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Exploring the Electromagnetic Interactions in the 2T + 3S Dimensions without Compactification: The Lorentz Force and the Electromagnetic Stress Tensor

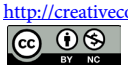
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Abstract

We used the five-dimensional formulation of classical electrodynamics in 2T + 3S dimensions and derived the Lorentz Force Equation in the relativistic covariant two-time paradigm. We find that the effective-charge idea introduced earlier in the five-dimensional formulation of classical electrodynamics brings the derived Lorentz Force Equation closer to the four-dimensional one. We also compute the components of the electromagnetic energy tensor in the 2T + 3S dimensions and identify the new additional features.

Keywords

Extra Time Dimension, Electromagnetic Interactions, Lorentz Force Equation, Electromagnetic Stress Tensor, Covariant Lagrangian Formulation

1. Introduction

Recently, Zahir [1] considered a two-time (characterized by distinct causality speeds c_1 and c_2) and three-space-dimensional Minkowski space (2T + 3S). He derived relativistic coordinate and velocity transformation formulas and expressions for a new effective speed limit $c_e = \sqrt{c_1^2 + c_2^2/k^2}$ where k is a scale factor connecting the two times t_1 and t_2. Previously, Velev [2] explored coordinate and kinematic transformations in a detailed analysis of flat space-time with extra time dimensions. However, Velev [2] used the same speed of causality c for all time dimensions, including the extra ones. This at least creates one calculational problem—we cannot take the limit for c assigned to the extra times to zero to get back the familiar four-dimensional (*i.e.*, those of Einstein's Special Theory of Relativity

([3] and references therein)) formulations. There was no way to distinguish one c from the other [2].

In our familiar space-time structure, we do not observe extra time dimensions; thus, higher-dimensional physics considers compactification. Zahir [1] also discussed compactification in a two-time-dependent Schrödinger-like equation with an infinite square-well potential. After compactifying the extra time dimension on a closed-loop topology with a period matching the Planck time, the solution generated interference among additional quantum states with ultra-short oscillation periods. Zahir [4] also considered compactification in another related work on the Klein-Gordon-type equation in 2T + 3S dimensions. Tyagi and Al-Shahristani [5] developed the Dirac equations within Zahir's paradigm but did not discuss compactification of the extra time dimension.

In this paper, we use Zahir's extension of classical electromagnetic fields in 2T + 3S dimensions without compactification [6]. In this extension, Zahir [6] investigated how Maxwell's equations were modified in a five-dimensional "Lorentz" covariant analysis and its possible implications. He postulated a bold physical space-time environment defined by two times, each associated with two interactions mediated by two massless intermediaries, with two distinct speeds of causality. Zahir [6] identified the 2T + 3S space-time environment with the hadronic Color-Space, where the massless gluon was postulated to move with an invariant intrinsic speed c_2, different from the speed of photons c_1. This provided the conceptual foundation for the two-time paradigm. Zahir [6] also made a preliminary estimate of the gluon speed from the R-values of $e^+ e^-$ collisions using a theoretical effective-charge formula.

In this paper, we explore two specific examples of classical electrodynamics in the two-time paradigm in a relativistic covariant formalism: a) the Lorentz Force Equation and b) the five-dimensional electromagnetic stress-energy formulations. Throughout the paper, we derive results in sufficient detail to benefit early-career researchers. Our objective is not only to explore new concepts and develop theoretical formulations that yield interesting results. We also aim to highlight problems, limitations, and open issues to advance the theory through further research.

Section 1 introduces the topic, Section 2 summarizes the 2T + 3S space-time structure, and Section 3 presents the five-dimensional electromagnetic field tensors [6]. In Section 4, we formulate the five-dimensional covariant Lagrangian and derive the Lorentz Force Equation. In Section 5, we present the calculated expressions for certain components of the electromagnetic stress-energy tensor in 2T + 3S dimensions, and in Section 6, we emphasize the requirement for a quantum formulation. We conclude in Section 7.

2. Summary of the 2T + 3S Space-Time Structure

As noted earlier, the different speeds of causality for times t_1 and t_2 are given by c_1 and c_2. Thus, the space-time variables in the (2T + 3S) space are (c_1t_1, c_2t_2, x, y, z). We consider Minkowski-like flat space with metric signature (+, +, −, −, −) such

that the invariant space-time interval ds is given by $\mathrm{d}s^2 = c_1^2\mathrm{d}t_1^2 + c_2^2\mathrm{d}t_2^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$. With these two different speeds of causality, c_1 and c_2, we emphasize that although t_1 and t_2 are both "time-like" variables, this time (t_1) is not the same as that time (t_2). We denote the dimensions of t_1 and t_2 by T and t, respectively, and the dimensions of x, y, and z by L. So, the dimensions of c_1 and c_2 are LT^{-1} and Lt^{-1}, respectively.

1) Simplified Coordinate Transformation and Expressions for Five-velocity and Five-momenta

After a detailed calculation, Zahir [1] derived the 2T relativistic coordinate transformation for the reference frame K′ moving at a uniform speed relative to reference frame K. He considered the standard configuration, and the motion of K′ is along the x coordinate only, such that at $t_1 = 0$ and $t_2 = 0$, the coordinate axes of K and K′ coincide. K′ is moving with uniform velocities v and w defined with respect to times t_1 and t_2, respectively. So, if x_0 is the coordinate of the origin of K′ at any time,

$$v = \mathrm{d}x_0/\mathrm{d}t_1;\ w = \mathrm{d}x_0/\mathrm{d}t_2$$

We also have $w = \dfrac{\mathrm{d}x_0}{\mathrm{d}t_2} = \dfrac{\mathrm{d}x_0}{\mathrm{d}t_1}\cdot\dfrac{\mathrm{d}t_1}{\mathrm{d}t_2} = v\cdot k$, where $k = \dfrac{\mathrm{d}t_1}{\mathrm{d}t_2}$. This implies that $t_1 = k\cdot t_2$. Here, k is a constant and is assumed to be defined globally (see five space-time derivatives later in the text), not just along the particle trajectory. If t_1 has dimension T and t_2 has dimension t, then k has dimension T/t. The expressions for coordinate transformations are,

$$x' = \frac{x - vt_1}{\sqrt{1-\dfrac{v^2}{c_e^2}}};\ t_1' = \frac{t_1 - \dfrac{xv}{c_e^2}}{\sqrt{1-\dfrac{v^2}{c_e^2}}};\ t_2' = \frac{t_2 - \dfrac{xw}{k^2c_e^2}}{\sqrt{1-\dfrac{w^2}{k^2c_e^2}}} = \frac{t_2 - \dfrac{xv}{kc_e^2}}{\sqrt{1-\dfrac{v^2}{c_e^2}}}$$

$$c_e^2 = c_1^2 + c_2^2/k^2 = c_1^2\left(1 + c_2^2/c_1^2k^2\right) = c_1^2\left(1+\lambda^2\right)$$

where $\lambda = \dfrac{c_2}{c_1k}$ and we will use this variable later in the text (1)

When $c_2 = 0, c_e = c_1$ and the coordinate transformation equations are similar to the ones in the familiar four-dimensional expressions.

We can also define two proper times $\mathrm{d}\tau_1$ and $\mathrm{d}\tau_2$ and other related variables ($\bar{x}$ is a three-dimensional space vector with components x, y, z).

$$\mathrm{d}s^2 = c_1^2\mathrm{d}t_1^2 + c_2^2\mathrm{d}t_2^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

$$\mathrm{d}\tau_1^2 = \mathrm{d}s^2/c_1^2 = \mathrm{d}t_1^2\gamma_1^{-2};\ \mathrm{d}\tau_1 = \gamma_1^{-1}\mathrm{d}t_1$$

$$\mathrm{d}\tau_2^2 = \mathrm{d}s^2/c_2^2 = \mathrm{d}t_2^2\gamma_2^{-2};\ \mathrm{d}\tau_2 = \gamma_2^{-1}\mathrm{d}t_2 \quad (2)$$

$$\gamma_1 = \frac{1}{\sqrt{1-\beta_1^2\left(1-\dfrac{1}{\beta_2^2}\right)}};\ \gamma_2 = \frac{1}{\sqrt{1-\beta_2^2\left(1-\dfrac{1}{\beta_1^2}\right)}};\ \bar{v}_1 = \frac{\mathrm{d}\bar{x}}{\mathrm{d}t_1};\ \bar{v}_2 = \frac{\mathrm{d}\bar{x}}{\mathrm{d}t_2}$$

$$v_1 = |\bar{v}_1|,\ v_2 = |\bar{v}_2|;\ \beta_1^2 = \frac{v_1^2}{c_1^2},\ \beta_2^2 = \frac{v_2^2}{c_2^2};\ \rho = \frac{c_2}{c_1},\ \frac{\mathrm{d}t_1}{\mathrm{d}t_2} = k = \frac{v_2}{v_1};\ \lambda = \frac{c_2}{c_1k} = \frac{\rho}{k}$$

i) Five Velocities in 2T + 3S Dimensions

We can define two types of five velocities ($\alpha = 1,2,3,4,5$)

$$\left(u_1\right)_\alpha = \frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau_1}; \left(u_2\right)_\alpha = \frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau_2} \ \left(x_1 = c_1 t_1,\ x_2 = c_2 t_2,\ x_3 = x,\ x_4 = y,\ x_5 = z\right)$$

Explicitly, some of the quantities are,

$$\left(u_1\right)_1 = c_1\gamma_1;\ \left(u_1\right)_2 = c_2\gamma_1\frac{1}{k};\ \overline{u}_1 = \gamma_1\overline{v}_1 \tag{3}$$

The other types of quantities are,

$$\left(u_2\right)_1 = c_1 k\gamma_2;\ \left(u_2\right)_2 = c_2\gamma_2;\ \overline{u}_2 = \gamma_2\overline{v}_2 \tag{4}$$

It can be easily proved (for each type),

$$\begin{aligned}\left(u_1\right)_\alpha\left(u_1\right)^\alpha &= \left(u_1\right)_1^2 + \left(u_1\right)_2^2 - \left(u_1\right)_3^2 - \left(u_1\right)_4^2 - \left(u_1\right)_5^2 = c_1^2\\ \left(u_2\right)_\alpha\left(u_2\right)^\alpha &= \left(u_2\right)_1^2 + \left(u_2\right)_2^2 - \left(u_2\right)_3^2 - \left(u_2\right)_4^2 - \left(u_2\right)_5^2 = c_2^2\end{aligned} \tag{5}$$

ii) Energy-Momentum Five Vectors

Just as there are two types of five-velocities, there are two types of energy-momentum five vectors.

$\left(p_1\right)_\alpha = m_0\left(u_1\right)_\alpha$

$\left(p_2\right)_\alpha = m_0\left(u_2\right)_\alpha \left(\alpha = 1,2,3,4,5\right)$ and m_0 is something like **mass in five dimension**

More explicitly, the space components are

$\overline{p}_1 = m_0\overline{u}_1 = m_0\gamma_1\overline{v}_1$

$\overline{p}_2 = m_0\overline{u}_2 = m_0\gamma_2\overline{v}_2$

It is obvious that $\left(p_1\right)^\alpha\left(p_1\right)_\alpha = m_0^2c_1^2$ and $\left(p_2\right)^\alpha\left(p_2\right)_\alpha = m_0^2c_2^2$

For more energy-momentum formulations in five-dimensions, see ref. (1) (6)

We can derive a relation $\rho\gamma_1 = \gamma_2 k$ and with this we can obtain $\beta_1\gamma_1 = \beta_2\gamma_2$. Also, using $\overline{v}_2 = \frac{\mathrm{d}\overline{x}}{\mathrm{d}t_2} = \frac{\mathrm{d}\overline{x}}{\mathrm{d}t_1}\frac{\mathrm{d}t_1}{\mathrm{d}t_2} = \overline{v}_1 k$ and $v_2 = \left|\overline{v}_2\right| = k\left|\overline{v}_1\right| = kv_1$ we can have another relation, $\rho\beta_2 = k\beta_1$.

iii) Proper Time Using c_e

We can also define another type of proper time using c_e,

$$\mathrm{d}\tau_e = \frac{\mathrm{d}s}{c_e}$$

The three proper times are related as

$$c_e\mathrm{d}\tau_e = c_1\mathrm{d}\tau_1 = c_2\mathrm{d}\tau_2$$

We can find a relation between γ_{1e} and γ_1 remembering that $\beta_1 = \frac{v_1}{c_1}$, $\beta_2 = \frac{v_2}{c_2}$, $v_2 = kv_1$

$$\gamma_1 = \frac{1}{\sqrt{1-\beta_1^2\left(1-\frac{1}{\beta_2^2}\right)}} = \frac{c_1}{c_e}\gamma_{1e} \tag{7}$$

$$\gamma_2 = \frac{\rho}{k}\gamma_1 = \frac{c_2}{kc_e}\gamma_{1e} \text{ where } \gamma_{1e} = \frac{1}{\sqrt{1-\frac{v_1^2}{c_e^2}}} \tag{8}$$

Using the various relations given above, it can be shown that

$$\mathrm{d}\tau_e = \frac{\mathrm{d}t_1}{\gamma_{1e}} = \frac{k\mathrm{d}t_2}{\gamma_{1e}}$$

The corresponding five velocities and momenta are ($\mu = 1,2,3,4,5$),

$$\begin{aligned} u_e^\mu &= \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau_e};\ u_e^\mu u_{e\mu} = c_e^2 \\ p_e^\mu &= m_0 u_e^\mu;\ p_e^\mu p_{e\mu} = m_0 c_e^2 \end{aligned} \tag{9}$$

More explicitly, we have,

$$u_e^1 = \frac{c_1\mathrm{d}t_1}{\mathrm{d}\tau_e} = c_1\gamma_{1e};\ u_e^2 = \frac{c_2\mathrm{d}t_2}{\mathrm{d}\tau_e} = \frac{c_2}{k}\gamma_{1e};\ \bar{u}_e = \frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau_e} = \frac{\mathrm{d}\bar{x}}{\mathrm{d}t_1}\gamma_{1e} = \bar{v}_1\gamma_{1e} \tag{10}$$

We use them in **Appendix A**.

2) Critical comments: Analyzing some expressions, matrix form for proper Lorentz transformation and group characterization of five-dimensional invariant transformations

The formulations presented in the preceding subsection a) are developed for five-dimensional invariant transformations. The crucial point is that we expanded space-time from four-dimensional Minkowski space (*i.e.*, 1T + 3S) to five-dimensional space-time (*i.e.*, 2T + 3S) by adding an extra time dimension with a distinct speed of causality, c_2. However, it is interesting to explore how much of the four-dimensional formulation is restored by taking the limit $c_2 \to 0$, or taking the $k \to 0$ limit. We must be careful when reproducing limiting results in later formulations of the theory. Often, we arrive at the final expressions by going through intermediate steps where we may divide by k or c_2. However, mathematically, we cannot perform such divisions if $k = 0$ or $c_2 = 0$. In fact, to obtain 1T + 3S-dimensional results, it may be preferable to use the $c_2 \to 0$ limit. See comments in Equation (1) box above.

Let's look at γ_1 first:

$$\gamma_1 = \frac{1}{\sqrt{1-\beta_1^2\left(1-\frac{1}{\beta_2^2}\right)}}$$

To take the limit $c_2 \to 0$, we need to consider the actual expression involving β_2 which is $\left(1/\beta_2\right)^2$. However, $\frac{1}{\beta_2^2} = \left(\frac{c_2}{v_2}\right)^2 = 0$ as $c_2 \to 0$.

So,

$$\gamma_1 = \frac{1}{\sqrt{1-\beta_1^2\left(1-\frac{1}{\beta_2^2}\right)}} \to \frac{1}{\sqrt{1-\beta_1^2}}$$

Now consider,

$$\gamma_2 = \frac{1}{\sqrt{1-\beta_2^2\left(1-\frac{1}{\beta_1^2}\right)}} = \frac{\frac{1}{\beta_2}}{\sqrt{\frac{1}{\beta_2^2}-1+\frac{1}{\beta_1^2}}} = \frac{0}{\sqrt{0-1+\frac{1}{\beta_1^2}}} = 0 \quad \text{as} \quad c_2 \to 0 \quad \text{(see above)}$$

It is more convenient to re-express some of the terms as,

$$\gamma_1 = \frac{c_1}{c_e}\gamma_{1e}\,;\, \gamma_2 = \frac{c_2}{c_e}\frac{1}{k}\gamma_{1e}\,;\, c_e = \sqrt{c_1^2 + c_2^2/k^2} = c_1\left(1+\lambda^2\right)^{1/2}$$

$$\gamma_{1e} = \frac{1}{\sqrt{1-\frac{v_1^2}{c_e^2}}} = \frac{1}{\sqrt{1-\beta_{1e}^2}};\ \beta_{1e} = \frac{v_1}{c_e}$$

We should make it clear that k and ρ have the same dimension and λ is dimensionless.

As $c_e \to c_1$, we get the same results as above in the limit $c_2 \to 0$. This also makes all the second components of the five-velocities and five-momenta zero in the same limit. *We emphasize that there are no singularities or imaginary numbers appearing in the limiting process.*

Another point we would like to clarify is the consequence of using the relation (let's call it the "scaling relation between times t_1 and t_2", with k as the scale factor),

$$w = \frac{\mathrm{d}x_0}{\mathrm{d}t_2} = \frac{\mathrm{d}x_0}{\mathrm{d}t_1}\cdot\frac{\mathrm{d}t_1}{\mathrm{d}t_2} = v\cdot k \text{, where } k = \frac{\mathrm{d}t_1}{\mathrm{d}t_2}\text{. This implies that } t_1 = k\cdot t_2\,. \quad (11)$$

This relation of Equation (11) is not essential for constructing the relativistic space-time transformation in the 2T + 3S dimensions. The most general five-dimensional relativistic coordinate-transformation formulations are given in the boxes of Equations (10)-(12) in ref. [7]. In **Appendix A** of [7], we demonstrated the invariance of the five-dimensional coordinate norm using the software tool Maple™ [8] without requiring the relations expressed in Equation (11). The scaling relation clarifies the two times in the formulation and supports real-world implications. In the real world, we have only one clock system representing t_1, and t_2 is not measurable. So, it is meaningful to use v as the uniform relative speed of the moving reference frame (K′) with respect to the stationary one (K) in a standard configuration. However, the real-world interpretation of w remains unclear. The linear scaling relation in Equation (11) is the simplest one to use, and it provides a way to express t_2 and w with respect to t_1 and v, subject to a value for k to be determined. Other functional forms, such as nonlinear ones, would complicate the formulations.

The coordinate transformation formulas in Equation (1) are obtained by using the scaling relation of Equation (11) that combines terms involving t_1 and t_2, to some extent, and gives the effective velocity c_e. They also preserve the invariance of the norms of the space-time coordinates between K and K′ in the five-dimensional Minkowski space. However, this does not work for a general five-vector defined in the 2T + 3S paradigm. Using only $w = k.v$ (and not combining the t_1

and t_2 terms using the $t_1 = kt_2$ relation), we can simplify the expressions of the elements of the general transformation matrix in the boxes of Equations (10)-(12) in reference [7] and construct the general five-dimensional Lorentz-like transformation matrix Λ from the Maple™ [8] output in **Appendix B** of reference [7]. After a tedious calculation, the matrix Λ is obtained as,

$$\Lambda = \begin{bmatrix} \frac{1}{\eta_0^2}\left(1+\lambda^2\gamma_e^{-1}\right)\gamma_e & \frac{\lambda}{\eta_0^2}\left(1-\gamma_e^{-1}\right)\gamma_e & -\frac{1}{\eta_0}\beta_e\gamma_e & 0 & 0 \\ \frac{\lambda}{\eta_0^2}\left(1-\gamma_e^{-1}\right)\gamma_e & \frac{1}{\eta_0^2}\left(\lambda^2+\gamma_e^{-1}\right)\gamma_e & -\frac{\lambda}{\eta_0}\beta_e\gamma_e & 0 & 0 \\ -\frac{1}{\eta_0}\beta_e\gamma_e & -\frac{\lambda}{\eta_0}\beta_e\gamma_e & \gamma_e & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Here,

$$\eta_0 = \sqrt{1+\lambda^2};\ \beta_e = \frac{v}{c_e};\ \gamma_e = \frac{1}{\sqrt{1-\beta_e^2}}$$

We would like to confirm that if we further consider the $t_1 = kt_2$ relation, the elements of the transformation matrix Λ simplify to the expressions in Equation (1).

If X is the five-vector with components $(X_1, X_2, X_3, X_4, X_5) = (c_1t_1, c_2t_2, x, y, z)$ represented as a column vector in reference frame K, and X' with components $(X_1', X_2', X_3', X_4', X_5') = (c_1t_1', c_2t_2', x', y', z')$ having a column vector representation is the corresponding five-vector in reference frame K′, then

$$X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu}$$

In matrix form we have, $X' = \Lambda X$ and $X'^{\mathrm{T}} = X^{\mathrm{T}}\Lambda^{\mathrm{T}}$. Here T stands for transpose.

Let Y and Y' be another similar pair of vectors,

$$Y'^{\mu} = \Lambda^{\mu}{}_{\nu} Y^{\nu}$$

In matrix form we have, $Y' = \Lambda Y$ and $Y'^{\mathrm{T}} = Y^{\mathrm{T}}\Lambda^{\mathrm{T}}$.

then, for the invariant relation under the five-dimensional Lorentz transformation

$$X^{\mu}Y_{\mu} = X'^{\mu}Y'_{\mu} \tag{13}$$

we compute as follows, $X^{\mu}Y_{\mu} = X^{\mu}g_{\mu\nu}Y^{\nu}$ and in matrix form it is $X^{\mathrm{T}}gY$ where g is the matrix form of the metric tensor.

Similarly, $X'^{\mu}Y'_{\mu} \Rightarrow$ (in matrix form) $X'^{\mathrm{T}}gY' = X^{\mathrm{T}}\Lambda^{\mathrm{T}}g\Lambda Y = X^{\mathrm{T}}gY$ (Equation (13)) implying,

$$\Lambda^{\mathrm{T}}g\Lambda = g \tag{14}$$

It can be shown (see **Appendix B** at the end of this paper) that Λ in Equation (12) indeed satisfies the condition in Equation (14). Thus, it represents a homogeneous Lorentz transformation in five dimensions of the 2T + 3S paradigm. In four dimensions, we denote the symmetry group of this transformation as SO(1,

3); here, it is SO(2, 3).

This observation compares with the findings of Khadka and Karki [9] and Khadka [10], although they did not consider the 2T + 3S space-time structure. They considered a conventional Minkowski space-time with a single speed of causality. Khadka [10] developed a matrix formulation of Lorentz transformations restricted to two-space-dimensional planar motion in two cases: 1) relative velocity of reference frames is collinear with the spatial position vector and 2) generalized to arbitrary boost directions within the plane. They found that the planar Lorentz transformation equations obtained for the 2S +1T conventional Minkowski space-time with metric signature (+, +, −) could not be expressed in a compact matrix form in a "straightforward" manner. However, they took an alternative approach by embedding the system in a higher-dimensional space-time (2S + 2T) with metric signature (+, +, −, −) and represented the Lorentz transformation in a matrix form l that satisfied the invariance condition $l^{\mathrm{T}} g l = g$, where g is the corresponding metric tensor matrix defined as $g_{11} = g_{22} = +1$; $g_{33} = g_{44} = -1$; $g_{ij} = 0$, $i \neq j$. As Khadka [10] notes, l in his case and Λ in ours can be used to transform velocities, momenta, spin, and other physical quantities under planar Lorentz boosts or 2T + 3S-dimensional relativistic transformations, respectively. For example, Λ can transform spin in the Color Space [6] and explore scattering experiments with polarized photon beams on polarized protons.

In this paper, we keep the extra time dimension as a valid dimension (*i.e.*, not compactified); in earlier works [1] [4], this extra dimension was compactified following the Kaluza-Klein theory [11]. In this technique, the extra dimension (space or time) is curled up on a circle of radius r_0 that is very small (*i.e.*, of the order of the Planck time or Planck length). Because of the loop's periodicity, we expand all physical dependence on the compactified variable in a Fourier series. This yields a series of new physical quantities, such as fields with a tower of masses. For pedagogical purposes, we provided calculational details in ref. [4] using massive scalar and Abelian vector fields. Thus, higher-dimensional theories in physics are often exotic, with lofty predictions that remain experimentally unverified.

Before we continue to the next section, it is worth mentioning some 2T physics research that has appeared in the literature over the last two decades. Outlining a novel gauge symmetry principle, Bars [12] stressed the need for a 2T paradigm to develop a new formulation of physics in space-time to explain that "phenomena described by one-time physics in 3 + 1 dimensions appear as various 'shadows' of the same phenomena that occur in 4 + 2 dimensions with one extra space and one extra time dimension (more generally, d + 2)". Previously, Bars [13] further emphasized, "Thus, in the 2T setting, the distinguished 1T which we call 'time' is a gauge-dependent concept." In the same article [13], Bars also claimed that "2T-physics has mainly been developed in the context of particles, including spin and supersymmetry, but some advances have also been made with strings and p-branes, and insights for M-theory have already emerged." Bars also claimed [14] that Kaluza-Klein type compactification is "the wrong idea for extra time dimensions". The 2T paradigm we discuss here is much simpler: a 2T + 3S-dimensional

flat space with two distinct causal speeds that more closely reflect reality. However, two-time physics has also faced conceptual and theoretical challenges. Some researchers ([15] and references therein) have discussed technical/mathematical issues of "on determinism and well-posedness in multiple time dimensions" in relation to the initial value problem, "due to failure of uniqueness." We do not know how these matters may affect the 2T paradigm of this research, and we leave them as topics for future research.

3. Five-Dimensional Electro-Magnetic Field Tensor

Using one ansatz about the second component of the five-potential (A_μ), Zahir [6] derived the anti-symmetric field tensors $F_{\mu\nu}$ and $F^{\mu\nu}$ as given below. In terms of the five-potential A_μ, we have $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$$A_\mu \to A_1, A_2, -\bar{A} \to \varphi, \frac{k}{\rho}\varphi, -A_x, -A_y, -A_z \text{ for } \mu = 1,2,3,4,5, \text{respectively.} \quad (15)$$

The standard relations $\bar{E} = -\frac{1}{c_1}\frac{\partial \bar{A}}{\partial t_1} - \bar{\nabla}\varphi$ and $\bar{B} = \bar{\nabla} \times \bar{A}$ were also assumed. In addition, the relation $t_1 = k \cdot t_2$ was used in defining the five derivatives (see the paragraph just above the Equation (1) in [6]). Quantities with a bar on top or in bold font imply three-vectors. While deriving the five-dimensional Maxwell's equation, Zahir [6] also used the second ansatz about the second component of the five-current J'_μ and the five-dimensional Lorentz condition $\partial_\mu A^\mu = 0$ (see the paragraph above Equation (3) in [6] and Equation (20) below in this paper).

In the matrices below, μ and ν are the row and column indices, taking values from 1 to 5.

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} \end{bmatrix} = \begin{bmatrix} 0 & 0 & E_x & E_y & E_z \\ 0 & 0 & \frac{k}{\rho}E_x & \frac{k}{\rho}E_y & \frac{k}{\rho}E_z \\ -E_x & -\frac{k}{\rho}E_x & 0 & -B_z & B_y \\ -E_y & -\frac{k}{\rho}E_y & B_z & 0 & -B_x \\ -E_z & -\frac{k}{\rho}E_z & -B_y & B_x & 0 \end{bmatrix} \quad (16)$$

$$F^{\mu\nu} = \begin{bmatrix} F^{11} & F^{12} & F^{13} & F^{14} & F^{15} \\ F^{21} & F^{22} & F^{23} & F^{24} & F^{25} \\ F^{31} & F^{32} & F^{33} & F^{34} & F^{35} \\ F^{41} & F^{42} & F^{43} & F^{44} & F^{45} \\ F^{51} & F^{52} & F^{53} & F^{54} & F^{55} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -E_x & -E_y & -E_z \\ 0 & 0 & -\frac{k}{\rho}E_x & -\frac{k}{\rho}E_y & -\frac{k}{\rho}E_z \\ E_x & \frac{k}{\rho}E_x & 0 & -B_z & B_y \\ E_y & \frac{k}{\rho}E_y & B_z & 0 & -B_x \\ E_z & \frac{k}{\rho}E_z & -B_y & B_x & 0 \end{bmatrix} \quad (17)$$

$$g^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$F^{\alpha\beta} = g^{\alpha\nu} F_{\nu\delta} g^{\delta\beta}$ (summation over repeated indices is understood)

The antisymmetric field tensor $F^{\mu\nu}$ has 10 independent components. Because of the specific ansatz mentioned earlier, one component is identically zero; six are the components of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$; and three are the components of $(k/\rho)\boldsymbol{E}$. We can also introduce the Levi-Civita epsilon symbol to define the field tensor $\tilde{F}^{\mu\nu}$ which is the dual of $F^{\mu\nu}$ as,

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

The homogeneous Maxwell's equations can be obtained in a covariant form as,

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0 \tag{18}$$

or

$$\partial_\lambda F_{\mu\nu} + \partial_\lambda F_{\mu\nu} + \partial_\lambda F_{\mu\nu} = 0 \tag{19}$$

where λ, μ, and ν are all different, with possible values (1, 2, 3, 4, 5) for each. In ref. [6], the latter method was used. The inhomogeneous Maxwell's equations were derived in a covariant form via

$$\partial_\mu F^{\mu\nu} = 4\pi J'^{\nu} \tag{20}$$

We used another ansatz regarding the second component of the charge-current five-vector J'^{ν},

$$J'^{1} = \rho_0; J'^{2} = \frac{k}{\rho}\rho_0; \overline{\mathrm{J}}' = \frac{\overline{\mathrm{J}}}{c_1}$$

The Equations (18) and (20) demonstrate the lack of a perfect symmetry that would have been possible "if there existed a 'magnetic current density' J_m^β which would serve as the source of the dual field tensor $\tilde{F}^{\alpha\beta}$" (see ref. [16], p. 160). So far, no magnetic charge (*i.e.*, monopoles) has been found experimentally.

4. Covariant Lagrangian Formulation and Derivation of the Lorentz Force Using the Five-Dimensional Electromagnetic Field Tensors in the 2T + 3S Dimension

In this section, we formulate the equation of motion for a charged particle of mass m in five-dimensional space interacting with the five-dimensional electromagnetic field, following Zahir [6] in the two-time paradigm. We will take the invariant relativistic Lagrangian approach [16] [17].

First, we consider a free-particle Lagrangian in five dimensions (mass = m_0). The action is given by,

$$A=\alpha\int \mathrm{d}s=\alpha\int\sqrt{\mathrm{d}x_{\mu}\mathrm{d}x^{\mu}}=\alpha\int\sqrt{c_1^2\mathrm{d}t_1^2+c_2^2\mathrm{d}t_2^2-\mathrm{d}x^2-\mathrm{d}y^2-\mathrm{d}z^2}$$

$$A=\alpha\int\sqrt{c_1^2+c_2^2/k^2-\mathrm{d}x^2/\mathrm{d}t_1^2-\mathrm{d}y^2/\mathrm{d}t_1^2-\mathrm{d}z^2/\mathrm{d}t_1^2}\,\mathrm{d}t_1$$

$$A=\alpha c_e\int\sqrt{1-\left(\frac{\overline{v}_1}{c_e}\right)^2}\,\mathrm{d}t_1=\alpha c_e\int\sqrt{1-\frac{v_1^2}{c_e^2}}\mathrm{d}t_1 \tag{21}$$

So, the Lagrangian is, $L=\alpha c_e\sqrt{1-\frac{v_1^2}{c_e^2}}$

This essentially is the kinetic energy T of a free particle. In the non-relativistic limit when $v_1\ll c_e$, it should be equal to $\frac{1}{2}m_o v_1^2$

We expand L,

$$L=T\approx\alpha c_e-\frac{1}{2}\frac{\alpha}{c_e}v_1^2$$

"Apart from first term which is constant and is irrelevant for equations of motion" (see ref. [16], p. 136), this matches the non-relativistic kinetic energy ($\frac{1}{2}m_o v_1^2$) if $\alpha=-c_e m_o$.

Thus, the free particle Action is,

$$A=-m_o c_e\int \mathrm{d}s=-m_o c_e\sqrt{g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}}$$

We could also integrate over $\mathrm{d}t_2$ and get the same result. However, $\mathrm{d}t_1$ or $\mathrm{d}t_2$ are not invariant in four- or five-dimensional relativity. We have the invariant proper times $\mathrm{d}\tau_1,\mathrm{d}\tau_2$ or $\mathrm{d}\tau_e$. But, dividing $\mathrm{d}x^{\mu}$ by any of the proper times, we get the five velocities that are constrained as,

$$u_{1\mu}u_1^{\mu}=c_1^2;\ u_{2\mu}u_2^{\mu}=c_2^2;\ u_{e\mu}u_e^{\mu}=c_e^2 \tag{22}$$

Then, to obtain the Lagrange equation, we need to perform the variational calculus subject to this additional constraint (*i.e.*, with a Lagrange multiplier). We avoid this by assuming the world line $x^{\mu}(\lambda')$ of the particle to depend on an invariant parameter λ', which, as a function of τ_i, increases monotonically with τ_i but is otherwise arbitrary [16] [17].

Then, ds can be rewritten as,

$$\mathrm{d}s=\sqrt{\mathrm{d}x_{\mu}\mathrm{d}x^{\mu}}=\sqrt{g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}}=\sqrt{g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda'}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda'}}\,\mathrm{d}\lambda'=\sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}\,\mathrm{d}\lambda'$$

$$\dot{x}^{\mu}=\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda'} \tag{23}$$

The quantities to be varied are the coordinates x^{μ} and the derivatives $\dot{x}^{\mu}$. At the end, after deriving the Lagrange equation, $\frac{\mathrm{d}}{\mathrm{d}\lambda'}\frac{\partial L}{\partial\dot{x}^{\mu}}-\frac{\partial L}{\partial x^{\mu}}=0$, we identify the parameter λ' with τ_i and restore $\sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}\,\mathrm{d}\lambda'=c_i\mathrm{d}\tau_i;i=1,2,e$.

Based on the above discussions, we can now add the interaction term,

$$\frac{q}{c_e}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda'}A_\mu \tag{24}$$

The complete action integral for a charged particle interacting with the electromagnetic fields in 2T + 3S dimensions is as follows (see [16] and [17] for the rest of the details):

$$A=-\int\left\{mc_e\sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda'}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda'}}+\frac{q}{c_e}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda'}A_\mu\right\}\mathrm{d}\lambda' \tag{25}$$

The term inside the curly bracket is the Lagrangian *L*.

The Lagrange equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda'}\frac{\partial L}{\partial\dot{x}^\mu}-\frac{\partial L}{\partial x^\mu}=0$$

$$mc_e\frac{\mathrm{d}}{\mathrm{d}\lambda'}\left[\frac{\dot{x}_\mu}{\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\right]+\frac{q}{c_e}\frac{\mathrm{d}A_\mu}{\mathrm{d}\lambda'}-\frac{q}{c_e}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda'}\frac{\partial A_\nu}{\partial x^\mu}=0 \tag{26}$$

Now, we can choose $\lambda'=\tau_1,\tau_2$ or τ_e. It can be shown that τ_1 or τ_2 gives the same result at the end. So, we consider two cases: $\lambda'=\tau_1$ and τ_e.

Case 1: $\lambda'=\tau_1$

Here, we can use

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau_1}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau_1}=u_1^\mu u_{1\mu}=c_1^2$$

$$\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau_1}=\frac{\partial A_\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau_1}$$

Transposing terms, we finally get,

$$\begin{aligned}m_o\frac{\mathrm{d}u_{1\mu}}{\mathrm{d}\tau_1}&=\frac{qc_1}{c_e^2}u_1^\nu\left(\frac{\partial A_\nu}{\partial x^\mu}-\frac{\partial A_\mu}{\partial x^\nu}\right)=\frac{qc_1}{c_e^2}u_1^\nu F_{\mu\nu}\\ \frac{\mathrm{d}p_{1\mu}}{\mathrm{d}\tau_1}&=\frac{qc_1}{c_e^2}u_1^\nu F_{\mu\nu}\end{aligned} \tag{27}$$

Next, we explore the space coordinates in Equation (27).

Let $\mu=3$, (*i.e.*, the x coordinate)

$$\frac{\mathrm{d}p_{13}}{\mathrm{d}\tau_1}=\frac{qc_1}{c_e^2}u_1^\nu F_{3\nu}=\frac{qc_1}{c_e^2}\left(u_1^1F_{31}+u_1^2F_{32}+u_1^3F_{33}+u_1^4F_{34}+u_1^5F_{35}\right)$$

Using (see Equation (3))

$$u_1^1=c_1\gamma_1;\ u_1^2=c_2\gamma_1\frac{1}{k};\ \bar{u}_1=\gamma_1\bar{v}_1$$

and tensor elements from the matrices in Equations (16)-(17), we get,

$$-\frac{\mathrm{d}p_{1x}}{\mathrm{d}\tau_1}=\frac{qc_1}{c_e^2}\left\{c_1\gamma_1\left(-E_x\right)+c_2\gamma_1\frac{1}{k}\left(-\frac{k}{\rho}E_x\right)+\gamma_1v_{1y}\left(-B_z\right)+\gamma_1v_{1z}B_y\right\};\rho=\frac{c_2}{c_1}$$

$$\frac{\mathrm{d}p_{1x}}{\mathrm{d}\tau_1}=\frac{qc_1}{c_e^2}\left\{2c_1\gamma_1E_x+\gamma_1\left(\bar{v}_1\times\bar{B}\right)_x\right\}$$

Since $\mathrm{d}\tau_1 = \mathrm{d}t_1/\gamma_1$, we finally have,

$$\frac{\mathrm{d}p_{1x}}{\mathrm{d}t_1} = \frac{qc_1^2}{c_e^2}\left\{2E_x + \frac{1}{c_1}\left(\overline{v}_1 \times \overline{B}\right)_x\right\} \tag{28}$$

The expressions for $\mu = 4, 5$ (*i.e.*, the y, z coordinates) are similar. Combining them all,

$$\frac{\mathrm{d}\overline{p}_1}{\mathrm{d}t_1} = \frac{qc_1^2}{c_e^2}\left\{2\overline{E} + \frac{1}{c_1}\left(\overline{v}_1 \times \overline{B}\right)\right\};\ \overline{p}_1 = m_o\overline{v}_1\gamma_1 \tag{29}$$

Remember, $c_e^2 = c_1^2 + c_2^2/k^2 = c_1^2\left(1+\lambda^2\right)$ where $\lambda = \frac{c_2}{c_1 k}$; $\frac{c_1^2}{c_e^2} = \frac{1}{1+\lambda^2}$

Thus, we get,

$$\frac{\mathrm{d}\overline{p}_1}{\mathrm{d}t_1} = \frac{q}{1+\lambda^2}\left\{2\overline{E} + \frac{1}{c_1}\left(\overline{v}_1 \times \overline{B}\right)\right\}$$

In ref. [6], we proposed that q should be replaced by the effective charge,

$$\tilde{q} = \left(1+\frac{k^2}{\rho^2}\right)q = \left(1+\frac{1}{\lambda^2}\right)q = \frac{1+\lambda^2}{\lambda^2}q$$

So, if we start Equation (25) with the effective charge $\tilde{q}$, we will get the Lorentz Force Equation as,

$$\frac{\mathrm{d}\overline{p}_1}{\mathrm{d}t_1} = \frac{2q}{\lambda^2}\overline{E} + \frac{q}{\lambda^2 c_1}\left(\overline{v}_1 \times \overline{B}\right);\ \overline{p}_1 = m_o\overline{v}_1\gamma_1 \tag{30}$$

Case 2: $\lambda' = \tau_e$

Here, we can use

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau_e}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau_e} = u_e^\mu u_{e\mu} = c_e^2 \quad \text{(see Equation (22))} \tag{31}$$

$$\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau_e} = \frac{\partial A_\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau_e}$$

Transposing terms, we finally get from Equation (27),

$$\begin{aligned} m_o\frac{\mathrm{d}u_{e\mu}}{\mathrm{d}\tau_e} &= \frac{q}{c_e}u_e^\nu\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right) = \frac{q}{c_e}u_e^\nu F_{\mu\nu} \\ \frac{\mathrm{d}p_{e\mu}}{\mathrm{d}\tau_e} &= \frac{q}{c_e}u_e^\nu F_{\mu\nu} \end{aligned} \tag{32}$$

We find that the space components of the above equation are exactly those in Equation (29), and the expression for the Lorentz Force Equation is the same as in Equation (30). See **Appendix A** for details of the calculation.

We can make a couple of comments here:

1) It is not clearly understandable what the meaning of the leftover factor $1/\lambda^2$ is. However, if we define the interaction term of the action in Equation (24) as (λ is dimensionless),

$$\lambda^2\frac{q}{c_e}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda'}A_\mu$$

the Equation (23) will not involve λ^2 anymore.

2) The factor 2 in the first term of the right-hand side in Equations (29), (30), unlike in the four-dimensional case, comes from the two components of the scalar potential (see Equation (11)) and due to the specific ansatz for it in the definition of the five-potential A_μ. To see how the formulations appear in four-dimensional electrodynamics, taking the $k \to 0$ limit is not meaningful, as k enters the formulations both from the five-potential (see Equation (11)) and from the five-dimensional space-time structure due to $\frac{\mathrm{d}t_1}{\mathrm{d}t_2} = k$. Similarly, c_2 enters the equations of motion from the space-time structure, kinematics, and electrodynamic formulations and thus presents a challenge to the limit $c_2 \to 0$. Still, taking the limit $c_2 \to 0$ provides more encouraging results as in this limit $c_e = c_1$. For example, if we consider Equation (29) and take $c_e = c_1$, the Lorentz equation becomes

$$\frac{\mathrm{d}\overline{p}_1}{\mathrm{d}t_1} = q\left\{2\overline{E} + \frac{1}{c_1}\left(\overline{v}_1 \times \overline{B}\right)\right\};\ \overline{p}_1 = m_o \overline{v}_1 \gamma_1$$

It is like the four-dimensional equation, except for the factor 2, which we can attribute to the two scalar components of the five-potential. For the sake of illustration, let's investigate the definition of the effective charge,

$$\tilde{q} = \left(1 + \frac{k^2}{\rho^2}\right)q = \left(1 + \frac{1}{\lambda^2}\right)q$$

We should remember that $\rho = c_2/c_1$ and has the same dimension as k. If we $c_2 \to 0$ or $k \to 0$ individually, we get one kind of result, and if we take the limits simultaneously, we get another.

For $k \to 0$, $\tilde{q} \to q$. For $c_2 \to 0$, $\rho \to 0$, $\tilde{q} \to \infty$.

For, $\rho \to 0$ and $k \to 0$ simultaneously,

$\frac{k^2}{\rho^2} = \frac{2k}{2\rho} = \frac{2}{2} = 1$ (a kind of modified L'Hospital's rule!). This applies to $\frac{k}{\rho}$, too.

This illustration stresses the need for extra caution when taking limits to derive four-dimensional results from five-dimensional ones. However, we emphasize that the additional λ-dependent factor and the extra factor of 2 in the electric term of the Lorentz force equation are features of the proposed ansatzes, and we note that simply taking parameter limits does not fully recover the four-dimensional expressions.

5. Electromagnetic Stress and Energy Density: The New Formulation in the 2T + 3S Dimensions

The electromagnetic stress-energy tensor $T^{\mu\nu}$ is given by [16],

$$T^{\mu\nu} = \frac{1}{4\pi}\left\{g^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right\} \tag{33}$$

Using the electromagnetic field-tensor elements from Section 3, we derive the desired results after a tedious calculation. We define,

$$X^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu}$$

$$Y = F_{\alpha\beta} F^{\alpha\beta}$$

We used the antisymmetric property of the field tensor and calculated the trace to derive the expression for Y. We get,

$$X^{11} = E^2 ; X^{22} = \frac{k^2}{\rho^2} E^2$$

$$Y = 2B^2 - 2\left(1 + \frac{k^2}{\rho^2}\right) E^2$$

Finally, the Equation (33) gives

$$T^{11} = \frac{1}{8\pi}\left\{B^2 + \left(1 - \frac{k^2}{\rho^2}\right) E^2\right\}$$

$$T^{22} = \frac{1}{8\pi}\left\{B^2 - \left(1 - \frac{k^2}{\rho^2}\right) E^2\right\}$$

where, $E^2 = \bar{E} \cdot \bar{E}, B^2 = \bar{B} \cdot \bar{B}$ (34)

Remember, in four-dimensional electrodynamics we have,

$$T^{00} \Rightarrow T^{11} = \frac{1}{8\pi}\left\{B^2 + E^2\right\}$$

It is not yet clear how we can aggregate T^{11} and T^{22} to provide a meaningful interpretation.

6. Leading to the Quantum Formulations from the Classical Field Theory

We have made a bold proposition in ref. [6] identifying the five-dimensional 2T + 3S space with the Color-Space. We tied the extra time dimension to gluons moving at an invariant intrinsic speed c_2. In ref. [6], we estimated values for c_2 from the R-values in $e^+ e^-$ collisions for $k \approx 1$ using a limit on λ derived from the R-values. In a separate paper, Elliott, Massarczyk and Zahir [18] used the five-dimensional Lorentz-invariance model of [1] developed in the 2T paradigm. They also explored different experimental limits on λ including Lorentz-violating parameters from the Standard-Model Extension.

In this paper, we take a classical approach to investigate the Lorentz Force Equation and the electromagnetic energy density in the proposed five-dimensional Color Space. However, the hadronic Color-Space dimension, about 10^{-15} meters, warrants a formal quantum-mechanical treatment. Until such theoretical work, beyond the scope of this paper, is completed, we cannot suggest any experimental test of this theory related to the Color-Space.

7. Discussions and Conclusions

We investigated five-dimensional classical electrodynamics in 2T + 3S-dimen-

sional spacetime. We derived the Lorentz Force Equation (LFE) in a covariant Lagrangian formulation. After applying the effective-charge concept and the formula given in Section 4, the LFE approaches the four-dimensional form. We do not know how to explain the extra term $1/\lambda^2$. However, as outlined in the previous section, we can address it by redefining the interaction term. We also note a factor of 2 in the first term because the ansatz defines an extra component of the five-potential. In Section 5, we calculated the electromagnetic energy density. However, it is unclear how to combine T^{11} and T^{22} to interpret the result meaningfully. Following the proposal in ref. [6], if we identify the 2T + 3S spacetime with the hadronic Color-Space (10^{-15} meters), the classical approach in this paper requires a quantum formulation. Once this quantum formulation is complete, we can experimentally verify the predictions. Further work remains for future studies. We acknowledge that stability, causality, boundary conditions, and the quantum interpretation of a second time dimension remain open questions beyond the scope of this paper.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendices

Appendix A

Next, as before, we explore the space coordinate of Equation (32).

Let $\mu = 3$, (*i.e.*, the x coordinate)

$$\frac{\mathrm{d}p_{e3}}{\mathrm{d}\tau_e} = \frac{q}{c_e} u_e^{\nu} F_{3\nu} = \frac{q}{c_e}\left(u_e^1 F_{31} + u_e^2 F_{32} + u_e^3 F_{33} + u_e^4 F_{34} + u_e^5 F_{35}\right)$$

Using items from Equation (10) and tensor elements from the matrix in (16), we get,

$$-\frac{\mathrm{d}p_{ex}}{\mathrm{d}\tau_e} = -\frac{\mathrm{d}p_{ex}}{\mathrm{d}t_1}\gamma_{1e} = \frac{q}{c_e}\left\{c_1\gamma_{1e}\left(-E_x\right) + c_2\gamma_{1e}\frac{1}{k}\left(-\frac{k}{\rho}E_x\right) + \gamma_{1e}v_{1y}\left(-B_z\right) + \gamma_{1e}v_{1z}B_y\right\}; \rho = \frac{c_2}{c_1}$$

$$\frac{\mathrm{d}p_{ex}}{\mathrm{d}t_1} = \frac{qc_1}{c_e}\left\{2E_x + \frac{1}{c_1}\left(\overline{v}_1 \times \overline{B}\right)_x\right\}$$

Since $p_{ex} = m_o\left(\overline{v}_1\right)_x \gamma_{1e} = m_o\left(\overline{v}_1\right)_x \gamma_1 \frac{c_e}{c_1} = p_{1x}\frac{c_e}{c_1}$, we finally have,

$$\frac{\mathrm{d}p_{1x}}{\mathrm{d}t_1} = \frac{qc_1^2}{c_e^2}\left\{2E_x + \frac{1}{c_1}\left(\overline{v}_1 \times \overline{B}\right)_x\right\}$$

The expressions for $\mu = 4$ and 5 (*i.e.*, the x- and y-coordinates) are similar. Combining them all,

$$\frac{\mathrm{d}\overline{p}_1}{\mathrm{d}t_1} = \frac{qc_1^2}{c_e^2}\left\{2\overline{E} + \frac{1}{c_1}\left(\overline{v}_1 \times \overline{B}\right)\right\}$$

$$\overline{p}_1 = m_o\overline{v}_1\gamma_1$$

It is the same as the result for $\lambda' = \tau_1$ (see Equation (29)). As before,

$$c_e^2 = c_1^2 + c_2^2/k^2 = c_1^2\left(1+\lambda^2\right) \text{ where } \lambda = \frac{c_2}{c_1 k}$$

$$\frac{c_1^2}{c_e^2} = \frac{1}{1+\lambda^2}$$

Appendix B

$$\Lambda=\begin{bmatrix}\frac{1}{\eta_0^2}\left(1+\lambda^2\gamma_e^{-1}\right)\gamma_e & \frac{\lambda}{\eta_0^2}\left(1-\gamma_e^{-1}\right)\gamma_e & -\frac{1}{\eta_0}\beta_e\gamma_e & 0 & 0\\ \frac{\lambda}{\eta_0^2}\left(1-\gamma_e^{-1}\right)\gamma_e & \frac{1}{\eta_0^2}\left(\lambda^2+\gamma_e^{-1}\right)\gamma_e & -\frac{\lambda}{\eta_0}\beta_e\gamma_e & 0 & 0\\ -\frac{1}{\eta_0}\beta_e\gamma_e & -\frac{\lambda}{\eta_0}\beta_e\gamma_e & \gamma_e & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{g}=\left[g^{\alpha\beta}\right]=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1\end{bmatrix}$$

$$\Lambda^{\mathrm{T}}\boldsymbol{g}=\begin{bmatrix}\frac{1}{\eta_0^2}\left(1+\lambda^2\gamma_e^{-1}\right)\gamma_e & \frac{\lambda}{\eta_0^2}\left(1-\gamma_e^{-1}\right)\gamma_e & \frac{1}{\eta_0}\beta_e\gamma_e & 0 & 0\\ \frac{\lambda}{\eta_0^2}\left(1-\gamma_e^{-1}\right)\gamma_e & \frac{1}{\eta_0^2}\left(\lambda^2+\gamma_e^{-1}\right)\gamma_e & \frac{\lambda}{\eta_0}\beta_e\gamma_e & 0 & 0\\ -\frac{1}{\eta_0}\beta_e\gamma_e & -\frac{\lambda}{\eta_0}\beta_e\gamma_e & -\gamma_e & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1\end{bmatrix}$$

Define $M=\Lambda^{\mathrm{T}}\boldsymbol{g}\Lambda$.

We check some values of the elements of M using the Maple™ software in the following output and confirm that

$$M=g$$

```
> T1 := (1 + L^2/G)·G/E^2;
```

$$T1:=\frac{1+L^2\sqrt{-B^2+1}}{\sqrt{-B^2+1}\,E^2} \quad (1)$$

```
> E := sqrt(1 + L^2);
```

$$E:=\sqrt{L^2+1} \quad (2)$$

```
> T2 := L·(1 - 1/G)·G/E^2;
```

$$T2:=\frac{L\left(1-\sqrt{-B^2+1}\right)}{\sqrt{-B^2+1}\,\left(L^2+1\right)} \quad (3)$$

```
> T3 := B·G/E;
```

$$T3:=\frac{B}{\sqrt{-B^2+1}\,\sqrt{L^2+1}} \quad (4)$$

```
> S := T1·T1 + T2·T2 - T3·T3;
```

$$S:=\frac{\left(1+L^2\sqrt{-B^2+1}\right)^2}{\left(-B^2+1\right)\left(L^2+1\right)^2}+\frac{L^2\left(1-\sqrt{-B^2+1}\right)^2}{\left(-B^2+1\right)\left(L^2+1\right)^2}-\frac{B^2}{\left(-B^2+1\right)\left(L^2+1\right)} \quad (5)$$

```
> simplify(S);
```

$$1 \quad (6)$$

Legends: $L=\lambda, G=\gamma_e, E=\eta_0, B=\beta_e$

$$S=M_{11}=1$$

```
> G := 1/sqrt(1 - B^2);
```

$$G := \frac{1}{\sqrt{-B^2+1}} \tag{7}$$

```
> T4 := L·T3;
```

$$T4 := \frac{L\,B}{\sqrt{-B^2+1}\,\sqrt{L^2+1}} \tag{8}$$

```
> S2 := T3·T3 + T4·T4 - G·G;
```

$$S2 := -\frac{1}{-B^2+1} + \frac{B^2}{(-B^2+1)\,(L^2+1)} + \frac{L^2\,B^2}{(-B^2+1)\,(L^2+1)} \tag{9}$$

```
> S2S := Simplify(S2);
```

$$S2S := Simplify\left(-\frac{1}{-B^2+1} + \frac{B^2}{(-B^2+1)\,(L^2+1)} + \frac{L^2\,B^2}{(-B^2+1)\,(L^2+1)}\right) \tag{10}$$

```
> simplify(S2S);
```

$$Simplify(-1) \tag{11}$$

$$S2=S2S=M_{33}=-1$$

```
> simplify(S2);
```

$$-1 \tag{12}$$

```
> simplify(S);
```

$$1 \tag{13}$$

```
> S3 := T1·(-T3) + T2·(-T4) + T3·G;
```

$$S3 := \frac{B}{(-B^2+1)\sqrt{L^2+1}} - \frac{\left(1+L^2\sqrt{-B^2+1}\right)B}{(-B^2+1)\,(L^2+1)^{3/2}} - \frac{L^2\left(1-\sqrt{-B^2+1}\right)B}{(-B^2+1)\,(L^2+1)^{3/2}} \tag{14}$$

```
> simplify(S3);
```

$$0 \tag{15}$$

```
> T5 := ((L^2 + 1/G)·G)/E^2;
```

$$T5 := \frac{L^2+\sqrt{-B^2+1}}{\sqrt{-B^2+1}\,(L^2+1)} \tag{16}$$

```
> S22 := T2·T2 + T5·T5 - T4·T4;
```

$$S22 := \frac{L^2\left(1-\sqrt{-B^2+1}\right)^2}{(-B^2+1)\,(L^2+1)^2} - \frac{L^2\,B^2}{(-B^2+1)\,(L^2+1)} + \frac{\left(L^2+\sqrt{-B^2+1}\right)^2}{(-B^2+1)\,(L^2+1)^2} \tag{17}$$

```
> simplify(S22);
```

$$1 \tag{18}$$

```
> S21 := T2·T1 + T2·T5 - T3·T4;
```

$$S21 := \frac{\left(1+L^2\sqrt{-B^2+1}\right)L\left(1-\sqrt{-B^2+1}\right)}{(-B^2+1)\,(L^2+1)^2} + \frac{L\left(1-\sqrt{-B^2+1}\right)\left(L^2+\sqrt{-B^2+1}\right)}{(-B^2+1)\,(L^2+1)^2} - \frac{B^2\,L}{(-B^2+1)\,(L^2+1)} \tag{19}$$

```
> simplify(S21);
```

$$0 \tag{20}$$

```
> S31 := (-T3)·T1 + (-T4)·T2 + T3·G;
```

$$S31 := \frac{B}{(-B^2+1)\sqrt{L^2+1}} - \frac{\left(1+L^2\sqrt{-B^2+1}\right)B}{(-B^2+1)\,(L^2+1)^{3/2}} - \frac{L^2\left(1-\sqrt{-B^2+1}\right)B}{(-B^2+1)\,(L^2+1)^{3/2}} \tag{21}$$

```
> simplify(S31);
```

$$0 \tag{22}$$

```
>
```

$$S3=M_{13}=0$$

$$S22=M_{22}=1$$

$$S21=M_{21}=0$$

$$S31 = M_{31} = 0$$

Interestingly, the exact formulas for β and λ (they involve k and c_2) were not needed in the simplification steps to obtain the elements of M (*i.e.*, g).