

The Wigner Quasiprobability and Its Integration over the Radius for Coherent States

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Abstract

The first part is a systematical representation of the Wigner quasiprobability in representations by canonical variables and pairs of complex conjugate variables with particular attention to the reconstruction of the density operator from the Wigner quasiprobability to show their equivalence. All formulae are derived from basic postulates of quantum mechanics. Problems of operator orderings are touched. The second part considers the integration over the amplitude in the Wigner quasiprobability and its specialization for coherent states and for even and odd coherent states and provides phase distributions which are presented in graphics that are new in their kind. It is demonstrated that the Heitler number phase uncertainty relation is incorrect. The problem of the transition from quantum to classical distribution functions by $\hbar \rightarrow 0$ is discussed with the conjecture that the inertial point of the distributions plays a basic role.

Keywords

Glauber-Sudarshan Quasiprobability, Husimi-Kano Quasiprobability, Operator Ordering, Coherent States, Even and Odd Coherent States, Convolution, Heitler Uncertainty Relation, Convolutions

Notation

We apply a very rational notation for the trace of a quantum-mechanical operator A by $\langle A \rangle \equiv \text{Trace}(A)$. Then the expectation value $\bar{A}$ of an operator A to a density operator ϱ is $\bar{A} \equiv \langle \varrho A \rangle$ and we see which density operator is meant if ϱ is specified before by definition.

1. Introduction

The main object of our investigation is the Wigner quasiprobability in transition

from canonical variables (q, p) to a pair of complex conjugate variables (α, α^*) in quantum theory for one mode of the electromagnetic field. Each mode of the electromagnetic field is represented by a harmonic oscillator and the whole electromagnetic field is thought to be decomposed into a linear superposition of independent (mutually orthogonal) harmonic oscillators that is already possible in classical treatment of the field. To make their number countable infinite one has as it is well known to impose boundary conditions onto the electromagnetic field. The complete representation of a quantum-mechanical state is then possible by a density operator ϱ in a Hilbert space which, in general, describes a mixture of pure states $\psi(t)$. In particular, we intend to develop systematically the relations for the reconstruction of the density operator from the Wigner quasiprobability to demonstrate their equivalence concerning the content of information. Then we discuss the integration over the radius of the Wigner quasiprobability and specialize this for coherent states and for even and odd coherent states with graphical representation in both cases. This brings into play the problem of the relations between amplitude and phase or of action and angle in quantum theory and the definition of a phase uncertainty which we discuss.

For present article the following stages of the high period of development of rigorous quantum mechanics are of most importance. In 1926 the wave function $\psi(t)$ was introduced by Schrödinger under “Quantization of an eigenvalue problem” in 4 reports [1] together with an equation for their time evolution (4-th report) which now bears his name. This equation was postulated in an analogy to classical Hamiltonian mechanics with correspondence of classical Poisson brackets to quantum-mechanical commutation relations which form a Lie algebra called Heisenberg-Weyl algebra [2]-[5]. The states described by Schrödinger, Heisenberg and the other founders of rigorous quantum mechanics are what is now called pure states. The notion of a quantum-mechanical state was generalized in 1932 by von Neumann [6] and is called a density operator. Its time evolution is described by an equation which bears now the name von Neumann equation. We do not write it down because we do not use it in this article. Almost at the same time in 1932 Wigner [7] (see also review of Tatarskii [8] and book of Kim and Noz [9], the last including the republished article of Wigner) introduced a probability function as an analogue to classical distribution functions which is now called Wigner quasiprobability (or Wigner function). In the following we use the more convenient notations of Dirac [10] by abstract states $|\psi\rangle$ and co-states $\langle\psi|$ (or bra’s and ket’s) instead of Schrödingers concrete realizations by wave functions $\psi(x)$.

Classical quantities are substituted in quantum mechanics by linear operators acting onto the states which form a Hilbert space of pure states. As postulate the pairs of classical canonical variables (q, p) are substituted by pairs of Hermitean operators (Q, P) satisfying the commutation relation (I identity operator)

$$[Q, P] \equiv QP - PQ = i\hbar I, \quad Q = Q^\dagger, \quad P = P^\dagger, \quad (1.1)$$

and, more generally, for ν independent modes

$$[Q_i, P_j] = i\hbar \delta_{ij}, [Q_i, Q_j] = [P_i, P_j] = 0, ((i, j) = 1, \dots, \nu). \quad (1.2)$$

The basic quantities of interest are then the expectation values $\bar{A} \equiv \langle \varrho A \rangle$ of linear operators A to density operators ϱ and their measurability if A corresponds to a classical quantity. The density operators ϱ themselves are special linear operators (normalized and positive definite ones) acting in Hilbert space.

In Section 2 we begin with the mathematical description and define the Wigner quasiprobability of one mode of a harmonic oscillator in classical canonical variables (q, p) and in pairs of complex conjugate variables (α, α^*) the last connected with the coherent states $|\alpha\rangle$ introduced by Glauber [11] in 1963. In Section 3 we discuss the displacement operator and the parity operator in variables (q, p) and (α, α^*) which are closely connected with the structure of the Wigner quasiprobability. In Section 4 we investigate the important problem of the reconstruction of the density operator from the Wigner quasiprobability and derive in detail the corresponding formulae. In Section 5 we discuss very shortly the transition to other quasiprobabilities and touch in this way the problem of operator ordering of functions of the basic canonical operators (Q, P) and pairs of annihilation and creation operators $(a, a^\dagger)$. The further Sections are devoted to the integration of the Wigner quasiprobability over the amplitude and to specialize the obtained distributions over the phase to coherent states and to even and odd coherent states introduced by Malkin and Manko [12] in 1979. The results are presented in graphics and lead to a new definition of the phase uncertainty. The basis for this definition is the uniform distribution of the phase. Discussing the limiting transition $\hbar \rightarrow 0$ to classical mechanics in Section 10 we conjectured that all quasiprobabilities for pure states with the same inertial point of the distribution lead exactly to the same classical distribution in form of delta functions displaced from the coordinate origin by the 2D-vector of the inertial point of the quantum distributions (i.e., quasiprobabilities).

Present article is a continuation of our article [13]. The main new results in comparison to cited article are the explicit calculation of the Wigner phase distribution for coherent states and the Wigner quasiprobability and its integration over the radius for even and odd coherent states and its graphical representations.

2. The Wigner Quasiprobability of One Mode of the Electromagnetic Field

In this Section we introduce the Wigner quasiprobability in two variants as $W(q, p)$ in canonical variables (q, p) of Hamilton mechanics and as $W(\alpha, \alpha^*)$ in pairs of complex conjugate variables (α, α^*) which became possible after the introduction of the coherent states and restrict us to one mode of a harmonic oscillator. The generalization to several modes are in many parts quite formal.

The Wigner quasiprobability $W(q, p)$ was introduced in quantum mechanics by Wigner [7] as a normalized probability function (his notation is $P(x, p)$ in canonical coordinates (x, p)) which for one pair of canonical variables

(q, p) of the phase space in Hamilton mechanics (or one mode) takes on the following form in the now more common notation of Dirac by “bra” and “ket” vectors $\langle \dots |$ and $|\dots\rangle$, respectively

$$W(q, p) = \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} dx \exp\left(i \frac{2px}{\hbar}\right) \langle q-x | \psi \rangle \langle \psi | q+x \rangle, \quad \int dq \wedge dp W(q, p) = 1. \quad (2.1)$$

Herein, $\langle q | \psi \rangle \equiv \psi(q)$ is the Schrödinger wave function of a state in q -representation and $\langle \psi | q \rangle \equiv \psi^*(q) = (\psi(q))^*$ its complex conjugate function. From this follows

$$\begin{aligned} W(q) &\equiv \int_{-\infty}^{+\infty} dp W(q, p) = \langle q | \psi \rangle \langle \psi | q \rangle \geq 0 \Rightarrow \int_{-\infty}^{+\infty} dq W(q) = \langle \psi | \psi \rangle, \\ W(p) &\equiv \int_{-\infty}^{+\infty} dq W(q, p) = \langle p | \psi \rangle \langle \psi | p \rangle \geq 0 \Rightarrow \int_{-\infty}^{+\infty} dp W(p) = \langle \psi | \psi \rangle, \end{aligned} \quad (2.2)$$

where the states $|q\rangle$ and (later) $|p\rangle$ are related by a pair of complex Fourier transformations (see below). The full normalization of $W(q, p)$ as given in (2.1) follows then from the assumption of normalization of the wave function $\langle \psi | \psi \rangle = 1$ in the Schrödinger theory. The relations in the first line of (2.2) are easily to obtain from (2.1) but Wigner does not show how he derived the relations in second line, likely also by Fourier transformation. The definition of $W(q, p)$ was likely made to formulate the Schrödinger wave mechanics together with the Heisenberg uncertainty relations which were well confirmed to this time by a probability function with the possibility to make the limiting transition $\hbar \rightarrow 0$ to classical (not only statistical) mechanics. In contrast, however, to classical probability functions the function $W(q, p)$ may take on also negative values and an everywhere non-negative such function was not found and obviously does not exist without extension of the classical phase space of variables (q, p) by additional hidden variables. The maximum what could be and has been achieved by Wigner was that $W(q)$ and $W(p)$ are positive definite functions and therefore are commonly considered as genuine probability functions whereas $W(q, p)$ is often called a quasiprobability.

The generalization of the Wigner quasiprobability $W(q, p)$ to mixed states with density operator ϱ of one mode is

$$\begin{aligned} W(q, p) &= \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} dx \exp\left(i \frac{2px}{\hbar}\right) \langle q-x | \varrho | q+x \rangle \\ &= \left\langle \varrho \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} dx \exp\left(i \frac{2px}{\hbar}\right) | q+x \rangle \langle q-x | \right\rangle, \end{aligned} \quad (2.3)$$

with the density operator in the following general form

$$\begin{aligned} \varrho &= \sum_k \varrho_k |\psi_k\rangle \langle \psi_k|, \quad \varrho_k = \varrho_k^* > 0, \quad \langle \psi_k | \psi_l \rangle = \delta_{k,l} \Rightarrow \\ \varrho &= \varrho^\dagger, \quad \varrho |\psi_k\rangle = \varrho_k |\psi_k\rangle, \quad \langle \varrho \rangle = \sum_k \varrho_k = 1. \end{aligned} \quad (2.4)$$

Thus the density operator ϱ is a positive definite Hermitean operator, therefore, with positive eigenvalues ϱ_k to orthonormalized eigenstates $|\psi_k\rangle$ and

with trace equal to 1. For pure states it simplifies to $\varrho = |\psi\rangle\langle\psi|$.

The states $|q\rangle$ and $|p\rangle$ are introduced as the right-hand eigenstates of the operators Q and P to real eigenvalues q and p , respectively, and $\langle q|$ and $\langle p|$ as their left-hand eigenstates to the same eigenvalues, correspondingly

$$Q|q\rangle = q|q\rangle, P|p\rangle = p|p\rangle, \langle q|Q = q\langle q|, \langle p|P = p\langle p|, \quad (2.5)$$

normalized by delta functions and satisfying completeness relations according to

$$\langle q'|q\rangle = \delta(q - q'), \langle p'|p\rangle = \delta(p - p'), \int_{-\infty}^{+\infty} dq |q\rangle\langle q| = \int_{-\infty}^{+\infty} dp |p\rangle\langle p| = I. \quad (2.6)$$

If we set together with (2.6) and in addition

$$\begin{aligned} P|q\rangle &= i\hbar \frac{\partial}{\partial q} |q\rangle, \langle q|P = -i\hbar \frac{\partial}{\partial q} \langle q| \equiv P' \langle q|, \langle q|Q = q\langle q| \equiv Q' \langle q|, \\ Q|p\rangle &= -i\hbar \frac{\partial}{\partial p} |p\rangle, \langle p|Q = i\hbar \frac{\partial}{\partial p} \langle p| \equiv Q'' \langle p|, \langle p|P = p\langle p| \equiv P'' \langle p|, \end{aligned} \quad (2.7)$$

then from the commutation relations (1.1) follow the commutation relations for Q' and P' as well as for Q'' and P'' as follows

$$\begin{aligned} Q' \equiv q, P' \equiv -i\hbar \frac{\partial}{\partial q} &\Rightarrow Q'P' - P'Q' \equiv -i\hbar \left(q \frac{\partial}{\partial q} - \frac{\partial}{\partial q} q \right) = i\hbar, \\ Q'' \equiv i\hbar \frac{\partial}{\partial p}, P'' \equiv p &\Rightarrow Q''P'' - P''Q'' \equiv i\hbar \left(\frac{\partial}{\partial p} p - p \frac{\partial}{\partial p} \right) = i\hbar, \end{aligned} \quad (2.8)$$

where here the multiplication and differentiation operators act only to the right onto states. This means that we have the isomorphisms (sign $\sim$) with action to the right

$$i\hbar I = QP - PQ \sim Q'P' - P'Q' \sim Q''P'' - P''Q'' = i\hbar I, \quad (2.9)$$

with some consequences, for example from $\langle q|P = -i\hbar \frac{\partial}{\partial q} \langle q|$

$$\langle q|P|\psi\rangle = -i\hbar \frac{\partial}{\partial q} \langle q|\psi\rangle \Rightarrow q\langle q|p\rangle = \langle q|Q|p\rangle = -i\hbar \frac{\partial}{\partial p} \langle q|p\rangle. \quad (2.10)$$

Herein, the last relation is a differential equation for $\langle q|p\rangle$ with the solution proportional to $\exp\left(i \frac{qp}{\hbar}\right)$. The most convenient choice of the factor is

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(i \frac{qp}{\hbar}\right). \quad (2.11)$$

It is now easy to see that the states $|p\rangle$ and $|q\rangle$ are connected by the following pair of Fourier transformation and its inversion

$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dq \exp\left(i \frac{pq}{\hbar}\right) |q\rangle, |q\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \exp\left(-i \frac{qp}{\hbar}\right) |p\rangle. \quad (2.12)$$

Using preceding relations one may prove the alternative representation of the Wigner quasiprobability (we remind that $\langle A \rangle$ means the trace of an arbitrary linear operator A)

$$\begin{aligned}
 W(q, p) &= \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} dy \exp\left(i \frac{2qy}{\hbar}\right) \langle p+y | \varrho | p-y \rangle \\
 &= \left\langle \varrho \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} dy \exp\left(i \frac{2qy}{\hbar}\right) | p-y \rangle \langle p+y | \right\rangle.
 \end{aligned}
 \quad (2.13)$$

A further equivalent expression for the Wigner quasiprobability which is symmetric in the operators Q and P is the following [13]

$$W(q, p) = \left\langle \varrho \exp\left(-Q \frac{\partial}{\partial q} - P \frac{\partial}{\partial p}\right) \right\rangle \delta(q) \delta(p). \quad (2.14)$$

In particular, this form is favorable, to a representation of the Wigner quasiprobability by a pair of complex conjugate variables. Furthermore it is important for the transition to other quasiprobabilities such as the Glauber-Sudarshan quasiprobability $P(q, p)$ and the Husimi-Kano quasiprobability $Q(q, p)$ (see Section 5).

In quantum optics the free electromagnetic field is described by superpositions of fictitious harmonic oscillators of frequency ω from which each of them belongs to a Hamilton operator H of the form

$$H = \frac{\omega}{2} (Q^2 + P^2) = H^?, \quad (2.15)$$

as direct translation of the classical Hamilton function $H(q, p) = \frac{\omega}{2} (q^2 + p^2)$ into quantum mechanics. By transition from the Hermitean operators (Q, P) to a pair of complex conjugated operators $(a, a^\dagger)$ according to

$$a \equiv \frac{Q + iP}{\sqrt{2\hbar}}, \quad a^\dagger \equiv \frac{Q - iP}{\sqrt{2\hbar}}, \quad Q = \sqrt{\frac{\hbar}{2}} (a + a^\dagger), \quad P = -i\sqrt{\frac{\hbar}{2}} (a - a^\dagger), \quad (2.16)$$

with commutation relations

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = I, \quad (2.17)$$

one arrives at the following representation of the Hamilton operator (2.15) with eigenstates $|n\rangle$ to eigenvalues according to

$$H = \frac{\hbar\omega}{2} (aa^\dagger + a^\dagger a) \equiv \hbar\omega \left(N + \frac{1}{2}I\right), \quad H|n\rangle = \hbar\omega \left(n + \frac{1}{2}\right)|n\rangle, \quad (n = 0, 1, 2, \dots). \quad (2.18)$$

The operator N is called the number operator and $|n\rangle$ are its orthonormalized eigenstates called number (or Fock) states to eigenvalues n according to the relations

$$N \equiv a^\dagger a, \quad N|n\rangle = n|n\rangle, \quad \langle m|n\rangle = \delta_{m,n}, \quad \sum_{n=0}^{\infty} n|n\rangle\langle n| = N, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = I. \quad (2.19)$$

The last of these relations is the completeness relation of the number states. The action of the operators $(a, a^\dagger)$ onto the number states possesses the form

$$\begin{aligned}
 a|n\rangle &= \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \\
 \langle n|a &= \sqrt{n+1}\langle n+1|, \quad \langle n|a^\dagger = \sqrt{n}\langle n-1|.
 \end{aligned}
 \quad (2.20)$$

Due to these properties of action of the operators a and $a^\dagger$ from the left onto the number states $|n\rangle$ to the right they are called annihilation and creation operators, respectively. The representation of the normalized number states $|n\rangle$ by the states $|q\rangle$ and $|p\rangle$ and its inversion is possible by the relations

$$\begin{aligned}\langle q|n\rangle &= \frac{1}{(\pi\hbar)^{\frac{1}{4}}} \exp\left(-\frac{q^2}{2\hbar}\right) \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{q}{\sqrt{\hbar}}\right) = \langle n|q\rangle^* = \langle q|n\rangle^*, \\ \langle p|n\rangle &= \frac{1}{(\pi\hbar)^{\frac{1}{4}}} \exp\left(-\frac{p^2}{2\hbar}\right) \frac{(-i)^n}{\sqrt{2^n n!}} H_n\left(\frac{p}{\sqrt{\hbar}}\right) = \langle n|p\rangle^* = (-1)^n \langle p|n\rangle^*,\end{aligned}\quad (2.21)$$

where $H_n(z)$ denotes the Hermite polynomials in their standard form. All this is well-known and can be developed from first postulates of Section 1 in a straightforward way (e.g., A. Bohm [14]). The Wigner quasiprobability $W(q, p)$ for general density operators ϱ in representation by the number states is presented in Appendix A.

The introduction of coherent states $|\alpha\rangle$ as normalized right-hand eigenstates of the annihilation operator a or left-hand eigenstates of the creation operator $a^\dagger$ of the pair $(a, a^\dagger)$ by Glauber [11] according to

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \langle\alpha|a^\dagger = \alpha^*\langle\alpha|, \quad |\alpha\rangle = \exp\left(-\frac{\alpha\alpha^*}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (\alpha \in \mathbb{C}), \quad (2.22)$$

where α are arbitrary complex numbers allows to make a transition of the Wigner quasiprobability $W(q, p)$ to a representation by the pair of complex conjugate variables (α, α^*) as follows. In analogy to (2.16) we make the variable transformation

$$\alpha = \frac{q + ip}{\sqrt{2\hbar}}, \quad \alpha^* = \frac{q - ip}{\sqrt{2\hbar}}, \quad q = \sqrt{\frac{\hbar}{2}}(\alpha + \alpha^*), \quad p = -i\sqrt{\frac{\hbar}{2}}(\alpha - \alpha^*), \quad (2.23)$$

from which follows

$$\begin{aligned}\frac{\partial}{\partial q} &= \frac{1}{\sqrt{2\hbar}} \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha^*} \right), \quad \frac{\partial}{\partial p} = \frac{i}{\sqrt{2\hbar}} \left(\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \alpha^*} \right), \\ \frac{\partial}{\partial \alpha} &= \sqrt{\frac{\hbar}{2}} \left(\frac{\partial}{\partial q} - i \frac{\partial}{\partial p} \right), \quad \frac{\partial}{\partial \alpha^*} = \sqrt{\frac{\hbar}{2}} \left(\frac{\partial}{\partial q} + i \frac{\partial}{\partial p} \right).\end{aligned}\quad (2.24)$$

Thus one finds

$$Q \frac{\partial}{\partial q} + P \frac{\partial}{\partial p} = a \frac{\partial}{\partial \alpha} + a^\dagger \frac{\partial}{\partial \alpha^*}. \quad (2.25)$$

Formally, in the calculus with exterior n -forms ω^n (Arnold [15]) one finds in the transition from real coordinates (x, y) to complex conjugate coordinates $(z, z^*) = (x + iy, x - iy)$ for the 2-form $\omega^2 \equiv dx \wedge dy = \frac{i}{2} dz \wedge dz^*$ with the right-hand side as a symbolic expression. However if one makes the transition from (z, z^*) to special coordinates, e.g., to polar coordinates $(z, z^*) = (|z|e^{i\varphi}, |z|e^{-i\varphi})$ the exterior form makes the right transition to its representation in these special

coordinates, *i.e.* in this example to $\frac{i}{2}dz \wedge dz^* = |z|d|z| \wedge d\varphi$ and we find

$$dq \wedge dp = -i\frac{\hbar}{2}(d\alpha + d\alpha^*) \wedge (d\alpha - d\alpha^*) = 2\hbar \frac{i}{2}d\alpha \wedge d\alpha^*. \quad (2.26)$$

On the other side if one denotes the two-dimensional delta function as $\delta(x)\delta(y) \equiv \delta(z, z^*)$ one gets

$$\delta(q)\delta(p) = \delta\left(\sqrt{2\hbar}\left(\frac{\alpha + \alpha^*}{2}\right)\right)\delta\left(-i\sqrt{2\hbar}\left(\frac{\alpha - \alpha^*}{2}\right)\right) = \frac{1}{2\hbar}\delta(\alpha, \alpha^*). \quad (2.27)$$

Using now the representation of the Wigner quasiprobability in the form (2.14) with the identity (2.23) and furthermore (2.26) and (2.27) one may define the Wigner quasiprobability in representation by a pair of complex conjugate variables (α, α^*) as follows [11]

$$\begin{aligned} W(q, p) &= \left\langle \varrho \exp\left(-Q\frac{\partial}{\partial q} - P\frac{\partial}{\partial p}\right) \right\rangle \delta(q)\delta(p) \\ &= \left\langle \varrho \exp\left(-a\frac{\partial}{\partial \alpha} - a^*\frac{\partial}{\partial \alpha^*}\right) \right\rangle \frac{\delta(\alpha, \alpha^*)}{2\hbar} \equiv \frac{W(\alpha, \alpha^*)}{2\hbar}. \end{aligned} \quad (2.28)$$

The factor $\frac{1}{2\hbar}$ on the right-hand side between $W(q, p)$ and $W(\alpha, \alpha^*)$ guarantees the normalization of the last in the integration measure $\frac{i}{2}d\alpha \wedge d\alpha^*$ according to

$$\int dq \wedge dp W(q, p) = \int \frac{i}{2}d\alpha \wedge d\alpha^* W(\alpha, \alpha^*) = 1, \quad (2.29)$$

and relation (2.28) using (2.23) enables the mutual transformation of one into the other form.

The Schrödinger wave function $\langle q|\alpha\rangle$ and its analogue $\langle p|\alpha\rangle$ of the coherent states $|\alpha\rangle$ one may calculate using the completeness relation for the number states $|n\rangle$ in (2.19) together with (2.21) as follows [13]

$$\begin{aligned} \langle q|\alpha\rangle &= \sum_{n=0}^{\infty} \langle q|n\rangle \langle n|\alpha\rangle \\ &= \frac{1}{(\hbar\pi)^{\frac{1}{4}}} \exp\left(-\frac{q^2}{2\hbar} - \frac{\alpha\alpha^*}{2}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha}{\sqrt{2}}\right)^n H_n\left(\frac{q}{\sqrt{\hbar}}\right) \\ &= \frac{1}{(\hbar\pi)^{\frac{1}{4}}} \exp\left(-\frac{q^2}{2\hbar} + \sqrt{\frac{2}{\hbar}}q\alpha - \frac{\alpha(\alpha + \alpha^*)}{2}\right), \\ \langle p|\alpha\rangle &= \sum_{n=0}^{\infty} \langle p|n\rangle \langle n|\alpha\rangle \\ &= \frac{1}{(\hbar\pi)^{\frac{1}{4}}} \exp\left(-\frac{p^2}{2\hbar} - \frac{\alpha\alpha^*}{2}\right) \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\frac{\alpha}{\sqrt{2}}\right)^n H_n\left(\frac{p}{\sqrt{\hbar}}\right) \\ &= \frac{1}{(\hbar\pi)^{\frac{1}{4}}} \exp\left(-\frac{p^2}{2\hbar} - i\sqrt{\frac{2}{\hbar}}p\alpha + \frac{\alpha(\alpha - \alpha^*)}{2}\right), \end{aligned} \quad (2.30)$$

where the well-known generating function of Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(z) = \exp(2tz - t^2), \quad (2.31)$$

was used. From this follows

$$\begin{aligned} \langle q|\alpha\rangle\langle\alpha|q\rangle &= \frac{1}{\sqrt{\hbar\pi}} \exp\left(-\frac{(q-\bar{Q})^2}{\hbar}\right), \quad \bar{Q} = \sqrt{\frac{\hbar}{2}}(\alpha + \alpha^*) = \sqrt{\frac{\hbar}{2}}(\bar{a} + \bar{a}^\dagger), \\ \langle p|\alpha\rangle\langle\alpha|p\rangle &= \frac{1}{\sqrt{\hbar\pi}} \exp\left(-\frac{(p-\bar{P})^2}{\hbar}\right), \quad \bar{P} = -i\sqrt{\frac{\hbar}{2}}(\alpha - \alpha^*) = -i\sqrt{\frac{\hbar}{2}}(\bar{a} - \bar{a}^\dagger). \end{aligned} \quad (2.32)$$

Arbitrary two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are mutually non-orthogonal according to

$$\langle\beta|\alpha\rangle = \exp\left(-\frac{\alpha\alpha^* + \beta\beta^*}{2} + \alpha\beta^*\right). \quad (2.33)$$

The coherent states $|\alpha\rangle; \alpha \in \mathbb{C}$ in their entirety are (over-) complete [11] expressed by

$$\frac{1}{\pi} \int \frac{i}{2} d\alpha \wedge d\alpha^* |\alpha\rangle\langle\alpha| = I, \quad (2.34)$$

that is easily to prove using the completeness relations for the number states. Over-completeness means that a coherent state may be represented as an integral over the entirety of (all) coherent states which can explicitly demonstrated by a relation for the displaced number states.

The Wigner quasiprobability $W(q, p)$ is restricted by the relation

$$-\frac{1}{\pi\hbar} \leq W(q, p) \leq \frac{1}{\pi\hbar}. \quad (2.35)$$

A proof is given in Appendix B.

Since we consider only one mode of the electromagnetic field we cannot include the description of the polarization properties of the vectorial electromagnetic field for which it is necessary to consider at least two modes of the same frequency and wave vector with different polarization (e.g. with linear or circular ones) and their correlation that leads to a 2×2 -matrix of 4 extended (Wigner-)quasiprobabilities from which one obtains a polarization matrix by integrations over the variables (q, p) or (α, α^*) and may discuss partial polarization of light with unpolarized light and fully polarized light in different forms as limiting cases.

The relations of this Section are more or less common knowledge but we wrote down them for a convenient citation in the following Sections.

3. Parity and Displacement Operator in Representation by Real Canonical and by a Pair of Complex Conjugate Variables

In this Section we introduce the parity operator Π which contributes to illuminate

the structure of the Wigner quasiprobability and, furthermore, the displacement operator $D(q, p)$ which is generally important for the transformations of states and operators.

The parity operator Π can be introduced in quantum optics as Hermitean operator by its action onto the basic states $|q\rangle$ and $|p\rangle$ of one mode as follows (e.g., Bishop and Vourdas [16])

$$\Pi|q\rangle = |-q\rangle \Leftrightarrow \Pi|p\rangle = |-p\rangle \Rightarrow \Pi^2 = I, ((q, p) \in \mathbb{R}). \quad (3.1)$$

Both ways are equivalent for the definition of Π and one follows from the other using the completeness relations (2.6), e.g.

$$\Pi|p\rangle = \int_{-\infty}^{+\infty} dq \Pi|q\rangle \langle q|p\rangle = \int_{-\infty}^{+\infty} dq |-q\rangle \langle q|p\rangle = \int_{-\infty}^{+\infty} dq |-q\rangle \langle -q|-p\rangle = |-p\rangle. \quad (3.2)$$

Furthermore, it follows the Hermitecity

$$\langle q'|\Pi|q\rangle = \langle q'|-q\rangle = \delta(q' + q) = \langle -q|q'\rangle = \langle q|\Pi^\dagger|q'\rangle. \quad (3.3)$$

This means that Π is at once a Hermitean and a unitary operator and squared provides the identity operator

$$\Pi = \Pi^\dagger = \Pi^{-1}, \Pi^2 = I. \quad (3.4)$$

The action of the parity operator onto the number states $|n\rangle$ is obtained using the completeness relation (2.6) and $\langle q|n\rangle = (-1)^n \langle -q|n\rangle$

$$\Pi|n\rangle = \int_{-\infty}^{+\infty} dq |q\rangle \langle q|n\rangle = (-1)^n \int_{-\infty}^{+\infty} dq |-q\rangle \langle -q|n\rangle = (-1)^n |n\rangle, \quad (3.5)$$

with the consequence

$$\Pi = \sum_{n=0}^{\infty} (-1)^n |n\rangle \langle n|. \quad (3.6)$$

Relation (3.5) shows that the only eigenvalues $+1$ and -1 of Π are highly degenerate to eigenstates which are general superpositions of even number states $|n\rangle = |2m\rangle$ and odd number states $|n\rangle = |2m+1\rangle$, respectively, and that it possesses the representations

$$\Pi = \sum_{m=0}^{\infty} (|2m\rangle \langle 2m| - |2m+1\rangle \langle 2m+1|). \quad (3.7)$$

The two subspaces of the Hilbert space to eigenvalues $+1$ and -1 of the parity operator are formed by the even and the odd number states of the harmonic oscillator.

The action of the parity operator onto the coherent states $|\alpha\rangle$ defined in (2.22) is described by

$$\Pi|\alpha\rangle = \sum_{n=0}^{\infty} \Pi|n\rangle \langle n|\alpha\rangle = \sum_{n=0}^{\infty} (-1)^n |n\rangle \langle n|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n|-\alpha\rangle = |-\alpha\rangle. \quad (3.8)$$

It is similar to the action of the parity operator onto the states $|q\rangle$ and $|p\rangle$. The representation of the parity operator by the coherent states is obtained then more indirectly by conjecture from following calculation

$$\begin{aligned}
& \frac{1}{\pi} \int \frac{i}{2} d\alpha \wedge d\alpha^* |\alpha\rangle \langle -\alpha| \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n |m\rangle \langle n|}{\sqrt{m!n!}} \underbrace{\frac{1}{\pi} \int \frac{i}{2} d\alpha \wedge d\alpha^* \exp(-\alpha\alpha^*) \alpha^m \alpha^{*n}}_{=n! \delta_{m,n}} \\
&= \sum_{n=0}^{\infty} (-1)^n |n\rangle \langle n| = \Pi \Rightarrow \langle \Pi \rangle = \frac{1}{2},
\end{aligned} \tag{3.9}$$

in analogy to (2.34) and its trace is derived as limiting case of the sum of the geometric series. Thus there are found the following expressions for the parity operator

$$\begin{aligned}
\Pi &= \int_{-\infty}^{+\infty} dq |q\rangle \langle -q| = \int_{-\infty}^{+\infty} dq |q\rangle \langle -p| \\
&= \frac{1}{\pi} \int \frac{i}{2} d\alpha \wedge d\alpha^* |\alpha\rangle \langle -\alpha| = \sum_{n=0}^{\infty} (-1)^n |n\rangle \langle n|.
\end{aligned} \tag{3.10}$$

In the following we need a well-known theorem for the product of two exponentials with operators A and B as arguments if the commutator $[A, B]$ commutes with both operators A and B of the form (e.g., [11])

$$\exp(A) \exp(B) = \exp\left(A + B + \frac{1}{2}[A, B]\right), \text{ if } [[A, B], A] = [[A, B], B] = 0. \tag{3.11}$$

It is a special case of the Baker-Campbell-Hausdorff formula for the multiplication of elements in arbitrary Lie groups if their Lie algebra is given (e.g., B. Hall, [17]). In quantum optics it is often used for the disentanglement of exponential operators with linear combinations of the operators (Q, P) or $(a, a^\dagger)$ in the argument. Furthermore, without restriction of the generality of the operators A and B the following known operator identity holds

$$\begin{aligned}
\exp(A) B \exp(-A) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \frac{(-1)^{n-k} n!}{k!(n-k)!} A^k B A^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\left[A, \left[A, \dots \left[A, B \right] \dots \right] \right]}_{n \times} \underbrace{\quad}_{n \times} \\
&= B + [A, B] + \frac{1}{2}[A, [A, B]] + \dots,
\end{aligned} \tag{3.12}$$

which may be proved by complete induction $n \rightarrow n+1$ for the inner sum.

Next we define the unitary displacement operator $D(q, p)$ in representation by two real parameters (q, p) and equivalently $D(\alpha, \alpha^*)$ by two complex conjugate parameters (α, α^*) related according to (2.15) by (essentially but not in form, already considered by Weyl [3], see also [11])

$$\begin{aligned}
D(q, p) &\equiv \exp\left(-\frac{i}{\hbar}(qP - pQ)\right) = \exp(\alpha a^\dagger - \alpha^* a) \equiv D(\alpha, \alpha^*), \\
(D(q, p))^\dagger &= D(-q, -p) = (D(q, p))^{-1}, \\
(D(\alpha, \alpha^*))^\dagger &= D(-\alpha, -\alpha^*) = (D(\alpha, \alpha^*))^{-1}.
\end{aligned} \tag{3.13}$$

These are essentially the operators of the Heisenberg-Weyl group with the operators of the Heisenberg algebra in the exponents. They obey the multiplication law

$$\begin{aligned} D(q, p)D(q', p') &= \exp\left(-i\frac{qp' - pq'}{2\hbar}\right)D(q+q', p+p'), \\ D(\alpha, \alpha^*)D(\beta, \beta^*) &= \exp\left(\frac{\alpha\beta^* - \beta\alpha^*}{2}\right)D(\alpha+\beta, \alpha^*+\beta^*). \end{aligned} \quad (3.14)$$

Using (3.12) one finds

$$\begin{aligned} D(q, p)Q(D(q, p))^\dagger &= Q - qI, \quad D(q, p)P(D(q, p))^\dagger = P - pI, \\ D(\alpha, \alpha^*)a(D(\alpha, \alpha^*))^\dagger &= a - \alpha I, \quad D(\alpha, \alpha^*)a^\dagger(D(\alpha, \alpha^*))^\dagger = a^\dagger - \alpha^* I, \end{aligned} \quad (3.15)$$

that justifies their names as displacement operators. Generally, we call the displacement of an operator A according to the definition

$$A(q, p) \equiv D(q, p)A(D(q, p))^\dagger = D(\alpha, \alpha^*)A(D(\alpha, \alpha^*))^\dagger \equiv A(\alpha, \alpha^*), \quad (3.16)$$

the displaced operator A . It is immediately to see that the displaced operators $A(q, p)$ possesses the same trace as the operators A and the same dimensionality.

$$\langle A(q, p) \rangle = \langle A \rangle, \quad \langle A(\alpha, \alpha^*) \rangle = \langle A \rangle. \quad (3.17)$$

Next we consider some further properties of the displacement operator. Two forms of “disentanglements” of the displacement operators derived from the identity (3.11) and using the commutation relations are

$$\begin{aligned} D(q, p) &= \exp\left(i\frac{qp}{2\hbar}\right)\exp\left(-\frac{i}{\hbar}qP\right)\exp\left(\frac{i}{\hbar}pQ\right), \\ D(\alpha, \alpha^*) &= \exp\left(-\frac{\alpha\alpha^*}{2}\right)\exp(\alpha a^\dagger)\exp(-\alpha^* a). \end{aligned} \quad (3.18)$$

Their traces can be found by the following small calculation using the completeness relations (2.6)

$$\begin{aligned} \langle D(q, p) \rangle &= \exp\left(i\frac{pq}{2\hbar}\right)\int_{-\infty}^{+\infty} dq' \int_{-\infty}^{+\infty} dp' \langle q' | \exp\left(-\frac{i}{\hbar}qP\right) | p' \rangle \langle p' | \exp\left(\frac{i}{\hbar}pQ\right) | q' \rangle \\ &= \exp\left(i\frac{pq}{2\hbar}\right)\int_{-\infty}^{+\infty} dq' \int_{-\infty}^{+\infty} dp' \exp\left(\frac{i}{\hbar}(pq' - qp')\right) \langle q' | p' \rangle \langle p' | q' \rangle \\ &= 2\pi\hbar\delta(q)\delta(p) \Rightarrow \\ &\quad \langle D(\alpha, \alpha^*) \rangle = \pi\delta(\alpha, \alpha^*). \end{aligned} \quad (3.19)$$

From this follows using (3.14)

$$\begin{aligned} \langle D(q, p)(D(q', p'))^\dagger \rangle &= 2\pi\hbar\delta(q-q')\delta(p-p'), \\ \langle D(\alpha, \alpha^*)(D(\beta, \beta^*))^\dagger \rangle &= \pi\delta(\alpha-\beta, \alpha^*-\beta^*). \end{aligned} \quad (3.20)$$

A consequence is that an arbitrary operator A may be expanded into an integral over the displacement operators with corresponding kernel functions in front of them that leads to

$$A = \frac{1}{2\pi\hbar} \int dq \wedge dp \left\langle A(D(q, p))^\dagger \right\rangle D(q, p),$$

$$A = \frac{1}{\pi} \int \frac{i}{2} d\alpha \wedge d\alpha^* \left\langle A(D(\alpha, \alpha^*))^\dagger \right\rangle D(\alpha, \alpha^*). \quad (3.21)$$

The introduction of the displacement operator $D(\alpha, \alpha^*)$ provides the possibility to represent the coherent states $|\alpha\rangle$ defined in (2.22) by the action of $D(\alpha, \alpha^*)$ onto the vacuum state $|0\rangle$ according to [11]

$$|\alpha\rangle = D(\alpha, \alpha^*)|0\rangle, \quad \langle\alpha| = \langle 0|(D(\alpha, \alpha^*))^\dagger,$$

$$|\alpha\rangle = \exp\left(-\frac{\alpha\alpha^*}{2}\right) \exp(\alpha a^\dagger) \underbrace{\exp(-\alpha^* a)|0\rangle}_{=|0\rangle} = \exp\left(-\frac{\alpha\alpha^*}{2}\right) \exp(\alpha a^\dagger)|0\rangle, \quad (3.22)$$

where the vacuum state $|0\rangle$ itself appears then also as a coherent state generated by the special displacement operator $D(\alpha=0, \alpha^*=0)=I$ from $|0\rangle$. The equivalence of both possible definitions of $|\alpha\rangle$ in (2.22) and in (3.22) is easily to establish.

4. Displaced Parity Operator and Reconstruction of Density Operator from Wigner Quasiprobability

For the following it is important to consider the displacement of the parity operator Π in two forms with variables (q, p) and (α, α^*) . In particular, this leads to formulae how one may reconstruct the density operator from the Wigner quasiprobability and shows directly the equivalence of both quantities concerning their information contents.

According to the general definition (3.16) of the displacement of an arbitrary operator A the displaced parity operator Π in the two forms of variables is defined by

$$\Pi(q, p) \equiv D(q, p)\Pi(D(q, p))^\dagger = \int_{-\infty}^{+\infty} dq' D(q, p)|q'\rangle\langle -q'| (D(q, p))^\dagger,$$

$$\Pi(\alpha, \alpha^*) \equiv D(\alpha, \alpha^*)\Pi(D(\alpha, \alpha^*))^\dagger$$

$$= \frac{1}{\pi} \int \frac{i}{2} d\beta \wedge d\beta^* D(\alpha, \alpha^*)|\beta\rangle\langle -\beta| (D(\alpha, \alpha^*))^\dagger \quad (4.1)$$

$$= \Pi(q, p).$$

The equality $\Pi(q, p) = \Pi(\alpha, \alpha^*)$ follows from the equality of $D(q, p) = D(\alpha, \alpha^*)$ in both definitions. We consider now the two variants of variables separately and begin with the variables (q, p) .

For this purpose we calculate first the following parts of $\Pi(q, p)$ in (4.1) applying the disentangled form (3.18) of the displacement operator $D(q, p)$

$$\begin{aligned}
D(q, p)|q'\rangle &= \exp\left(i\frac{qp}{2\hbar}\right)\exp\left(-\frac{i}{\hbar}qP\right)\exp\left(\frac{i}{\hbar}pQ\right)|q'\rangle \\
&= \exp\left(i\frac{p(q+2q')}{2\hbar}\right)\exp\left(q\frac{\partial}{\partial q'}\right)|q'\rangle \\
&= \exp\left(i\frac{p(q+2q')}{2\hbar}\right)|q+q'\rangle, \\
\langle -q'|(D(q, p))^\dagger &= \exp\left(-i\frac{p(q-2q')}{2\hbar}\right)\langle q-q'|.
\end{aligned} \tag{4.2}$$

Using these relations one obtains for the displaced parity operator (we substitute here the integration variable q' in (4.1) by $q' \rightarrow x$)

$$\begin{aligned}
\Pi(q, p) &= \int_{-\infty}^{+\infty} dx D(q, p)|x\rangle\langle -x|(D(q, p))^\dagger \\
&= \int_{-\infty}^{+\infty} dx \exp\left(i\frac{2px}{\hbar}\right)|q+x\rangle\langle q-x|,
\end{aligned} \tag{4.3}$$

with the trace

$$\langle \Pi(q, p) \rangle = \int_{-\infty}^{+\infty} dx \exp\left(i\frac{2px}{\hbar}\right) \delta(2x) = \frac{1}{2}, \tag{4.4}$$

which is and has to be the same as the trace $\langle \Pi \rangle = \frac{1}{2}$. By comparison of this expression with the Wigner quasiprobability in the form (2.3) one finds the following representation of $W(q, p)$ by the displaced parity operator $\Pi(q, p)$

$$W(q, p) = \frac{1}{\pi\hbar} \langle \varrho \Pi \rangle. \tag{4.5}$$

Our aim is the reconstruction of the density operator ϱ from the Wigner quasiprobability $W(q, p)$. For this purpose we calculate from (4.3) the product of two displaced parity operators with different arguments with the result

$$\begin{aligned}
&\Pi(q, p)\Pi(q', p') \\
&= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' \exp\left(i\frac{2(px+p'x')}{\hbar}\right)|q+x\rangle\langle q-x|q'+x'\rangle\langle q'-x'| \\
&= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' \delta(q-x-q'-x') \exp\left(i\frac{2(px+p'x')}{\hbar}\right)|q+x\rangle\langle q'-x'| \\
&= \int_{-\infty}^{+\infty} dx \exp\left(i\frac{2((p-p')x+p'(q-q'))}{\hbar}\right)|q+x\rangle\langle 2q'-q+x|.
\end{aligned} \tag{4.6}$$

Calculating the trace over this product we obtain

$$\begin{aligned}
\langle \Pi(q, p)\Pi(q', p') \rangle &= \int_{-\infty}^{+\infty} dx \exp\left(i\frac{2((p-p')x+p'(q-q'))}{\hbar}\right) \delta(2(q-q')) \\
&= \frac{1}{2} \delta(q-q') \int_{-\infty}^{+\infty} dx \exp\left(i\frac{2(p-p')x}{\hbar}\right) \\
&= \frac{1}{2} \delta(q-q') 2\pi \delta\left(\frac{2(p-p')}{\hbar}\right) \\
&= \frac{\pi\hbar}{2} \delta(q-q') \delta(p-p').
\end{aligned} \tag{4.7}$$

This relation can be used to make an expansion of an arbitrary operator into an integral over the displaced parity operator $\Pi(q, p)$ with coefficients similar to the widely analogous case (3.21). A difference between $D(q', p')$ and $\Pi(q', p')$ is that the first is a unitary operator whereas the second is a Hermitean operator and that in first we have the adjoint operator instead of the same operator in the second relation. Since the Wigner quasiprobability is already determined by (4.5) one finds that the following proposition

$$\varrho = \lambda \int dq \wedge dp W(q, p) \Pi(q, p), \quad (4.8)$$

with an unknown factor λ which is to determine is sufficient to solve the inversion problem according to

$$\begin{aligned} W(q', p') &= \frac{1}{\pi \hbar} \langle \varrho \Pi(q', p') \rangle \\ &= \frac{\lambda}{\pi \hbar} \int dq \wedge dp W(q, p) \Pi(q, p) \Pi(q', p') \\ &= \frac{\lambda}{2} W(q', p') \Rightarrow \lambda = 2. \end{aligned} \quad (4.9)$$

Thus the factor $\lambda = 2$ is found and one may easily check the trace of the density operator using (4.4)

$$\langle \varrho \rangle = 2 \int dq \wedge dp W(q, p) \langle \Pi(q, p) \rangle = \int dq \wedge dp W(q, p) = 1, \quad (4.10)$$

that is correct. It is also possible first to gain the factor $\lambda = 2$ from postulated trace $\langle \varrho \rangle = 1$ in (4.4) and then to verify the Formula (4.5) for $W(q, p)$. Thus it was proved that the pair of Wigner quasiprobability and density operator described by

$$W(q, p) = \frac{1}{\pi \hbar} \langle \varrho \Pi(q, p) \rangle \Leftrightarrow \varrho = 2 \int dq \wedge dp W(q, p) \Pi(q, p), \quad (4.11)$$

are equivalent concerning their information content. From the equality of (4.5) to (2.14) for arbitrary ϱ follows

$$\frac{1}{\pi \hbar} \Pi(q, p) = \exp \left(-Q \frac{\partial}{\partial q} - P \frac{\partial}{\partial p} \right) \delta(q) \delta(p). \quad (4.12)$$

Thus the pair of transformation and its inversion (4.11) can be also represented by

$$\begin{aligned} W(q, p) &= \left\langle \varrho \exp \left(-Q \frac{\partial}{\partial q} - P \frac{\partial}{\partial p} \right) \right\rangle \delta(q) \delta(p) \Leftrightarrow \\ \varrho &= 2\pi \hbar \int dq \wedge dp W(q, p) \exp \left(-Q \frac{\partial}{\partial q} - P \frac{\partial}{\partial p} \right) \delta(q) \delta(p). \end{aligned} \quad (4.13)$$

The second of these equalities can be transformed furthermore with advantage for calculations as follows

$$\begin{aligned} \varrho &= 2\pi \hbar \int dq \wedge dp \delta(q) \delta(p) \exp \left(Q \frac{\partial}{\partial q} + P \frac{\partial}{\partial p} \right) W(q, p) \\ &= 2\pi \hbar \left\{ \exp \left(Q \frac{\partial}{\partial q} + P \frac{\partial}{\partial p} \right) W(q, p) \right\}_{(q=0, p=0)}. \end{aligned} \quad (4.14)$$

This relation shows that using Taylor series of the exponential functions the density operator can be reconstructed from Wigner quasiprobability by a kind of its derivatives which is called symmetrically ordered (notation $S\{\dots\}$) for the content in braces where the relation $\exp(\mu Q + \nu P) \equiv S\{\exp(\mu Q + \nu P)\}$ and following $(\mu Q + \nu P)^n \equiv S\{(\mu Q + \nu P)^n\}$ with differential operators substituted by variables could even be taken as starting point for the introduction of symmetrical ordering (see also next Section). Expectation values to classical phase-space functions $f(q, p)$ correspond in quantum mechanics to expectation values of symmetrically ordered functions $f(Q, P)$ according to (4.11)

$$\begin{aligned} \varrho S\{f(Q, P)\} &= \int dq \wedge dp W(q, p) f(q, p) \\ &= 2 \int dq \wedge dp W(q, p) \Pi S\{f(Q, P)\}. \end{aligned} \quad (4.15)$$

We discuss now the problem of the reconstruction of the density operator ϱ from the Wigner quasiprobability $W(\alpha, \alpha^*)$ in complex variables. It is easy to see that the displaced parity operators are equal in both representations

$$\Pi(q, p) = D(q, p) \Pi(D(q, p))^\dagger = D(\alpha, \alpha^*) \Pi(D(\alpha, \alpha^*))^\dagger = \Pi(\alpha, \alpha^*), \quad (4.16)$$

as result of the equality of the displacement operators $D(q, p) = D(\alpha, \alpha^*)$ in both representations. Then relation (2.28) combined with Equation (4.5) leads to

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \langle \varrho \Pi(\alpha, \alpha^*) \rangle. \quad (4.17)$$

Furthermore, due to (2.28) one has the following analogous representations of the Wigner quasiprobability to (2.14)

$$W(\alpha, \alpha^*) = \left\langle \varrho \exp\left(-a \frac{\partial}{\partial \alpha} - a^\dagger \frac{\partial}{\partial \alpha^*}\right) \right\rangle \delta(\alpha, \alpha^*). \quad (4.18)$$

Relation (4.17) already admits to write down the analogous relations to (4.11) of definition of the Wigner quasiprobability from the density operator and the reconstruction of the density operator from the Wigner quasiprobability in the new variables

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \langle \varrho \Pi(\alpha, \alpha^*) \rangle, \quad \varrho = 2 \int \frac{i}{2} d\alpha \wedge d\alpha^* W(\alpha, \alpha^*) \Pi(\alpha, \alpha^*). \quad (4.19)$$

We continue now with a more detailed investigation of the displaced parity operator $\Pi(\alpha, \alpha^*)$ and insert the form (3.10) of the parity operator Π

$$\Pi(\alpha, \alpha^*) = \frac{1}{\pi} \int \frac{i}{2} d\beta \wedge d\beta^* D(\alpha, \alpha^*) |\beta\rangle \langle -\beta| (D(\alpha, \alpha^*))^\dagger. \quad (4.20)$$

Using now

$$\begin{aligned} D(\alpha, \alpha^*) |\beta\rangle &= D(\alpha, \alpha^*) D(\beta, \beta^*) |0\rangle \\ &= \exp\left(\frac{\alpha\beta^* - \beta\alpha^*}{2}\right) D(\alpha + \beta, \alpha^* + \beta^*) |0\rangle \\ &= \exp\left(\frac{\alpha\beta^* - \beta\alpha^*}{2}\right) |\alpha + \beta\rangle, \end{aligned}$$

$$\langle -\beta | (D(\alpha, \alpha^*))^\dagger = \exp\left(\frac{\alpha\beta^* - \beta\alpha^*}{2}\right) \langle \alpha - \beta |, \quad (4.21)$$

and inserting this in (4.20) we arrive at the representation

$$\Pi(\alpha, \alpha^*) = \frac{1}{\pi} \int \frac{i}{2} d\beta \wedge d\beta^* \exp(\alpha\beta^* - \beta\alpha^*) |\alpha + \beta\rangle \langle \alpha - \beta|, \quad (4.22)$$

with trace

$$\begin{aligned} \langle \Pi(\alpha, \alpha^*) \rangle &= \frac{1}{\pi} \int \frac{i}{2} d\beta \wedge d\beta^* \exp(\alpha\beta^* - \beta\alpha^*) \langle \alpha - \beta | \alpha + \beta \rangle \\ &= \frac{1}{\pi} \int \frac{i}{2} d\beta \wedge d\beta^* \exp(-2\beta\beta^*) = \frac{1}{2}. \end{aligned} \quad (4.23)$$

From (4.17) follows then for the Wigner quasiprobability

$$W(\alpha, \alpha^*) = \frac{2}{\pi^2} \int \frac{i}{2} d\beta \wedge d\beta^* \exp(\alpha\beta^* - \beta\alpha^*) \langle \alpha - \beta | \varrho | \alpha + \beta \rangle. \quad (4.24)$$

This is the analogue to the representations (2.3) and (2.13) of the Wigner quasiprobability.

If one uses the representation (3.10) for the parity operator then one finds for the displaced parity operator

$$\begin{aligned} \Pi(q, p) &= \sum_{n=0}^{\infty} (-1)^n D(q, p) |n\rangle \langle n| (D(q, p))^\dagger, \\ \Pi(\alpha, \alpha^*) &= \sum_{n=0}^{\infty} (-1)^n D(\alpha, \alpha^*) |n\rangle \langle n| (D(q, p))^\dagger \equiv \sum_{n=0}^{\infty} (-1)^n |\alpha, n\rangle \langle \alpha, n|, \end{aligned} \quad (4.25)$$

where $|\alpha, n\rangle$ are by definition the displaced number states¹

$$|\alpha, n\rangle \equiv D(\alpha, \alpha^*) |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger - \alpha^* I)^n |\alpha\rangle. \quad (4.26)$$

For the Wigner quasiprobability then follows

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \langle \alpha, n | \varrho | \alpha, n \rangle. \quad (4.27)$$

The displaced number states $|\alpha, n\rangle$ include the coherent states as special case $|\alpha\rangle = |\alpha, 0\rangle$ and possess interesting properties and were investigated, for example, in [18].

5. Operator Ordering and Related Quasiprobabilities

There are often used other quasiprobabilities such as the Glauber-Sudarshan quasiprobability $P(\alpha, \alpha^*)$ and the Husimi-Kano quasiprobability $Q(\alpha, \alpha^*)$ according to

$$\varrho \equiv \int \frac{i}{2} d\beta \wedge d\beta^* P(\beta, \beta^*) |\beta\rangle \langle \beta|, \quad Q(\alpha, \alpha^*) \equiv \frac{\langle \alpha | \varrho | \alpha \rangle}{\pi} \quad (5.1)$$

¹The states $D(q, p) |n\rangle$ are the same as $|\alpha, n\rangle$ only variables (α, α^*) expressed by variables (q, p) according to (2.23).

with $|\alpha\rangle$ and $|\beta\rangle$ coherent states. The first quasiprobability is here indirectly defined and its formula is merely the reconstruction of the density operator from the quasiprobability. The normalization of both quasiprobabilities is easily to see by forming the trace over the first equation and integrating the second over the two-dimensional phase space. The connection between both quasiprobabilities can be established calculating $\frac{\langle\alpha|\varrho|\alpha\rangle}{\pi}$ from the first equation

$$\begin{aligned} Q(\alpha, \alpha^*) &\equiv \frac{\langle\alpha|\varrho|\alpha\rangle}{\pi} = \frac{1}{\pi} \int \frac{i}{2} d\beta \wedge d\beta^* P(\beta, \beta^*) \langle\alpha|\beta\rangle \langle\beta|\alpha\rangle \\ &= \frac{1}{\pi} \int \frac{i}{2} d\beta \wedge d\beta^* \exp\left(-(\alpha - \beta)(\alpha^* - \beta^*)\right) P(\beta, \beta^*) \\ &= \frac{1}{\pi} \exp(-\alpha\alpha^*) * P(\alpha, \alpha^*), \end{aligned} \quad (5.2)$$

where “ $*$ ” between two functions denotes the convolution of the two functions (see Appendix C). The relations between different quasiprobabilities by convolutions are connected with different operator ordering of $(a, a^\dagger)$ in functions of them.

A convenient starting point for the introduction of other interesting quasiprobabilities from the Wigner quasiprobability is the representation (2.14) if we there apply the operator identity (3.11) to the operator $\exp\left(-a\frac{\partial}{\partial\alpha} - a^\dagger\frac{\partial}{\partial\alpha^*}\right)$. Then one finds if we make the transition to normal ordering (all parts with the creation operator $a^\dagger$ stand in front of all parts with the annihilation operator a)

$$\begin{aligned} W(\alpha, \alpha^*) &= \left\langle \varrho \exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right) \exp\left(-a\frac{\partial}{\partial\alpha}\right) \right\rangle \exp\left(\frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) \delta(\alpha, \alpha^*) \\ &= \left\langle \varrho \exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right) \exp\left(-a\frac{\partial}{\partial\alpha}\right) \right\rangle \frac{2}{\pi} \exp(-2\alpha\alpha^*) \end{aligned} \quad (5.3)$$

where identities (C.14) and (C.10) are used. It is a special case of the much more general identities for convolutions

$$\begin{aligned} &\exp\left(-a\frac{\partial}{\partial\alpha} - a^\dagger\frac{\partial}{\partial\alpha^*}\right) (f(\alpha, \alpha^*) * g(\alpha, \alpha^*)) \\ &= f(\alpha, \alpha^*) * \exp\left(-a\frac{\partial}{\partial\alpha} - a^\dagger\frac{\partial}{\partial\alpha^*}\right) g(\alpha, \alpha^*) \\ &= g(\alpha, \alpha^*) * \exp\left(-a\frac{\partial}{\partial\alpha} - a^\dagger\frac{\partial}{\partial\alpha^*}\right) f(\alpha, \alpha^*), \end{aligned} \quad (5.4)$$

with arbitrary functions $f(\alpha, \alpha^*)$ and $g(\alpha, \alpha^*)$ and where

$\exp\left(-a\frac{\partial}{\partial\alpha} - a^\dagger\frac{\partial}{\partial\alpha^*}\right)$ can be substituted by other forms of its operator ordering.²

²With less amount of writing one first easily proves the one-dimensional case

$\exp\left(-a\frac{\partial}{\partial x}\right) (f(x) * g(x)) = f(x) * \exp\left(-a\frac{\partial}{\partial x}\right) g(x)$, generalizes this to the two-dimensional case and makes then the transition to a pair of complex conjugate coordinates and generalizes the scalar function $\exp\left(-a\frac{\partial}{\partial x}\right)$ to an operator function.

From (5.3) together with (5.4) follows the equivalent representation of $W(\alpha, \alpha^*)$

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \exp(-2\alpha\alpha^*) * \left\langle \varrho \exp\left(-a^\dagger \frac{\partial}{\partial \alpha^*}\right) \exp\left(-a \frac{\partial}{\partial \alpha}\right) \right\rangle \delta(\alpha, \alpha^*). \quad (5.5)$$

Furthermore, using anti-normal ordering (all parts with operators a stand in front of all parts with operators $a^\dagger$) of the operators within braces in the expression (4.18) one obtains the following relation of $Q(\alpha, \alpha)$ to $W(\alpha, \alpha^*)$ calculating

$$\begin{aligned} & \frac{2}{\pi} \exp(-2\alpha\alpha^*) * W(\alpha, \alpha^*) \\ &= \exp\left(\frac{1}{2} \frac{\partial^2}{\partial \alpha \partial \alpha^*}\right) \left\langle \varrho \exp\left(-a \frac{\partial}{\partial \alpha} - a^\dagger \frac{\partial}{\partial \alpha^*}\right) \right\rangle \delta(\alpha, \alpha^*) \\ &= \exp\left(\frac{1}{2} \frac{\partial^2}{\partial \alpha \partial \alpha^*}\right) \left\langle \varrho \exp\left(-a \frac{\partial}{\partial \alpha}\right) \exp\left(-a^\dagger \frac{\partial}{\partial \alpha^*}\right) \right\rangle \exp\left(-\frac{1}{2} \frac{\partial^2}{\partial \alpha \partial \alpha^*}\right) \delta(\alpha, \alpha^*) \\ &= \frac{1}{\pi} \int \frac{i}{2} d\beta \wedge d\beta^* \left\langle \varrho \exp\left(-a \frac{\partial}{\partial \alpha}\right) \right\rangle |\beta\rangle \langle \beta| \exp\left(-a^\dagger \frac{\partial}{\partial \alpha^*}\right) \delta(\alpha, \alpha^*) \\ &= \frac{1}{\pi} \int \frac{i}{2} d\beta \wedge d\beta^* \langle \beta | \varrho | \beta \rangle \delta(\alpha - \beta, \alpha^* - \beta^*) = \frac{\langle \alpha | \varrho | \alpha \rangle}{\pi} = Q(\alpha, \alpha^*). \end{aligned} \quad (5.6)$$

If we use the special case of the convolution (C.13)

$$\frac{2}{\pi} \exp(-2\alpha\alpha^*) * \frac{2}{\pi} \exp(-2\alpha\alpha^*) = \frac{1}{\pi} \exp(-\alpha\alpha^*), \quad (5.7)$$

then from (5.2) and (5.6) the last included we find the convolution relations

$$\begin{aligned} Q(\alpha, \alpha^*) &= \frac{2}{\pi} \exp(-2\alpha\alpha^*) * W(\alpha, \alpha^*) = \exp\left(\frac{1}{2} \frac{\partial^2}{\partial \alpha \partial \alpha^*}\right) W(\alpha, \alpha^*) \\ &= \frac{1}{\pi} \exp(-\alpha\alpha^*) * P(\alpha, \alpha^*) = \exp\left(\frac{\partial^2}{\partial \alpha \partial \alpha^*}\right) P(\alpha, \alpha^*), \\ W(\alpha, \alpha^*) &= \frac{2}{\pi} \exp(-2\alpha\alpha^*) * P(\alpha, \alpha^*) = \exp\left(\frac{1}{2} \frac{\partial^2}{\partial \alpha \partial \alpha^*}\right) P(\alpha, \alpha^*). \end{aligned} \quad (5.8)$$

Written in real variables (q, p) these convolution relations are

$$\begin{aligned} Q(q, p) &= \frac{1}{\pi \hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) * W(q, p) = \exp\left\{\frac{\hbar}{4} \left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\right)\right\} W(q, p) \\ &= \frac{1}{2\pi \hbar} \exp\left(-\frac{q^2 + p^2}{2\hbar}\right) * P(q, p) = \exp\left\{\frac{\hbar}{2} \left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\right)\right\} P(q, p), \\ W(q, p) &= \frac{1}{\pi \hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) * P(q, p) = \exp\left\{\frac{\hbar}{4} \left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\right)\right\} P(q, p), \end{aligned} \quad (5.9)$$

where we used the transformations (2.23) and (2.24).

Together with the more explicit representation of the Glauber-Sudarshan quasiprobability (2.28) and collected altogether with the analogous representations of

the other quasiprobabilities one obtains

$$\begin{aligned} P(\alpha, \alpha^*) &= \left\langle \varrho \exp\left(-a^\dagger \frac{\partial}{\partial \alpha^*}\right) \exp\left(-a \frac{\partial}{\partial \alpha}\right) \right\rangle \delta(\alpha, \alpha^*), \\ W(\alpha, \alpha^*) &= \left\langle \varrho \exp\left(-a^\dagger \frac{\partial}{\partial \alpha^*} - a \frac{\partial}{\partial \alpha}\right) \right\rangle \delta(\alpha, \alpha^*), \\ Q(\alpha, \alpha^*) &= \left\langle \varrho \exp\left(-a \frac{\partial}{\partial \alpha}\right) \exp\left(-a^\dagger \frac{\partial}{\partial \alpha^*}\right) \right\rangle \delta(\alpha, \alpha^*). \end{aligned} \quad (5.10)$$

In these representations act different operators onto the same function which is the two-dimensional delta function but the operators themselves are ordered in different ways. We do not write down here the representations in real variables (q, p) .

Equations (5.8) and (5.10) are suited for the further derivation of a large amount of other representations and expressions for special density operators ϱ . We would like to do it but cannot for different reason, mainly for the necessary place and time and for the further aim of this work. We make only a few remarks. One could make the linear interpolation between the different quasiprobabilities by a parameter on the level of their Fourier transforms and their inversion. It is necessary to remind that the other quasiprobabilities derived from the Wigner quasiprobability do not possess an own meaning independently of the calculation of expectation values $\langle \varrho A \rangle$ of operators A and changes of the quasiprobabilities have as necessary consequence to change the functions $A(\alpha, \alpha^*)$ with which they has to be made it in case that these are not functions of a or $a^\dagger$ alone (in other case it is simpler). Sometimes this seems to be forgotten. These formulae are also suited as basis to unroll the whole problems of different operator ordering of the annihilation and creation operators in functions of them and to derive the corresponding formulae, in particular, for normal, symmetrical and anti-normal ordering for which we used in our earlier papers the symbols $\mathcal{N}\{\dots\}, \mathcal{S}\{\dots\}, \mathcal{A}\{\dots\}$ which are not (linear) operators, for example, due to

$\mathcal{N}\{a^{\dagger k} a^l\} = \mathcal{N}\{a^l a^{\dagger k}\} \equiv a^{\dagger k} a^l$. We want to give in addition only the normal ordering of an arbitrary linear operator A . A good representation of many of the mentioned problems one finds in the monograph of Peřina [19] (see also monograph of Peřinová, Lukš and Peřina [20]).

The expansion of an arbitrary linear operator in powers $a^{\dagger k} a^l$ in dual way possesses the form

$$A = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle A a_{k,l} \rangle a^{\dagger k} a^l = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle A a^{\dagger k} a^l \rangle a_{k,l}, \quad (5.11)$$

with the operators $a_{k,l}$ defined by

$$a_{k,l} \equiv \sum_{j=0}^{\min\{k,l\}} \frac{(-1)^j |l-j\rangle \langle k-j|}{j! \sqrt{(k-j)!(l-j)!}}. \quad (5.12)$$

It can be also used as expansion of an operator A in the basis operators $a_{k,l}$ with coefficients determined by means of $a^{\dagger k} a^l$. A proof is given in Appendix D.

6. Integration of Wigner Quasiprobability over the Radius

If one integrates the Wigner quasiprobability over the radius or what is the same over the amplitude of the variables one obtains a function of the phase φ , for example, in the representation by complex variables (α, α^*)

$$\int_0^{+\infty} d|\alpha| |\alpha| W(|\alpha| e^{i\varphi}, |\alpha| e^{-i\varphi}) \equiv W(\varphi), \quad \int_0^{2\pi} d\varphi W(\varphi) = 1. \quad (6.1)$$

It is generally a 2π -periodic function and therefore the normalization can be written over an arbitrary of the 2π -periodic ranges (λ arbitrary real number)

$$W(\varphi) = W(\varphi \pm 2k\pi), (k = 0, 1, 2, \dots), \quad \int_{-\pi+\lambda}^{+\pi+\lambda} d\varphi W(\varphi) = 1. \quad (6.2)$$

In the following we call $W(\varphi)$ the Wigner phase distribution. The normalization of the Wigner phase distribution $W(\varphi)$ follows automatically from the normalization of the Wigner quasiprobability. The integration over the radius can be also made in the representation of the Wigner quasiprobability by real variable (q, p) if one introduces for them polar coordinates (see next Section). A representation of many problems of the phase in (classical and quantum) optics with many citations is given in the the monograph of Peřinová, Lukš and Peřina [20].

Let us say a few words about the mathematical terminology in the theory of probability (e.g., Prokhorov and Rozanov [21]). The Wigner distribution $W(\varphi)$ corresponds then to a probability density often denoted by $P(\xi)$ ($-\infty < \xi < +\infty$) and $W(\varphi)d\varphi$ corresponding to $P(\xi)d\xi$ and is the probability to find $P(\xi)$ in an interval of length $d\xi$ around ξ in $P(\xi)$. Mostly, there is additionally introduced a monotonically increasing function $F(x)$ which is the integrated probability density $P(\xi)$ and which would correspond in our case to an integrated function $W(\varphi)$ denoting it here for a short moment by $\Phi(\chi)$ as follows

$$F(x) = \int_{-\infty}^x d\xi P(\xi), \quad \lim_{x \rightarrow -\infty} F(x) = 0, \quad \lim_{x \rightarrow +\infty} F(x) = 1 \Leftrightarrow \\ \Phi(\chi) = \int_{\varphi_0}^{\chi+\varphi_0} d\varphi W(\varphi), \quad \Phi(0) = 0, \quad \Phi(2\pi) = 1. \quad (6.3)$$

It is called there the distribution function. Our function $\Phi(\chi)$ would depend in addition from the chosen initial point φ_0 of the integration and in case that $W(\varphi)$ possesses intervals with negative probabilities it would not be monotonically increasing. The classical probability theory does not consider periodical probabilities which necessarily then have to possess also intervals of negative probabilities. Thus our $\Phi(\varphi)$ is not fully analogous to the function $F(x)$ and we do no more mention or use it. Furthermore, the letter P as notation for probability would be in conflict with other use of it in present paper.

In Appendix A we derived representations for the Wigner quasiprobability if the state is given by the matrix elements $\langle m | \varrho | n \rangle$ of the density operator ϱ by the number states. Starting from the representation (A.7) of the Wigner quasiprobability one can make the following first steps of the integration over the radius or amplitude

$$\begin{aligned}
W(\varphi) &= \int_0^{+\infty} d|\alpha| |\alpha| W(|\alpha| e^{i\varphi}, |\alpha| e^{-i\varphi}) \\
&= \int_0^{+\infty} d|\alpha| |\alpha| \frac{2}{\pi} \exp(-2|\alpha|^2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m|\varrho|n\rangle}{\sqrt{m!n!}} L_{n,m}(2|\alpha| e^{i\varphi}, 2|\alpha| e^{-i\varphi}) \\
&= \frac{2}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m|\varrho|n\rangle}{\sqrt{m!n!}} \int_0^{+\infty} d|\alpha| |\alpha| \exp(-2|\alpha|^2) e^{i(n-m)\varphi} \\
&\quad \cdot \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} (2|\alpha|)^{m-j} (2|\alpha|)^{n-j},
\end{aligned} \tag{6.4}$$

where $L_{m,n}(z, z^*)$ are the Laguerre 2D polynomials defined in (A.6) and given there together with their explicit representation. The definite integral which is now still to solve is from a known type and is to find in tables of integrals. We may write its result in the form

$$2^{m+n-2j} \int_0^{+\infty} d|\alpha| \exp(-2|\alpha|) |\alpha|^{m+n+1-2j} = \frac{1}{4} 2^{\frac{m+n}{2}-j} \left(\frac{m+n}{2} - j \right)! \tag{6.5}$$

Inserting this into (6.4) we found

$$\begin{aligned}
W(\varphi) &= \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m|\varrho|n\rangle}{\sqrt{m!n!}} e^{i(n-m)\varphi} \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} 2^{\frac{m+n}{2}-j} \left(\frac{m+n}{2} - j \right)! \\
&\equiv \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m|\varrho|n\rangle e^{i(n-m)\varphi} c_{m,n},
\end{aligned} \tag{6.6}$$

where we introduced the following abbreviation for real-valued numbers $c_{m,n}$

$$c_{m,n} \equiv \sqrt{m!n!} \sum_{j=0}^{\{m,n\}} \frac{(-1)^j 2^{\frac{m+n}{2}-j} \left(\frac{m+n}{2} - j \right)!}{j!(m-j)!(n-j)!} = c_{n,m}, \tag{6.7}$$

which obviously are symmetric in the indices (m, n) . Equation (6.6) is the complex representation of the one-dimensional Fourier series of the 2π -periodic function $W(\varphi)$ which with $k = n - m$ takes on the form

$$W(\varphi) = \sum_{k=-\infty}^{+\infty} e^{ik\varphi} \underbrace{\frac{1}{2\pi} \sum_{n \geq (0,k)} \langle n-k|\varrho|n\rangle c_{n-k,n}}_{\equiv a_k}, \quad ((j, k) \equiv \max(j, k)), \tag{6.8}$$

with each Fourier coefficients a_k as an infinite sum over matrix elements $\langle n-k|\varrho|n\rangle$ multiplied by coefficients.

At first we check the normalization of $W(\varphi)$ using $\int_0^{2\pi} d\varphi e^{i(n-m)\varphi} = 2\pi \delta_{m,n}$ and the binomial formula and find it correct according to

$$\begin{aligned}
\int_0^{2\pi} d\varphi W(\varphi) &= \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m|\varrho|n\rangle 2\pi \delta_{m,n} c_{m,n} = \sum_{n=0}^{\infty} \langle n|\varrho|n\rangle c_{n,n} \\
&= \sum_{n=0}^{\infty} \langle n|\varrho|n\rangle \sum_{j=0}^n \frac{(-1)^j 2^{n-j} n!}{j!(n-j)!} = \sum_{n=0}^{\infty} \langle n|\varrho|n\rangle = 1.
\end{aligned} \tag{6.9}$$

Now we devote us to the calculation of the coefficients $c_{m,n}$. They can be expressed by the Jacobi polynomials $P_n^{(\alpha, \beta)}(z)$ (e.g., Szegő [22] and Bateman and Erdélyi [23]) which possess the following two basic explicit expressions and are

programmed in “Mathematica”

$$\begin{aligned} P_n^{(\alpha, \beta)}(z) &= \frac{(n+\alpha)!}{(n+\alpha+\beta)!} \sum_{j=0}^n \frac{(2n+\alpha+\beta-j)!}{j!(n-j)!(n+\alpha-j)!} \left(\frac{z-1}{2}\right)^{n-j} \\ &= \left(\frac{z+1}{2}\right)^n \sum_{j=0}^n \frac{(n+\alpha)!(n+\beta)!}{j!(n-j)!(n+\alpha-j)!(j+\beta)!} \left(\frac{z-1}{z+1}\right)^{n-j}. \end{aligned} \quad (6.10)$$

We consider the sum part of $c_{m,n}$ which may be presented in following ways

$$\begin{aligned} \sum_{j=0}^{\{m,n\}} \frac{(-1)^j 2^{\frac{m+n}{2}-j} \left(\frac{m+n}{2}-j\right)!}{j!(m-j)!(n-j)!} &= \frac{\frac{m-n}{2}\pi}{\sin\left(\frac{m-n}{2}\pi\right)} \sum_{j=0}^{\{m,n\}} \frac{2^{\frac{m+n}{2}-j}}{j!(m-j)!(n-j)! \left(j-\frac{m+n}{2}\right)!} \\ &= \frac{2^{\frac{m-n}{2}} \left(\frac{m-n}{2}\right)!}{m!} P_n^{\left(\frac{m+n}{2}, m-n\right)}(3) \\ &= \frac{2^{\frac{m+n}{2}} \left(\frac{m-n}{2}\right)!}{m!} P_n^{\left(\frac{m+n}{2}, \frac{m+n}{2}-1\right)}(0), \end{aligned} \quad (6.11)$$

using the Jacobi polynomials and were obtained already in [13] for such sums.

Together with the factors $\sqrt{m!n!}$ in (6.7) this provides the following table of numerical values of first coefficients $c_{m,n}$ (see **Table 1**).

For comparison let us give also the corresponding results for the Husimi-Kano quasiprobability $Q(\alpha, \alpha^*)$ which are much simpler to obtain than for the Wigner quasiprobability $W(\alpha, \alpha^*)$. This quasiprobability is defined by (see 5.1)

$$Q(\alpha, \alpha^*) = \frac{\langle \alpha | \varrho | \alpha \rangle}{\pi}, \quad \int \frac{i}{2} d\alpha \wedge d\alpha^* Q(\alpha, \alpha^*) = 1. \quad (6.12)$$

In polar coordinates together with the number representation of the density operator this leads to

$$Q(|\alpha|e^{i\varphi}, |\alpha|e^{-i\varphi}) = \frac{1}{\pi} \exp(-|\alpha|^2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m | \varrho | n \rangle}{\sqrt{m!n!}} e^{i(n-m)\varphi} |\alpha|^{m+n}. \quad (6.13)$$

The integration over the amplitude with measure $d|\alpha|d\alpha$ in polar coordinates provides

$$\begin{aligned} Q(\varphi) &\equiv \int_0^{+\infty} d|\alpha| |\alpha| Q(|\alpha|e^{i\varphi}, |\alpha|e^{-i\varphi}) \\ &= \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m | \varrho | n \rangle}{\sqrt{m!n!}} e^{i(n-m)\varphi} c_{m,n}(1) \end{aligned} \quad (6.14)$$

with the following abbreviation $c_{m,n}(1)$ in analogy to (6.7)

$$c_{m,n}(1) \equiv \frac{\left(\frac{m+n}{2}\right)!}{\sqrt{m!n!}} = c_{n,m}(1) \leq 1. \quad (6.15)$$

The argument (1) in $c_{m,n}(1)$ results from the ordering parameter.

In most cases the given number of initial coefficients in these two tables are not sufficient to get good approximations for $W(\varphi)$ or for $Q(\varphi)$, for example, for coherent states $|\beta\rangle$ with $|\beta| \gg 1$. In such cases remains the exact calculations of these functions that is often difficult and troublesome but which is made in present article for two categories of states in the next sections. Furthermore, we remind that $Q(\varphi)$ does not possess a meaning for itself since it has to be seen in every case in connection with the ordering of the operators for which one wants to calculate expectation values. For the Wigner quasiprobability $W(q, p)$ this is the symmetrical ordering. The numerical results for the first coefficients $c_{m,n}$ in (6.7) and $c_{m,n}(1)$ in (6.15) are represented in **Table 1** and **Table 2**.

To the end of this Section we want to make some remarks to the booklet [24] of H. Paul where the integration over the radius of the Glauber-Sudarshan quasiprobability $P(q, p)$ (not the Wigner quasiprobability although denoted there with letter W from German “Wahrscheinlichkeit”) is dealt with to get a phase function. It possesses shortages. The starting point being the Glauber-Sudarshan quasiprobability are singular functions for all pure states. The most harmless singularities of them are the delta functions for coherent states. Likely, he (P.) considers the Glauber-Sudarshan quasiprobability as nearest to the classical probability function that is right and choose it as best starting point for the calculations. Then he makes an amplification of the states but what is amplified is practically not exactly to describe generally due to the singularities and we also cannot do it for the same reason but maybe this is not necessary at all. From the amplified state there is apparently no way to calculate back to the state from which one started since the entropy should become greater in the process of the amplification. A sketch of the experimental arrangement for the amplification is not given. However, in the amplified state some spatial or time parameter from device does not play a role. As limiting case for the amplified states he gives as approximation a time-independent result for the phase reconstruction which exactly corresponds to the first diagonals above and below to the main diagonal in **Table 2** with the numbers 1 in this diagonal which last corresponds to the uniform phase distribution and is absent. This is highly doubtful and in addition an incomplete result since it belongs to the Husimi-Kano quasiprobability $Q(q, p)$ but his calculation remains as one may see at the Glauber-Sudarshan quasiprobability $P(q, p)$. These calculations are not given in detail and therefore are difficult to reconstruct. The mentioned diagonals in **Table 2** and below the main diagonal correspond to the values $k = +1$ and $k = -1$ in Formula (6.8). For the function $W(\varphi)$ and in the formalism of Susskind Glogower [25] and Carruthers and Nieto [26] (the last republished by Editor Man’ko in [27]) this corresponds to the operators E_- and E_+ . The coefficients above and below the main diagonal in **Table 2** are to find also in **Table 1** and below the main diagonal but there in jumps of two units. The main diagonals in **Table 1** and **Table 2** with values 1 which as said corresponds to a uniform phase distribution remains to be not taken into account. Furthermore, in [24] it is not made any effort to reconstruct the phase properties of

the state before the amplification that apparently was an aim of the amplification. Shortly said, we cannot find any trace or parameter in his result related to quasiprobability $Q(q, p)$ which remained from his amplification of the primary state that alone is already strange. A full calculation of a radius-integrated quasiprobability for a special case as we discuss it in next Section for the coherent states is also absent in cited work.

Table 1. Coefficients $c_{m,n}$ in $W(\varphi)$ according to Formula (6.7).

$m \backslash n$	0	1	2	3	4	5	6	7	8	9
0	1	1.25331	1.41421	1.53499	1.63299	1.71617	1.78885	1.85368	1.91237	1.96612
1	1.25331	1	0.88623	0.81650	0.76750	0.73030	0.70062	0.67612	0.65537	0.63746
2	1.41421	0.88623	1	1.08540	1.15470	1.21352	1.26491	1.31075	1.35225	1.39026
3	1.53499	0.81650	1.08540	1	0.93999	0.89443	0.85809	0.82807	0.80267	0.78072
4	1.63299	0.76750	1.15470	0.93999	1	1.05094	1.09545	1.13514	1.17108	1.20400
5	1.71617	0.73030	1.21352	0.89443	1.05094	1	0.95937	0.92582	0.89741	0.87287
6	1.78885	0.70062	1.26491	0.85809	1.09545	0.95937	1	1.03624	1.06904	1.09910
7	1.85368	0.67612	1.31075	0.82807	1.13514	0.92582	1.03624	1	0.96931	0.94281
8	1.91237	0.65537	1.35225	0.80267	1.17108	0.89741	1.06904	0.96931	1	1.02811
9	1.96612	0.63746	1.39026	0.78072	1.20400	0.87287	1.09910	0.94281	1.02811	1

Table 2. Coefficients $c_{m,n}(1)$ in $Q(\varphi)$ according to Formula (6.15).

$m \backslash n$	0	1	2	3	4	5	6	7	8	9
0	1	0.88623	0.70711	0.54270	0.40825	0.30338	0.22361	0.16384	0.11952	0.08689
1	0.88623	1	0.93999	0.81650	0.67838	0.54772	0.43349	0.33806	0.26067	0.19920
2	0.70711	0.93999	1	0.95937	0.86603	0.75082	0.63246	0.52135	0.42258	0.33793
3	0.54270	0.81650	0.95937	1	0.96931	0.89443	0.79637	0.69007	0.58531	0.48795
4	0.40825	0.67838	0.86603	0.96931	1	0.97535	0.91287	0.82775	0.73193	0.63408
5	0.30338	0.54772	0.75082	0.89443	0.97535	1	0.97941	0.92582	0.85071	0.76376
6	0.22361	0.43349	0.63246	0.79637	0.91287	0.97941	1	0.98232	0.93541	0.86825
7	0.16384	0.33806	0.52135	0.69007	0.82775	0.92582	0.98232	1	0.98451	0.94281
8	0.11952	0.26067	0.42258	0.58531	0.73193	0.85071	0.93541	0.98451	1	0.98621
9	0.08689	0.19920	0.33793	0.48795	0.63408	0.76376	0.86825	0.94281	0.98621	1

In a new trial to understand the discussion of Paul I can definitely say that the result of his amplification is a convolution of the Glauber-Sudarshan quasiprobability $P(\alpha, \alpha^*)$ of the primary state with the normalized Gaussian function $\frac{1}{\pi} \exp(-\alpha \alpha^*)$ (namely the jump from $P(\alpha, \alpha^*)$ to $Q(\alpha, \alpha^*)$). Since he remains with his calculations within $P(\alpha, \alpha^*)$ it should be interpreted as the Glauber-Sudarshan quasiprobability $P(\alpha, \alpha^*)$ of the amplified state although his obtained coefficients belong to the Husimi-Kano quasiprobability $Q(q, p)$ (see **Table 2**). This amplified state is then a mixed state with higher entropy than that of the primary state and thus no more a pure state. Nevertheless, a reconstruction of the primary state from the function $P(\alpha, \alpha^*)$ of the amplified state seems to be possible, in principle, by the inverse convolution with the function $\frac{1}{\pi} \exp(+\alpha \alpha^*)$. However, this should be very difficult and troublesome due to convergence problems and is only possible by means of the theory of Generalized functions. As parameter of the amplifier device or the time of its action in cited booklet is only the constant $\hbar$ of the convolution (in corresponding real variables (q, p)) to see. All this projected to the Wigner quasiprobability $W(\alpha, \alpha^*)$ of the primary state with the same convolution due to its amplification leads to a large smoothing of $W(\alpha, \alpha^*)$ greater than that in the usual transition from $W(\alpha, \alpha^*)$ to $Q(\alpha, \alpha^*)$. The most is difficult to accept for us.

7. Wigner Phase Distribution for Coherent States and Definition of the Phase Uncertainty

The Wigner quasiprobabilities $W(q, p)$ and $W(\alpha, \alpha^*)$ for coherent states

$$|\beta\rangle = \left| \frac{q_0 + ip_0}{\sqrt{2\hbar}} \right\rangle \text{ are}$$

$$W(q, p) = \frac{1}{\pi\hbar} \exp\left(-\frac{(q-q_0)^2 + (p-p_0)^2}{\hbar}\right),$$

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \exp(-2(\alpha - \beta)(\alpha^* - \beta^*)). \quad (7.1)$$

They are known (e.g., [11]) and can be calculated in different ways from the formulae in Sections 2 - 4.

We calculate now the integration over the radius and make it from $W(q, p)$. Expressed by the real parameters (q_0, p_0) or $s = \sqrt{q_0^2 + p_0^2}$ and $|\beta|$ and angle χ it follows

$$\beta = \frac{q_0 + ip_0}{\sqrt{2\hbar}} = |\beta| e^{i\chi}, \quad \beta^* = \frac{q_0 - ip_0}{\sqrt{2\hbar}} = |\beta| e^{-i\chi}, \quad s^2 = q_0^2 + p_0^2 = 2\hbar|\beta|^2$$

$$q_0 = \sqrt{\frac{\hbar}{2}}(\beta + \beta^*) \equiv s \cos(\chi), \quad p_0 = -i\sqrt{\frac{\hbar}{2}}(\beta - \beta^*) \equiv s \sin(\chi). \quad (7.2)$$

The integration over the radius $r = \sqrt{q^2 + p^2}$ provides

$$\begin{aligned}
W(\varphi) &\equiv \int_0^{+\infty} dr r W(r \cos(\varphi), r \sin(\varphi)) \\
&= \frac{1}{\pi \hbar} \int_0^{+\infty} dr r \exp \left(-\frac{(r \cos(\varphi) - s \cos(\chi))^2 + (r \sin(\varphi) - s \sin(\chi))^2}{\hbar} \right) \\
&= \frac{1}{\pi \hbar} \int_0^{+\infty} dr r \exp \left(-\frac{r^2 - 2rs \cos(\varphi - \chi) + s^2}{\hbar} \right) \\
&= \exp \left(-\frac{s^2 \sin^2(\varphi - \chi)}{\hbar} \right) \frac{1}{\pi \hbar} \int_0^{+\infty} dr r \exp \left(-\frac{(r - s \cos(\varphi - \chi))^2}{\hbar} \right) \\
&= \frac{1}{2\pi} \left\{ \exp \left(-\frac{s^2}{\hbar} \right) + \sqrt{\pi} \frac{s}{\sqrt{\hbar}} \cos(\varphi - \chi) \exp \left(-\frac{s^2 \sin^2(\varphi - \chi)}{\hbar} \right) \right. \\
&\quad \left. \cdot \operatorname{erfc} \left(-\frac{s}{\sqrt{\hbar}} \cos(\varphi - \chi) \right) \right\},
\end{aligned} \tag{7.3}$$

or “ s ” expressed by the parameter $|\beta|$ of the coherent state

$$\begin{aligned}
W(\varphi) &= \frac{1}{2\pi} \left\{ \exp(-2|\beta|^2) + \sqrt{2\pi} |\beta| \cos(\varphi - \chi) \exp(-2|\beta|^2 \sin^2(\varphi - \chi)) \right. \\
&\quad \left. \cdot \operatorname{erfc}(-\sqrt{2} |\beta| \cos(\varphi - \chi)) \right\}.
\end{aligned} \tag{7.4}$$

One finds the integral in (7.3) in tables of integrals and it is not very difficult to calculate it by one-self. The function $\operatorname{erfc}(z)$ is the well-known complementary error function or complementary error integral to $\operatorname{erf}(z)$ in the new form (e.g., in program “Mathematica”; the old form, e.g., in Bateman and Erdélyi [23] is defined without the factor $\frac{2}{\sqrt{\pi}}$ in front of the error integral)

$$\begin{aligned}
\operatorname{erfc}(z) &\equiv \frac{2}{\sqrt{\pi}} \int_z^\infty dt \exp(-t^2) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{z^2}{2}\right) D_{-1}(\sqrt{2}z) = 1 - \operatorname{erf}(z), \\
F(z) &\equiv \exp(-z^2) \int_0^z dt \exp(t^2) = \frac{\sqrt{\pi}}{2} \exp(-z^2) \operatorname{erf}(iz), \\
\operatorname{erfc}(-z) &= 1 - \operatorname{erf}(-z) = 1 + \operatorname{erf}(z), \quad \operatorname{erfi}(z) = -i \operatorname{erf}(iz),
\end{aligned} \tag{7.5}$$

where $D_\nu(z)$ are the functions of the parabolic cylinder. It is also possible to express the error integral by other known functions such as the Dawson integral $F(z)$ or the function $w(z)$ (see Temme [28]).

The radius-integrated Wigner quasiprobability $W(\varphi)$ calculated in (7.3) is a 2π -periodic function which due to normalization of the Wigner quasiprobability $W(q, p)$ is normalized according to (6.1). In considered simple case of one coherent state it depends apart from $|\beta|$ only on the phase difference $\varphi - \chi$. Due to normalization the part of the area of the distribution $W(\varphi)$ above $\frac{1}{2\pi}$ is exactly equal to the part which is now absent under the line $\frac{1}{2\pi}$. Therefore, the line $W(\varphi) = \frac{1}{2\pi}$ is a standard which can be used for the definition of a phase

uncertainty $\Delta\varphi$ by $\theta(x)$ is Heaviside's step function)

$$\Delta\varphi = \int_0^{+2\pi} d\varphi \theta\left(W(\varphi) - \frac{1}{2\pi}\right), \quad \theta(x) \equiv \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \quad (7.6)$$

It measures the sum of interval lengths where the phase distribution is larger than $\frac{1}{2\pi}$ which last may be considered as the uniform phase distribution $W(\varphi) = \frac{1}{2\pi}$ and which is the distribution for all phase-invariant states. These are, for example, the vacuum state $|0\rangle$ and all number states $|n\rangle$. No other line than $\frac{1}{2\pi}$ can be chosen as such standard because in this case there would exist genuine phase distributions $W(\varphi)$ which do not possess a phase uncertainty at all that means $\Delta\varphi = 0$ or such which possess a phase uncertainty of the full period length that means $\Delta\varphi = 2\pi$. The phase uncertainty of the uniform distribution $W(\varphi)$ itself remains undetermined but possesses for coherent states in the limiting case $|\beta| \rightarrow 0$ the phase uncertainty $\Delta\varphi = \pi$. The coherent states $|\beta\rangle$ and the coherent phase states $|\varepsilon\rangle$ possess the same phase uncertainty $\Delta\varphi = \pi$ in this limiting case which last was found in [13] for $\varepsilon \rightarrow 0$.

The radius-integrated Wigner quasiprobability for coherent states is shown in **Figure 1** for different values of the parameter $|\beta|$ of coherent states and with parameter $\chi = 0$. **Figure 2** shows in a spatial graphics the changes of the phase uncertainty $\Delta\varphi$ in dependence on the parameter $|\beta|$ of the coherent state for $0 \leq |\beta| \leq 4.0$. The plane $W(\varphi) = \frac{1}{2\pi}$ (in blue color) show us where the phase distribution is larger or smaller than $\frac{1}{2\pi}$. Additionally, it is drawn the plane $W(\varphi) = 0$ and the Wigner phase distribution for coherent states is fully non-negative the same as was already found for coherent phase states [13]. The ridge $\varphi - \chi = 0$ of the phase distribution $W(\varphi)$ approaches for $|\beta| \geq 1$ quickly to a straight line proportional with increasing parameter to $|\beta|$ of the coherent state. This is caused by the two sum terms in braces in (7.4). The first sum term determines the phase distribution for small $|\beta| \ll 1$ and describes a Gaussian distribution. It goes quickly to zero with increasing $|\beta|$ and for $|\beta| \gg 1$ is only essentially the second sum term in braces in (7.4) which describes a non-Gaussian distribution (see **Figure 1**). Due to $\lim_{x \rightarrow \infty} \operatorname{erfc}(-x) = 2$ its ridge approaches for $|\beta| \gg 1$ to the line $W(0) = \sqrt{\frac{2}{\pi}} |\beta| \approx 0.79788 |\beta|$. As seen from **Figure 2**, in the limiting case $|\beta| \rightarrow 0$ the phase uncertainty $\Delta\varphi$ determined by the width of the phase distribution above $\frac{1}{2\pi}$ goes to π that is the half of the full interval of periodicity 2π of the distribution. More difficult is it to determine this uncertainty in dependence on the parameter $|\beta|$ of the coherent states in analytic form because from setting $W(\varphi) = \frac{1}{2\pi}$ in (7.4) the remaining implicit function $f(|\beta|, \varphi) = 0$ cannot be resolved analytically in the form of a function $\varphi = g(|\beta|)$.

However, it is easy to present its border contour graphically that is made in **Figure 3**.

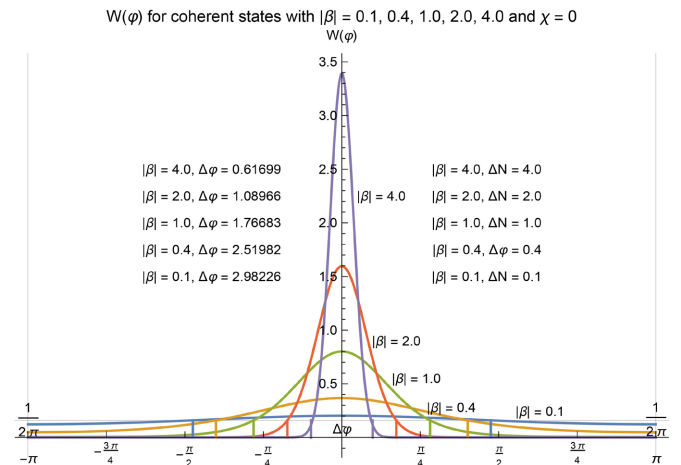


Figure 1. Radius-integrated Wigner quasiprobability for coherent states with different parameters. One may feel approximately that the areas present above and absent below the line $\frac{1}{2\pi}$ of the distributions are equal that due its normalization is exactly the case. One can also see how the phase uncertainty $\Delta\varphi$ may be constructed from the distribution curves by the intersections with the level $\frac{1}{2\pi}$ of the uniform phase distribution.

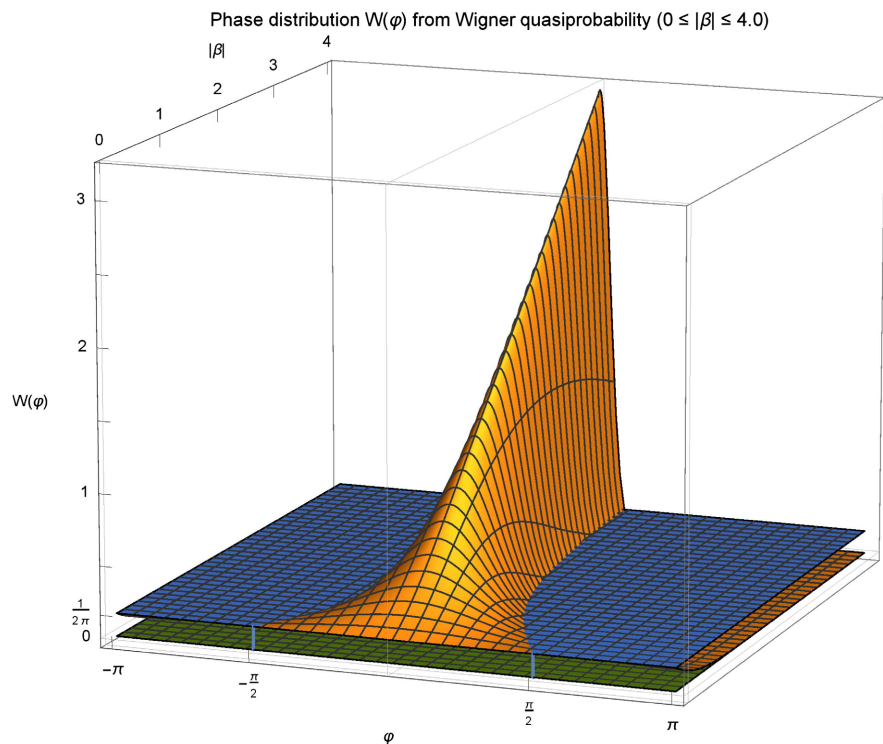


Figure 2. Radius-integrated Wigner quasiprobability for coherent states in the range $0 \leq |\beta| \leq 4.0$. It can be seen that the phase uncertainty $\Delta\varphi$ in the limiting case $|\beta| \rightarrow 0$ goes to the value π .

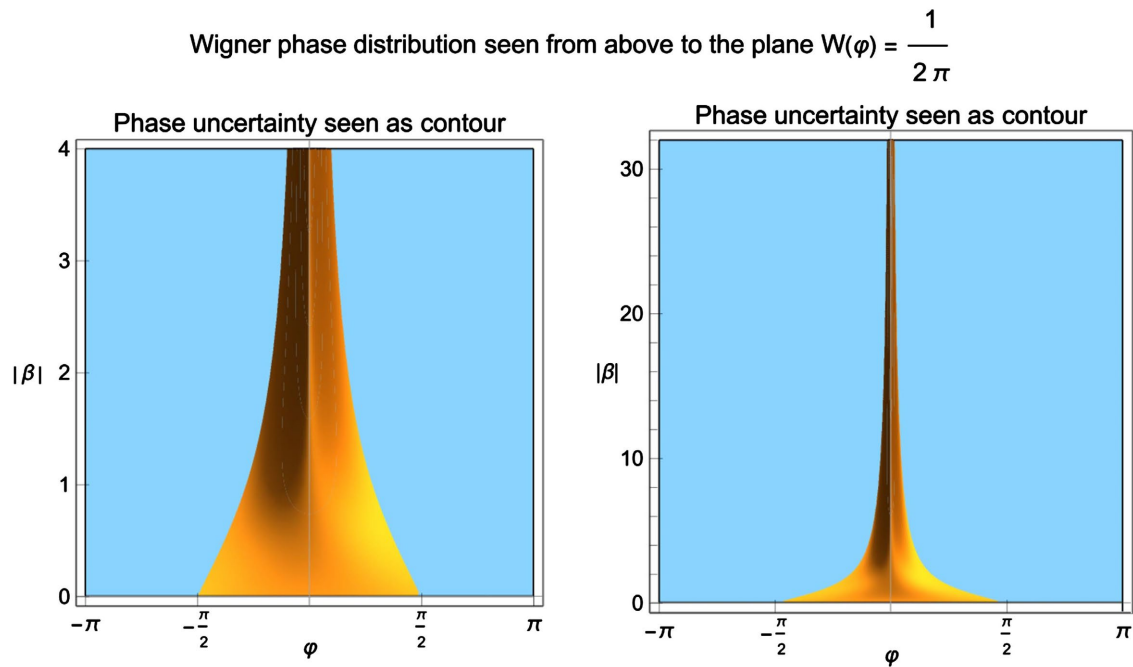


Figure 3. Wigner phase distribution seen from above onto the plane $W(\varphi) = \frac{1}{2\pi}$. It shows the border contour where $W(\varphi)$ is greater than $\frac{1}{2\pi}$ (in orange) to the range where it is smaller than $\frac{1}{2\pi}$ (in blue). The difference between the two partial picture is the range of values $|\beta|$ shown. One may make similar but not identical pictures in their kind also from a viewpoint below the plane $W(\varphi) = \frac{1}{2\pi}$ if before this we cancel the plane $W(\varphi) = 0$. The analytic form of the border is hardly to determine.

As it is well known coherent states $|\beta\rangle$ possess the following expectation values of the number operator N and its uncertainty

$$\bar{N} \equiv \langle N \rangle = |\beta|^2, \quad \overline{N^2} \equiv \langle N^2 \rangle = \bar{N}^2 + \bar{N}, \quad \Delta N \equiv \sqrt{\overline{N^2} - \bar{N}^2} = \sqrt{\bar{N}} = |\beta|. \quad (7.7)$$

It goes to zero with $|\beta| \rightarrow 0$. The width of the Wigner phase distribution, however, does not go to infinity and possesses the value π . This allows doubts that the uncertainty product $\Delta N \Delta \varphi \geq 1$ of Heitler [29] (p. 65, Equation (37')) is correct. We delay a discussion of this problem to the later Section 9.

8. Even and Odd Coherent States

Even and odd coherent states to our knowledge were first introduced by Malkin and Man'ko [12], (p. 35). Whereas the usual coherent states of a harmonic oscillator are now already common knowledge in quantum theory the even and odd coherent states are less trivial but are suited to demonstrate less common properties of states than it is possible for coherent states. It is, however, not clear how they can be realized experimentally.

The even coherent states $|\beta_+\rangle$ and the odd coherent states $|\beta_-\rangle$ can be defined as the following superpositions of a coherent state $|\beta\rangle$ with its antipode

coherent state $|\beta\rangle$ and are expressed by number states

$$\begin{aligned} |\beta_+\rangle &= \frac{|\beta\rangle + |-\beta\rangle}{\sqrt{2(1 + \exp(-2\beta\beta^*))}} = \frac{1}{\sqrt{\text{ch}(\beta\beta^*)}} \sum_{m=0}^{\infty} \frac{\beta^{2m}}{\sqrt{(2m)!}} |2m\rangle, \\ |\beta_-\rangle &= \frac{|\beta\rangle - |-\beta\rangle}{\sqrt{2(1 - \exp(-2\beta\beta^*))}} = \frac{1}{\sqrt{\text{sh}(\beta\beta^*)}} \sum_{m=0}^{\infty} \frac{\beta^{2m+1}}{\sqrt{(2m+1)!}} |2m+1\rangle. \end{aligned} \quad (8.1)$$

They are normalized and mutually orthogonal to each other according to

$$\langle\beta_+|\beta_+\rangle=1, \quad \langle\beta_-|\beta_+\rangle=0, \quad \langle\beta_-|\beta_-\rangle=1, \quad \langle\beta_+|\beta_-\rangle=0. \quad (8.2)$$

The limiting cases for $\beta \rightarrow 0$ are number states $|n\rangle$ to $n=0$ and $n=1$

$$\lim_{\beta \rightarrow 0} |\beta_+\rangle = |0\rangle, \quad \lim_{\beta \rightarrow 0} |\beta_-\rangle = \sqrt{\frac{\beta}{\beta^*}} |1\rangle = \frac{\beta}{|\beta|} |1\rangle, \quad (8.3)$$

that means the number state $|0\rangle$ which is the vacuum state and the number state $|1\rangle$. Both these limiting cases possess a uniform phase distribution $W(\varphi) = \frac{1}{2\pi}$.

The even and odd coherent states are right-hand eigenstates of the squared annihilation operator a^2

$$a^2 |\beta_+\rangle = \beta^2 |\beta_+\rangle, \quad a^2 |\beta_-\rangle = \beta^2 |\beta_-\rangle, \quad (8.4)$$

and are mutually transformed into each other by the operator a

$$a |\beta_+\rangle = \beta \sqrt{\frac{\text{sh}(\beta\beta^*)}{\text{ch}(\beta\beta^*)}} |\beta_-\rangle, \quad a |\beta_-\rangle = \beta \sqrt{\frac{\text{ch}(\beta\beta^*)}{\text{sh}(\beta\beta^*)}} |\beta_+\rangle. \quad (8.5)$$

The Wigner quasiprobability in the representation by complex variables $W(\alpha, \alpha^*)$ for even and odd coherent states possesses the form

$$\begin{aligned} W(\alpha, \alpha^*) &= \frac{1}{\pi(1 \pm \exp(-2\beta\beta^*))} \left\{ \exp(-2(\alpha - \beta)(\alpha^* - \beta^*)) + \exp(-2(\alpha + \beta)(\alpha^* + \beta^*)) \right. \\ &\quad \left. \pm \exp(-2\beta\beta^*) \left(\exp(-2(\alpha + \beta)(\alpha^* - \beta^*)) + \exp(-2(\alpha - \beta)(\alpha^* + \beta^*)) \right) \right\}. \end{aligned} \quad (8.6)$$

We call the sum terms in the first line basic terms and that in second line cross terms. The cross terms are absent for a statistical mixing of coherent states where the formulae are then very analogous to that for a single coherent state. The number of cross term enlarges more quickly than that of basic terms if one considers the superposition of more than two coherent states and their form complicates if one does not have a symmetrical separation to the coordinate origin and not the same amplitude.

Using the substitutions (7.2) and taking into account relation (2.28) between $W(q, p)$ and $W(\alpha, \alpha^*)$ one arrives at the Wigner quasiprobability represented by real variables (q, p)

$$W(q, p) = \frac{\exp\left(-\frac{q^2 + p^2}{\hbar}\right)}{\pi\hbar\left(1 \pm \exp\left(-\frac{q_0^2 + p_0^2}{\hbar}\right)\right)} \left\{ \exp\left(-\frac{q_0^2 + p_0^2}{\hbar}\right) \operatorname{ch}\left(\frac{2(q_0 q + p_0 p)}{\hbar}\right) \right. \\ \left. \pm \cos\left(\frac{2(p_0 q - q_0 p)}{\hbar}\right) \right\}. \quad (8.7)$$

One may check the limiting cases. For $(q_0^2 + p_0^2) \rightarrow 0$ one finds for the even coherent state (sign plus) the Wigner quasiprobability for the vacuum state $|0\rangle$ and for the odd coherent state (sign minus) the Wigner quasiprobability for the number state $|1\rangle$, correspondingly, where in last case one has to make an expansion of the functions up the first-order sum terms. For high values $\sqrt{q_0^2 + p_0^2} \gg 1$ the differences between even and odd coherent states vanish more and more and both cases approaches to each other that can be best seen from Formula (8.7) where the terms with $\pm$ in front in numerator and denominator become smaller with approaching to the limit and there remains only an equal formula for both cases.

Figure 4 shows the Wigner quasiprobability $W(q, p)$ for even and odd coherent states with different parameters (q_0, p_0) generated from Formula (8.7). The amount of separation of the two superposed states possess the value $2|\beta|$ in the parametrization $|\beta|e^{i\chi}$ and with $|\beta| = \sqrt{2\hbar(q_0^2 + p_0^2)}$ (clearly, we set $\hbar = 1$ in the figures). Due to chosen $\chi = 0$ this separation is in direction of the variable q as it is seen from the pictures. The limiting case $(q_0, p_0) \rightarrow (0, 0)$ corresponding to absent separation provides the Wigner quasiprobability for the vacuum state $|0\rangle$ and for the number state $|1\rangle$, respectively, and thus affirms the correctness of the formula used for the calculation. The parameter values $q_0 = 2.0, p_0 = 0$ are in an intermediate range between the two limiting cases. The parameter values $q_0 = 7.5, p_0 = 0$ illustrate already the behavior of the quasiprobabilities to the limiting case of large separation of the two superposed coherent states. Here one has three separated ranges where $W(q, p)$ is essentially different from zero. The range in the center originates from the cross terms with $\langle -\beta | \beta \rangle$ and $\langle \beta | -\beta \rangle$ in the density operator which would not be present if instead the even and the odd coherent state we would consider a statistical mixture with $\varrho = \frac{1}{2}(|\beta\rangle\langle\beta| + |-\beta\rangle\langle-\beta|)$ of two coherent states.

The Wigner phase distribution $W(\varphi)$ is obtained by integration over the radius in (8.7) or the amplitude in (8.6) similar as it was made in Section 6 for the common coherent states. The integrals to solve in this way are a little similar but more general and are known, in principle, but altogether this is complicated and susceptible for errors. We used for both variants from (8.7) and from (8.6) the program “Mathematica” and got in first variant a mixed representation by the error integral and the Dawson integral and in second variant by the error integral alone which we could simplify essentially to the following representation (upper

signs even and lower signs odd coherent states)³.

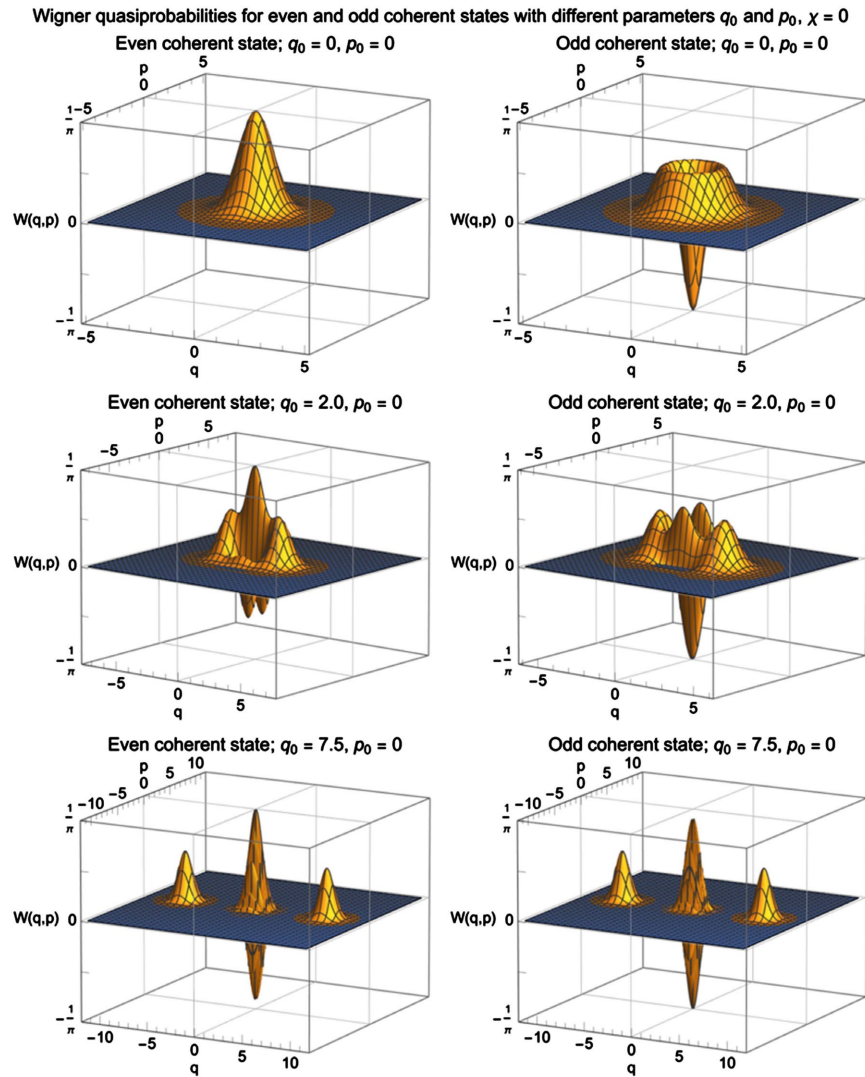


Figure 4. Wigner quasiprobabilities for even and odd coherent states with different values of the separation parameters (q_0, p_0) . The separation parameter is chosen to get separation of the two involved coherent states in direction of the variable q . The distribution peaks at the center with positive and negative values result from the cross terms. To see a higher resolution or to make analogous pictures for other values of the parameters it is recommended to program Equation (8.7) with “Mathematica” and may see this then in every wanted perspective. In particular the peaks in the center are not well resolved here in their complicated structure in case of $q_0 \gg 1$.

³I was at first confused that the first term $\pm \frac{1}{2\pi}$ in my simplification possesses different signs where only the positive sign $\frac{1}{2\pi}$ describes a uniform phase distribution but the pictures which I made with this formula show that for both cases are obtained reasonable ones. A good representation with separation of only one term with positive sign $\frac{1}{2\pi}$ for the separated uniform distribution and including all further terms in the remaining parts I could not find.

$$W(\varphi) = \frac{1}{2\pi} \left\{ \pm 1 + \frac{\sqrt{2\pi} |\beta| \exp(-2|\beta|^2 \sin^2(\varphi - \chi))}{(1 \pm \exp(-2|\beta|^2))} \cdot \left(\cos(\varphi - \chi) \operatorname{erf}(\sqrt{2} |\beta| \cos(\varphi - \chi)) \pm i \sin(\varphi - \chi) \operatorname{erf}(i\sqrt{2} |\beta| \sin(\varphi - \chi)) \right) \right\}. \quad (8.8)$$

Figure 5 shows the phase distribution $W(\varphi)$ in a basic interval of length 2π for different values of the parameter $|\beta|$ and with constant $\chi = 0$. The maxima and minima are in both cases greater for the even coherent states in comparison to the odd coherent states but approaches to each other for high values $|\beta|$. It was fully unexpected that the distributions for the even coherent states possess negative values $W(\varphi) < 0$ whereas for odd coherent states they are positive in all cases. Apparently, the negative values of the distribution for even coherent states remain up to arbitrary high values of $|\beta|$ though they become then more and more small in their amplitude. The maximum of this negativity was found for the parameter nearby $|\beta| = 1.25$. Along the possible presence of negative parts in the Wigner phase distribution $W(\varphi)$ it is worth to consider the even and odd coherent states because the common coherent states and the coherent phase states [13] do not show this possibility and though it is not clear how one may realize these states experimentally.

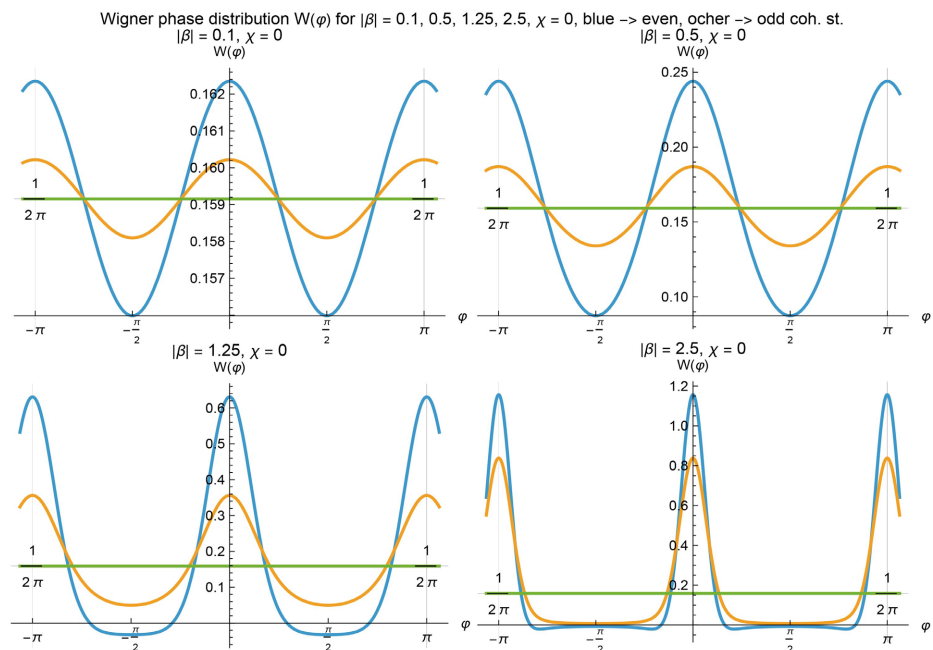


Figure 5. Wigner phase distributions $W(\varphi)$ for even (blue) and odd (ocher) coherent states. The parameters are $|\beta| = 0.1, 0.5, 1.25, 2.5$, $\chi = 0$ and $2|\beta|$ is the separation of the two involved coherent states. The green line is the homogeneous distribution $W(\varphi) = \frac{1}{2\pi}$. The even coherent states only possess ranges of negativity of the distribution with the largest minimum for $|\beta| \approx 1.25$. Pay attention that in the two upper figures the abscissa is not at zero level!

Figure 6 shows the Wigner phase distribution in dependence on the (separation) parameter $|\beta|$ for even coherent states and **Figure 7** the same for odd coherent states. On first view it seems that the distributions are mainly concentrated on three ranges but in reality these are only two since with choosing $\chi = 0$ the second range falls onto the borders $\varphi = \pm\pi$ of the basic interval of periodicity of length 2π of every distribution $W(\varphi)$. This would change if one chooses, e.g., $\chi = \pm\frac{\pi}{2}$. The even and odd coherent states $|\beta_{\pm}\rangle$ remain invariant against a rotation about an angle $\chi = \pi$ in the complex plane of values $\beta = |\beta|e^{i\chi}$ up to phase factors and the Wigner quasiprobability remains exactly invariant with it. This causes in considered very special kind of states an exact periodicity of π in the distribution $W(\varphi)$. It is also seen in these pictures that in the limiting case $|\beta| \rightarrow 0$ the total length of intervals of $|\beta|$ where $W(\varphi) > \frac{1}{2\pi}$ goes to π each separate interval of both to $\frac{\pi}{2}$. The negativity of $W(\varphi)$ in the distribution for even coherent states are not to see in **Figure 6** and it is difficult to find a view-point in that kind of pictures for this purpose but **Figure 5** demonstrates this sufficiently.

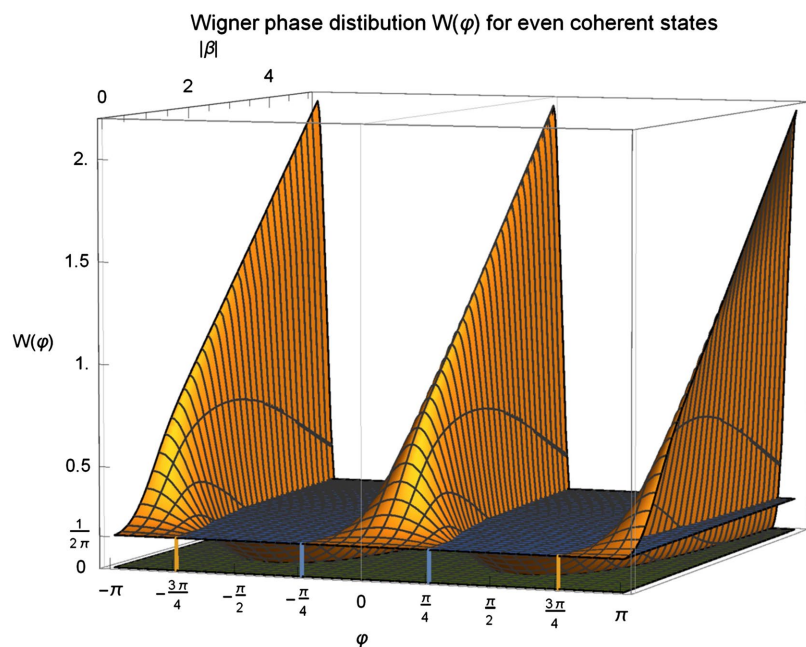


Figure 6. Wigner phase distribution for even coherent states with $0 \leq |\beta| \leq 4$. There are drawn two planes, the plane at $\frac{1}{2\pi}$ (in blue) and the plane at 0 (in green but looks more like brown). In principle, there are only two ranges where the distribution is essentially above $\frac{1}{2\pi}$ but due to additional symmetry $\beta \leftrightarrow -\beta$ and the choice $\chi = 0$ the second range falls onto the boundary of the chosen interval $-\pi \leq \varphi \leq +\pi$ of length 2π and is cut up. It was also difficult to find a perspective which could show the negativity of the distributions in small ranges without great disadvantage for the whole picture.

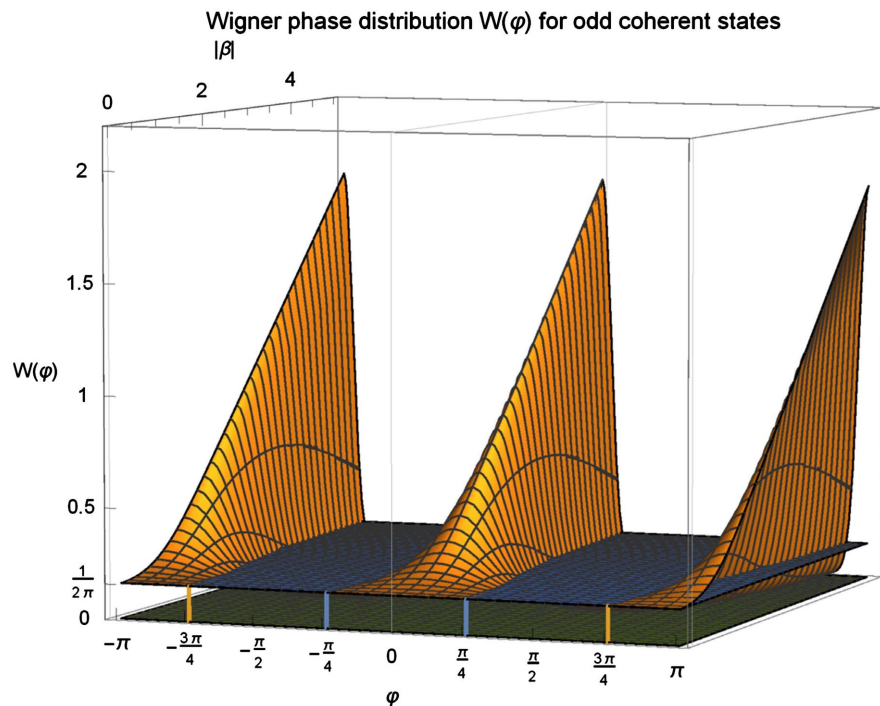


Figure 7. Wigner distribution for odd coherent states with $0 \leq |\beta| \leq 4$. It can be said almost the same as to **Figure 6** but the phase distributions for odd coherent states does not show negativity in comparison of both pictures.

The case of even and odd coherent states $|\beta_{\pm}\rangle$ is convenient to discuss less known properties of the Wigner quasiprobability than for the common coherent states because they are described only by one continuous complex parameter which is the separation parameter 2β of the two involved coherent states in comparison to the only parameter β for the common coherent states.

9. About the Number-Phase Uncertainty Relation of Heitler

The number-phase uncertainty relation $\Delta N \Delta \Phi \geq 1$ of Heitler [29] (p. 65, Equation (37'))⁴ cannot be generally true because the number uncertainty of number states $|n\rangle$ ($n = 0, 1, 2, \dots$) is zero and the phase uncertainty for phase distributions which are 2π -periodic cannot become infinite in these cases and I am not the first who found this out (see [26] temporarily overlooked by me). Heitler writes that the phase is then entirely undetermined and, obviously, he identified this indeterminacy with infinity in a way that the uncertainty relation is satisfied then as limiting case. However, it is now evident that it has to be finite in all cases and in the examples for categories of states which I considered up to now and where such limiting case is possible at all (coherent states, coherent phase states and even and odd coherent states) I found that it is exactly equal to value π , the half of the

⁴I denote here the phase uncertainty by $\Delta \Phi$ instead of $\Delta \varphi$ in sections before because my definition (7.6) is essentially different from that which are mostly meant when Heitler's uncertainty relation is discussed in literature.

interval of periodicity. How can this be when this uncertainty relation is generally derived for Hermitean operators A and B which satisfy the commutation relation $\Delta A \Delta B = cI$ with c a (real) number and I the identity operator acting in considered Hilbert space of states. Then it should be analogous to the Heisenberg uncertainty relations for the Hermitean operators Q and P as position and momentum (but in quantum optics for fictitious inner canonical conjugate operators combined with the annihilation and creation operators). This is only possible if A and B are not perfect Hermitean operators. It seems that Heitler considers his $e^{i\Phi}$ as a unitary operator and therefore Φ as a Hermitean operator.

The status of the operator $e^{i\Phi}$ became clear later than the first appearance of Heitler's book (1954) from the paper of Susskind and Glogower on (1964) introducing operators C and S for genuinely Hermitean operators corresponding to $\cos(\varphi)$ and $\sin(\varphi)$ and continued in the article of Carruthers and Nieto [26] (or [27] for last) who also show as mentioned that the uncertainty relation of Heitler is not true (Equation (5.13) and discussion around it). Their argument is that number and phase do not form a pair of operators with the ordinary commutation relations for the derivation of uncertainty relations. Essentially and inseparable from this seems to be that the operators which play a role in Heisenberg's and Heitler's uncertainty relations possess different spectra of eigenvalues, both continuous in first case and one discrete and one continuous in second case. One may add to this that already in classical mechanics the transformation from (q, p) to action-angle variables (j, φ) possesses a defect as a canonical transformation in the coordinate origin where the angle is undefined and has to be excluded from the canonical transformation that was found long ago. The unsolvable problem to find a perfect Hermitean phase operator Φ which satisfies the commutation relation $[N, \Phi] = iI$ is surely connected with this defect. The action operator though a perfect Hermitean operator possess a discrete spectrum of eigenvalues to ortho-normalizable eigenfunctions of the harmonic oscillator in contrast to the operators in the Heisenberg uncertainty relations with continuous spectra. The phase operator on the other side has to possess a continuous 2π -periodic spectrum of eigenvalues to non-normalizable eigenstates. According to Equations (6.6) - (6.8) the operator $\Phi(\varphi)$ which determines the Wigner phase distribution $W(\varphi)$ is

$$\Phi(\varphi) = \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{i(n-m)\varphi} c_{m,n} |n\rangle \langle m| = \Phi^\dagger(\varphi) \Rightarrow W(\varphi) = \langle \varphi | \Phi(\varphi) \rangle. \quad (9.1)$$

Though it is a Hermitean operator, however, the commutation relation with the number operator

$$[N, \Phi(\varphi)] = \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{i(n-m)\varphi} c_{m,n} (n-m) |n\rangle \langle m| = -i \frac{\partial}{\partial \varphi} \Phi(\varphi), \quad (9.2)$$

does not lead to an operator proportional to I and, moreover, depends on φ . Thus the different requirements to the operators in the Heisenberg and in the Heitler uncertainty relations cannot lead to full analogies that can be formulated in very different but equivalent ways.

We mention here that a unitary phase operator $e^{i\Phi}$ can be constructed in a finite- (s) -dimensional Hilbert space which is practically closed to a ring where after the s -th dimension follow again the 1-st dimension (Pegg and Barnett [30]; see also [20] with further citations). For an analogy it should be necessary here to determine what are the analogues to the states in the infinite-dimensional space, for example, the analogues for the coherent states.

10. Transition from Wigner Quasiprobability to Classical Distribution Functions by $\hbar \rightarrow 0$

Whereas the development of quantum theory was an adventure in unknown territory the transition back from quantum theory to classical theory as the limiting case $\hbar \rightarrow 0$ may appear as an easy problem. However, this is not a trivial problem in each case and we want to discuss this on hand of the Wigner quasiprobability $W(q, p)$ as a distribution over the phase space. We felt confronted with this problem, in particular, when we looked to the Wigner quasiprobability for even and odd coherent states (8.7). Furthermore, there should be a substantial difference making the transition $\hbar \rightarrow 0$ for pure states and for mixed states with density operators ϱ which are not representable as a product $|\psi\rangle\langle\psi|$ where $|\psi\rangle$ are pure states in their abstract form. We remark that for the transition from quantum to classical mechanics the Wigner quasiprobability can be used only in its representation by real variables (q, p) since in its representation $W(\alpha, \alpha^*)$ by complex variables (α, α^*) the constant $\hbar$ is adsorbed in the definition of these variables (see (2.23)).

Before we apply an approach to the two-dimensional phase space to which we restrict us we explain it for the one-dimensional case of a function $f(x)$ with Fourier transform $\tilde{f}(y)$ according to

$$\tilde{f}(y) \equiv \int_{-\infty}^{+\infty} dx f(x) e^{-iyx}, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy \tilde{f}(y) e^{ixy}. \quad (10.1)$$

One may make an expansion of the function $f(x)$ in a series of the delta function and its derivatives as follows

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} dx' f(x') \delta(x-x') \\ &= \int_{-\infty}^{+\infty} dx' f(x') \exp\left(-x' \frac{\partial}{\partial x}\right) \delta(x) \\ &= \tilde{f}\left(-i \frac{\partial}{\partial x}\right) \delta(x) \\ &= \int_{-\infty}^{+\infty} dx' f(x') \sum_{k=0}^{\infty} \frac{(-x')^k}{k!} \frac{\partial^k}{\partial x^k} \delta(x) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \underbrace{\int_{-\infty}^{+\infty} dx' f(x') x'^k}_{\equiv f_k} \delta^{(k)}(x), \end{aligned} \quad (10.2)$$

where we applied that $\exp\left(-y \frac{\partial}{\partial x}\right)$ is the displacement operator which displaces

the variable x of a function $f(x)$ to $x - y$. We call such expansions moment series because the coefficients are determined by the moments f_k of the function $f(x)$ [31]. Their convergence is weak convergence of generalized functions. Such expansions in two-dimensional case form also the basis for limiting transitions $\hbar \rightarrow 0$ in the Wigner quasiprobability $W(q, p)$.

A classical distribution $f(q, p)$ should be positive definite. The most singular positive definite 2D functions are the delta functions $\delta(q - q_0)\delta(p - p_0)$ as generalized functions with arbitrary real parameters (q_0, p_0) . One may expect that they correspond to the pure states of quantum mechanics. The Fourier transform $\tilde{W}(u, v)$ of the Wigner quasiprobability $W(q, p)$ we define and give it together with its inversion according to

$$\begin{aligned}\tilde{W}(u, v) &= \int dq \wedge dp W(q, p) \exp(-i(uq + vp)) \Leftrightarrow \\ W(q, p) &= \frac{1}{(2\pi)^2} \int du \wedge dv \tilde{W}(u, v) \exp(i(qu + pv)).\end{aligned}\quad (10.3)$$

First we consider the coherent states for which the Wigner quasiprobability $W(q, p)$ given in (7.1) and its Fourier transform $\tilde{W}(u, v)$ are

$$\begin{aligned}W(q, p) &= \frac{1}{\pi\hbar} \exp\left(-\frac{(q - q_0)^2 + (p - p_0)^2}{\hbar}\right), \\ \tilde{W}(u, v) &= \exp\left(-i(q_0 u + p_0 v) - \frac{\hbar}{4}(u^2 + v^2)\right).\end{aligned}\quad (10.4)$$

Therefore one finds according to (10.2) using (10.3)

$$\begin{aligned}W(q, p) &= \tilde{W}\left\{-i\left(\frac{\partial}{\partial q} + \frac{\partial}{\partial p}\right)\right\} \delta(q) \delta(p) \\ &= \exp\left\{-\left(q_0 \frac{\partial}{\partial q} + p_0 \frac{\partial}{\partial p}\right) + \frac{\hbar}{4}\left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\right)\right\} \delta(q) \delta(p),\end{aligned}\quad (10.5)$$

from which follows for coherent states

$$\begin{aligned}\lim_{\hbar \rightarrow 0} W(q, p) &= \exp\left\{-\left(q_0 \frac{\partial}{\partial q} + p_0 \frac{\partial}{\partial p}\right)\right\} \delta(q) \delta(p) = \delta(q - q_0) \delta(p - p_0) \\ &= \delta(q - \bar{Q}) \delta(p - \bar{P}),\end{aligned}\quad (10.6)$$

or for $W(\alpha, \alpha^*)$ where the parameter $\hbar$ is not explicitly to see

$$\lim_{\hbar \rightarrow 0} W(\alpha, \alpha^*) = \delta(\alpha - \bar{a}, \alpha^* - \bar{a}^*). \quad (10.7)$$

Clearly, in this simple case the direct limiting transition

$$\lim_{\hbar \rightarrow 0} W(q, p) = \lim_{\hbar \rightarrow 0} \frac{1}{\pi\hbar} \exp\left(-\frac{(q - q_0)^2 + (p - p_0)^2}{\hbar}\right) = \delta(q - q_0) \delta(p - p_0) \quad (10.8)$$

is also sufficiently known from illustrations of the delta function as limiting case of Gaussian distributions.

We now consider the limiting transition $\hbar \rightarrow 0$ for the even and odd coherent

states using their Wigner quasiprobability which was derived in (8.7). In case of $q_0^2 + p_0^2 \neq 0$ the term $\exp\left(-\frac{q_0^2 + p_0^2}{\hbar}\right)$ in the denominator may be neglected in the limiting transition. The result of this limiting transition from (8.7) seems to be then

$$\lim_{\hbar \rightarrow 0} W(q, p) = \frac{1}{2} \{ \delta(q - q_0) \delta(p - p_0) + \delta(q + q_0) \delta(p + p_0) \}. \quad (10.9)$$

It originates fully from the terms in first line of (8.7) and may be obtained in analogy to the derivation for coherent states but the Fourier transforms of the states are not used here. The contribution of the term in second line vanishes in the limiting transition $\hbar \rightarrow 0$ meaning that there is no difference between even and odd coherent states in the classical limit. Its Fourier transform is quite complicated as the calculation with program “Mathematica” shows. However, that this term vanishes can be seen in more simple way again from the identity

$$\lim_{\hbar \rightarrow 0} \frac{1}{\pi \hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) = \lim_{\hbar \rightarrow 0} \exp\left\{ \frac{\hbar}{4} \left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2} \right) \right\} \delta(q) \delta(p) = \delta(q) \delta(p), \quad (10.10)$$

in connection with the function $\cos\left(\frac{2(p_0 q - q_0 p)}{\hbar}\right)$ which changes for $\hbar \rightarrow 0$

“infinitely often” its sign at $q = p = 0$ with a vanishing average value. Multiplied by $\delta(q) \delta(p)$ this suggest the vanishing of this term, a derivation which is not fully exact but can be surely substituted by a more correct one in the sense of the theory of Generalized functions.

It was unexpected that in the limiting procedure to classical mechanics there remains a separation of two two-dimensional delta functions in the phase space of variables (q, p) . This problem is removed if one could prove with certainty that the even and odd coherent states cannot be realized but then one has the problem to determine which states at all are not realizable, for example, then all superpositions of coherent states. To set here a boundary between realizable and not realizable states seems here to be a much greater problem.

To demonstrate the principal difference in the limiting transition to classical mechanics by $\hbar \rightarrow 0$ between pure and mixed states we consider a harmonic oscillator in thermal equilibrium with the medium at temperature T with the density operator ϱ in energy representation $(\bar{E} = \hbar \omega (\bar{N} + \frac{1}{2}))$ according to Gibbs statistics [11] (κ Boltzmann constant)

$$\varrho = \frac{\exp\left(-\frac{\hbar \omega N}{\kappa T}\right)}{\left\langle \exp\left(-\frac{\hbar \omega N}{\kappa T}\right) \right\rangle} = \frac{\sum_{n=0}^{\infty} e^{-nx} |n\rangle \langle n|}{\sum_{n=0}^{\infty} e^{-nx}} = (1 - e^{-x}) \sum_{n=0}^{\infty} e^{-nx} |n\rangle \langle n|,$$

$$x \equiv \frac{\hbar \omega}{\kappa T}, \quad \bar{N} = \frac{1}{e^x - 1} = \frac{1}{\exp\left(\frac{\hbar \omega}{\kappa T}\right) - 1},$$

$$\delta N \equiv N - \bar{N}, \quad \overline{(\delta N)^2} = \bar{N}(1 + \bar{N}), \quad \Delta N = \sqrt{\bar{N}(1 + \bar{N})}. \quad (10.11)$$

Its Wigner quasiprobability calculated from (2.1) is

$$W(q, p) = \frac{1}{(2\bar{N} + 1)\hbar\pi} \exp\left(-\frac{q^2 + p^2}{(2\bar{N} + 1)\hbar}\right), \quad 2\bar{N} + 1 = \frac{e^x + 1}{e^x - 1}. \quad (10.12)$$

Thus with the meaning of x one has

$$\begin{aligned} \lim_{\hbar \rightarrow 0} W(q, p) &= \lim_{\hbar \rightarrow 0} \frac{\exp\left(\frac{\hbar\omega}{\kappa T}\right) - 1}{\left(\exp\left(\frac{\hbar\omega}{\kappa T}\right) + 1\right)\hbar\pi} \exp\left[-\frac{\left(\exp\left(\frac{\hbar\omega}{\kappa T}\right) - 1\right)(q^2 + p^2)}{\left(\exp\left(\frac{\hbar\omega}{\kappa T}\right) + 1\right)\hbar}\right] \\ &= \frac{\omega}{2\pi\kappa T} \exp\left(-\frac{\omega(q^2 + p^2)}{2\kappa T}\right). \end{aligned} \quad (10.13)$$

This is a normalized Gaussian distribution function of canonical variables (q, p) .

The von-Neumann entropy $S(\bar{N})$ of the thermal states (10.11) is

$$S(\bar{N}) \equiv -\varrho \log(\varrho) = (1 + \bar{N}) \log(1 + \bar{N}) - \bar{N} \log(\bar{N}) > 0. \quad (10.14)$$

It is greater than that for pure states $\varrho = |\psi\rangle\langle\psi|$ which is exactly equal to zero.

The explicit calculation of the Fourier transform of the Wigner quasiprobability or sometimes of the Wigner quasiprobability itself of a given state for the purpose of the transition $\hbar \rightarrow 0$ is often very difficult. For coherent states we found that the result of the transition to the classical distribution by $\hbar \rightarrow 0$ is described by (10.6) where $q_0 \equiv \bar{Q}$ and $p_0 \equiv \bar{P}$ are the expectation values of the operators Q and P . In this connection we formulate the following conjecture:

The classical limit $\hbar \rightarrow 0$ of the Wigner quasiprobability (and of all other quasiprobabilities) for all pure quantum states $\varrho = |\psi\rangle\langle\psi|$ is in every case expected to be a shifted normalized two-dimensional delta function of the kind

$$\lim_{\hbar \rightarrow 0} W(q, p) = \delta(q - \bar{Q})\delta(p - \bar{P}), \quad (10.15)$$

where $\bar{Q}$ and $\bar{P}$ are the expectation values of the operators Q and P and the two-dimensional vector with components $(\bar{Q}, \bar{P})$ corresponds to the inertial point of the distribution over the phase plane of variables (q, p) .

For a proof one may take then any of the derived representations of the Wigner quasiprobability as starting point. A specifics of the pure states in comparison to mixed states is the relation

$$\varrho^2 = \varrho \Leftrightarrow |\psi\rangle\langle\psi|\varrho|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|. \quad (10.16)$$

Nevertheless, we could not find up to now a satisfying (*i.e.* correct) proof of the above conjecture. It is easy to see that all other quasiprobabilities possess the same limit in classical case $\hbar \rightarrow 0$ as that of the Wigner quasiprobability $W(q, p)$ and provided the conjecture is true for the Wigner quasiprobability it is also true for the other quasiprobabilities of the type of $Q(q, p)$ and $P(q, p)$. The other

quasiprobabilities of the type of $Q(q, p)$ and $P(q, p)$ and their linear interpolation on the level of their Fourier transforms are connected with the Wigner quasiprobability by a convolution with a normalized Gaussian function. As can be seen from the convolutions in (5.9) which contain $\hbar$ as linear factor in the exponents that for $\hbar \rightarrow 0$ this leads to the convolution of the Wigner quasiprobability $W(q, p)$ with the delta function $\delta(q)\delta(p)$ (*i.e.*, the identity element in the algebra of convolutions) and does not change the other quasiprobabilities. For mixed states as we have seen in (10.13) one obtains in classical limit continuous normalized distribution functions because the constant $\hbar$ comes a second time into the distribution by the quantum-statistical distribution law.

11. Conclusions

In this article we continued our investigations to the phase properties of quantum states which we had begun in [13]. The main object for this purpose was the Wigner quasiprobability $W(q, p)$ for one mode of a fictitious harmonic oscillator of the radiation field in dependence on the canonical variables of phase space (q, p) or of a pair of complex conjugate variable (α, α^*) . The reconstruction of the density operator from the Wigner quasiprobability was derived. The integration of the Wigner quasiprobability over the radius (or amplitude) providing a normalized phase distribution $W(\varphi)$ over a basic interval of length 2π with 2π -periodicity was considered. This was specialized for two categories of states, the coherent states and the even and odd coherent states. The uniform distribution $W(\varphi) = \frac{1}{2\pi}$ over a basic interval of length 2π is the starting point for the definition of a phase uncertainty since due to normalization each non-uniform phase distribution has to possess a part greater than $\frac{1}{2\pi}$ and a part smaller than $\frac{1}{2\pi}$ and both parts the greater part and the now absent part below $\frac{1}{2\pi}$ have to be exactly equal. Therefore the sum of intervals where the phase distribution $W(\varphi)$ is greater than $\frac{1}{2\pi}$ is an appropriate measure for the phase uncertainty.

It was found that the Heitler uncertainty relation for the product of number uncertainty with phase uncertainty cannot be true since all number states $|n\rangle$ possess a number uncertainty equal to zero and their phase uncertainty though undetermined but cannot become infinite as Heitler suggest it. This fact was already discovered before by Carruthers and Nieto. Their argument is that number and phase do not form a pair of operators with the ordinary commutation relations for the derivation of uncertainty relations. We discussed this and found too that all other formulations are somehow connected with this defect of the phase operator to be a “perfect” Hermitean operator. We added to this that already in classical mechanics the transformation of (q, p) to action-angle variables (j, φ) possesses a defect in the coordinate origin of the complex plane which has to be excluded in the transformation.

With the exception of the squeezed coherent states or the displaced squeezed vacuum states which is the same and including the coherent states as states with two-dimensional Gaussian distributions of the Wigner quasiprobability (higher than quadratic exponents in (q, p) in such functions cannot be present due to (2.21)) all other pure states possess ranges of negativity in the Wigner quasiprobability. For the Wigner phase function $W(\varphi)$ it was not clear up to now whether or not it is positive definite in all cases. We found by its graphical representation that this function possesses intervals of negativity for the even coherent states but not for the odd coherent states and clearly also not for the common coherent states of Glauber.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Schrödinger, E. (1926) Quantisierung als Eigenwertproblem. *Annalen der Physik*, **386**, 109-139. <https://doi.org/10.1002/andp.19263861802>
- [2] Heisenberg, W. (1925) Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen. *Zeitschrift für Physik*, **33**, 879-893. <https://doi.org/10.1007/bf01328377>
- [3] Weyl, H. (1928) Gruppentheorie und Quantenmechanik [The Theory of Groups and Quantum Mechanics]. Dover.
- [4] Born, M. and Jordan, P. (1925) Zur Quantenmechanik. *Zeitschrift für Physik*, **34**, 858-888. <https://doi.org/10.1007/bf01328531>
- [5] Heisenberg, W. (1927) Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik*, **43**, 172-198. <https://doi.org/10.1007/bf01397280>
- [6] von Neumann, J. (1932) Mathematische Grundlagen der Quantenmechanik. Springer.
- [7] Wigner, E. (1932) On the Quantum Correction for Thermodynamic Equilibrium. *Physical Review*, **40**, 749-759. <https://doi.org/10.1103/physrev.40.749>
- [8] Tatarskii, V.I. (1983) The Wigner Representation of Quantum Mechanics. *Uspekhi Fizicheskikh Nauk*, **139**, 587-619. <https://doi.org/10.3367/ufnr.0139.198304b.0587>
- [9] Kim, Y.S. and Noz, M.E. (1991) Phase Space Picture of Quantum Mechanics. World Scientific. <https://doi.org/10.1142/1197>
- [10] Dirac, P.A.M. (1930) The Principles of Quantum Mechanics. Oxford University Press.
- [11] Glauber, R.J. (1963) Coherent and Incoherent States of the Radiation Field. *Physical Review*, **131**, 2766-2788. <https://doi.org/10.1103/physrev.131.2766>
- [12] Malkin, I.A. and Man'ko, V.I. (1979) Dinamicheskiye simmetrii i kogerentnyye sostoyaniya kvantovykh system [Dynamic Symmetries and Coherent States of Quantum Systems]. Nauka.

- [13] Wünsche, A. (2026) Wigner Quasiprobability of Coherent Phase States. *Physics*, **8**, Article 37. <https://doi.org/10.3390/physics8020037>
- [14] Bohm, A. (1993) Quantum Mechanics. 3rd Edition, Springer.
- [15] Arnol'd, V.I. (1989) Matematicheskiye metody klassicheskoi mekhaniki [Mathematical Methods of Classical Mechanics]. 3rd Edition, Na'uka.
- [16] Bishop, R.F. and Vourdas, A. (1994) Displaced and Squeezed Parity Operator: Its Role in Classical Mappings of Quantum Theories. *Physical Review A*, **50**, 4488-4501. <https://doi.org/10.1103/physreva.50.4488>
- [17] Hall, B.C. (2003) Lie Groups, Lie Algebras, and Representations. Springer.
- [18] Wunsche, A. (1991) Displaced Fock States and Their Connection to Quasiprobabilities. *Quantum Optics: Journal of the European Optical Society Part B*, **3**, 359-383. <https://doi.org/10.1088/0954-8998/3/6/005>
- [19] Peřina, J. (1974) Coherence of Light. Van Nostrand Reinhold Company.
- [20] Perinová, V., Lukš, A. and Perina, J. (1998) Phase in Optics. World Scientific. <https://doi.org/10.1142/3541>
- [21] Prokhorov, Y.V. and Rozanov, Y.A. (1973) Teoriya veroyatnostey [Probability Theory]. Nauka.
- [22] Szegő, G. (1975) Orthogonal Polynomials. 4th Edition, American Mathematical Society.
- [23] Bateman, H. and Erdélyi, A. (1963) Higher Transcendental Functions, Vol. 2. Mac-Graw-Hill.
- [24] Paul, H. (1969) Lasertheorie, Band II. Akademie-Verlag. (In German)
- [25] Susskind, L. and Glogower, J. (1964) Quantum Mechanical Phase and Time Operator. *Physics Physique Fizika*, **1**, 49-61. <https://doi.org/10.1103/physicsphysiquefizika.1.49>
- [26] Carruthers, P. and Nieto, M.M. (1968) Phase and Angle Variables in Quantum Mechanics. *Reviews of Modern Physics*, **40**, 411-440. <https://doi.org/10.1103/revmodphys.40.411>
- [27] Man'ko, V.I. (1972) Kogerentnyye sostoyaniya v kvantovoi teorii [Coherent States in Quantum Theory]. Izd. Mir. (In Russian)
- [28] Temme, N.M. (2010) Error Functions, Dawson's and Fresnel Integrals. In: Olver, F.W., Lozier, D.W., Boisvert, R. and Clark, C.W., Eds., *NIST Handbook of Mathematical Functions*, Cambridge University Press, p. 159.
- [29] Heitler, W. (1954) The Quantum Theory of Radiation. 3rd Edition, Clarendon Press.
- [30] Pegg, D.T. and Barnett, S.M. (1989) Phase Properties of the Quantized Single-Mode Electromagnetic Field. *Physical Review A*, **39**, 1665-1675. <https://doi.org/10.1103/physreva.39.1665>
- [31] Wünsche, A. (2021) Boundary Conditions for Sturm-Liouville Equation. *Advances of Pure Mathematics*.
- [32] Wünsche, A. (2025) Operational Calculus with Hermite Polynomials and a Quite General Gaussian Integral. *Journal of Modern Physics*, **16**, 719-753. <https://doi.org/10.4236/jmp.2025.165040>
- [33] Wünsche, A. (2001) Hermite and Laguerre 2D Polynomials. *Journal of Computational and Applied Mathematics*, **133**, 665-678. [https://doi.org/10.1016/s0377-0427\(00\)00681-6](https://doi.org/10.1016/s0377-0427(00)00681-6)
- [34] Wünsche, A. (2003) The Role of Jacobi Polynomials in the Theory of Hermite and Laguerre 2D Polynomials. *Journal of Computational and Applied Mathematics*, **153**,

521-529.

- [35] Wünsche, A. (2015) Generating Functions for Products of Special Laguerre 2D and Hermite 2D Polynomials. *Applied Mathematics*, **6**, 2142-2168.
<https://doi.org/10.4236/am.2015.612188>

Appendix A: Wigner Quasiprobability in Representation by Number States

In this Appendix we derive the Wigner quasiprobability $W(q, p)$ for a general density operator ϱ in representation by the number states $|n\rangle$. The number representation of the general density operator ϱ can be written

$$\varrho = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |m\rangle \langle m| \varrho |n\rangle \langle n|. \quad (\text{A.1})$$

Starting from definition (2.1) of the Wigner quasiprobability and using the Schrödinger wave function $\langle q|n\rangle$ of number states one gets in a first step

$$\begin{aligned} W(q, p) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m| \varrho |n\rangle \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} dx \exp\left(i \frac{2px}{\hbar}\right) \langle q-x|m\rangle \langle n|q+x\rangle \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m| \varrho |n\rangle}{\sqrt{2^{m+n}} m! n! \pi \hbar} \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} dx \exp\left(i \frac{2px}{\hbar}\right) \exp\left(-\frac{(x-q)^2 + (x+q)^2}{2\hbar}\right) \\ &\quad \cdot (-1)^m H_m\left(\frac{x-q}{\sqrt{\hbar}}\right) H_n\left(\frac{x+q}{\sqrt{\hbar}}\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m| \varrho |n\rangle}{\sqrt{2^{m+n}} m! n!} \frac{(-1)^n}{\pi \hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) \\ &\quad \cdot \frac{1}{\sqrt{\pi \hbar}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-ip)^2}{\hbar}\right) H_m\left(\frac{x-q}{\sqrt{\hbar}}\right) H_n\left(\frac{x+q}{\sqrt{\hbar}}\right), \end{aligned} \quad (\text{A.2})$$

where relation $H_m(-z) = (-1)^m H_m(z)$ for the Hermite polynomials was used. The integral can be calculated using the following quite general integral derived in [32] ((6.20) there)

$$\begin{aligned} &\frac{1}{\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} dx \exp\left(-\frac{(x-x_0)^2}{a^2}\right) H_m\left(\frac{x-x_1}{a}\right) H_n\left(\frac{x-x_2}{a}\right) \\ &= a^{m+n} \sum_{j=0}^{\{m,n\}} \frac{m! n!}{j!(m-j)!(n-j)!} \left(\frac{2}{a^2}\right)^{m+n-j} (x_0-x_1)^{m-j} (x_0-x_2)^{n-j} \\ &= \sqrt{2^{m+n}} \sum_{j=0}^{\{m,n\}} \frac{m! n!}{j!(m-j)!(n-j)!} \left(\sqrt{\frac{2}{a^2}} (x_0-x_1)\right)^{m-j} \left(\sqrt{\frac{2}{a^2}} (x_0-x_2)\right)^{n-j}. \end{aligned} \quad (\text{A.3})$$

Using it with $a = \sqrt{\hbar}$, $x_0 = ip$, $x_1 = -x_2 = q$ this leads to

$$\begin{aligned} W(q, p) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m| \varrho |n\rangle}{\sqrt{2^{m+n}} m! n!} \frac{(-1)^m}{\pi \hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) \\ &\quad \cdot \sum_{j=0}^{\{m,n\}} \frac{m! n!}{j!(m-j)!(n-j)!} \left(\frac{\hbar}{2}\right)^j (ip-q)^{m-j} (ip+q)^{n-j} \\ &= \frac{1}{\pi \hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m| \varrho |n\rangle}{\sqrt{m! n!}} \\ &\quad \cdot \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m! n!}{j!(m-j)!(n-j)!} \left(\sqrt{\frac{2}{\hbar}} (q-ip)\right)^{m-j} \left(\sqrt{\frac{2}{\hbar}} (q+ip)\right)^{n-j}. \end{aligned} \quad (\text{A.4})$$

This may be written in the shorter form

$$W(q, p) = \frac{1}{\pi\hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m|\varrho|n\rangle}{\sqrt{m!n!}} L_{m,n}\left(\sqrt{\frac{2}{\hbar}}(q-ip), \sqrt{\frac{2}{\hbar}}(q+ip)\right), \quad (\text{A.5})$$

where the special Laguerre-2D polynomials $L_{m,n}(z, z^*)$ are used which are defined by [33]-[35]

$$\begin{aligned} L_{m,n}(z, z^*) &= \exp\left(-\frac{\partial^2}{\partial z \partial z^*}\right) z^m z^{*n} = \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} z^{m-j} z^{*n-j} \\ &= (-1)^n n! z^{m-n} L_n^{m-n}(zz^*) = (-1)^m m! z^{*n-m} L_m^{n-m}(zz^*), \\ L_{m,n}(z, z^*) &= L_{n,m}(z^*, z) = (L_{m,n}(z, z^*))^*, \quad L_{m,n}(\kappa z, \kappa^{-1} z^*) = \kappa^{m-n} L_{m,n}(z, z^*). \end{aligned} \quad (\text{A.6})$$

They are related as given to the standard (1D-) Laguerre polynomials $L_n^\alpha(z)$ and possess many symmetries and other advantageous properties.

Using the variable transformation (2.23) and taking into account (2.28) one arrives at the following representation of the Wigner quasiprobability by complex variables (α, α^*) and of the density operator by number states

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \exp(-2\alpha\alpha^*) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle m|\varrho|n\rangle}{\sqrt{m!n!}} L_{n,m}(2\alpha, 2\alpha^*). \quad (\text{A.7})$$

It is easy to specialize (A.5) or (A.7) for pure states $|\psi\rangle$ by $\varrho = |\psi\rangle\langle\psi|$ and more specially for number states $|\psi\rangle = |n\rangle$ by

$$\begin{aligned} W(q, p) &= \frac{1}{\pi\hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) \frac{1}{n!} L_{n,n}\left(\sqrt{\frac{2}{\hbar}}(q-ip), \sqrt{\frac{2}{\hbar}}(q+ip)\right) \\ &= \frac{1}{\pi\hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) (-1)^n L_n\left(\frac{2}{\hbar}(q^2 + p^2)\right). \end{aligned} \quad (\text{A.8})$$

For coherent states $|\beta\rangle$ one has to apply the Generating functions for the Laguerre 2D polynomials

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{s^m t^n}{m!n!} L_{m,n}(z, z^*) &= \exp\left(-\frac{\partial^2}{\partial z \partial z^*}\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{s^m t^n}{m!n!} z^m z^{*n} \\ &= \exp\left(-\frac{\partial^2}{\partial z \partial z^*}\right) \exp(sz + tz^*) \\ &= \exp(sz + tz^* - st), \end{aligned} \quad (\text{A.9})$$

and finds

$$\begin{aligned} W(q, p) &= \frac{1}{\pi\hbar} \exp\left(-\frac{q^2 + p^2}{\hbar}\right) \exp(-\beta\beta^*) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta^m \beta^{*n}}{m!n!} L_{m,n}\left(\sqrt{\frac{2}{\hbar}}(q-ip), \sqrt{\frac{2}{\hbar}}(q+ip)\right) \\ &= \frac{1}{\pi\hbar} \exp\left(-\frac{(q-q_0)^2 + (p-p_0)^2}{\hbar}\right), \end{aligned} \quad (\text{A.10})$$

where the substitutions (7.3) has been used. This result is easier to obtain from the Wigner quasiprobability $W(\alpha, \alpha^*)$ in representation by the pair of complex conjugate coordinates and possesses the form

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \exp(-2(\alpha - \beta)(\alpha^* - \beta^*)). \quad (\text{A.11})$$

Appendix B: Proof for the Bounds of the Wigner Quasiprobability $W(q, p)$

A compact proof for the bounds of the Wigner quasiprobability $W(q, p)$ can be given using the Bunyakovski-Cauchy-Scharz inequality for the traces of operators A and B

$$\langle A^\dagger B \rangle \langle B^\dagger A \rangle \leq \langle A^\dagger A \rangle \langle B^\dagger B \rangle, \quad (\text{B.1})$$

as follows

$$\begin{aligned} (W(q, p))^2 &= \frac{1}{(\pi\hbar)^2} \varrho \Pi(q, p) \Pi(q, p) \varrho \\ &= \frac{1}{(\pi\hbar)^2} \left\langle \underbrace{\sqrt{\varrho}}_{A^\dagger} \underbrace{\Pi(q, p) \sqrt{\varrho}}_B \right\rangle \left\langle \underbrace{\sqrt{\varrho} \Pi(q, p)}_{B^\dagger} \underbrace{\sqrt{\varrho}}_A \right\rangle \\ &\leq \frac{1}{(\pi\hbar)^2} \langle \varrho \rangle \sqrt{\varrho} (\Pi(q, p))^2 \sqrt{\varrho} \\ &= \frac{1}{(\pi\hbar)^2} \langle \varrho \rangle \left\langle \sqrt{\varrho} D(q, p) \Pi_{=I}^2(D(q, p))^{-1} \sqrt{\varrho} \right\rangle \\ &= \frac{\langle \varrho \rangle}{(\pi\hbar)^2} (D(q, p))^{-1} \sqrt{\varrho} \sqrt{\varrho} D(q, p) \\ &= \frac{\langle \varrho \rangle^2}{(\pi\hbar)^2} = \frac{1}{(\pi\hbar)^2}, \end{aligned} \quad (\text{B.2})$$

from which follows

$$-\frac{1}{\pi\hbar} \leq W(q, p) \leq \frac{1}{\pi\hbar}. \quad (\text{B.3})$$

There are examples that the upper and lower bound are really taken on in special cases.

Since the density operator ϱ is a positive definite operator its square root $\sqrt{\varrho}$ may be defined (but must not necessarily) with the positive square roots of all its eigenvalues to the mutually orthogonal eigenvectors of ϱ .

Appendix C: Two-Dimensional Functions in Representation by Pairs of Complex Conjugate Variable and Convolutions

In this Appendix we consider Fourier transformations of functions of two complex conjugate variables $(z = x + iy, z^* = x - iy)$ and $(w = u + iv, w^* = u - iv)$ according to

$$f(x, y) \equiv f(z, z^*), \quad \tilde{f}(u, v) \equiv \tilde{f}(w, w^*), \quad (\text{C.1})$$

without changing the function symbols f and $\tilde{f}$. Then the two-dimensional Fourier transformation and its inversion

$$\begin{aligned} \tilde{f}(u, v) &= \int dx \wedge dy f(x, y) \exp(-i(ux + vy)), \\ f(x, y) &= \frac{1}{(2\pi)^2} \int du \wedge dv \tilde{f}(u, v) \exp(i(xu + yv)), \end{aligned} \quad (\text{C.2})$$

makes the transition to

$$\begin{aligned} \tilde{f}(w, w^*) &= \int \frac{i}{2} dz \wedge dz^* f(z, z^*) \exp\left(-\frac{i}{2}(w^*z + wz^*)\right), \\ f(z, z^*) &= \frac{1}{(2\pi)^2} \int \frac{i}{2} dw \wedge dw^* \tilde{f}(w, w^*) \exp\left(\frac{i}{2}(z^*w + zw^*)\right). \end{aligned} \quad (\text{C.3})$$

We consider now the convolution of two functions $f(z, z^*)$ and $g(z, z^*)$ (notation $f(z, z^*) * g(z, z^*)$) and transform it using the operators of displacement of the arguments of function and find in this way by comparison with (C.3)

$$\begin{aligned} h(z, z^*) &= f(z, z^*) * g(z, z^*) \\ &\equiv \int \frac{i}{2} dz' \wedge dz'^* f(z', z'^*) g(z - z', z^* - z'^*) \\ &= \int \frac{i}{2} dz' \wedge dz'^* f(z', z'^*) \exp\left(-z' \frac{\partial}{\partial z} - z'^* \frac{\partial}{\partial z^*}\right) g(z, z^*) \\ &= \tilde{f}\left(-i2 \frac{\partial}{\partial z^*}, -i2 \frac{\partial}{\partial z}\right) g(z, z^*). \end{aligned} \quad (\text{C.4})$$

It is expressed by the Fourier transform $\tilde{f}(w, w^*)$ of $f(z, z^*)$ with differential operators in its arguments acting onto the function $g(z, z^*)$.

For the Fourier transform $\tilde{h}(w, w^*)$ of the convolution $h(z, z^*)$ we find

$$\begin{aligned} \tilde{h}(w, w^*) &= \int \frac{i}{2} dz \wedge dz^* \left(\int \frac{i}{2} dz' \wedge dz'^* f(z', z'^*) g(z - z', z^* - z'^*) \right) \exp\left(-\frac{i}{2}(w^*z + wz^*)\right) \\ &= \int \frac{i}{2} dz' \wedge dz'^* f(z', z'^*) \exp\left(-\frac{i}{2}(w^*z' + wz'^*)\right) \\ &\quad \cdot \int \frac{i}{2} dz \wedge dz^* g(z - z', z^* - z'^*) \exp\left(-\frac{i}{2}(w^*(z - z') + w(z^* - z'^*))\right) \\ &= \tilde{f}(w, w^*) \tilde{g}(w, w^*). \end{aligned} \quad (\text{C.5})$$

This is the well-known result that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the involved functions which is true for arbitrary dimension. Since the factors in the Fourier transform of a convolution are commutable the factors in the original functions in the convolution are also commutable, *i.e.*

$$f(z, z^*) * g(z, z^*) = g(z, z^*) * f(z, z^*). \quad (\text{C.6})$$

Furthermore, it is easy to see that convolutions are associative, that means

$$\begin{aligned} (f(z, z^*) * g(z, z^*)) * h(z, z^*) &= f(z, z^*) * (g(z, z^*) * h(z, z^*)) \\ &\equiv f(z, z^*) * g(z, z^*) * h(z, z^*), \end{aligned} \quad (\text{C.7})$$

and it is not necessary to set brackets between the factors. We specialize (C.4) here for the important example of a normalized Gaussian function $f(z, z^*)$ with parameter a^2

$$f(z, z^*) = \frac{1}{\pi a^2} \exp\left(-\frac{zz^*}{a^2}\right) \Leftrightarrow \tilde{f}(w, w^*) = \exp\left(-\frac{a^2 ww^*}{4}\right), (a^2 > 0). \quad (\text{C.8})$$

From (C.4) follows in this way for arbitrary functions $g(z, z^*)$

$$\frac{1}{\pi a^2} \exp\left(-\frac{zz^*}{a^2}\right) * g(z, z^*) = \exp\left(a^2 \frac{\partial^2}{\partial z \partial z^*}\right) g(z, z^*). \quad (\text{C.9})$$

In the limiting case $a^2 \rightarrow 0 \Rightarrow f(z, z^*) \rightarrow \delta(z, z^*)$ follows

$$\delta(z, z^*) * g(z, z^*) = g(z, z^*), \quad (\text{C.10})$$

meaning that $\delta(z, z^*)$ is the unit in the algebra of convolutions, here of functions of two variables. Together with a second normalized Gaussian function $g(z, z^*)$ with parameter b^2

$$g(z, z^*) = \frac{1}{\pi b^2} \exp\left(-\frac{zz^*}{b^2}\right) \Leftrightarrow \tilde{g}(w, w^*) = \exp\left(-\frac{b^2 ww^*}{4}\right), (b^2 > 0), \quad (\text{C.11})$$

we find for their convolution $h(z, z^*) = f(z, z^*) * g(z, z^*)$ according to (C.5) or by applying (C.4)

$$\tilde{h}(w, w^*) = \tilde{f}(w, w^*) \tilde{g}(w, w^*) = \exp\left(-\frac{(a^2 + b^2) ww^*}{4}\right), \quad (\text{C.12})$$

with the correspondence

$$\begin{aligned} h(z, z^*) &= f(z, z^*) * g(z, z^*) \\ &= \frac{1}{\pi a^2} \exp\left(-\frac{zz^*}{a^2}\right) * \frac{1}{\pi b^2} \exp\left(-\frac{zz^*}{b^2}\right) \\ &= \frac{1}{\pi(a^2 + b^2)} \exp\left(-\frac{zz^*}{a^2 + b^2}\right) \\ &= \exp\left(a^2 \frac{\partial^2}{\partial z \partial z^*}\right) \frac{1}{\pi b^2} \exp\left(-\frac{zz^*}{b^2}\right). \end{aligned} \quad (\text{C.13})$$

In particular, in the limiting case $b^2 \rightarrow 0 \Rightarrow g(z, z^*) \rightarrow \delta(z, z^*)$ follows from this relation

$$\exp\left(a^2 \frac{\partial^2}{\partial z \partial z^*}\right) \delta(z, z^*) = \frac{1}{\pi a^2} \exp\left(-\frac{zz^*}{a^2}\right). \quad (\text{C.14})$$

The application of the derived relations is convenient for the transition from the Wigner quasiprobabilities to other quasiprobabilities via different operator

ordering and vice versa⁵.

Appendix D: Proof to the Expansion of Operators in Normally Ordered Powers $a^{\dagger k} a^l$

We prove the expansion in powers in normally ordered annihilation and creation operators $(a, a^{\dagger})$

$$A = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle A a_{k,l} \rangle a^{\dagger k} a^l, \quad a_{k,l} \equiv \sum_{j=0}^{\{k,l\}} \frac{(-1)^j |l-j\rangle \langle k-j|}{j! \sqrt{(k-j)!(l-j)!}}. \quad (\text{D.1})$$

From this relation follows the number-state representation of A

$$\langle m | A | n \rangle = \sum_{j=0}^{\{m,n\}} a_{m-j,n-j} \frac{\sqrt{m!n!}}{j!}. \quad (\text{D.2})$$

This is a finite relation between $\langle m | A | n \rangle$ and $a_{k,l}$ which can be successively converted to $a_{k,l}$ for low indices from which the above given relation can be conjectured.

For the purpose of the proof of (D.1) we calculate the traces $\langle a^{\dagger k} a^l a_{m,n} \rangle$

$$\begin{aligned} \langle a^{\dagger k} a^l a_{m,n} \rangle &= \sum_{j=0}^{\{m,n\}} \left\langle \frac{(-1)^j |n-j\rangle \langle m-j|}{j! \sqrt{(m-j)!(n-j)!}} a^{\dagger k} a^l \right\rangle \\ &= \sum_{j=0}^{\{m,n\}} \frac{(-1)^j \langle m-j | a^{\dagger k} a^l | n-j \rangle}{j! \sqrt{(m-j)!(n-j)!}} \\ &= \delta_{m-k,n-l} \frac{1}{(m-k)!} \underbrace{\sum_{j=0}^{k-m} \frac{(-1)^j (m-k)!}{j! (m-k-j)!}}_{(-1+1)^{m-k}} \\ &= \delta_{m-k,n-l} \delta_{m-k,0} = \delta_{k,m} \delta_{l,n}. \end{aligned} \quad (\text{D.3})$$

By multiplication of (D.1) with $a_{m,n}$ and forming the trace one gets an identity which affirms its correctness.

⁵In continuation of the formula from positive $a^2 > 0$ to negative $a^2 < 0$ for the the inverse convolution one has to be cautious since $f(z, z^*)$ and $g(z, z^*)$ in (C.8) or (C.11) become infinitely increasing functions for $zz^* \rightarrow \infty$ which may be only dealt with as generalized functions that means as linear functionals over certain suited spaces of basis functions to overcome divergence problems. This concerns, however, only the signs of the factors $\frac{1}{\pi a^2}$ or $\frac{1}{\pi b^2}$ in front of the other parts for which one finds that it would be better then to choose it as $\frac{1}{\pi |a^2|}$ and $\frac{1}{\pi |b^2|}$, respectively.