The field equations, as presented in the second 2014 paper [22].

In our model, the Universe is an $M_4$ manifold associated not to one single metric, but to two: $g_{\mu\nu}^{(+)}$ and $g_{\mu\nu}^{(-)}$, the former linked to species of positive mass and energy, the latter to species of negative mass and energy. From these metrics, one can build the associated Ricci tensors, $R_{\mu\nu}^{(+)}$ and $R_{\mu\nu}^{(-)}$. A system of two coupled field equations was then proposed:[20]

$$R_{\mu\nu}^{(+)} - \frac{1}{2}R^{(+)}g_{\mu\nu}^{(+)} = \chi\left(T_{\mu\nu}^{(+)} + \widehat{T}_{\mu\nu}^{(-)}\right) \tag{2a}$$

interaction tensors with «hat»

$$R_{\mu\nu}^{(-)} - \frac{1}{2}R^{(-)}g_{\mu\nu}^{(+)} = -\chi\left(\widehat{T}_{\mu\nu}^{(+)} + T_{\mu\nu}^{(-)}\right) \tag{2b}$$

where the tensors $T_{\mu\nu}^{(+)}$ and $T_{\mu\nu}^{(+)}$ represent positive and negative energy contents (and positive and negative mass contents as well). Previously, in 1957, Bondi[23] did study the possibility of introducing negative masses into the Einsteinian model with a single metric. As a result, positive masses attracted everything and negative masses repelled everything. It then led to a phenomenon that was called "runaway": when a mass $+m$ met a mass $-m$, the positive mass fled, pursued by the negative mass, both undergoing a continuous acceleration, with conservation of energy, since one of the energies was positive and the other negative.

**Figure 28.** These equations, in the original manuscript.

While in the submitted version, hats marked the fundamental difference between these interaction tensors $\hat{T}_{\mu\nu}^{(+)}$ and $\hat{T}_{\mu\nu}^{(-)}$ and the field tensors $T_{\mu\nu}^{(+)}$ and $T_{\mu\nu}^{(-)}$. See Figure 28.

Thus, upon receiving his registered letter with acknowledgment of receipt at our home address—dated January 4, 2019 [24] and pointing to the article he had just posted on his IHES webpage [23]—the first thing we did was to state that we were fully aware of this requirement for compatibility with Bianchi conditions; we noted that simply identifying interaction tensors with field tensors imposed the degeneracy $g_{\mu\nu}^{(+)} = g_{\mu\nu}^{(-)}$, a condition that had already been satisfied in the non-stationary case (assuming homogeneity and isotropy) in the form of a generalized energy conservation law (see **Appendix 1**). Regarding the stationary solution, we explained that, following discussions with the referee for the journal Progress in Physics, we had secured acceptance for an article—just published in the January 2019 issue [20]—presenting a solution consistent with the Newtonian approximation; we forwarded a copy of this article to him. At the same time, we proposed a meeting to discuss the points raised in his article.

However, in his registered letter with acknowledgment of receipt [24], we find, see Figure 29.

> **Copies:** I reserve the right to send copies of this letter to all the persons whom you cite on your website, in your videos, and in your letters, as well as to any person interested in the "Janus model".

Figure 29. Damour's 2019 paper recipients [24].

It turned out, in fact, that a great many people within the French and international scientific communities, as well as in the press, had already received a copy of his article. We then proposed a meeting to T. Damour to examine these various points and shared with him the January 2019 article [20], which presented a construction of interaction tensors satisfying the constraints considered in the Newtonian limit. We received no response to this proposal or to the document over the following three years.

The details of the calculations regarding this construction of interaction tensors—which satisfy the Bianchi conditions in the Newtonian approximation—are given in **Appendix 2**. For an object composed of negative-mass matter, the matter tensor of the negative sector takes the form:

$$\bar{T}_{\mu}^{\nu} = \mathrm{diag}\left(\bar{\rho}, -\frac{\bar{p}}{\bar{c}^2}, -\frac{\bar{p}}{\bar{c}^2}, -\frac{\bar{p}}{\bar{c}^2}\right) \tag{97}$$

The compatibility condition in the Newtonian limit is then satisfied if the interaction tensor of the positive sector is given the form:

$$\bar{K}_{\mu}^{\nu} = \mathrm{diag}\left(\bar{\rho}, +\frac{\bar{q}}{\bar{c}^2}, +\frac{\bar{q}}{\bar{c}^2}, +\frac{\bar{q}}{\bar{c}^2}\right) \tag{98}$$

With $\bar{q} = -\bar{p}$. In other words, if the pressure terms of the interaction tensor

are equal and opposite to those of the corresponding matter tensor.

The two TOV equations then lead, in the Newtonian limit, to the same Euler equation. With

$$M(r) = 4\pi \tag{99}$$

We get:

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}r} = -\frac{G\bar{M}(r)\bar{\rho}}{r^2} \tag{100}$$

and, for the interaction tensor:

$$\frac{\mathrm{d}\bar{q}}{\mathrm{d}r} = \frac{G\bar{M}(r)\bar{\rho}}{r^2} \tag{101}$$

Avec

$$\bar{q} = -\bar{p} \tag{102}$$

The two relations then become compatible.

It is particularly worth noting that T. Damour's document of December 28, 2022, itself acknowledges that the modification introduced in 2019 eliminates the contradiction between the two Newtonian equations and leads to a single Newtonian structure equation. See **Figure 30**.

> It is true that if one formally takes the Newtonian limit $1/c^2 \to 0$ in Eqs. (7), these two equations become compatible, because they both reduce to the single Newtonian structure equation (6).

**Figure 30.** Sentence from the article posted on the IHES website by T. Damour on December 12, 2022 [27].

However, the scope of the result must be clarified. We do not claim here to have demonstrated the identity of the two exact TOV equations to all relativistic orders. The result established is their compatibility in the Newtonian limit, which is precisely the regime used here to describe the diffuse negative-mass structures considered in the model.

### 12.4. On the Management of Superdense Objects

In his 2022 paper, T. Damour acknowledges—three years after receiving the 2019 article presenting the elements discussed in the previous section—that the two structure equations become compatible in the Newtonian limit. Indeed, he writes that as $\frac{1}{c^2} \to 0$, both equations become identical to the single Newtonian structure equation. However, he maintains his objection regarding the relativistic regime, citing the example of a neutron star, for which the $\frac{1}{c^2}$ correction terms can no longer be neglected.

He cites the model's inability to describe supermassive objects. This is incorrect for two reasons. In the case of neutron stars, the first field equation—through the exterior and interior metrics—yields the required geometry. However, there is no

obligation to formulate the interaction tensor appearing in the second equation, given that pursuing this theoretical work leads to no confrontation with observational data. Indeed, our instruments do not detect negative-energy photons. Finally, developments within the Janus model show that its negative sector is populated solely by vast, low-density, spheroidal conglomerates made of light elements—devoid of structures such as galaxies, stars, or planets. Consequently, they contain no high-density objects. Conversely, the construction of the positive sector's Newtonian metric—generated by the interaction term—predicts a dimming of distant background objects, thereby offering a potential observational confirmation.

### 12.5. Fourth Objection: Variational Derivation of the System

In his December 2022 document, T. Damour raises two objections to the derivation proposed in the 2019 article. The first concerns the introduction of the matrix

$$\varphi = \mathrm{diag}\left(1,-1,-1,-1\right) \tag{103}$$

which he considers incompatible with general covariance. The second concerns the relation used in this derivation,

$$\delta g^{-}_{\mu\nu} = -\delta g^{+}_{\mu\nu} \tag{104}$$

which, according to him, would not allow for obtaining two independent field equations via variation.

It is not necessary, in the present article, to defend this 2019 variational construction. Since then, the problem has been revisited independently. The derivation published in 2024 in the European Physical Journal C starts from a bimetric action and defines the interaction tensors by varying with respect to the two metrics. The present article goes a step further by demonstrating the close relationship between this construction and the class of bimetric actions introduced by Damour and Kogan. As shown in sections 6 through 9, the essential difference considered here lies in the relative sign of the two gravitational coupling constants. Damour and Kogan use constants of the same sign, whereas the Janus system considered here corresponds to the choice

$$\bar{\chi} = -\chi \tag{105}$$

The criticism regarding the constant matrix and the relationship between variations used in 2019 does not, therefore, constitute an objection to the variational derivation employed in the current formulation. It concerns an earlier derivation that is no longer necessary for the definition of the system studied here.

### 12.6. Objection Regarding the Direction of Gravitational Interactions

In his critiques of the Janus model, T. Damour identifies a contradiction regarding the nature of gravitational interactions between positive and negative masses. In his article dated December 28, 2022, his reasoning involves requiring a single four-velocity field—associated with a test particle—to simultaneously satisfy two geo-

desic equations, constructed respectively from the connections associated with the metrics $g^+$ and $g^-$. In the document dated December 28, 2022 [29], T. Damour explicitly states that "the same [...] four-velocity field" must satisfy, in turn, geodesic Equations (5) and (6), which are associated with the two metrics.

This prescription is inconsistent with the foundational assumptions of the Janus model. In that model, the two species of particles are associated with two different metrics. Positive-mass test particles follow the geodesics of the metric $g^+$ the solution to the first field equation—whereas negative-mass test particles follow the geodesics of the metric $g^-$ the solution to the second field equation. A single test particle is therefore not assumed to follow the geodesics of both metrics simultaneously.

The reasoning leading to the contradiction put forward by T. Damour becomes clear when he takes the Newtonian limit. By successively applying the connections associated with the two metrics to the same four-velocity field $u_\mu^+$, he obtains two accelerations of opposite signs.

$$\frac{\mathrm{d}v_+^i}{\mathrm{d}t} = +\partial_i U \qquad \frac{\mathrm{d}v_+^i}{\mathrm{d}t} = -\partial_i U \tag{106}$$

...then interprets this opposition as a contradiction within the model. Yet this contradiction arises precisely from the simultaneous application of the two geodesic prescriptions to the same test particle. It does not occur when each type of particle is subjected to the metric associated with it in the definition of the model.

The most direct way to determine the nature of gravitational interactions is to refer to the exterior metric solutions obtained under spherical symmetry, as was done in the preceding sections. In the weak-field limit of the corresponding Schwarzschild-type metrics, the sign of the Newtonian term directly determines the direction of the acceleration of the test particle in the sector under consideration.

This yields the four situations:

(+ source, + test particule) ⟶ Attraction

(− source, − test particule) ⟶ Attraction

(+ source, − test particule) ⟶ Repulsion

(− source, + test particule) ⟶ Repulsion

These four results reproduce the interaction laws that constitute one of the founding hypotheses of the Janus model: masses of the same sign attract each other, and masses of opposite signs repel each other.

The origin of the contradiction raised in T. Damour's analysis can thus be unambiguously identified. It lies not in the sign of the forces derived from the system's metric solutions, but in the assumption that a single test-matter motion must simultaneously satisfy the geodesic equations associated with both metrics. This assumption is not part of the definition of the Janus model.

## 13. Conclusions

Since 2019, the two texts by T. Damour addressing the alleged inconsistency of

the Janus model have constituted—to the best of our knowledge—the most frequently cited explicitly developed mathematical critiques challenging the model's consistency.

Among these criticisms is T. Damour's assertion that the system of two coupled field equations upon which this model rests cannot be derived from an action.

Such a derivation has since been published. However, the present work sheds additional, particularly simple light on this issue. In 2002, Damour and Kogan themselves proposed an action describing the coupled dynamics of two metrics. Their construction was framed within a different context—that of non-linear bigravity—and was in no way intended to construct the Janus cosmological model. We have shown here that by considering, within a construction of this type, two gravitational coupling constants that are equal in absolute value but opposite in sign, the structure of the system of two coupled field equations characteristic of the Janus model emerges naturally.

This result does not imply that the Janus model should be viewed merely as an application or a historically derived variant of the work of Damour and Kogan. Its genesis is independent, and it addresses a different set of physical issues. However, the comparison drawn in this article demonstrates that the objection—claiming the structure of its field equations is inherently incompatible with a Lagrangian formulation—cannot be sustained.

We review the origins of the Janus model, tracing the path from the introduction of negative masses and the "runaway" problem to the construction of a bimetric system in which positive- and negative-mass particles are associated with two distinct metrics. This structure yields the desired interaction laws: masses of the same sign attract one another, while masses of opposite signs repel. The runaway effect inherent in introducing negative masses within a monometric framework is eliminated.

We also recall the key cosmological and astrophysical properties that motivated the model's development, as well as the various qualitative agreements with observations achieved during its formulation. Taken in isolation, these elements do not constitute a validation of the model. However, they demonstrate that it yields consequences amenable to observational testing, thereby justifying an examination of its mathematical consistency and physical predictions based on its effective equations.

This consistency has been examined here in two situations possessing symmetries high enough to allow for an explicit analysis. In the non-stationary, homogeneous, and isotropic case, the constraints arising from the two Bianchi identities lead to a generalized conservation law and to coupled cosmological solutions; thus, they do not reveal the a priori incompatibility that had been cited as an argument against the model. In the stationary, spherically symmetric case, an analysis of the Tolman-Oppenheimer-Volkoff equations demonstrates how the structure of the interaction tensors ensures the hydrostatic compatibility of the two sectors within the Newtonian limit under consideration.

We then re-examine the following criticisms raised by T. Damour. The objection that the two Bianchi identities are necessarily incompatible is countered by the existence of an explicit cosmological solution that simultaneously satisfies the constraints they impose. Finally, the apparent contradiction regarding the direction of the forces stems from the requirement that the same four-velocity field of a test-matter particle simultaneously satisfy the geodesic equations associated with both metrics. This is precisely how the argument is formulated in the document dated December 28, 2022.

This latter prescription is not among the foundational hypotheses of the Janus model. A positive-mass test particle follows the geodesics of the metric $g_{\mu\nu}$, whereas a negative-mass test particle follows those of the metric $\overline{g}_{\mu\nu}$. Directly determining the nature of the interactions from spherically symmetric exterior metric solutions thus unambiguously yields attraction between masses of the same sign and repulsion between masses of opposite signs. In the December 2022 text, by contrast, the contradictory accelerations are obtained by successively applying the two connections to the same test-matter motion.

With this clarification, we hope that the Janus Cosmological Model can now be examined by the specialist community based on its equations, its solutions, and how they compare with observations.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Appendix 1

**Generalized law of energy conservation for unsteady conditions and in homogeneous, isotropic media.**

1 – **In classical cosmology**:

In general relativity, when the symmetry assumptions of homogeneity and isotropy are applied to a non-stationary solution, the metric takes the FLRW form, and the field equation yields a system of two second-order differential equations involving the scale factor and its first and second time derivatives.

$$G_0^0 - \left(\frac{3k}{a^2} + \frac{3\dot{a}^2}{c^2a^2}\right) = \chi\rho \tag{1}$$

$$G_i^i = -\left(\frac{k}{a^2} + \frac{\dot{a}^2}{c^2a^2} + \frac{2\ddot{a}}{c^2a}\right) = -\chi\frac{p}{c^2} \tag{2}$$

We must then derive a compatibility condition from this system, which will be the equivalent of the Bianchi condition relating to the symmetries of this solution. With:

$$\chi = -\frac{8\pi G}{c^2} \tag{3}$$

$$\frac{8\pi G}{c^2}\rho = \frac{3k}{a^2} + \frac{3\dot{a}^2}{c^2a^2} \tag{4}$$

$$\frac{8\pi G}{c^2}\left(\frac{p}{c^2}\right) = -\left(\frac{k}{a^2} + \frac{\dot{a}^2}{c^2a^2} + \frac{2\ddot{a}}{c^2a}\right) \tag{5}$$

By adding and then subtracting these two equations, we obtain:

$$\frac{4\pi G}{c^2}\left(\rho + \frac{3p}{c^2}\right) = -\frac{3\ddot{a}}{c^2a} \tag{6a}$$

$$\frac{4\pi G}{c^2}\left(\rho + \frac{p}{c^2}\right) = \frac{k}{a^2} + \frac{a'^2 - a\ddot{a}}{c^2a^2} \tag{6b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{c^2}\frac{\dot{a}}{a}\right) = \frac{k}{a^2} - \frac{4\pi G}{c^2}\left(\rho + \frac{p}{c^2}\right) \tag{7}$$

We now differentiate (4) with respect to time, and obtain:

$$\frac{8\pi G}{c^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{6k}{a^3}\dot{a} + \frac{6\dot{a}}{a}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{c^2}\frac{\dot{a}}{a}\right) \tag{8}$$

Inserting (7) into (8) and rearranging we obtain the differential identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho a^3\right) + \frac{p}{c^2}\frac{\mathrm{d}a^3}{\mathrm{d}t} = 0 \tag{9}$$

Let us take an element of volume in the co-moving coordinates. Its geometry content will be proportional to $a^3$, from the Robertson-Walker metric. This, if $V(t)$ denotes the actual volume content of the element considered, $M = \rho V$ would measure the mass content and (9) could be written as:

$$\mathrm{d}M + \frac{1}{c^2} p\mathrm{d}V = 0 \tag{10}$$

Since $c^2 M$ measures the energy content of the element and $p\mathrm{d}V$ the work done against the pressure forces, we see that the energy balance under the cosmic evolution, is preserved. If we prescribe the equation of state of the fluid $p = p(\rho)$, we can obtain from (9) the dependance of $a$ upon $\rho$. Indeed we may write Equation (9) in the alternative form:

$$\frac{\mathrm{d}a}{a} + \frac{1}{3}\left(\frac{\mathrm{d}\rho}{\rho + p(\rho)/c^2}\right) = 0 \tag{11}$$

It is this relationship that ensures the compatibility of the system of equations (1) + (2)—that is, the mathematical consistency of the solution, or the equivalent of the Bianchi condition for the field equation—within this particular symmetry.

In the denominator, we have a sum of two terms. Pressure is simply a volumetric energy density, just like the term $\rho c^2$. It is the volumetric energy density that determines cosmic geometry and dynamics. The relative importance of the term $p$ compared to the term $\rho c^2$ determines the type of dynamics under consideration. If we treat the matter fluid as a perfect fluid, and if $\langle v^2 \rangle$ is the mean square velocity of thermal motion within it, the pressure is given by:

$$p_m = \frac{\rho_m \langle v^2 \rangle}{3} \tag{12}$$

If the medium is ionized, radiation pressure is added to this material pressure:

$$p_r = \frac{\rho_r c^2}{3} \tag{13}$$

In the general case, pressures are additive:

$$p = \frac{\rho_m \langle v^2 \rangle}{3} + \frac{\rho_r c^2}{3} \tag{14}$$

In the matter-dominated era, pressure terms are negligible compared to $\rho c^2$. In the radiation-dominated era, radiation pressure is comparable to $\rho c^2$. In the calculation, both simulations can be handled by expressing the pressure terms in the form:

$$p = \beta \frac{\rho c^2}{3}$$

$\beta = 0$ corresponds to the matter-dominated phase and $\beta = 1$ to the radiation-dominated phase. Equation (11) becomes:

$$\frac{\mathrm{d}a}{a} + \frac{1}{3}\left(\frac{\mathrm{d}\rho}{\rho(1+\beta/3)}\right) = \frac{\mathrm{d}a}{a} + \frac{\mathrm{d}\rho}{(3+\beta)\rho} = 0 \tag{15}$$

This results in two distinct dynamics.

Matter-dominated era:

$$E = \rho c^2 a^3 = Cst \tag{16}$$

Radiation-dominated era:

$$E = \rho c^2 a^4 = Cst \tag{17}$$

Or, overall

$$E = \rho c^2 a^{3+\beta} \tag{18}$$

We have added the factor $c^2$ because this condition represents the conservation of energy. For a dust-filled universe—modeled as masses $m$—the decrease in volumetric energy density is solely due to expansion. The individual energy $mc^2$ of the masses remains constant. During the radiation-dominated era, it is the "photon gas" that expands. However, since these photons constitute the very fabric of space, their wavelength $\lambda$ stretches in tandem with the scale factor $a$; consequently, their individual energy $hc/\lambda$ varies as $1/a$. This explains the exponent 4 in Equation (17).

With $E$ representing energy, the system becomes:

$$-\left(\frac{k}{a^2}+\frac{\dot{a}^2}{c^2a^2}\right)=\chi\frac{E}{3c^2a^{3+\beta}} \tag{19}$$

$$-\left(\frac{k}{a^2}+\frac{\dot{a}^2}{c^2a^2}+\frac{2\ddot{a}}{c^2a}\right)=-\chi\frac{\beta E}{3c^2a^{3+\beta}} \tag{20}$$

Which leads to the equation:

$$a^{2+\beta}\ddot{a}=\frac{\chi(1+\beta)E}{6}=-\frac{4\pi G(1+\beta)E}{3c^2} \tag{21}$$

Consequently, in general relativity, if $E>0$, then $\ddot{a}<0$.

When $\beta=0$, we obtain the Friedmann equation.

When $\beta=1$, we have:

$$a^3\ddot{a}=-\frac{8\pi GE}{3c^2} \tag{22}$$

This equation does not prejudge, a priori, which value to choose for the curvature index:

$$k\in\{-1,0,+1\} \tag{23}$$

In classical cosmology, a zero value is generally chosen. In this case:

$$a(t)=\left(\frac{32\pi GE}{3c^2}\right)^{1/4}\sqrt{t} \tag{24}$$

Let us consider (which will be justified later):

$$k=-1 \tag{25}$$

As we have:

$$\dot{a}=c\sqrt{-\left(\frac{\chi E}{3c^2a^{1+\beta}}+k\right)} \tag{26}$$

He is coming with:

$$A = \frac{8\pi GE}{3c^2} \tag{27}$$

$$\dot{a} = \frac{\sqrt{c^2a^2 + A}}{a} \tag{28}$$

$$\mathrm{d}t = \frac{a\mathrm{d}a}{\sqrt{c^2a^2 + A}} \tag{29}$$

By taking:

$$(t \to 0) \Rightarrow (a \to 0) \tag{30}$$

$$t = \frac{1}{c^2}\left(\sqrt{c^2a^2 + A} - \sqrt{A}\right) \tag{31}$$

$$a(t) = \sqrt{c^2t^2 + 2\sqrt{\frac{8\pi GE}{3c^2}}t} \tag{32}$$

When $t$ is very small:

$$t \ll \sqrt{\frac{32\pi GE}{3c^6}} \tag{33}$$

Then:

$$a(t) \cong \left(\frac{32\pi GE}{3c^2}\right)^{1/4}\sqrt{t} \tag{34}$$

Near the origin, the three solutions—for the three values of the curvature index $k$ exhibit the same asymptotic behavior.

**Noticed**:

Let us now take relation (18)— $E = \rho c^2 a^{3+\beta}$ —as a hypothesis. That is to say:

$$\rho = \frac{E}{c^2a^{3+\beta}} \tag{35}$$

To which is added:

$$\frac{p}{c^2} = \frac{\beta E}{3c^2a^{3+\beta}} \tag{36}$$

From the system of Equations (1) + (2), we can move to (1) + (6a), which becomes:

$$\dot{a} = c\sqrt{-\left(\frac{\chi E}{3c^2a^{1+\beta}} + k\right)} \tag{37}$$

$$\frac{\chi E(1+\beta)}{6a^{2+\beta}} = \ddot{a} \tag{38}$$

Differentiating (37) yields:

$$\ddot{a} = \frac{c(1+\beta)\left(\frac{\chi E}{3c^2}\right)a^{-(2+\beta)}\dot{a}}{2\sqrt{-\left(\frac{\chi E}{3c^2a^{1+\beta}} + k\right)}} = \frac{(1+\beta)\chi Ea^{-(2+\beta)}}{6} \tag{39}$$

That is identical to equation (38). Thus, the consistency of the system is ensured.

2 – **In the Janus Cosmological Model**:

In the first part of the article, we opted to construct the coupled system of field equations using a variant of the Damour-Kogan action [11]. In this case, the system is written as follows:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \chi\left(T_{\mu\nu} + \overline{K}_{\mu\nu}\right) \tag{40a}$$

$$\overline{R}_{\mu\nu} - \frac{1}{2} \overline{R}\overline{g}_{\mu\nu} = -\chi\left(\overline{T}_{\mu\nu} + K_{\mu\nu}\right) \tag{40b}$$

where $K_{\mu\nu}$ and $\overline{K}_{\mu\nu}$ are the interaction tensors. In reference [18], a non-stationary solution exhibiting isotropy and homogeneity was sought, introducing specific forms for the interaction tensors in which $\varphi$ and $\phi$ are functions of time:

$$\overline{K}_{\mu\nu} = \varphi \overline{T}_{\mu\nu} \tag{41a}$$

$$K_{\mu\nu} = \phi T_{\mu\nu} \tag{41b}$$

We will seek solutions derived from symmetry hypotheses—isotropy and homogeneity—that is, FLRW metric solutions, using the same time variable $x°$ for both:

$$\mathrm{d}s^2 = \mathrm{d}x^{\circ 2} - a^2\left(\frac{\mathrm{d}u^2}{1-ku^2} + u^2\mathrm{d}\theta^2 + u^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{42a}$$

$$\mathrm{d}\overline{s}^2 = \mathrm{d}x^{\circ 2} - \overline{a}^2\left(\frac{\mathrm{d}u^2}{1-\overline{k}u^2} + u^2\mathrm{d}\theta^2 + u^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{42b}$$

Let:

$$H = \frac{\dot{a}}{a} \qquad \overline{H} = \frac{\dot{\overline{a}}}{\overline{a}} \tag{43}$$

We write the equations of state, with $\beta \in \{O,1\}$, in the form:

$$p = \frac{\beta}{3}\rho c^2 = \frac{\beta}{3}\varepsilon \qquad \overline{p} = \frac{\beta}{3}\overline{\rho c}^2 = \frac{\beta}{3}\overline{\varepsilon} \tag{44}$$

To simplify the calculation, let us set:

$$T_{00} + \overline{K}_{00} = X = \varepsilon + \varphi\overline{\varepsilon} \tag{45}$$

Thus, our first field equation is written as:

$$G_0^0 = -\left(\frac{3k}{a^2} + \frac{3\dot{a}^2}{c^2a^2}\right) = \chi X \tag{46}$$

And the second:

$$-\left(\frac{k}{a^2} + \frac{\dot{a}^2}{c^2a^2} + \frac{2\ddot{a}}{c^2a}\right) = -\frac{\chi\beta}{3}X \tag{47}$$

Let's set

$$F = \frac{k}{a^2} + \frac{\dot{a}^2}{c^2a^2} \tag{48}$$

Equation (47) becomes:

$$F+\frac{2\ddot{a}}{c^2a}=\frac{\chi\beta}{3}X \tag{49}$$

$$-\frac{\chi X}{3}+\frac{2\ddot{a}}{c^2a}=\frac{\chi\beta}{3}X \tag{50}$$

$$\frac{\ddot{a}}{c^2a}=\frac{\chi}{6}(1+\beta)X \tag{51}$$

Let us now start with the first of the two equations of the system, Equation (43), which becomes:

$$-3F=\chi X \tag{52}$$

Let's derive it:

$$-3\dot{F}=\chi\dot{X} \tag{53}$$

$$\dot{F}=-\frac{2k\dot{a}}{a^3}+\frac{2\dot{a}\ddot{a}}{c^2a^2}-\frac{2\dot{a}^3}{c^2a^3} \tag{54}$$

With (43):

$$\dot{F}=2H\left[-\frac{k}{a^2}-\frac{\dot{a}^2}{c^2a^2}+\frac{\ddot{a}}{c^2a}\right] \tag{55}$$

But:

$$-\frac{k}{a^2}-\frac{\dot{a}^2}{c^2a^2}=-F \tag{56}$$

Hence:

$$\dot{F}=2H\left(-F+\frac{\ddot{a}}{c^2a}\right) \tag{57}$$

Using (51) and (52) we get:

$$-F+\frac{\ddot{a}}{c^2a}=\frac{\chi}{3}X+\frac{\chi}{6}(1+\beta)X \tag{59}$$

$$-F+\frac{\ddot{a}}{c^2a}=\frac{\chi}{6}(3+\beta)X \tag{60}$$

Thus, Equation (57) becomes:

$$\dot{F}=\frac{\chi}{3}(3+\beta)HX \tag{61}$$

Let us substitute it into (53).

$$\dot{X}+(3+\beta)HX=0 \tag{62}$$

That's to say:

$$\frac{\mathrm{d}X}{X}+(3+\beta)\frac{\mathrm{d}a}{a}=0 \tag{63}$$

Or:

$$Xa^{3+\beta}=Cst \tag{64}$$

But $X = \varepsilon + \varphi\bar{\varepsilon}\cdot I$ :

$$\dot{\varepsilon} + \varphi\dot{\bar{\varepsilon}} + \dot{\varphi}\bar{\varepsilon} + (3+\beta)H\varepsilon + (3+\beta)H\varphi\bar{\varepsilon} = 0 \tag{65}$$

As $\varepsilon = \rho c^2$; $\bar{\varepsilon} = \bar{\rho}\bar{c}^2$

$$C_+ = c^2\dot{\rho} + \varphi\bar{c}^2\dot{\bar{\rho}} + \dot{\varphi}\bar{c}^2\bar{\rho} + (3+\beta)H\left(\rho c^2 + \varphi\bar{\rho}\bar{c}^2\right) = 0 \tag{66}$$

We must now perform the same calculation in the negative sheet.

$$\bar{X} = \bar{\varepsilon} + \phi\varepsilon \tag{67}$$

$$-3\left(\frac{\bar{k}}{\bar{a}^2} + \frac{\dot{\bar{a}}^2}{\bar{c}^2\bar{a}^2}\right) = -\chi\bar{X} \tag{68}$$

$$-\left(\frac{\bar{k}}{\bar{a}^2} + \frac{\dot{\bar{a}}^2}{\bar{c}^2\bar{a}^2} + \frac{2\ddot{\bar{a}}}{\bar{c}^2\bar{a}}\right) = \frac{\chi\beta}{3}\bar{X} \tag{69}$$

$$\bar{F} = \frac{\chi}{3}\bar{X} \tag{70}$$

$$\frac{\chi\bar{X}}{3} + \frac{2\ddot{\bar{a}}}{\bar{c}^2\bar{a}} - = \frac{\chi\beta}{3}\bar{X} \tag{71}$$

$$\frac{\ddot{\bar{a}}}{\bar{c}^2\bar{a}} = -\frac{\chi}{6}(1+\beta)\bar{X} \tag{72}$$

Let's derive:

$$\dot{\bar{F}} = 2\bar{H}\left(-\bar{F} + \frac{\ddot{\bar{a}}}{\bar{c}^2\bar{a}}\right) \tag{73}$$

$$-\bar{F} = -\frac{\chi}{3}\bar{X} \tag{74}$$

$$-\bar{F} + \frac{\ddot{\bar{a}}}{\bar{c}^2\bar{a}} = -\frac{\chi}{6}(3+\beta)\bar{X} \tag{75}$$

$$\dot{\bar{F}} = -\frac{\chi}{3}(3+\beta)\bar{H}\bar{X} \tag{76}$$

$$-3\dot{\bar{F}} = -\chi\dot{\bar{X}} \tag{77}$$

$$\chi(3+\beta)\bar{H}\bar{X} = -\chi\dot{\bar{X}} \tag{78}$$

$$\dot{\bar{X}} + (3+\beta)\bar{H}\bar{X} = 0 \tag{79}$$

$$\frac{\mathrm{d}\bar{X}}{\bar{X}} + (3+\beta)\frac{\mathrm{d}\bar{a}}{\bar{a}} = 0 \tag{80}$$

Or:

$$\bar{X}\bar{a}^{3+\beta} = Cst \tag{81}$$

$$\dot{\bar{\varepsilon}} + \phi\dot{\varepsilon} + \dot{\phi}\varepsilon + (3+\beta)\bar{H}\bar{\varepsilon} + (3+\beta)\bar{H}\phi\varepsilon = 0 \tag{82}$$

As $\varepsilon = \rho c^2$; $\bar{\varepsilon} = \bar{\rho}\bar{c}^2$

$$C_- = \bar{c}^2\dot{\bar{\rho}} + \phi c^2\dot{\rho} + \dot{\phi}\rho c^2 + (3+\beta)\bar{H}\left(\bar{\rho}\bar{c}^2 + \phi\rho c^2\right) = 0 \tag{83}$$

We had a system of two pairs of second-order differential equations. The com-

patibility of the first two (derived from the first field equation) yielded Equation (66). Let us call this equation $C_+$. The compatibility of the next two (derived from the second field equation) has thus yielded Equation (83). Let us call this equation $C_-$ To ensure the consistency of the four equations, we must therefore determine the time-dependent functions $\varphi$ and $\phi$. Let us identify:

$$C_+ = 0 \quad \text{et} \quad \lambda C_- = 0 \tag{84}$$

$$\dot{\varepsilon} + \varphi\dot{\bar{\varepsilon}} + \dot{\varphi}\bar{\varepsilon} + (3+\beta)H\varepsilon + (3+\beta)H\varphi\bar{\varepsilon} = 0 \tag{85}$$

$$\lambda\left[\dot{\bar{\varepsilon}} + \phi\dot{\varepsilon} + \dot{\phi}\varepsilon + (3+\beta)\bar{H}\bar{\varepsilon} + (3+\beta)\bar{H}\phi\varepsilon\right] = 0$$

We will therefore perform a term-by-term identification. The coefficient of $\dot{\varepsilon}$ yields:

$$1 = \lambda\phi \tag{86}$$

The coefficient $\dot{\bar{\varepsilon}}$ gives

$$\varphi = \lambda \tag{87}$$

Thus:

$$\varphi\phi = 1 \tag{88}$$

The two time-dependent functions $\varphi$ and $\phi$ are inverses of each other. The coefficient gives:

$$(3+\beta)H = \lambda\left[\dot{\phi} + (3+\beta)\bar{H}\phi\right] \tag{89}$$

But $\lambda = \varphi$. Hence:

$$(3+\beta)H = \varphi\dot{\phi} + (3+\beta)\bar{H}\varphi\phi \tag{90}$$

But $\varphi\phi = 1$ thus:

$$(3+\beta)H = \frac{\dot{\phi}}{\phi} + (3+\beta)\bar{H} \tag{91}$$

$$\frac{\dot{\phi}}{\phi} = (3+\beta)\left(H - \bar{H}\right) = (3+\beta)\left(\frac{\dot{a}}{a} - \frac{\dot{\bar{a}}}{\bar{a}}\right) \tag{92}$$

$$\phi = C\left(\frac{a}{\bar{a}}\right)^{3+\beta} \tag{93}$$

Hence:

$$\varphi = \frac{1}{C}\left(\frac{\bar{a}}{a}\right)^{3+\beta} \tag{94}$$

Since the integration constant is absorbed in the normalization of the interaction tensors, we will take $C = 1$.

Let's go back to our equation $C_+$.

$$\dot{\varepsilon} + \varphi\dot{\bar{\varepsilon}} + \dot{\varphi}\bar{\varepsilon} + (3+\beta)H\varepsilon + (3+\beta)H\varphi\bar{\varepsilon} = 0 \tag{95}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon + \varphi\bar{\varepsilon}\right) + (3+\beta)H\left(\varepsilon + \varphi\bar{\varepsilon}\right) = 0 \tag{96}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\varepsilon+\varphi\overline{\varepsilon}\right)+\left(3+\beta\right)\frac{\mathrm{d}}{\mathrm{d}t}\ln a=0 \tag{97}$$

But:

$$\varepsilon+\varphi\overline{\varepsilon}=\frac{\varepsilon a^{(3+\beta)}+\overline{\varepsilon}\,\overline{a}^{(3+\beta)}}{a^{(3+\beta)}} \tag{98}$$

Whence:

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\varepsilon a^{(3+\beta)}+\overline{\varepsilon}\,\overline{a}^{(3+\beta)}\right)=0 \tag{99}$$

$$\boxed{\rho c^2 a^{(3+\beta)}+\overline{\rho}\,\overline{c}^{\,2}\overline{a}^{(3+\beta)}=E=Cst} \tag{100}$$

Let's go back to our equation.

In the matter dominated era:

$$\rho c^2 a^3+\overline{\rho}\,\overline{c}^{\,2}\overline{a}^{3}=E=Cst \tag{101}$$

The radiation dominated era:

$$\rho c^2 a^4+\overline{\rho}\,\overline{c}^{\,2}\overline{a}^{4}=E=Cst \tag{102}$$

Thus we get, for matter dominated era:

$$\dot{a}=\sqrt{-k-\frac{8\pi GE}{3a}} \tag{103a}$$

$$\ddot{a}=-\frac{4\pi GE}{3a^2} \tag{104a}$$

$$\dot{\overline{a}}=\sqrt{-\overline{k}+\frac{8\pi GE}{3\overline{a}}} \tag{103b}$$

$$\ddot{\overline{a}}=\frac{4\pi GE}{3\overline{a}^{\,2}} \tag{104b}$$

## Appendix 2

Dual numbering: This calculation is, in fact, an adaptation of the classic calculation presented in Chapter 14 of the book "Introduction to General Relativity" by Adler, Schiffer, and Bazin [31]. In what follows, the equations from the article are numbered with digits enclosed in parentheses. These are followed by the corresponding equations from Chapter 14 of the book, enclosed in square brackets. For instance, Equation (15c) is followed by Equations ([14], eq. 12c).

Satisfaction of Bianchi conditions in the stationary case and in the Newtonian approximation.

In the following, tensors are always defined in their "old form":

$$T_{\mu}^{\nu}=\begin{pmatrix}\rho & 0 & 0 & 0\\ 0 & -p/c^2 & 0 & 0\\ 0 & 0 & -p/c^2 & 0\\ 0 & 0 & 0 & -p/c^2\end{pmatrix} \tag{1}$$

And not in their "modern form":

$$T_{\mu}^{\nu}=\begin{pmatrix}\rho c^2 & 0 & 0 & 0\\ 0 & -p & 0 & 0\\ 0 & 0 & -p & 0\\ 0 & 0 & 0 & -p\end{pmatrix} \tag{2}$$

In order to be able to make use of the symmetry

$$\chi=-\frac{8\pi G}{c^2}=-\bar{\chi}=\frac{8\pi\bar{G}}{c^2} \tag{3}$$

This means that $G$ and $\bar{G}$ have opposite signs but are not equal in absolute value. Assuming a stationary state and spherical symmetry, we can consider the two cases where only one of the two types of mass—positive or negative—is present. When the mass is positive, the system is described as follows:

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\chi T_{\mu\nu} \tag{4}$$

$$\bar{R}_{\mu\nu}-\frac{1}{2}\bar{R}\bar{g}_{\mu\nu}=-\chi K_{\mu\nu} \tag{5}$$

The solution to (1)—comprising the exterior and interior metrics—is the classical Schwarzschild solution. To construct the solution to Equation (2), we would need to define the form of the interaction tensor $K_{\mu\nu}$. This would allow us to construct the metrics $\bar{g}_{\mu\nu}^{ext}$ and $\bar{g}_{\mu\nu}^{int}$. However, this would yield geodesics that cannot be compared with observational data. Consequently, such work would be pointless.

The situation is different when considering the field created by a negative mass, which corresponds to the system:

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\chi\bar{K}_{\mu\nu} \tag{6}$$

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} = -\chi\bar{T}_{\mu\nu} \tag{7}$$

Which will be written in the form:

$$R_{\mu\nu} = \chi\left(\bar{K}_{\mu\nu} - \frac{1}{2}\bar{K}g_{\mu\nu}\right) \tag{8}$$

$$\bar{R}_{\mu\nu} = -\chi\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{T}\bar{g}_{\mu\nu}\right) \tag{9}$$

This situation corresponds to the "Great Repeller" object, located at the center of the 100-million-light-year-diameter void known as the Dipole Repeller [21]. We conjecture that a negative mass lies at its center, modeled as a sphere filled with negative mass of constant density $\bar{\rho}$. We will need to derive the form of the interaction tensor $\bar{K}_{\mu\nu}$, followed by the solutions $g_{\mu\nu}^{ext}$ and $g_{\mu\nu}^{int}$. Light rays originating from the background—from distant sources—will undergo a negative gravitational lensing effect in this region, which will serve to attenuate their luminosity. This effect can then be compared with observational data. Under spherical symmetry and stationary conditions, the metrics take the form:

$$\mathrm{d}s^2 = \mathrm{e}^{\nu}\mathrm{d}x^{\circ 2} - \mathrm{e}^{\lambda}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) \tag{10}$$

$$\mathrm{d}\bar{s}^2 = \mathrm{e}^{\bar{\nu}}\mathrm{d}x^{\circ 2} - \mathrm{e}^{\bar{\lambda}}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) \tag{11}$$

Attention will focus on the compatibility between the two metric solutions—between these two geometries—which entails the need to identify the two hydrodynamic equations, specifically the TOV-type (Tolman-Oppenheimer-Volkoff) equations derived from them. The solutions to the second field Equation (7), describing the geometry of the negative sector, are in every respect analogous to the pair of Schwarzschild exterior and interior metric solutions. Taking as the source tensor:

$$\bar{T}_{\mu}^{\nu} = \mathrm{diag}\left(\bar{\rho}\bar{c}^2, -\bar{p}, -\bar{p}, -\bar{p}\right) \tag{12}$$

That's to say:

$$\bar{T}_{\mu\mu} = \mathrm{diag}\left(\bar{\rho}\bar{c}^2\mathrm{e}^{\bar{\nu}}, \bar{p}\mathrm{e}^{\bar{\lambda}}, r^2\bar{p}, r^2\sin^2\theta\bar{p}\right) \quad ([14], \text{eq. 8}) \tag{13}$$

Alongside our own equation numbering, we will indicate in square brackets the corresponding numbering from Chapter 14 of reference [31]. The covariant components of the Ricci tensor, derived from the form of the metric (11), are:

$$\bar{R}_{00} = \mathrm{e}^{\bar{\nu}-\bar{\lambda}}\left[-\frac{\bar{\nu}''}{2} + \frac{\bar{\nu}'\bar{\lambda}'}{4} - \frac{\bar{\nu}'^2}{4} - \frac{\bar{\nu}'}{r}\right] \quad ([14], \text{eq. 11a}) \tag{14a}$$

$$\bar{R}_{11} = \frac{\bar{\nu}''}{2} - \frac{\bar{\nu}'\bar{\lambda}'}{4} + \frac{\bar{\nu}'^2}{4} - \frac{\bar{\lambda}'}{r} \quad ([14], \text{eq. 11b}) \tag{14b}$$

$$\bar{R}_{22} = \mathrm{e}^{-\bar{\lambda}}\left[1 + \frac{\bar{\nu}'r}{2} - \frac{\bar{\lambda}'r}{2}\right] - 1 \quad ([14], \text{eq. 11c}) \tag{14c}$$

By calculating the right-hand sides from the source tensor (13), one obtains the classical system of equations:

$$e^{-\bar{\lambda}}\left[-\frac{\bar{\nu}''}{2}+\frac{\bar{\nu}'\bar{\lambda}'}{4}-\frac{\bar{\nu}'^2}{4}-\frac{\bar{\nu}'}{r}\right]=-\frac{\chi}{2}e^{\bar{\nu}}\left(\bar{\rho}+\frac{3\bar{p}}{\bar{c}^2}\right) \quad ([14], \text{eq. 12a}) \tag{15a}$$

$$e^{-\bar{\nu}}\left[\frac{\bar{\nu}''}{2}-\frac{\bar{\nu}'\bar{\lambda}'}{4}+\frac{\bar{\nu}'^2}{4}-\frac{\bar{\lambda}'}{r}\right]=-\frac{\chi}{2}\left(\bar{\rho}-\frac{\bar{p}}{\bar{c}^2}\right) \quad ([14], \text{eq. 12b}) \tag{15b}$$

$$e^{-\bar{\lambda}}\left[1+\frac{\bar{\nu}'r}{2}-\frac{\bar{\lambda}'r}{2}\right]-1=-\frac{\chi}{2}r^2\left(\bar{\rho}-\frac{\bar{p}}{\bar{c}^2}\right) \quad ([14], \text{eq. 12c}) \tag{15c}$$

The two unknowns in this system of differential equations are the functions $\bar{\lambda}(r)$ and $\bar{\nu}(r)$. The problem of finding a solution is then (standardly) reduced to finding the single function $\bar{\nu}(r)$ by setting:

$$-\bar{g}_{11}=e^{-\bar{\lambda}}=1-\frac{2\bar{m}(r)}{r} \quad ([14], \text{eq. 15}) \tag{16}$$

This is to match the external metric $\bar{g}_{\mu\nu}^{ext}$, which is:

$$d\bar{s}^2=\left(1-\frac{2\bar{m}_0}{r}\right)dx^{\circ 2}-\frac{dr^2}{1-\frac{2\bar{m}_0}{r}}-r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right) \tag{17C}$$

By combining equation (16) ([14], eq. 15) with the system (15a) ([14], eq. 12a), (15b) ([14], eq. 12b), (15c) ([14], eq. 12c), what is known as a "geometric mass" emerges:

$$\bar{m}=-\frac{4\pi r^3}{3c^2}G\bar{\rho}>0 \quad ([14], \text{eq. 18}) \tag{18}$$

This reflects the fact that negative masses attract negative masses (compare with metric (11), where $\bar{m}_0>0$). From this system, using standard manipulations, one obtains a TOV-type equation:

$$\bar{p}'=-\frac{\left(\bar{\rho}\bar{c}^2+\bar{p}\right)\left(\bar{m}-4\pi G\bar{p}r^3/c^4\right)}{r\left(r-2\bar{m}\right)} \quad ([14], \text{eq. 22}) \tag{19}$$

This one, without the Newtonian approximation:

$$|\bar{p}|\ll|\bar{\rho}|\bar{c}^2 \qquad r\gg 2\bar{m} \qquad 4\pi G\bar{p}r^3/c^2\ll\bar{m} \tag{20}$$

If $\bar{M}(r)$ is the total Newtonian mass contained within a sphere of radius *r*:

$$\bar{M}(r)=\frac{4}{3}\pi r^3\bar{\rho}<0 \tag{21}$$

The TOV equation then becomes the Euler equation for negative mass.

$$\bar{p}'=-\frac{G\bar{M}(r)\bar{\rho}}{r^2} \quad ([14], \text{eq. 24}) \tag{22}$$

We shall not proceed any further with the solution to this second field equation. While we could indeed go so far as to construct the form of the interior metric, doing so would be of purely formal interest, since the geodesics derived from the solutions $\bar{g}_{\mu\nu}$ cannot be compared with observational data.

However, the same does not apply to the metrics $g_{\mu\nu}^{ext}$ and $g_{\mu\nu}^{int}$,; this compels us to specify a form for the interaction tensor $\bar{K}_{\mu\nu}$—which acts as the source of

the field—in a manner consistent with the system's mathematical coherence, that is, leading to a TOV-type equation that can be identified with the one previously derived from the second equation.

It has not been possible to construct such a tensor $\bar{K}_{\mu\nu}$ in the most general form—that is, for a non-linear solution capable of describing objects far removed from the Newtonian regime. This will therefore remain an open problem for the mathematical geometer:

- *How can the interaction tensors $\bar{K}_{\mu\nu}$ and $K_{\mu\nu}$ which appear in the Janus field equations be constructed in their most general, non-linearized form, given that the form of such tensors is constrained by the requirement to satisfy the mathematical consistency of the system?*

Someone may eventually discover this most general, non-linear form. In the meantime, we shall content ourselves with asymptotic satisfaction under Newtonian conditions, while recalling the arguments of a physical nature that justify this approach:

- *If the source of the field is a very dense object with highly concentrated positive mass—such as a neutron star or one of the supermassive objects located at the centers of galaxies—then the Newtonian approximation is no longer valid. In this case, however, the theorist is not required to construct the corresponding interaction tensor $K_{\mu\nu}$ or the solution $g_{\mu\nu}$ to the second equation, since, in essence, such constructs cannot be compared against observational data in any scenario. At most, a mathematician might formulate a theorem guaranteeing the existence of this solution.*

However, if the source of the field is a negative mass, then the Newtonian approximation and the asymptotic satisfaction of the consistency condition suffice, since simulations indicate that these negative masses remain loosely concentrated, taking the form of immense conglomerates located at the centers of large voids in the large-scale structure.

This will therefore be the guiding thread for what follows. We will express the interaction tensor in the following form:

$$K_{\mu}^{\nu} = \mathrm{diag}\left(\bar{\rho}, -\frac{\bar{q}}{\bar{c}^2}, -\frac{\bar{q}}{\bar{c}^2}, -\frac{\bar{q}}{\bar{c}^2}\right) \tag{23}$$

Here, $\bar{q}(r)$ will be a function to be determined. We will have (we shall indicate, in square brackets, the corresponding numbers from Chapter 14 of reference [20] to make it easier for the reader to follow, as this is one of the tricky points in this calculation **Appendix**):

$$K_{\mu\mu} = \mathrm{diag}\left(\bar{\rho}\mathrm{e}^{\nu}, \frac{\bar{q}}{\bar{c}^2}\mathrm{e}^{\lambda}, r^2\frac{\bar{q}}{\bar{c}^2}, r^2\sin^2\theta\frac{\bar{q}}{\bar{c}^2}\right) \quad ([14], \text{eq. 8}) \tag{A24}$$

From this tensor, the field equation, and the coefficients of the Ricci tensor constructed from the metric, we obtain the system of three equations:

$$\mathrm{e}^{-\lambda}\left[-\frac{\nu''}{2}+\frac{\nu'\lambda'}{4}-\frac{\nu'^2}{4}-\frac{\nu'}{r}\right] = \frac{\chi}{2}\mathrm{e}^{\nu}\left(\bar{\rho}+3\frac{\bar{q}}{\bar{c}^2}\right) \quad ([14], \text{eq.12a}) \tag{25a}$$

$$e^{-\nu}\left[\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{\nu'^2}{4}-\frac{\lambda'}{r}\right]=\frac{\chi}{2}\left(\bar{\rho}-\frac{\bar{q}}{\bar{c}^2}\right) \quad ([14], \text{eq.12b}) \tag{25b}$$

$$e^{-\lambda}\left[1+\frac{\nu' r}{2}-\frac{\lambda' r}{2}\right]-1=\frac{\chi}{2}r^2\left(\bar{\rho}-\frac{\bar{q}}{\bar{c}^2}\right) \quad ([14], \text{eq.12c}) \tag{25c}$$

For the student unfamiliar with this calculation, we will detail the steps. By adding (15a) ([14], eq. 12a) and (15b) ([14], eq. 12b), we obtain:

$$-\chi\left(\bar{\rho}+\frac{\bar{q}}{\bar{c}^2}\right)=e^{-\lambda}\left(\frac{\lambda'+\nu'}{r}\right) \quad ([14], \text{eq. 13}) \tag{26}$$

Now, by combining (15c) ([14], eq. 12c) and (26) ([14], eq. 13) to extract $\bar{\rho}\bar{c}^2$ and $\bar{q}$, and by eliminating $\nu''$ from (15b) ([14], eq. 12b) and (15c) ([14], eq. 12c), we obtain the system of three equations:

$$\chi\bar{\rho}=e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2} \quad ([14], \text{eq. 14a}) \tag{27a}$$

$$\chi\frac{\bar{q}}{\bar{c}^2}=\frac{1}{r^2}-e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right) \quad ([14], \text{eq. 14b}) \tag{27b}$$

$$\frac{e^{\lambda}}{r^2}=\frac{1}{r^2}-\frac{\nu'^2}{4}+\frac{\nu'\lambda'}{4}+\frac{\lambda'+\nu'}{r}-\frac{\nu''}{2} \quad ([14], \text{eq. 14c}) \tag{27c}$$

We will now reintroduce a "geometric mass" by ensuring a match with the external metric $g_{\mu\nu}^{ext}$, which reflects a *repulsion* (a negative mass repels positive masses) and thus takes the form:

$$ds^2=\left(1+\frac{2m_0}{r}\right)dx^{\circ 2}-\frac{dr^2}{1+\frac{2m_0}{r}}-r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right) \text{ with } m_0>0 \tag{28}$$

Earlier, $\bar{m}(r)$ was the "geometric mass" acting on negative masses. Here, we introduce the geometric mass $m(r)$ acting on positive masses. We must therefore have:

$$e^{-\lambda}=1-\frac{2m(r)}{r} \text{ with } m(r)<0 \quad ([14], \text{eq. 15}) \tag{29}$$

If we calculate:

$$-\frac{1}{r^2}2m'=-\frac{1}{r^2}\left[r\left(1-e^{-\lambda}\right)\right]'=-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right) \quad ([14], \text{eq. 16}) \tag{30}$$

We see that the right-hand side of this equation is identical to the right-hand side of (27a) ([14], eq. 14a). But we have:

But:

$$\chi=-\frac{8\pi G}{c^2} \tag{31}$$

Thus:

$$m'=\frac{G\bar{\rho}}{c^2}4\pi r^2<0 \quad ([14], \text{eq. 17}) \tag{32}$$

Then:

$$m=\frac{4\pi r^3}{3c^2}G\bar{\rho}<0 \quad ([14], \text{eq. 18}) \tag{33}$$

Let us return to (27b) ([14], eq. 14b). Applying (29), ([14], eq. 15), we obtain:

$$\chi\bar{q}=\frac{2m}{r^3}-\mathrm{e}^{-\lambda}\frac{\nu'}{r}=\frac{2m}{r^3}-\left(1-\frac{2m}{r}\right)\frac{\nu'}{r} \tag{34}$$

So:

$$\nu'=2\frac{m+4\pi G\bar{q}r^3/c^4}{r\left(r-2m\right)} \quad ([14], \text{eq. 19}) \tag{35}$$

Comparing with (26) ([14], eq. 13) we obtain a simple relationship between $\nu'$ and $p'$

$$\frac{\bar{p}'}{\bar{c}^2}=-\frac{\nu'}{2}\left(\bar{\rho}+\frac{\bar{p}}{\bar{c}^2}\right) \quad ([14], \text{eq. 21}) \tag{36}$$

By differentiating (27b) ([14], eq. 14b) and combining it with (27c) ([14], eq. 14c), we obtain

$$\bar{q}'=-\frac{\left(\bar{\rho}\bar{c}^2+\bar{q}\right)\left(m+4\pi G\bar{q}r^3/c^4\right)}{r\left(r-2m\right)} \quad ([14], \text{eq. 21}) \tag{37}$$

That is to say, the equivalent of the TOV equation for the first field equation. We have:

$$m\left(r\right)=-\bar{m}\left(r\right) \tag{38}$$

We get:

$$\bar{q}'=-\frac{\left(\bar{\rho}\bar{c}^2+\bar{q}\right)\left(-\bar{m}+4\pi G\bar{q}r^3/c^4\right)}{r\left(r+2\bar{m}\right)} \tag{39}$$

Which cannot be identified with the TOV equation of the second equation, unless the Newtonian approximation is applied, yielding:

$$\bar{q}'=\frac{\bar{\rho}\bar{c}^2\bar{m}}{r^2}=\frac{G\bar{\rho}\bar{M}}{r^2} \tag{40}$$

To recover the equation:

$$\bar{p}'=-\frac{G\bar{M}\left(r\right)\bar{\rho}}{r^3} \quad ([14], \text{eq. 24}) \tag{41}$$

That is to say, Euler's hydrostatic equation obtained using the second field equation—representing an asymptotic compatibility of the system; restricted to the Newtonian approximation, the following is required:

$$\bar{q}=-\bar{p} \tag{42}$$

Which gives the interaction tensor the form:

$$\bar{K}_{\mu}^{\nu}=\begin{pmatrix}\bar{\rho} & 0 & 0 & 0\\ 0 & \frac{\bar{p}}{\bar{c}^2} & 0 & 0\\ 0 & 0 & \frac{\bar{p}}{\bar{c}^2} & 0\\ 0 & 0 & 0 & \frac{\bar{p}}{\bar{c}^2}\end{pmatrix} \tag{43}$$

In the Newtonian approximation, one then recovers (41) ([14], eq. 24). This is what was demonstrated in references [8] [20].

However, since this will lend itself to comparison with observations, we shall have to construct the linearized form of the interior metric.

As early as 1916, K. Schwarzschild had provided the expression for the interior metric. While the behavior of this metric has been extensively studied—particularly to reveal its geometric or physical criticality conditions—its application to the study of photon propagation through a distribution of transparent matter is not typically the aspect highlighted. In the Newtonian approximation, the interpretation becomes particularly straightforward: the trajectories of photons passing through such a transparent mass would undergo a deflection caused, within that mass, by the mass $M(r)$ contained inside a sphere of radius *r*.

$$M(r)=\frac{4\pi r^3\rho}{3} \tag{44}$$

Taking into account that this deflection would then be negative, we obtain:

$$\mathrm{d}s^2=\left(1+\frac{2GM(r)}{c^2r}\right)c^2\mathrm{d}t^2-\frac{\mathrm{d}r^2}{1+\dfrac{2GM(r)}{c^2r}}-r^2\mathrm{d}\Omega^2 \tag{45}$$

Taking linearization into account, the interior metric becomes:

$$\mathrm{d}s^2=\left(1+\frac{8\pi G\rho r^2}{3c^2}\right)c^2\mathrm{d}t^2-\left(1-\frac{8\pi G\rho r^2}{3c^2}\right)\mathrm{d}r^2-r^2\mathrm{d}\Omega^2 \tag{46}$$

Which matches the external metric, also linearized.

$$\mathrm{d}s^2=\left(1+\frac{2G\left|\bar{M}\right|}{c^2r}\right)c^2\mathrm{d}t^2-\left(1-\frac{2G\left|\bar{M}\right|}{c^2r}\right)\mathrm{d}r^2-r^2\mathrm{d}\Omega^2 \tag{47}$$