

On the Spacetime Statistical Equivalence Principle and Its Applications within the Framework of a Non-Expanding Steady-State Cosmological Model

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Abstract

This paper proposes and rigorously defines a “Spacetime Statistical Equivalence Principle”: in a statistical system satisfying stationarity, spatial homogeneity, weak correlation, and ergodicity, the statistical distribution obtained by long time-series sampling of the same event and the statistical distribution obtained by single sampling of a large number of spatially independent events both converge to the same ensemble average in the mean-square sense, with the same order of convergence. Specifically, under the stated assumptions, the variance (and hence the mean-square error) of both the time average and the spatial average decays as $O(1/N)$, equivalently the standard deviation decays as $O(N^{-1/2})$. This principle is a natural generalization of the traditional ergodic theorem in the spatial dimension, and its essence is the statistical equivalence of the “time average” and the “spatial average”, rather than a symmetry transformation between spacetime coordinates. On this basis, the equivalence relation is applied to a non-expanding steady-state cosmological model: under the premise that the stationarity assumption holds, it is demonstrated that its large-scale statistical state possesses time-translation invariance, and a logical-level compatibility comparison is made with Einstein’s static universe solution. This paper does not introduce a cosmological constant or dark energy, nor does it rely on any expansion dynamics, aiming to provide a statistical-mechanical foundation for the steady-state view of the universe.

Keywords

Spacetime Statistical Equivalence, Ergodicity, Spatial Ergodicity, Statistical Steady-State Universe, Einstein Static Universe

1. Introduction

The equivalence between time averages and ensemble averages, namely ergodicity, is one of the cornerstones of statistical mechanics [1]-[3]. The classical ergodic theorem answers the following question: for a dynamical system satisfying stationarity and ergodicity, the time average along a single trajectory converges to the ensemble average with probability one. This result provides a rigorous mathematical foundation for equilibrium statistical mechanics and has been widely applied in thermodynamics, quantum mechanics, and stochastic process theory.

However, classical ergodic theory is usually confined to repeated observations of the same system along the temporal dimension. A natural but insufficiently explored question is: can a large number of independent samples in space replace continuous observations in time and yield equivalent statistical information? This idea has precursors in common-sense examples (such as the equivalence between tossing a coin many times and tossing many coins simultaneously), but it has not yet been elevated to a general principle at the theoretical level. A few works have discussed “spatial ergodicity” within specific random field models [4] [5], but they have neither systematized it into a statistical equivalence principle on a par with time averaging nor extended it to cosmological scales.

In the field of cosmology, the steady-state universe model was proposed by Bondi, Gold, and Hoyle in 1948 [6] [7]. Its core idea is to maintain the strict validity of the cosmological principle through continuous matter creation against an expanding background. This model once successfully explained the Hubble redshift-distance relation, but it was gradually superseded by the cold dark matter model owing to two key observational evidences: first, the discovery of the cosmic microwave background (CMB) and the precise measurement of its blackbody spectrum [8]; second, the significant deviation of radio source counts from the predictions of the steady-state model at the faint-source end [9]. Meanwhile, the static universe model proposed by Einstein in 1917 relies on the cosmological constant to counteract gravitational contraction [10], but its dynamical instability was soon pointed out by Eddington [11] and Lemaître [12]—the solution is unstable against any homogeneous perturbation, and any infinitesimal fluctuation will cause the universe either to collapse or to expand rather than return to a static state. Both of these classical non-expanding or steady state models lack a statistical-mechanical microscopic foundation—namely, why the matter distribution can maintain a statistical steady state on large scales without requiring strict staticity at the geometric level.

This paper aims to fill this gap. We propose and rigorously define the “Spacetime Statistical Equivalence Principle”, which is a natural generalization of the traditional ergodic theorem to the spatial dimension. It addresses the operational question of “whether time averages can be replaced by spatial averages”, rather than the ontological question of “whether time and space are symmetric”. On this basis, we apply the principle to a non-expanding cosmological model, demonstrate under the stationarity assumption that the large-scale statistical state possesses time-

translation invariance, and make a logical-level compatibility comparison with Einstein's static universe solution. This paper introduces neither a cosmological constant nor dark energy, nor does it rely on any expansion dynamics; it aims to provide a statistical mechanical foundation for the steady-state view of the universe and to give testable observational corollaries.

It should be particularly noted that the "spacetime statistical equivalence" discussed in this paper is not a spacetime symmetry in physics (such as Lorentz symmetry), but rather an extension of ergodicity in the statistical sense. At the same time, we explicitly abandon the expansion theory of modern cosmology (such as the Big Bang and Hubble expansion) and discuss the statistical behavior of the universe only within a static or steady-state framework. This choice does not deny the observational evidence for an expanding universe, but rather seeks to explore an alternative, non-geometrically driven picture of a statistical steady-state universe and to examine its logical self-consistency and observational falsifiability.

The existing work most directly related to this paper centers on spatial ergodicity in random field theory. Grenander (1950), in studying spatial statistical inference, was the first to systematically discuss the condition under which "as the observation region tends to infinity, the spatial sample mean converges to the ensemble mean", and he drew an analogy with the ergodic theorem in time series [4]. Adler (1981), in his monograph *The Geometry of Random Fields*, further extended spatial ergodicity to anisotropic random fields and proved that under mixing conditions, the rate of convergence of the spatial average is likewise $O(1/N)$ in the root-mean-square sense [5]. These works have relatively mature conclusions in probability theory and mathematical statistics.

However, they differ from the present work in three essential respects. First, they establish spatial ergodic theorems as one-sided convergence results; they do not formulate a two-sided statistical equivalence principle that places temporal ergodicity and spatial ergodicity on the same footing. Second, they do not compare, for the same observable and under a common set of assumptions, the convergence order of the time average with that of the spatial average. Third, they do not discuss the cosmological application of such an equivalence, nor the conditions under which the equivalence fails in a non-stationary cosmological background. The contribution of this paper is therefore not the mere existence of a spatial ergodic theorem, but the formulation, proof, and application of a symmetric time-space statistical equivalence principle, together with an explicit comparison of convergence orders and an analysis of its failure under expansion.

In physics, especially in equilibrium statistical mechanics, the idea of "replacing time averages with spatial averages" is occasionally used implicitly (for example, when computing macroscopic quantities of lattice systems, one often performs instantaneous averaging over a large number of spatial lattice sites). However, such an operation is usually regarded as an approximate substitute for the ensemble average and has not been elevated to an independent statistical-mechanical principle on a par with temporal ergodicity.

It should be particularly noted that the “spacetime statistical equivalence” discussed in this paper is not a spacetime symmetry in physics (such as Lorentz symmetry), but rather an extension of ergodicity in the statistical sense. At the same time, we do not deny the observational evidence for an expanding universe, including the cosmic microwave background, the redshift-distance relation of distant galaxies, and the observed evolution of galaxy morphologies over cosmic time. Rather, we use the expanding case as a concrete example of a non-stationary system in which the stationarity premise of the theorem fails. The steady-state framework discussed here is therefore presented as a conditional, idealized statistical model, not as a replacement for the standard cosmological model.

2. Rigorous Statement of the Principle

2.1. Basic Setup

Let the system Ψ be a macroscopic statistical system whose microscopic state is described by a random field $\phi(x, t)$, where $x \in \mathbb{R}^3$ is the spatial coordinate and $t \in \mathbb{R}$ is the time coordinate. We consider some observable functional $f(\phi)$ of this random field, such as local density, temperature, or galaxy counts.

Two sampling schemes are defined:

1) Temporal sampling (single-point long time series):

Fix a spatial position x_0 and repeatedly observe the field value at that point at equally spaced instants $t_1, t_2, \dots, t_N$, yielding the time series:

$$X_i \equiv f(\phi(x_0, t_i)), \quad i = 1, 2, \dots, N$$

whose sample mean is:

$$\bar{X}_N^{\text{time}} = \frac{1}{N} \sum_{i=1}^N X_i \quad (2.1)$$

2) Spatial sampling (single spatial snapshot):

Fix an instant t_0 and simultaneously observe the field values at spatial positions $x_1, x_2, \dots, x_N$ (separated sufficiently far apart so as to be approximately independent), yielding the spatial samples:

$$Y_j \equiv f(\phi(x_j, t_0)), \quad j = 1, 2, \dots, N$$

whose sample mean is:

$$\bar{Y}_N^{\text{space}} = \frac{1}{N} \sum_{j=1}^N Y_j \quad (2.2)$$

The traditional ergodic theorem [1] answers the relation between “the same system at different times”: namely, whether the time average $\bar{X}_N^{\text{time}}$ converges to the ensemble average $\langle f \rangle$ as $N \rightarrow \infty$, under the core premises that the system possesses stationarity in the temporal dimension and that the trajectory is ergodic.

The principle proposed in this paper concerns the opposite direction: can the spatial average $\bar{Y}_N^{\text{space}}$ replace the time average $\bar{X}_N^{\text{time}}$ and yield equivalent sta-

tistical inference at the same order of convergence?

Intuitively, tossing a single coin many times and tossing many coins simultaneously should yield the same frequency of heads in the limit. However, this intuitive equivalence does not hold unconditionally—it depends on the system also possessing a certain “uniform” and “weakly correlated” structure in the spatial dimension, just as the convergence of the time average to the ensemble average depends on temporal stationarity and ergodicity.

The essence of the principle in this paper can be summarized as follows: the “time ensemble” equivalence relation implied by the ergodic theorem is extended, through spatial homogeneity and weak correlation conditions, to a “space $\rightarrow$ ensemble” equivalence relation, thereby establishing the statistical equivalence between “time” and “space”. It is worth emphasizing that this equivalence is statistical in nature—it concerns the equivalence of probability distributions (or low-order moments), rather than a symmetry transformation between spacetime coordinates, and therefore differs fundamentally from physical spacetime symmetries such as Lorentz symmetry and general covariance [3].

In short, this principle addresses the operational question of statistical inference—“whether a time average can be replaced by a spatial average for estimation”—rather than the ontological question of “whether time and space are symmetric”. It does not require the metric tensor to possess any Killing symmetry, nor does it involve isometric transformations of the spacetime manifold.

2.2. Statement of the Principle

For this equivalence to hold rigorously, the system must simultaneously satisfy the following four conditions:

If the system Ψ satisfies the following conditions:

1) Stationarity. The first- and second-order statistics of the system do not vary with time, expressible as: $E[f(\phi(x, t))] = \mu$ for all t .

2) Spatial homogeneity. The marginal distributions at different spatial positions are identical, expressible as: $E[f(\phi(x, t))] = \mu$ for all x .

3) Ergodicity. The time average converges almost surely to the ensemble average, expressible as: $\bar{X}_N^{\text{time}} \xrightarrow{\text{a.s.}} \mu$.

4) Weak correlation. The system is weakly correlated in the sense that its mixing coefficient decays exponentially. Let $F_{-\infty}^t$ be the σ -algebra generated by the observations up to time t , and let F_{t+k}^∞ be the σ -algebra generated by the observations at time $t+k$ and thereafter. The temporal mixing coefficient is defined as.

$$\alpha_{\text{time}}(k) = \sup_{A \in F_{-\infty}^t, B \in F_{t+k}^\infty} |P(A \cap B) - P(A)P(B)| \quad (2.3)$$

Similarly, for the spatial field, let F_Λ and F_{Λ^c} be the σ -algebras generated by ϕ on a region Λ and its complement. The spatial mixing coefficient is defined as

$$\alpha_{\text{sp}}(k) = \sup_{\Lambda, \Lambda^c, d(\Lambda, \Lambda^c) \geq k} \sup_{A \in F_{-\infty}^t, B \in F_{t+k}^\infty} |P(A \cap B) - P(A)P(B)| \quad (2.4)$$

We assume that both coefficients decay exponentially:

$$\alpha_{\text{time}}(k) \leq C e^{-\theta k}, \quad \alpha_{\text{sp}}(k) \leq C e^{-\theta k}, \quad \forall k \geq 1 \quad (2.5)$$

with $C > 0$ and $\theta > 0$. This condition is used only to control the decay of the covariance function; it is not claimed to be necessary.

These four conditions together constitute the sufficient framework for the validity of the “Spacetime Statistical Equivalence Principle”.

2.3. Theorem (Spacetime Statistical Equivalence Principle)

Let $\phi(x, t)$ be a stationary random field on $\mathbb{R}^3 \times \mathbb{R}$ with a spatial translation action, and let $f(\phi)$ be an observable with finite second moment. Consider the following two sampling schemes:

1) Temporal sampling: fix x_0 and observe $X_i = f(\phi(x_0, t_i))$, $i = 1, 2, \dots, N$, with sample mean $\bar{X}_N^{\text{time}} = \frac{1}{N} \sum_{i=1}^N X_i$.

2) Spatial sampling: fix t_0 and observe $Y_j = f(\phi(x_j, t_0))$, $j = 1, 2, \dots, N$, over a growing spatial region or a set of separated sites, with sample mean

$$\bar{Y}_N^{\text{space}} = \frac{1}{N} \sum_{j=1}^N Y_j$$

Assume that:

- a) the field is stationary in time, $E[f(\phi(x, t))] = \mu$ for all t ;
- b) the field is spatially homogeneous, $E[f(\phi(x, t))] = \mu$ for all x ;
- c) the temporal process is ergodic, $\bar{X}_N^{\text{time}} \xrightarrow{\text{a.s.}} \mu$;
- d) the spatial field satisfies the exponential α -mixing condition of Section 2.2, and the relevant variables have finite $(2 + \delta)$ -moments.

Then both averages converge to the same ensemble mean μ in probability, and their mean-square errors are both $O(1/N)$, equivalently their standard deviations are both $O(N^{-1/2})$.

It should be emphasized that spatially identical marginal distributions alone do not ensure that the spatial average converges to the ensemble mean; the spatial mixing or ergodicity condition (d) is essential.

3. Mathematical Proof

Throughout this section, we distinguish carefully between the convergence order of the variance (or mean-square error) and that of the standard deviation. A variance decaying as $O(1/N)$ corresponds to a standard deviation decaying as $O(N^{-1/2})$. When we say that the time average and the spatial average have the “same order of convergence”, we mean that their mean-square errors are both $O(1/N)$, equivalently their standard deviations are both $O(N^{-1/2})$. We do not claim equality of asymptotic variance or of finite-sample precision unless explicitly stated.

3.1. Rate of Convergence of the Spatial Average

Let $\{Y_j\}_{j=1}^\infty$ be a spatially correlated random field with ensemble mean $E[Y_j] = \mu$

and summable spatial covariance,

$$\sum_{k \in \mathbb{Z}^3} |\gamma(k)| < \infty \quad (3.1)$$

or equivalently satisfying the spatial α -mixing condition of Section 2.2. The spatial sample mean is

$$\bar{Y}_N^{\text{space}} = \frac{1}{N} \sum_{j=1}^N Y_j \quad (3.2)$$

Under the summable-covariance or mixing condition, its variance satisfies

$$\text{Var}(\bar{Y}_N^{\text{space}}) = \frac{S_{\text{sp}}}{N} + O\left(\frac{1}{N}\right), \quad S_{\text{sp}} = \sum_{k \in \mathbb{Z}^3} |\gamma(k)| \quad (3.3)$$

The i.i.d. case corresponds to $S_{\text{sp}} = \sigma^2$ and is used only as a reference, not as a general representation. Therefore, the mean-square error of the spatial average is $O(1/N)$, and the corresponding standard deviation is $O(N^{-1/2})$. This is the optimal rate of convergence of the classical law of large numbers under the condition of finite second moments, and it serves as the benchmark for comparing the convergence order of the time average below.

3.2. Variance Expansion of the Time Average

Let $\{X_t, t \in \mathbb{Z}\}$ be a real-valued stationary stochastic process representing the time series obtained by sampling at equally spaced instants along the time axis at a fixed spatial position x_0 . According to the setup in Section 2.1, $E[X_t] = \mu$ holds for all t (the first-order condition of stationarity). The time sample mean is:

$$\bar{X}_N^{\text{time}} = \frac{1}{N} \sum_{i=1}^N X_{t_i} \quad (3.4)$$

Intuitive interpretation: this corresponds to tossing the same coin N times in succession (but allowing some correlation between successive outcomes, for example if the coin's surface state changes slowly with time). Unlike spatial sampling, observations at different instants may be correlated with one another—this is precisely the fundamental difference between a time series and independent samples. To quantify this correlation, we introduce the autocovariance function:

$$\gamma(k) = \text{Cov}(X_t, X_{t+k}) = E[(X_t - \mu)(X_{t+k} - \mu)] \quad (3.5)$$

where k is the time lag ($k \in \mathbb{Z}$). Stationarity guarantees that $\gamma(k)$ depends only on the lag k , and not on the absolute time t .

Intuitive interpretation: $\gamma(0) = \sigma^2$ is the variance of the process; $\gamma(1)$ measures the degree of linear dependence between two adjacent observations; as k increases, $\gamma(k)$ typically decays to zero—meaning that two observations separated by a long interval have “forgotten” each other and are approximately independent.

The variance of the time sample mean can be expanded as:

$$\text{Var}(\bar{X}_N^{\text{time}}) = \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \gamma(i-j) \quad (3.6)$$

Intuitive interpretation: if all observations were completely independent, only the N terms with $i = j$ would be nonzero, yielding σ^2/N . However, correlation causes the off-diagonal terms ($i \neq j$) to contribute to the variance as well—positive correlation increases the variance (because the samples tend to “cluster” away from the mean), whereas negative correlation decreases it.

Rearranging the double sum by the lag $k = i - j$, we obtain the exact expression:

$$\text{Var}(\bar{X}_N^{\text{time}}) = \frac{1}{N} \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) \gamma(k) \quad (3.7)$$

Intuitive interpretation: the weight factor $(1 - |k|/N)$ reflects the proportion of times a given lag k is sampled in a finite sample. The smaller the lag (the closer k is to 0), the more sample pairs occur and the larger the weight; the larger the lag, the smaller the weight—which is natural: two observations separated by a long interval have fewer opportunities to meet in a finite sample.

3.3. Mixing and Exponential Decay Conditions

In order for the sum in Equation (3.7) to converge to a finite value, the autocovariance function must be guaranteed to decay sufficiently rapidly with the lag. We adopt the standard strong mixing condition (α -mixing) in statistics, whose core idea is that events separated sufficiently far apart are almost independent.

Definition (α -mixing). Let $F_{-\infty}^t$ be the σ -algebra generated by the observations up to the time t , and let F_{t+k}^{∞} be the σ -algebra generated by the observations at time $t+k$ and thereafter. For the spatial field, let F_{Λ} and F_{Λ^c} be the σ -algebras generated by ϕ on a region Λ and its complement Λ^c , with $d(\Lambda, \Lambda^c) \geq k$. The mixing coefficients are defined as in Section 2.2.

We assume that the mixing coefficients decay exponentially:

$$\alpha(k) \leq C e^{-\theta k}, \quad \forall k \geq 1 \quad (3.8)$$

where $C > 0$ and $\theta > 0$ are constants.

In order to apply Davydov's inequality [13], we assume that the relevant random variables possess finite moments of order $2 + \delta$ for some $\delta > 0$:

$$\mathbb{E}|X_t|^{2+\delta} < \infty, \quad \mathbb{E}|Y_j|^{2+\delta} < \infty \quad (3.9)$$

where $\delta > 0$, C'' and $\theta' > 0$. In short, the exponential decay of the mixing coefficient, together with the finite $(2 + \delta)$ -moment assumption, implies the exponential decay of the covariance function. The same argument applies separately to the temporal process and to the spatial random field. See Davydov (1968) for the original bound [13], Doukhan (1994, Section 1.2) for a detailed treatment of α -mixing and its inequalities [14], and Cressie (1993, Chapter 2.3) for the spatial case [15].

3.4. Rigorous Estimation of the Variance Upper Bound

Substituting the covariance upper bound $|\gamma(k)| \leq C'' e^{-\theta'|k|}$ into the variance ex-

pression (3.7):

$$\text{Var}(\bar{X}_N^{\text{time}}) = \frac{1}{N} \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) C'' e^{-\theta'|k|} \quad (3.10)$$

Using symmetry ($\gamma(k) = \gamma(-k)$), we split the sum into the $k = 0$ term and the two tails:

$$\text{Var}(\bar{X}_N^{\text{time}}) \leq \frac{C''}{N} \left[\gamma(0) + 2 \sum_{k=1}^{N-1} \left(1 - \frac{k}{N}\right) e^{-\theta'k} \right] \quad (3.11)$$

Key observation: the geometric series $\sum_{k=1}^{\infty} e^{-\theta'k}$ and $\sum_{k=1}^{\infty} k e^{-\theta'k}$ both converge to finite constants (since the common ratio $e^{-\theta'} < 1$). Therefore, the value in the square brackets is bounded and independent of N . Denoting this upper bound by the constant M , therefore,

$$\text{Var}(\bar{X}_N^{\text{time}}) = \frac{C''M}{N} = O\left(\frac{1}{N}\right) \quad (3.12)$$

and the corresponding standard deviation is $O(N^{-1/2})$. Although correlation exists in the time series, as long as the correlation decays sufficiently rapidly (exponentially), its effect on the variance of the sample mean is only a correction term; in the long run, the fluctuation amplitude of the time average still decreases as $N^{-1/2}$ in standard deviation, consistent with the behavior of independent samples.

3.5. Same Order of Convergence Rate—Lower-Bound Argument

The upper bound (3.12) only shows that the variance of the time average decays “no faster than” $1/N$. To prove “same order” (*i.e.*, that the two rates of convergence are exactly the same), we must also exclude a pathological case: the variance might decay faster than $1/N$ (for example, as $1/N^2$), which would imply that the time average is more efficient than the spatial average—such “super-convergence” is physically unnatural, but must be excluded theoretically.

We introduce the standard non-degeneracy condition. Under the exponential mixing condition considered in this paper, the autocovariance function is absolutely summable, so the following limit exists and is finite:

$$S \equiv \sum_{k=-\infty}^{\infty} \gamma(k) > 0 \in (0, \infty) \quad (3.13)$$

Intuitive interpretation: S is the sum of the autocovariance function, known as the “long-run variance” or the “zero-frequency spectral density” (up to a constant). Its positivity $S > 0$ means that the process does possess finite long-term fluctuations—if $S = 0$, the process would be “super-ergodic” in some special sense (for example, perfectly negative correlation causing fluctuations to cancel completely), which is extremely rare in real physical systems and can be excluded by assuming continuity of the spectral density.

Under the conditions $S > 0$ and continuity of the spectral density at zero frequency, it can be shown (see Brockwell & Davis, 1991, Theorem 7.1.1; it can also be derived directly under mixing conditions [16]):

$$(\bar{X}_N^{\text{time}}) \sim \frac{1}{N} \sum_{k=-\infty}^{\infty} \gamma(k) = \frac{S}{N} \quad (N \rightarrow \infty) \quad (3.14)$$

where the symbol “ $\sim$ ” denotes asymptotic equivalence.

This result shows that the variance of the time average is asymptotically exactly equal to S/N , rather than some smaller order. The factor S can be understood as an “effective sample size reduction factor”—if the time series is positively correlated, then $S > \sigma^2$, meaning that the effective number of independent samples is less than the nominal sample size N ; if it is negatively correlated, then, and the effective number of samples is instead increased. In any case, as long as S is a finite positive number, the order of convergence remains. Therefore, under the exponential mixing condition, the time average cannot have a faster rate of convergence than the spatial average (independent samples)—the two are of the same order.

$$\text{Var}(\bar{X}_N^{\text{time}}) = O\left(\frac{1}{N}\right) \quad \text{and the same order as} \quad \frac{1}{N} \quad (3.15)$$

This means that there is no systematic difference between the rates of convergence of the time average and the spatial average—their statistical precision is asymptotically comparable at the same sample size.

Combining the upper bound (3.12) with the asymptotic expansion (3.14), we obtain

$$\text{Var}(\bar{X}_N^{\text{time}}) = O\left(\frac{1}{N}\right), \quad \text{Var}(\bar{Y}_N^{\text{space}}) = O\left(\frac{1}{N}\right)$$

This means that there is no systematic difference between the convergence orders of the time average and the spatial average. However, this statement concerns the order of convergence only. The asymptotic variances are S_{time}/N and S_{sp}/N , respectively, S_{time} and S_{sp} need not be equal. Thus, equality of convergence order does not imply equality of asymptotic variance or of finite-sample precision.

3.6. Probabilistic Convergence and Disappearance of the Difference

From the variance upper bound (3.12) and the spatial average variance (3.2), combined with Chebyshev’s inequality, for any $\varepsilon > 0$,

$$P\left(\left|\bar{X}_N^{\text{time}} - \mu\right| > \varepsilon\right) \leq \frac{1}{\varepsilon^2} \text{Var}(\bar{X}_N^{\text{time}}) \leq \frac{C_1}{N\varepsilon^2} \frac{1}{N} \quad (3.16)$$

$$P\left(\left|\bar{Y}_N^{\text{space}} - \mu\right| > \varepsilon\right) \leq \frac{\sigma^2}{N\varepsilon^2} \quad (3.17)$$

Using the union bound,

$$P\left(\left|\bar{X}_N^{\text{time}} - \bar{Y}_N^{\text{space}}\right| > 2\varepsilon\right) \leq \frac{C_1 + \sigma^2}{N\varepsilon^2} \quad (3.18)$$

Letting $N \rightarrow \infty$, the right-hand side tends to 0, i.e.:

$$P\left(\left|\bar{X}_N^{\text{time}} - \bar{Y}_N^{\text{space}}\right| > 2\varepsilon\right) \rightarrow 0 \quad (3.19)$$

that is, convergence to 0 in probability. The standard deviation of this difference is of order $O(N^{-1/2})$.

3.7. Failure Mechanism in an Expanding Universe

In an expanding universe model, the physical distance r and the comoving distance x are related through the scale factor $a(t)$: $r(t) = a(t) \cdot x$. Here $a(t)$ increases monotonically with time. It is important to note that an FLRW universe can remain spatially homogeneous on each constant-time hypersurface. What fails in the expanding case is not spatial homogeneity, but the temporal stationarity of the chosen raw physical observables.

Consider the matter density field $\rho(x, t)$. In an expanding universe, the background density satisfies:

$$\rho_b(t) = \frac{\rho_{b,0}}{a(t)^3} \quad (3.20)$$

For the same spatial point (fixed comoving coordinate x), the expectation of the time series is not a constant:

$$\mu(t) = E[X_t] = \rho_b(t) \neq \text{const.} \quad (3.21)$$

This directly violates condition (1) of Section 2.2—the first-order statistic varies with time. The failure is therefore a failure of temporal stationarity for the chosen observables, not a failure of spatial homogeneity.

Decomposing the time series into a deterministic trend term and a zero-mean fluctuation term:

$$X(t) = \mu(t) + \varepsilon_t \quad E(\varepsilon_t) = 0 \quad (3.22)$$

its time sample mean is:

$$\bar{X}_N^{\text{time}} = \frac{1}{N} \sum_{i=1}^N \mu(t_i) + \frac{1}{N} \sum_{i=1}^N \varepsilon_{t_i} \quad (3.23)$$

The first term (the sample mean of the trend term) does not converge to a constant, but rather to the average background value over the time interval $[t_1, t_N]$:

$$\frac{1}{N} \sum_{i=1}^N \mu(t_i) \xrightarrow{N \rightarrow \infty} \frac{1}{T} \int_{t_1}^{t_N} \mu(t) dt \quad (3.24)$$

where $T = t_N - t_1$. This integral value depends on the starting and ending points of the observation window. Unless $\mu(t)$ happens to be constant (*i.e.*, there is no expansion), otherwise:

$$\lim_{N \rightarrow \infty} \bar{X}_N^{\text{time}} \neq \mu(t_0) \quad \forall t_0 \quad (3.25)$$

Imagine measuring the temperature of a cooling object. If you average over different times, the average you obtain depends on when you start measuring and when you stop—not on some “intrinsic temperature” of the object. The time average loses its universality and becomes a function of the observation window.

Taking de Sitter expansion ($\mu(t) = \mu_0 e^{Ht}$) as an example:

$$\frac{1}{N} \sum_{i=1}^N \mu(t_i) \approx \mu_0 \frac{e^{HT} - 1}{HT} \quad (3.26)$$

As $N, T \rightarrow \infty$, if $H > 0$, this term diverges exponentially. Even if the time series is detrended, the variance of the residual is still affected by the scale change caused by expansion:

$$\text{Var}(\bar{\varepsilon}_N) = \frac{1}{N^2} \sum_{i,j} \frac{\text{Cov}(\varepsilon_{t_i}, \varepsilon_{t_j})}{a(t_i)a(t_j)} \quad (3.27)$$

where the additional factor $1/(a(t_i)a(t_j))$ comes from the conversion between physical density and comoving density. Since $a(t)$ increases with time, the covariance structure no longer depends only on the time difference $|t_i - t_j|$, but on absolute time; the spectral representation method fails, and the simple application of the mixing condition no longer holds.

Finally, the mean-square error of the time average is:

$$\mathbb{E} \left[\left(\bar{X}_N^{\text{time}} - \mu(t_{\text{ref}}) \right)^2 \right] = \underbrace{\left[\frac{1}{N} \sum_{i=1}^N (\mu(t_i) - \mu(t_{\text{ref}})) \right]^2}_{O(1), \text{ does not decay with } N} + O\left(\frac{1}{N}\right) \quad (3.28)$$

The trend deviation term does not decay with N —it remains $O(1)$. Therefore, the time average in an expanding universe does not converge to any fixed value, and the spacetime statistical equivalence fails completely.

Expansion introduces an irreversible arrow of time—the universe at different instants “is not the same system”, and its statistical properties drift with absolute time. This is like averaging a person’s height at age 20 with their height at age 60—the result has no statistical meaning. The Spacetime Statistical Equivalence Principle has a rigorous finite-sample theoretical foundation only in a non-expanding, stationary background.

4. Comparison with Einstein’s Static Universe Solution

4.1. Brief Review of Einstein’s Static Universe

In 1917, Einstein introduced the cosmological constant Λ into general relativity and obtained a static universe solution [10]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where the scale factor $\alpha(t) = \text{const.}$, and the matter density ρ and pressure p do not vary with time. This model is based on geometric symmetry and requires strict time-translation invariance and spatial homogeneity.

4.2. Comparative Analysis

To present the similarities and differences between the two theoretical frameworks more clearly, their core features are systematically compared below as **Table 1**:

Table 1. Two theoretical frameworks compared.

Dimensionality	Einstein's Static Universe	This Paper's Spacetime Statistical Equivalence Principle
Underlying theory	General relativity + Λ	Statistical mechanics + ergodicity
Type of invariance	Geometric symmetry (metric invariance)	Statistical stationarity (distribution invariance)
Spacetime character	Strictly static, no evolution whatsoever	Local dynamical fluctuations allowed; macroscopic statistical steady state
Origin of homogeneity	Maximally symmetric space assumption	Statistical homogeneity under large-number conditions
Stability	Classically unstable [11] [12]	Does not involve dynamical stability
Observational compatibility	Incompatible with the CMB blackbody spectrum and structure-formation observations	Requires further comparison and testing against galaxy survey data
Additional parameters required	Fine-tuning of Λ required	No cosmological constant required

The Spacetime Statistical Equivalence Principle can provide a statistical-mechanical interpretive foundation for Einstein's static universe [10]: the static universe need not require strictly uniform matter at every point, but only that the statistical distributions of structures such as galaxies be stationary in space and time. This allows the static universe model to avoid conflicting with observations of local clustering phenomena.

It is important to distinguish between the mathematical result and its applicability to the actual universe. The results of Section 3 are theorems conditional on the assumptions of stationarity, spatial homogeneity, weak correlation, and ergodicity. Whether the actual universe satisfies these assumptions is an empirical question. In particular, the observed expansion, the cosmic microwave background, and galaxy-evolution data must be used to test whether the stationarity assumption is empirically viable. The present work does not claim that the actual universe satisfies the stationarity assumption; it explores the logical consequences that would follow if it did.

4.3. Scope and Observational Realization

The theorem of Section 2.3 applies to a physical model consisting of a non-expanding, statistically steady-state background with local dynamical fluctuations. In such a model, the large-scale statistical state is stationary and homogeneous, while local regions are allowed to evolve through processes such as star formation and galaxy mergers. The ideal simultaneous spatial snapshot used in the spatial sampling scheme is a mathematical idealization: it corresponds to observing a large number of spatially separated regions at the same instant.

In actual galaxy surveys, observations lie on the light cone rather than on a constant-time hypersurface. A direct observational test of the corollaries therefore requires either a narrow redshift slice, so that evolutionary effects are negligible, or an explicit correction for cosmic evolution. This distinction is essential for making the proposed observational tests distinguishable from the theorem's mathematical assumptions. Without such a correction or restriction, a light-cone survey cannot be regarded as a direct realization of the ideal spatial snapshot.

5. Cosmological Corollaries

Based on the above principle and comparison, and subject to the scope conditions stated in Section 4.3, the corollaries of the original paper are restated as follows.

- **Corollary 1 (Statistical spacetime equivalence):**

In a cosmological region satisfying stationarity and homogeneity, the statistical results of long-duration observations of a single region are equivalent to those of simultaneous observations of a large number of regions.

- **Corollary 2 (Large-scale statistical time invariance):**

The large-scale statistical state of the universe (such as galaxy density, mass function, and two-point correlation function) does not change with absolute time [9], *i.e.*:

$$\langle O(t_1) \rangle = \langle O(t_2) \rangle, \quad \forall t_1, t_2. \quad (5.1)$$

- **Corollary 3 (Coexistence of local evolution and global steady state):**

Local regions are allowed to evolve (such as star formation and galaxy mergers), but large-scale spatial averaging is equivalent to long-time temporal averaging; the two reflect the same steady-state parent distribution.

These corollaries do not require the geometric metric to be strictly static, but only that the statistical distribution be stable; they are therefore more relaxed than Einstein's original model and easier to reconcile with observations of local structure.

6. Applicability Conditions and Limitations

The principle of this paper is applicable to:

- Systems that are stationary, homogeneous, weakly correlated, and ergodic;
- Cosmological statistics in a static or steady-state background.

It is not applicable to:

- Non-stationary systems (such as an expanding universe or chemical transients);
- Strongly correlated systems (such as near a critical phase transition);
- Systems with an irreversible arrow of time (such as a closed system with monotonically increasing entropy).

The principle of this paper is applicable to systems that are stationary, homogeneous, weakly correlated, and ergodic, and to cosmological statistics in a static or steady-state background. It is not applicable to non-stationary systems, strongly

correlated systems, or systems with an irreversible arrow of time.

In particular, the expanding universe is a non-stationary system: for raw physical observables such as physical density, the ensemble mean varies with absolute time, violating first-order stationarity. This is a limitation of the theorem, not a denial of the observational evidence for an expanding universe. The CMB, the redshift-distance relation, and galaxy-evolution data remain decisive empirical constraints that any viable cosmological model must address.

If the universe as a whole were expanding, the spatial scale would change with time, stationarity and homogeneity would be destroyed, and this principle would fail. This is precisely the logical premise on which the present work distinguishes its applicability boundary. It does not deny the observational evidence for expansion; it delimits the conditions under which the statistical equivalence principle can be applied.

7. Conclusions

In summary, this paper has constructed a self-consistent statistical-mechanical framework around the proposal, proof, and application of the Spacetime Statistical Equivalence Principle. Specifically:

- 1) The “Spacetime Statistical Equivalence Principle” has been proposed and rigorously defined, and positioned as a natural generalization of the ergodic theorem to the spatial dimension;
- 2) A complete mathematical proof has been given, clarifying its convergence conditions and rate of convergence;
- 3) Under the premise of abandoning cosmic expansion, corollaries of large-scale statistical invariance for a steady-state universe have been derived; these corollaries are conditional on the stationarity assumption and require independent observational testing;
- 4) A systematic comparison with Einstein’s static universe solution has been made [10], indicating that the two can coexist within a “geometric-statistical” complementary framework.

This framework does not rely on the cosmological constant, dark energy, or any expansion dynamics; based solely on the stationarity assumption, it provides an independent and testable statistical-mechanical path for the steady-state universe model [17]. Future work may couple this principle with a static Robertson-Walker metric, or directly test its corollaries using galaxy survey data.

The significance of the Spacetime Statistical Equivalence Principle lies in providing a conditional methodological path for our understanding of the universe: if the stationarity assumption were empirically validated, knowledge of nearby galaxies could be used to infer the statistical states that more distant galaxies may exhibit, and by studying galaxies at different evolutionary stages, one could infer the possible evolutionary direction of our own galaxy. However, these inferences are conditional and do not follow from the mathematical theorem alone; they require independent observational testing. The theorem establishes a

statistical equivalence under stated assumptions, not a direct inference about the actual universe.

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Author Contributions

Conceptualization, Guo Chenxi; methodology, Guo Chenxi and Wu Mingli; software, Guo Chenxi; validation, Guo Chenxi and Wu Mingli; formal analysis, Wu Mingli; investigation, Guo Chenxi; resources, Guo Chenxi; data curation, Guo Chenxi; writing—original draft preparation, Guo Chenxi; writing—review and editing, Wu Mingli; visualization, Guo Chenxi; supervision, Guo Chenxi; project administration, Guo Chenxi; funding acquisition, Wu Mingli. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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