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Scattering States, Wave Packets, Transition Amplitudes, and Tunneling in the Energy Domain: A Time-Free Formulation of Quantum Processes

Michail Lazos[1,2*], Neda Maria Kaizumi[3], John R. Woodward[4]

[1]HAIQU (Hellenic AI, Quantum and Mathematical Technologies), Dubai, United Arab Emirates
[2]Manchester Innovation and Technology Academy, Dubai, United Arab Emirates
[3]European Quantum Academy, Oslo, Norway
[4]School of Mathematical and Computer Sciences, Heriot-Watt University Dubai, Dubai, United Arab Emirates
Email: *mlazos@hotmail.com, J.WOODWARD@hw.ac.uk

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Abstract

This paper develops an energy-domain formulation of time-independent non-relativistic quantum processes, including scattering, wave packets, transition amplitudes, and tunneling. A fixed-energy spectral component is written as $\Psi_E(x,z)=\psi_E(x)\chi_z(E)$, while the complete state $\Psi(x,z)$ is obtained by summing or integrating over the spectral components. The energy derivative is defined on the spectral phase $\chi_z(E)$, with the energy-dependent spatial eigenfunction $\psi_E(x)$ retained as an element of the spectral basis. For a specified reference energy E_{ref}, the spectral phase is $\chi_z(E)=\exp\left[-\mathrm{i}(E-E_{\text{ref}})^2(z-z_0)/\hbar\right]$. Fixed-energy reflection and transmission amplitudes, tunneling probabilities, probability currents, transition matrix elements, and selection rules retain their standard forms because this factor is a common global phase. For superpositions, however, the z-ordering is generated by $\left(\hat{H}_x-E_{\text{ref}}\hat{I}\right)^2$ and is generally not identical to the standard time evolution generated by $\hat{H}_x$. Continuum states are treated with generalized normalization, channel labels, and the appropriate spectral measure. A right-moving Gaussian free-particle packet is worked out explicitly, yielding analytic narrow-band expressions for its centre and spatial width as functions of z. Transition probabilities are defined as Born-rule probabilities conditioned on a specified interaction event; transition rates, lifetimes, delay times,

and tunneling times are not claimed to be reproduced without an operational clock. The paper therefore establishes the preservation of fixed-energy stationary observables and a consistent unitary spectral ordering for time-independent Hamiltonians, while clearly identifying the remaining limitations of the proposed dynamics.

Keywords

Quantum Mechanics, Quantum Foundations, Time-Free Quantum Mechanics, Energy-Domain Quantum Mechanics, Scattering Theory, Wave Packets, Transition Amplitudes, Quantum Tunneling, Quantum Processes, Energy-Based Evolution

1. Introduction

The status of time remains one of the central conceptual questions in quantum theory. In standard non-relativistic quantum mechanics, the state evolves with respect to an external parameter t according to

$$i\hbar \frac{\partial \left|\Psi(t)\right\rangle}{\partial t} = \hat{H}_x \left|\Psi(t)\right\rangle. \tag{1}$$

Position, momentum, angular momentum, and energy are represented by operators, whereas time normally remains a parameter used to order change. This asymmetry motivates relational and timeless approaches, including the Page-Wootters construction, relational dynamics, and timeless formulations of quantum gravity [1]-[6].

Quantum cosmology provides an important setting in which the absence of an external Schrödinger time is not optional but structural. The Wheeler-DeWitt equation is a constraint on the wave function of the universe rather than an evolution equation in an external time coordinate [7]-[11]. Recent applications continue to examine how relational or Bohmian histories may emerge from such timeless constraints [12] [13]. The present construction is not a model of quantum cosmology, but it belongs to the same broad discussion concerning whether change can be represented without assuming an external fundamental time.

In earlier work, an energy-domain phase prescription was proposed for stationary quantum states [14]. It was first applied to one-dimensional stationary systems and was later extended to three-dimensional quantum mechanics and the hydrogen atom [15]. Those studies recovered the standard spatial eigenfunctions, spectra, probability densities, expectation values, angular momentum structure, and spectral lines. These are fixed-energy or stationary results.

The present paper asks a more demanding question: what remains valid when the state contains several energy components or when the spatial problem describes scattering, tunneling, or a transition between channels? The answer must be stated carefully. At a fixed energy, an additional phase common to all spatial

components cancels from boundary conditions, probability currents, and amplitude ratios. The recovery of fixed-energy scattering and tunneling coefficients is therefore an exact consistency check, but it is not by itself a proof of full dynamical equivalence with Equation (1). The genuinely new mathematical content appears for superpositions, whose z-ordering is generated by the square of a referenced Hamiltonian and is generally distinct from standard time evolution.

The analysis has four specific aims. First, it separates the spectral label E from the ordering parameter z and defines the action of the energy derivative on the spectral phase. Second, it introduces a reference-energy convention that is covariant under additive shifts of the Hamiltonian. Third, it treats continuum states with channel labels, generalized normalization, and the appropriate spectral measure. Fourth, it constructs an explicit right-moving Gaussian packet and derives its centre and width as functions of z.

The scope is restricted to time-independent, self-adjoint, non-relativistic Hamiltonians. The paper establishes fixed-energy amplitudes, event-conditioned channel probabilities, and a unitary spectral ordering. It does not claim a universal replacement of explicitly time-dependent dynamics, nor does it derive transition rates, lifetimes, detector response times, Wigner-Smith delays, or tunneling traversal times without an operational clock.

The paper is organized as follows. Section 2 presents the spectral formulation and reference-energy convention. Section 3 distinguishes fixed-energy components from complete superpositions. Section 4 develops the continuum wavepacket construction and a Gaussian example. Section 5 and Section 6 treat fixed-energy scattering and barriers. Section 7 discusses event-conditioned transition probabilities, and Section 8 separates tunneling probability from tunneling time. Sections 9-11 examine the z-generator, its limitations, and the relation to other timeless approaches.

2. Spectral Formulation

The formulation separates the spatial spectral problem from the phase used to order the spectral components. A quantity of the form $\Psi_{E,\alpha}(x,z)$ denotes one spectral component, whereas $\Psi(x,z)$ denotes the complete superposition over the discrete and continuous parts of the spectrum.

2.1. Spatial Hamiltonian, Spectral Labels, and Continuum Normalization

Let $\hat{H}_x$ be a time-independent self-adjoint spatial Hamiltonian. Its discrete eigenstates satisfy

$$\hat{H}_x|n\rangle = E_n|n\rangle, \tag{2}$$

and its continuum generalized eigenstates satisfy

$$\hat{H}_x|E,\alpha\rangle = E|E,\alpha\rangle. \tag{3}$$

here E is a spectral label and α denotes degeneracy or channel data. For a

one-dimensional free particle, for example, $\alpha = +$ and $\alpha = -$ distinguish the right-moving and left-moving momentum branches. In an energy-normalized representation,

$$\langle E, \alpha | E', \alpha' \rangle = \delta_{\alpha\alpha'} \delta(E - E'), \tag{4}$$

with the understanding that continuum states are generalized, rather than square-normalizable, vectors. More generally, the continuum contribution is integrated with a spectral measure $\mathrm{d}\mu(E)$ that contains the relevant density-of-states or Jacobian factor [16] [17].

The corresponding completeness relation may be written schematically as

$$\sum_n |n\rangle\langle n| + \sum_\alpha \int_{\sigma_c} |E, \alpha\rangle\langle E, \alpha| \mathrm{d}\mu(E) = \hat{I}, \tag{5}$$

where σ_c denotes the continuous spectrum.

2.2. Reference Energy and Spectral Phase

Choose one reference energy E_{ref} for the physical problem and define

$$\varepsilon(E) = E - E_{\mathrm{ref}}, \quad \Delta z = z - z_0. \tag{6}$$

The spectral phase is

$$\chi_z(E) = \exp\left[-\frac{\mathrm{i}\varepsilon(E)^2 \Delta z}{\hbar} \right]. \tag{7}$$

The parameter z has dimensions $[z] = [\hbar] / [E]^2$ and is an auxiliary ordering parameter, not a laboratory clock reading.

The energy origin is part of the definition of the representation. Under an additive shift

$$\hat{H}_x \to \hat{H}_x + C\hat{I}, \quad E_{\mathrm{ref}} \to E_{\mathrm{ref}} + C, \tag{8}$$

the referenced energy $\varepsilon(E)$ and all z-dependent phases are unchanged. For scattering calculations, E_{ref} is chosen as the declared asymptotic threshold; for the free particle and the barrier examples below it is the zero of the asymptotic potential. For a bound problem, it may be chosen as a declared ground-state or threshold energy. Once this convention is fixed, the later fixed-energy examples write E for the referenced energy to avoid unnecessary notation.

For $\Delta z \neq 0$, define the spectral differential operator

$$\hat{D}_{E,z} = \frac{\mathrm{i}\hbar}{2\Delta z} \frac{\partial}{\partial E}. \tag{9}$$

Then

$$\hat{D}_{E,z} \chi_z(E) = \varepsilon(E) \chi_z(E). \tag{10}$$

This is a differential identity in the spectral coefficient. It is not an eigenvalue equation obtained by differentiating the full coordinate-space product $\psi_E(x) \chi_z(E)$. Indeed,

$$\hat{D}_{E,z} \left[\psi_E(x) \chi_z(E) \right] = \varepsilon(E) \psi_E(x) \chi_z(E) + \frac{\mathrm{i}\hbar}{2\Delta z} \frac{\partial \psi_E(x)}{\partial E} \chi_z(E). \tag{11}$$

The second term is generally non-zero. The operator $\hat{D}_{E,z}$ is therefore defined on the spectral coefficient $\chi_z(E)$, while the generalized eigenket $|E,\alpha\rangle$ is retained as the spectral basis element.

2.3. Complete State and Unitary *z*-Ordering

A fixed spectral component is denoted by

$$|\Psi_{E,\alpha}(z)\rangle = \chi_z(E)|E,\alpha\rangle, \tag{12}$$

whereas the complete state is

$$|\Psi(z)\rangle = \sum_{n=0}^{\infty} c_n \chi_z(E_n)|n\rangle + \sum_{\alpha}\int_{\sigma_c} c_\alpha(E)\chi_z(E)|E,\alpha\rangle \mathrm{d}\mu(E). \tag{13}$$

In the position representation, $\Psi_{E,\alpha}(x,z) = \psi_{E,\alpha}(x)\chi_z(E)$ and $\Psi(x,z) = \langle x|\Psi(z)\rangle$. Thus E labels a component, while z orders the complete superposition.

If the initial spectral coefficients are normalized,

$$\sum_n |c_n|^2 + \sum_\alpha \int_{\sigma_c} |c_\alpha(E)|^2 \,\mathrm{d}\mu(E) = 1, \tag{14}$$

then normalization is preserved because $|\chi_z(E)| = 1$. The complete state may be written as

$$|\Psi(z)\rangle = \hat{U}_z(z,z_0)|\Psi(z_0)\rangle, \tag{15}$$

with

$$\hat{U}_z(z,z_0) = \exp\left[-\frac{\mathrm{i}\left(\hat{H}_x - E_{\mathrm{ref}}\hat{I}\right)^2 \Delta z}{\hbar}\right]. \tag{16}$$

For self-adjoint $\hat{H}_x$, this operator is unitary.

2.4. Fixed-Energy Observables and Scope

For a single spectral component, $\chi_z(E)$ is a global phase. Therefore,

$$|\Psi_{E,\alpha}(x,z)|^2 = |\psi_{E,\alpha}(x)|^2, \tag{17}$$

and any expectation value or probability-current ratio formed entirely within the same fixed-energy component is unchanged. This observation explains the exact recovery of stationary bound-state quantities and fixed-energy scattering amplitudes.

For a superposition, relative phases are observable. The *z*-ordered state is generated by $\left(\hat{H}_x - E_{\mathrm{ref}}\hat{I}\right)^2$, whereas standard time evolution is generated by $\hat{H}_x$. Consequently, the two evolutions are not identical for a general broadband state. The present paper treats this difference as a physical limitation and investigates the consequences that can be established without identifying z with laboratory

time.

3. Quantum Processes under Spectral z-Ordering

3.1. Fixed-Energy Components and Complete States

A single component of referenced energy E is written as

$$\Psi_{E,\alpha}(x,z)=\psi_{E,\alpha}(x)\exp\left(-\frac{\mathrm{i}E^2\Delta z}{\hbar}\right), \tag{18}$$

where the reference-energy convention of Equation (6) has already been applied. The complete state is denoted by $\Psi(x,z)$ and is obtained by summing or integrating over the spectral labels.

For the discrete part of the spectrum,

$$\Psi_{\mathrm{d}}(x,z)=\sum_{n=0}^{\infty}c_n\psi_n(x)\exp\left(-\frac{\mathrm{i}E_n^2\Delta z}{\hbar}\right). \tag{19}$$

For the continuum,

$$\Psi_{\mathrm{c}}(x,z)=\sum_{\alpha}\int_{\sigma_{\mathrm{c}}}c_\alpha(E)\psi_{E,\alpha}(x)\exp\left(-\frac{\mathrm{i}E^2\Delta z}{\hbar}\right)\mathrm{d}\mu(E). \tag{20}$$

The channel label and measure are essential: an energy value alone may correspond to several propagation directions or scattering channels.

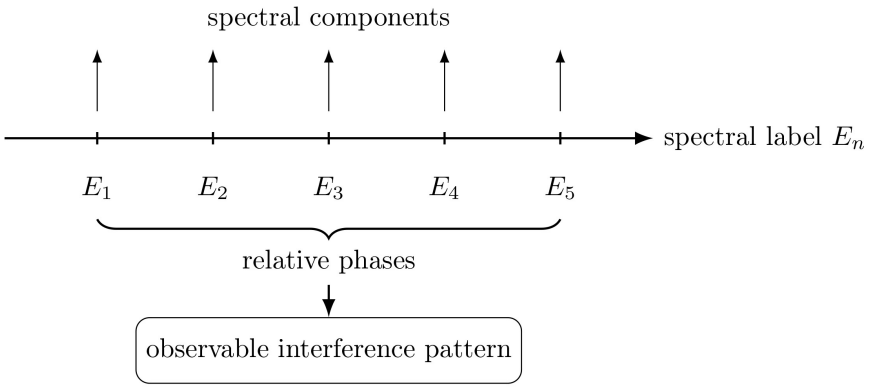

Figure 1. Spectral interpretation of a z-ordered superposition. A fixed-energy component carries only a global phase, whereas observable changes arise from relative phases among distinct spectral components.

3.2. Relative Phases and Observable Structure

The relative phase between two discrete components is

$$\Delta\phi_{nm}(z)=-\frac{\left(E_n^2-E_m^2\right)\Delta z}{\hbar}. \tag{21}$$

It is this relative phase, rather than the global phase of one fixed-energy state, that can change the probability density of a superposition.

The distinction between fixed-energy global phases and observable relative

phases is summarized in Figure 1.

The probability density is

$$\rho(x,z)=\left|\Psi(x,z)\right|^2. \tag{22}$$

The diagonal spectral terms are independent of z, while the off-diagonal terms contain the relative phases. Scattering and tunneling coefficients at a fixed energy remain stationary quantities; packet displacement, packet deformation, and interference between distinct energies belong to the z-ordered superposition.

3.3. Interpretive Claim

The contribution of the present framework is therefore limited but precise. It provides a spectral ordering of the complete state without assuming that z is a clock variable, and it separates three logically different objects: the stationary spatial eigenproblem, the global phase of one fixed-energy component, and the relative phases of a superposition. The framework does not imply that every quantity normally expressed per unit time has already been reconstructed.

4. Wave Packets in the Energy Domain

A localized one-dimensional packet requires more information than a Gaussian distribution in energy. The energy $E=\hbar^2k^2/(2m)$ does not distinguish k from $-k$, and a change from k-normalization to E-normalization introduces a Jacobian. We therefore construct the packet in momentum space, select a propagation branch explicitly, and then relate it to the spectral formulation.

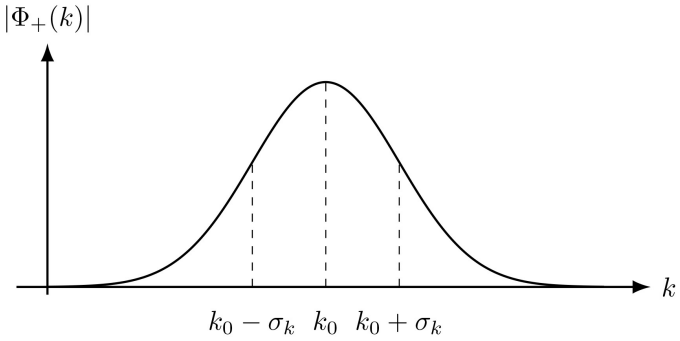

Figure 2. Gaussian momentum distribution on the right-moving branch. The branch label and momentum-space phase are required in addition to the energy distribution.

4.1. Continuum Packet and Spectral Measure

For free-particle momentum eigenstates normalized by $\langle k|k'\rangle=\delta(k-k')$, a right-moving packet is

$$\Psi_+(x,z)=\frac{1}{\sqrt{2}}\int_0^\infty \Phi_+(k)\,\mathrm{e}^{\mathrm{i}kx}\exp\left[-\frac{\mathrm{i}E(k)^2\,\Delta z}{\hbar}\right]\mathrm{d}k. \tag{23}$$

where

$$E(k)=\frac{\hbar^2k^2}{2m}. \tag{24}$$

The choice $k>0$ fixes the right-moving branch. In an energy-normalized form, the same state contains the measure factor $\mathrm{d}k/\mathrm{d}E=m/\left(\hbar^2k\right)$, or equivalently an energy-normalized coefficient that absorbs its square root. This is why a Gaussian $A(E)$ by itself does not completely specify a localized directional packet.

Choose

$$\Phi_+(k)=\mathcal{N}\exp\left[-\frac{(k-k_0)^2}{4\sigma_k^2}-\mathrm{i}kx_0\right],\ k>0. \tag{25}$$

where $\mathcal{N}$ normalizes the positive-k branch. When $k_0\gg\sigma_k$, the negligible negative-k tail allows the lower limit to be extended to $-\infty$ and $\mathcal{N}\simeq\left(2\sigma_k^2\right)^{-1/4}$.

The branch-restricted Gaussian momentum distribution is shown in Figure 2.

4.2. Explicit Gaussian Evolution

Define the z-dispersion function

$$\Omega_z(k)=\frac{E(k)^2}{\hbar}=\frac{\hbar^3k^4}{4m^2}. \tag{26}$$

For a narrow packet, write $q=k-k_0$ and expand to second order,

$$\Omega_z(k)\simeq\Omega_0+v_zq+\frac{1}{2}\beta_zq^2, \tag{27}$$

where

$$v_z=\left.\frac{\mathrm{d}\Omega_z}{\mathrm{d}k}\right|_{k_0}=\frac{\hbar^3k_0^3}{m^2}, \tag{28}$$

and

$$\beta_z=\left.\frac{\mathrm{d}^2\Omega_z}{\mathrm{d}k^2}\right|_{k_0}=\frac{3\hbar^3k_0^2}{m^2}. \tag{29}$$

Evaluation of the Gaussian integral gives a packet centred at

$$x_c(z)=x_0+\frac{\hbar^3k_0^3}{m^2}\Delta z, \tag{30}$$

with spatial variance

$$\sigma_x^2(z)=\frac{1}{4\sigma_k^2}+\left(\frac{3\hbar^3k_0^2}{m^2}\right)^2\sigma_k^2\Delta z^2. \tag{31}$$

The first term is the initial Fourier width. The second term is the leading narrow-k spreading produced by the curvature of the quadratic-energy phase. These formulas provide an explicit consequence of the proposed ordering rather than a purely qualitative statement.

For a packet centred at $E_0=\hbar^2k_0^2/(2m)$, the local identification

$$t_{\text{eff}} = 2E_0 \Delta z \tag{32}$$

reproduces the standard group displacement, $x_c - x_0 = (p_0/m)t_{\text{eff}}$. The full width in Equation (31), however, retains the curvature of $E^2(k)$ and displays the difference between exact z-ordering and standard time evolution. Standard packet evolution is recovered only in the stronger regime in which the contribution of $(E - E_0)^2 \Delta z/\hbar$ is negligible across the spectral width.

4.3. Probability Conservation

Because $\hat{U}_z$ is unitary,

$$\int_{-\infty}^{\infty} \left|\Psi_+(x,z)\right|^2 \mathrm{d}x = 1 \tag{33}$$

whenever the initial packet is normalized. For broad spectra the quartic phase in Equation (23) produces non-Gaussian deformation, and the complete integral must be evaluated numerically.

5. Scattering States

Scattering phenomena occupy a central position in quantum mechanics because they provide a direct connection between the mathematical description of a system and measurable experimental quantities. In a scattering experiment, a prepared state approaches an interaction region and is subsequently detected in one or more outgoing channels. The measurable quantities are obtained from the probability currents associated with these channels.

In the standard description, a scattering process is often presented through the time evolution of a localized wave packet. The packet approaches a potential, interacts with it, and separates into reflected and transmitted components. However, the reflection and transmission coefficients are normally calculated from stationary solutions of the Schrödinger equation. This is important for the present formulation because the stationary spatial equation does not require time as an independent variable [18] [19].

Within the energy-domain framework, the spatial scattering solutions remain unchanged. The incident, reflected, and transmitted components are obtained from the same stationary differential equation and satisfy the same boundary conditions as in standard quantum mechanics. The complete state differs only through the energy-dependent phase attached to each energy component.

5.1. Free Particle Scattering

Strictly speaking, a free particle does not scatter when the potential vanishes everywhere. The free-particle solutions are nevertheless essential because they describe the state sufficiently far from a localized interaction region, where the potential approaches a constant value. They therefore provide the incoming and outgoing asymptotic states used in scattering theory.

For a free particle,

$$V(x)=0,\ -\infty<x<\infty. \tag{34}$$

The stationary Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_E(x)}{\mathrm{d}x^2}=E\psi_E(x). \tag{35}$$

For positive energy, the general solution is

$$\psi_E(x)=A\mathrm{e}^{\mathrm{i}kx}+B\mathrm{e}^{-\mathrm{i}kx}, \tag{36}$$

where

$$k=\frac{\sqrt{2mE}}{\hbar}. \tag{37}$$

The component $\mathrm{e}^{\mathrm{i}kx}$ has positive momentum and represents propagation towards increasing x, while $\mathrm{e}^{-\mathrm{i}kx}$ has negative momentum and represents propagation towards decreasing x.

Within the energy-domain formulation, the complete state of definite energy is

$$\Psi_E(x,z)=\left(A\mathrm{e}^{\mathrm{i}kx}+B\mathrm{e}^{-\mathrm{i}kx}\right)\chi_z(E). \tag{38}$$

Both spatial components correspond to the same energy and therefore carry the same energy-domain phase. For a stationary scattering state, this factor is global and does not modify the relative amplitudes A and B.

The probability current associated with a one-dimensional state is

$$j(x,z)=\frac{\hbar}{2m\mathrm{i}}\left[\Psi_E^*(x,z)\frac{\partial\Psi_E(x,z)}{\partial x}-\Psi_E(x,z)\frac{\partial\Psi_E^*(x,z)}{\partial x}\right]. \tag{39}$$

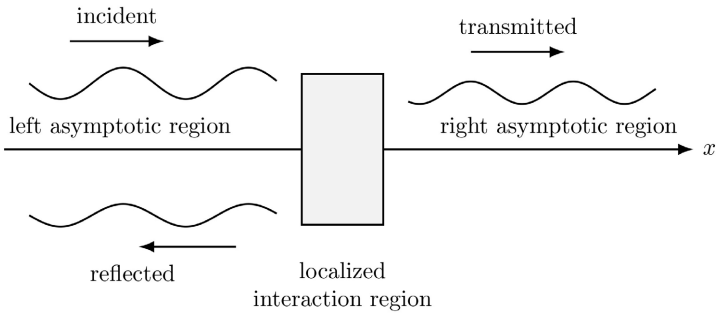

Figure 3. General one-dimensional scattering process. A state incident from the left interacts with a localized potential and produces reflected and transmitted outgoing components.

Since the energy-domain phase does not depend on position, it cancels from Equation (39). Consequently,

$$j(x,z)=\frac{\hbar}{2m\mathrm{i}}\left[\psi_E^*(x)\frac{\mathrm{d}\psi_E(x)}{\mathrm{d}x}-\psi_E(x)\frac{\mathrm{d}\psi_E^*(x)}{\mathrm{d}x}\right]. \tag{40}$$

For the right-moving plane wave $A\mathrm{e}^{\mathrm{i}kx}$, the current is

$$j_+=\frac{\hbar k}{m}|A|^2. \tag{41}$$

For the left-moving plane wave Be^{-ikx}, the current is

$$j_{-} = -\frac{\hbar k}{m}|B|^2. \tag{42}$$

The sign of the current gives the direction of propagation. The energy-domain phase does not change either current because it has unit magnitude and is common to both spatial components.

The incident, reflected, and transmitted asymptotic channels are illustrated in Figure 3.

5.2. Incoming and Outgoing States

We now consider a potential that approaches constant values far from the interaction region,

$$V(x) \to V_{\mathrm{L}}, \quad x \to -\infty, \tag{43}$$

and

$$V(x) \to V_{\mathrm{R}}, \quad x \to +\infty. \tag{44}$$

For an energy greater than both asymptotic potentials, the corresponding wavenumbers are

$$k_{\mathrm{L}} = \frac{\sqrt{2m(E - V_{\mathrm{L}})}}{\hbar}, \tag{45}$$

and

$$k_{\mathrm{R}} = \frac{\sqrt{2m(E - V_{\mathrm{R}})}}{\hbar}. \tag{46}$$

For a state incident from the left, the asymptotic spatial wavefunction is written as

$$\psi_E(x) \sim Ae^{ik_{\mathrm{L}}x} + Be^{-ik_{\mathrm{L}}x}, \quad x \to -\infty, \tag{47}$$

and

$$\psi_E(x) \sim Ce^{ik_{\mathrm{R}}x}, \quad x \to +\infty. \tag{48}$$

The coefficient A is the incident amplitude, B is the reflected amplitude, and C is the transmitted amplitude. No left-moving component is included in the right asymptotic region because we assume that no wave is incident from the right.

The corresponding energy-domain states are

$$\Psi_{\mathrm{L}}(x,z) = \left(Ae^{ik_{\mathrm{L}}x} + Be^{-ik_{\mathrm{L}}x}\right)\chi_z(E), \tag{49}$$

and

$$\Psi_{\mathrm{R}}(x,z) = Ce^{ik_{\mathrm{R}}x}\chi_z(E). \tag{50}$$

The same phase multiplies all three components because the scattering is elastic and the energy is conserved. Consequently, the matching conditions imposed on

the complete states reduce directly to the usual matching conditions for the spatial wavefunctions.

For a general two-sided scattering problem, waves may be incident from both the left and the right. The asymptotic state can then be written as

$$\psi_E(x) \sim A\mathrm{e}^{\mathrm{i}k_{\mathrm{L}}x} + B\mathrm{e}^{-\mathrm{i}k_{\mathrm{L}}x}, \quad x \to -\infty, \tag{51}$$

and

$$\psi_E(x) \sim C\mathrm{e}^{\mathrm{i}k_{\mathrm{R}}x} + D\mathrm{e}^{-\mathrm{i}k_{\mathrm{R}}x}, \quad x \to +\infty. \tag{52}$$

The amplitudes A and D describe the incoming channels, while B and C describe the outgoing channels. When the asymptotic wavenumbers are different, the amplitudes must be normalized with respect to probability flux. We therefore define

$$\begin{aligned} \tilde{A} &= \sqrt{k_{\mathrm{L}}}A, \quad \tilde{B} = \sqrt{k_{\mathrm{L}}}B, \\ \tilde{C} &= \sqrt{k_{\mathrm{R}}}C, \quad \tilde{D} = \sqrt{k_{\mathrm{R}}}D. \end{aligned} \tag{53}$$

To avoid unnecessary dependence on a matrix environment, we define the incoming and outgoing amplitude vectors as

$$\begin{aligned} \boldsymbol{a}_{\mathrm{in}}(E) &= \left(\tilde{A}, \tilde{D}\right)^{\mathrm{T}}, \\ \boldsymbol{a}_{\mathrm{out}}(E) &= \left(\tilde{B}, \tilde{C}\right)^{\mathrm{T}}. \end{aligned} \tag{54}$$

The relation between the incoming and outgoing channels is then

$$\boldsymbol{a}_{\mathrm{out}}(E) = \boldsymbol{S}(E)\boldsymbol{a}_{\mathrm{in}}(E). \tag{55}$$

For a real potential and elastic scattering, the flux-normalized scattering matrix is unitary,

$$\boldsymbol{S}^{\dagger}(E)\boldsymbol{S}(E) = \boldsymbol{I}. \tag{56}$$

This condition expresses probability-current conservation and remains unchanged in the energy-domain formulation.

5.3. Scattering Amplitudes

For incidence from the left, the reflection amplitude is defined as

$$r(E) = \frac{B}{A}. \tag{57}$$

The transmission amplitude is

$$t(E) = \frac{C}{A}. \tag{58}$$

The incident probability current is

$$j_{\mathrm{in}} = \frac{\hbar k_{\mathrm{L}}}{m}|A|^2. \tag{59}$$

The reflected probability current is

$$j_{\mathrm{ref}} = -\frac{\hbar k_{\mathrm{L}}}{m}|B|^2. \tag{60}$$

The transmitted probability current is

$$j_{\mathrm{tr}} = \frac{\hbar k_{\mathrm{R}}}{m}\left|C\right|^2. \tag{61}$$

The reflection coefficient is

$$R(E) = \frac{\left|j_{\mathrm{ref}}\right|}{j_{\mathrm{in}}} = \left|r(E)\right|^2, \tag{62}$$

and the transmission coefficient is

$$T(E) = \frac{j_{\mathrm{tr}}}{j_{\mathrm{in}}} = \frac{k_{\mathrm{R}}}{k_{\mathrm{L}}}\left|t(E)\right|^2. \tag{63}$$

For a real potential with no absorption, current conservation requires

$$R(E) + T(E) = 1. \tag{64}$$

The result is unchanged in the energy-domain formulation because the common phase cancels from all three probability currents.

The scattering of a localized packet is obtained by combining the stationary amplitudes over a range of energies. Let $\mathcal{D}_E$ denote the range of energies for which the relevant asymptotic channels are open, and let $A_0(E)$ represent the incident spectral distribution. The incident, reflected, and transmitted packets may be written as

$$\Psi_{\mathrm{in}}(x,z) = \int_{\mathcal{D}_E} A_0(E)\mathrm{e}^{\mathrm{i}k_{\mathrm{L}}(E)x}\chi_z(E)\mathrm{d}E. \tag{65}$$

$$\Psi_{\mathrm{ref}}(x,z) = \int_{\mathcal{D}_E} A_0(E)r(E)\mathrm{e}^{-\mathrm{i}k_{\mathrm{L}}(E)x}\chi_z(E)\mathrm{d}E. \tag{66}$$

and

$$\Psi_{\mathrm{tr}}(x,z) = \int_{\mathcal{D}_E} A_0(E)t(E)\mathrm{e}^{\mathrm{i}k_{\mathrm{R}}(E)x}\chi_z(E)\mathrm{d}E. \tag{67}$$

These expressions show that the functions $r(E)$ and $t(E)$ change the amplitude and phase of each energy component, while the factor

$$\chi_z(E)$$

determines the relative energy-domain phase between the different components.

5.4. Fixed-Energy Interpretation

The stationary Schrödinger equation, asymptotic plane waves, matching conditions, and probability currents retain their standard forms. For one state of definite referenced energy, the spectral phase is common to the incident, reflected, and transmitted components. It therefore cancels from the boundary conditions and from the ratios defining $R(E)$ and $T(E)$.

In the energy-domain formulation, localized incident and outgoing packets are constructed from channel-resolved stationary states using the spectral z-phase. Each energy component is weighted by its stationary amplitudes $r(E)$ and $t(E)$, while the complete reflected and transmitted packets are determined by interference across the spectral distribution. The stationary boundary-value prob-

lem therefore determines the energy-resolved amplitudes, and the spectral ordering determines the structure of their superposition.

6. Potential Scattering

We now apply the general scattering formalism developed in the previous section to two standard one-dimensional potentials. We begin with the potential step and then consider a finite rectangular barrier. These examples contain the main physical ingredients of quantum scattering: an incident state, reflected and transmitted components, continuity conditions at the boundaries, conservation of probability current, and penetration into a classically forbidden region [18] [19].

The potential step provides the simplest example of reflection and transmission at a discontinuity. The finite barrier extends the same calculation to two boundaries and leads directly to quantum tunneling. The two systems are therefore naturally connected. A semi-infinite step with $E < V_0$ produces an evanescent wave but no transmitted current, whereas a finite barrier allows a non-zero transmitted component to appear beyond the classically forbidden region.

In the energy-domain formulation, the spatial part of the problem remains governed by the stationary Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}+V(x)\psi(x)=E\psi(x). \tag{68}$$

For a state of definite energy, the complete state is written as

$$\Psi_E(x,z)=\psi_E(x)\chi_z(E). \tag{69}$$

The energy-domain factor in Eq. (69) is common to all spatial regions of an elastic fixed-energy scattering problem. It therefore cancels from the boundary conditions and from all ratios of probability currents.

6.1. Potential Step

We first consider the potential step

$$V(x)=0,\ \ x<0, \tag{70}$$

and

$$V(x)=V_0,\ \ x>0, \tag{71}$$

where $V_0>0$. A particle is incident from the left with energy E.

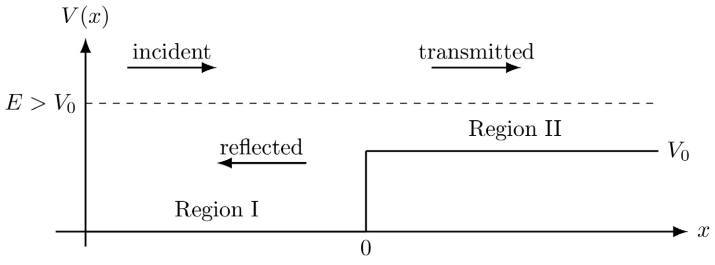

Figure 4. Scattering at a potential step for $E > V_0$. The incident state produces reflected and transmitted components, with different wavenumbers in the two spatial regions.

For $E > V_0$, both regions admit oscillatory solutions. The wavenumbers are

$$k_1 = \frac{\sqrt{2mE}}{\hbar}, \tag{72}$$

and

$$k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}. \tag{73}$$

The spatial wavefunction in the first region is

$$\psi_1(x) = A e^{ik_1 x} + B e^{-ik_1 x}, \quad x < 0, \tag{74}$$

and in the second region it is

$$\psi_2(x) = C e^{ik_2 x}, \quad x > 0. \tag{75}$$

Only a right-moving component is included in the second region because no wave is assumed to be incident from positive infinity.

The potential profile and the three scattering components for $E > V_0$ are shown in **Figure 4**.

For a finite discontinuity in the potential and a constant particle mass, the wavefunction and its first derivative must be continuous at $x = 0$. Therefore,

$$\psi_1(0) = \psi_2(0), \tag{76}$$

and

$$\psi_1'(0) = \psi_2'(0). \tag{77}$$

Substitution gives

$$A + B = C, \tag{78}$$

and

$$k_1(A - B) = k_2 C. \tag{79}$$

For $0 < E < V_0$, the first region remains oscillatory, but the second region is classically forbidden. We define

$$\kappa_2 = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \tag{80}$$

The physically acceptable solution in the second region is

$$\psi_2(x) = C e^{-\kappa_2 x}, \quad x > 0. \tag{81}$$

The single decaying exponential carries no probability current, and the particle is completely reflected by a semi-infinite step even though the probability density is non-zero for a finite distance inside the forbidden region. Applying the boundary conditions gives

$$r(E) = \frac{k_1 - i\kappa_2}{k_1 + i\kappa_2}, \tag{82}$$

so that

$$|r(E)|^2 = 1. \tag{83}$$

Consequently,

$$R(E)=1, \quad T(E)=0. \tag{84}$$

6.2. Reflection and Transmission

For $E>V_0$, the reflection amplitude is

$$r(E)=\frac{B}{A}, \tag{85}$$

and the transmission amplitude is

$$t(E)=\frac{C}{A}. \tag{86}$$

Solving Equation (78) and Equation (79), we obtain

$$r(E)=\frac{k_1-k_2}{k_1+k_2}, \tag{87}$$

and

$$t(E)=\frac{2k_1}{k_1+k_2}. \tag{88}$$

The reflection coefficient is

$$R(E)=\left(\frac{k_1-k_2}{k_1+k_2}\right)^2, \tag{89}$$

and the transmission coefficient is

$$T(E)=\frac{k_2}{k_1}\left|t(E)\right|^2=\frac{4k_1k_2}{\left(k_1+k_2\right)^2}. \tag{90}$$

Adding the two coefficients gives

$$R(E)+T(E)=1. \tag{91}$$

The physical behaviour of the coefficients becomes clearer when the energy is written in units of the step height,

$$\varepsilon=\frac{E}{V_0}. \tag{92}$$

For $\varepsilon>1$, the coefficients become

$$R(\varepsilon)=\left(\frac{\sqrt{\varepsilon}-\sqrt{\varepsilon-1}}{\sqrt{\varepsilon}+\sqrt{\varepsilon-1}}\right)^2, \tag{93}$$

and

$$T(\varepsilon)=\frac{4\sqrt{\varepsilon}\sqrt{\varepsilon-1}}{\left(\sqrt{\varepsilon}+\sqrt{\varepsilon-1}\right)^2}. \tag{94}$$

The resulting energy dependence of the reflection and transmission coefficients is plotted in Figure 5.

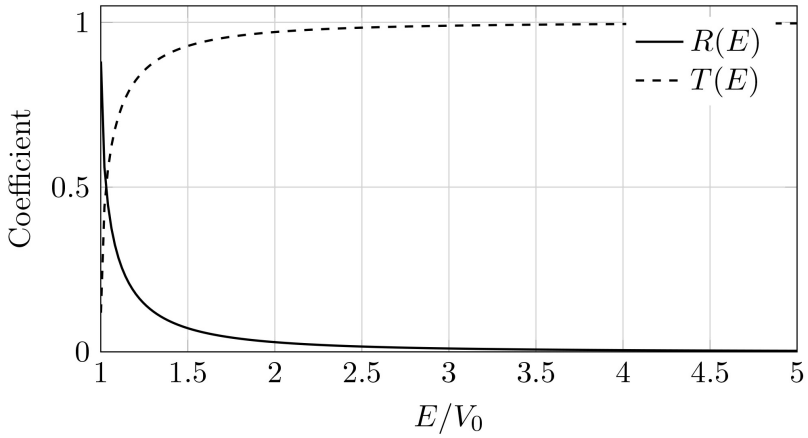

Figure 5. Reflection and transmission coefficients for a potential step with $E > V_0$. Close to the threshold, reflection dominates. At high incident energy, transmission approaches unity.

As $E \to V_0^+$, the transmitted wavenumber approaches zero and the particle is almost completely reflected. In the high-energy limit, $E \gg V_0$, the step becomes a small perturbation and $T(E) \to 1$.

6.3. Potential Barrier

We next consider the finite rectangular barrier

$$V(x) = 0, \quad x < 0, \tag{95}$$

$$V(x) = V_0, \quad 0 < x < a, \tag{96}$$

and

$$V(x) = 0, \quad x > a. \tag{97}$$

The barrier has height V_0 and width a. Unlike the potential step, the forbidden region is finite. This allows the evanescent state inside the barrier to connect to a propagating state on the far side.

For $E > V_0$, the spatial solutions are oscillatory in all three regions,

$$\psi_1(x) = Ae^{ikx} + Be^{-ikx}, \quad x < 0, \tag{98}$$

$$\psi_2(x) = Ce^{iqx} + De^{-iqx}, \quad 0 < x < a, \tag{99}$$

and

$$\psi_3(x) = Fe^{ikx}, \quad x > a, \tag{100}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}, \tag{101}$$

and

$$q = \frac{\sqrt{2m(E - V_0)}}{\hbar}. \tag{102}$$

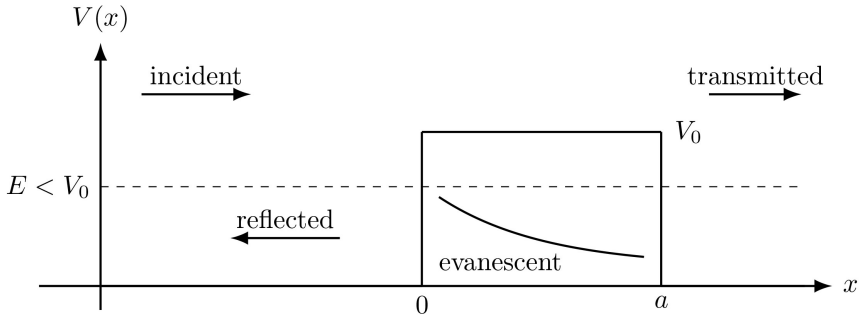

Figure 6. Rectangular potential barrier for $E < V_0$. The state is oscillatory outside the barrier and evanescent inside it. Since the forbidden region has finite width, a transmitted component exists for $x > a$.

For $0 < E < V_0$, we define

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \tag{103}$$

The spatial solutions are then

$$\psi_1(x) = Ae^{ikx} + Be^{-ikx}, \quad x < 0, \tag{104}$$

$$\psi_2(x) = Ce^{\kappa x} + De^{-\kappa x}, \quad 0 < x < a, \tag{105}$$

and

$$\psi_3(x) = Fe^{ikx}, \quad x > a. \tag{106}$$

Both exponential terms must be retained inside a finite barrier. Although one term increases with x, it remains finite over the interval $0 < x < a$ and is required to satisfy the boundary conditions at both interfaces.

The oscillatory and evanescent regions of the finite barrier are displayed in **Figure 6**.

The boundary conditions at $x = 0$ are

$$\psi_1(0) = \psi_2(0), \tag{107}$$

and

$$\psi_1'(0) = \psi_2'(0). \tag{108}$$

At $x = a$, the conditions are

$$\psi_2(a) = \psi_3(a), \tag{109}$$

and

$$\psi_2'(a) = \psi_3'(a). \tag{110}$$

For $E < V_0$, substitution of the three regional solutions gives

$$A + B = C + D, \tag{111}$$

$$ik(A - B) = \kappa(C - D), \tag{112}$$

$$Ce^{\kappa a} + De^{-\kappa a} = Fe^{ika}, \tag{113}$$

and

$$\kappa\left(C\mathrm{e}^{\kappa a}-D\mathrm{e}^{-\kappa a}\right)=\mathrm{i}kF\mathrm{e}^{\mathrm{i}ka}. \tag{114}$$

Solving these equations gives the transmission coefficient

$$T(E)=\left[1+\frac{V_0^2\sinh^2(\kappa a)}{4E(V_0-E)}\right]^{-1},\ 0<E<V_0. \tag{115}$$

The reflection coefficient is

$$R(E)=1-T(E). \tag{116}$$

For a thick or high barrier, $\kappa a\gg 1$, the transmission coefficient becomes

$$T(E)\simeq\frac{16E(V_0-E)}{V_0^2}\exp(-2\kappa a). \tag{117}$$

For $E>V_0$, the corresponding transmission coefficient is

$$T(E)=\left[1+\frac{V_0^2\sin^2(qa)}{4E(E-V_0)}\right]^{-1},\ E>V_0. \tag{118}$$

Transmission resonances occur whenever

$$qa=n\pi,\ n=1,2,3,\cdots, \tag{119}$$

for which $T(E)=1$.

6.4. Fixed-Energy Interpretation

At fixed energy, the same Schrödinger equation is solved in every region, the same boundary conditions are imposed, and the same probability currents define reflection and transmission. The common spectral phase cancels at each interface. Consequently, $r(E)$, $t(E)$, $R(E)$, and $T(E)$ are identical to the standard stationary results.

Probability-current conservation, total reflection from a semi-infinite forbidden region, finite-barrier transmission, exponential tunneling suppression, and above-barrier resonances are determined by the stationary spatial boundary-value problem. The z-dependence becomes non-trivial when the energy-resolved amplitudes are assembled into a packet, where $r(E)$ and $t(E)$ alter both the spectral weights and the relative phases.

7. Transition Amplitudes in the Energy Domain

This section considers a specified interaction that connects an initial stationary channel to one or more final channels. The result is a conditional Born-rule probability for the outcome of that interaction event, not a transition rate per unit time.

7.1. Initial and Final States

Let

$$\hat{H}_0|i\rangle=E_i|i\rangle,\quad \hat{H}_0|f\rangle=E_f|f\rangle, \tag{120}$$

and attach the referenced spectral phases

$$\begin{aligned}\left|\Psi_i(z)\right\rangle &= \mathrm{e}^{-\mathrm{i}E_i^2\Delta z/\hbar}\left|i\right\rangle,\\ \left|\Psi_f(z)\right\rangle &= \mathrm{e}^{-\mathrm{i}E_f^2\Delta z/\hbar}\left|f\right\rangle.\end{aligned} \tag{121}$$

Here and below the energies are measured from the reference declared in Section 2.2.

7.2. Matrix Elements

Write

$$\hat{H} = \hat{H}_0 + \hat{W}, \tag{122}$$

where $\hat{W}$ represents the specified interaction. The stationary coupling matrix element is

$$W_{fi} = \left\langle f\left|\hat{W}\right|i\right\rangle. \tag{123}$$

For the complete z-ordered states,

$$\mathcal{W}_{fi}(z) = \exp\left[\frac{\mathrm{i}\left(E_f^2 - E_i^2\right)\Delta z}{\hbar}\right]W_{fi}, \tag{124}$$

and therefore

$$\left|\mathcal{W}_{fi}(z)\right|^2 = \left|W_{fi}\right|^2. \tag{125}$$

The spectral ordering changes the phase of the amplitude but not the stationary coupling strength.

7.3. Event-Conditioned Born Probability

A matrix element alone is not a normalized probability. Suppose that the event under consideration is represented by the action of $\hat{W}$ and that $\hat{W}\left|\Psi_i(z)\right\rangle \neq 0$. The normalized post-event state is

$$\left|\Phi_i(z)\right\rangle = \frac{\hat{W}\left|\Psi_i(z)\right\rangle}{\sqrt{\left\langle\Psi_i(z)\left|\hat{W}^\dagger\hat{W}\right|\Psi_i(z)\right\rangle}}. \tag{126}$$

The Born-rule probability of detecting the system in a normalized final channel $\left|\Psi_f(z)\right\rangle$ is

$$P_{i\to f|W} = \left|\left\langle\Psi_f(z)\middle|\Phi_i(z)\right\rangle\right|^2 = \frac{\left|W_{fi}\right|^2}{\left\langle i\left|\hat{W}^\dagger\hat{W}\right|i\right\rangle}. \tag{127}$$

For a subspace of final outcomes $\mathcal{F}$ with projector $\hat{\Pi}_{\mathcal{F}}$, the corresponding probability is

$$P_{i\to\mathcal{F}|W} = \left\langle\Phi_i(z)\left|\hat{\Pi}_{\mathcal{F}}\right|\Phi_i(z)\right\rangle. \tag{128}$$

If the final projectors resolve the identity, these conditional probabilities sum to unity. Continuum final channels are treated by replacing the sum with the appropriate channel integral and spectral measure.

This definition is the standard Born rule applied after a specified interaction event. It is appropriate here because the framework specifies the relative weights of possible final channels but does not specify when the event occurs or how long it lasts. It therefore must not be confused with Fermi's golden-rule rate, a decay constant, or a lifetime.

7.4. Selection Rules and Physical Interpretation

A channel is forbidden when $W_{fi}=0$. Consequently, parity and angular-momentum selection rules remain those of the stationary matrix element. For an electric-dipole interaction,

$$\Delta\ell=\pm 1,\quad \Delta m=0,\pm 1, \tag{129}$$

and the initial and final states have opposite parity.

The energy-domain description separates the stationary coupling, the conditional channel probability, and the phase assigned to the complete states. No temporal sequence of intermediate states is required to define the conditional distribution over final channels. Rates, lifetimes, linewidths, and detector response remain outside the present construction because they require an operational duration.

8. Quantum Tunneling without Time

Quantum tunneling is one of the clearest examples of the difference between classical and quantum physics. In classical mechanics, a particle with energy smaller than the height of a potential barrier cannot enter the forbidden region. In quantum mechanics, the wavefunction does not vanish at the classical turning point. It continues into the barrier as an evanescent solution and, when the forbidden region has finite width, it may connect to a propagating state on the far side [18] [19].

The standard explanation often describes tunneling as a process that takes place during the temporal evolution of a wave packet. However, the measurable transmission probability is obtained from stationary spatial solutions, boundary conditions, and probability currents. This makes tunneling a particularly useful test of the energy-domain formulation developed in our earlier work [14] [15].

8.1. Barrier Penetration

Consider a particle of mass m moving in one spatial dimension. The stationary Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}+V(x)\psi(x)=E\psi(x). \tag{130}$$

In a region where $V(x)>E$, we define the local decay coefficient

$$\kappa(x)=\frac{\sqrt{2m\left[V(x)-E\right]}}{\hbar}. \tag{131}$$

For a rectangular barrier of constant height V_0 extending from $x=0$ to $x=a$, the decay coefficient is

$$\kappa = \frac{\sqrt{2m\left(V_0 - E\right)}}{\hbar}, \ 0 < E < V_0. \tag{132}$$

The spatial solution inside the barrier is

$$\psi_2\left(x\right) = C\mathrm{e}^{\kappa x} + D\mathrm{e}^{-\kappa x}, \ 0 < x < a. \tag{133}$$

The complete energy-domain state in the forbidden region is

$$\Psi_2\left(x,z\right) = \left(C\mathrm{e}^{\kappa x} + D\mathrm{e}^{-\kappa x}\right)\chi_z\left(E\right). \tag{134}$$

The probability density is therefore

$$\rho_2\left(x\right) = \left|C\right|^2 \mathrm{e}^{2\kappa x} + \left|D\right|^2 \mathrm{e}^{-2\kappa x} + 2\,\mathrm{Re}\left(CD^*\right). \tag{135}$$

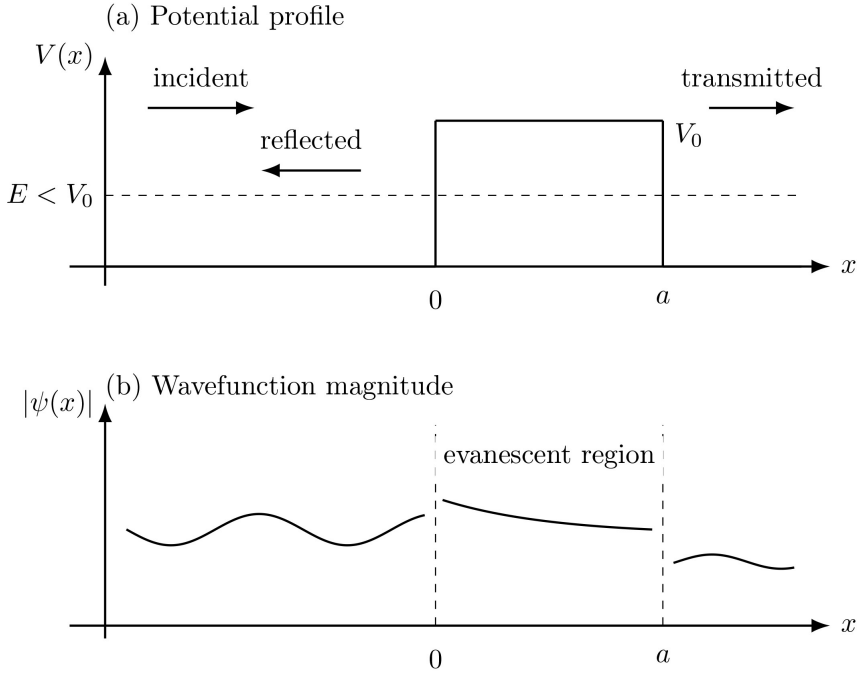

Figure 7. Barrier penetration for $E < V_0$. The upper panel shows the rectangular potential barrier and the incident, reflected, and transmitted components. The lower panel shows the corresponding wavefunction magnitude.

The energy-domain phase has disappeared because it has unit magnitude. The spatial density inside the barrier is therefore the same as in standard stationary quantum mechanics.

If the decreasing exponential is dominant, the wavefunction amplitude falls approximately as

$$\left|\psi_2\left(x\right)\right| \propto \mathrm{e}^{-\kappa x}, \tag{136}$$

while the corresponding probability density falls as

$$\left|\psi_2(x)\right|^2 \propto \mathrm{e}^{-2\kappa x}. \tag{137}$$

The characteristic penetration length is

$$\ell_\kappa = \frac{1}{\kappa} = \frac{\hbar}{\sqrt{2m(V_0 - E)}}. \tag{138}$$

The barrier profile and the corresponding attenuation of the wavefunction magnitude are shown in Figure 7.

The probability current inside the barrier is

$$j_2 = \frac{\hbar}{m}\mathrm{Im}\left[\psi_2^*(x)\frac{\mathrm{d}\psi_2(x)}{\mathrm{d}x}\right] = \frac{2\hbar\kappa}{m}\mathrm{Im}\left(CD^*\right). \tag{139}$$

A single real decaying exponential carries no current. The non-zero current through a finite barrier arises from the relative complex phase between the two exponential components.

8.2. Tunneling Probability

For the rectangular barrier, the spatial solutions outside the barrier are

$$\psi_1(x) = A\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-\mathrm{i}kx}, \quad x < 0, \tag{140}$$

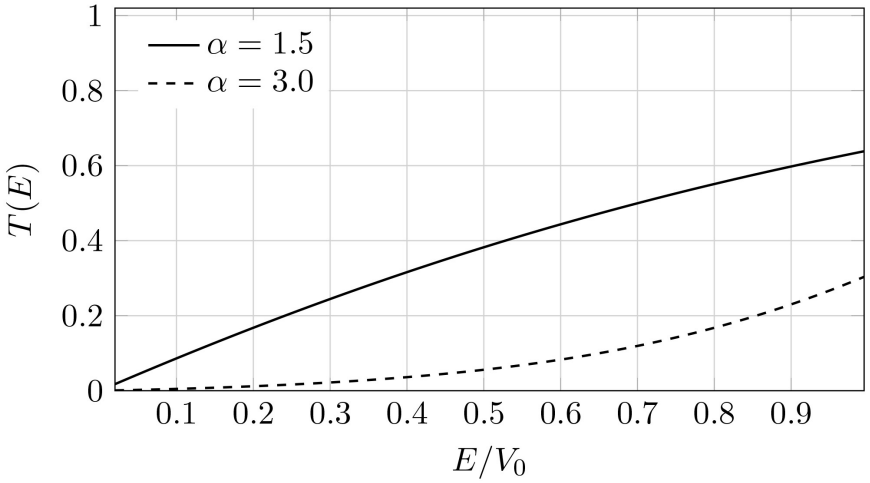

Figure 8. Exact tunneling probability for a rectangular barrier as a function of the normalized energy E/V_0. The parameter $\alpha = a\sqrt{2mV_0}/\hbar$ measures the effective barrier width.

and

$$\psi_3(x) = F\mathrm{e}^{\mathrm{i}kx}, \quad x > a, \tag{141}$$

where $k = \sqrt{2mE}/\hbar$. Since the asymptotic potentials are equal on both sides of the barrier, the incident and transmitted wavenumbers are the same. The tunneling probability is

$$T(E) = \left[1 + \frac{V_0^2 \sinh^2(\kappa a)}{4E(V_0 - E)}\right]^{-1}, \quad 0 < E < V_0. \tag{142}$$

The corresponding reflection probability is

$$R(E)=1-T(E). \tag{143}$$

For a high or wide barrier, $\kappa a \gg 1$, the transmission probability becomes

$$T(E)\simeq \frac{16E(V_0-E)}{V_0^2}\exp(-2\kappa a). \tag{144}$$

For a general smooth barrier, the leading WKB transmission probability is

$$T_{\mathrm{WKB}}(E)\simeq \exp\left[-2\int_{x_1}^{x_2}\kappa(x)\mathrm{d}x\right]. \tag{145}$$

The exact rectangular-barrier transmission probability is shown in **Figure 8** for two effective barrier widths.

For an incident wave packet with normalized spectral distribution $A(E)$, the total transmitted probability is obtained by averaging the fixed-energy coefficient over the packet,

$$P_{\mathrm{tr}}=\int_0^{V_0}\left|A(E)\right|^2 T(E)\mathrm{d}E. \tag{146}$$

If the packet also contains components with $E\geq V_0$, the corresponding above-barrier transmission coefficient must be included in the same spectral average. The barrier therefore acts as an energy-dependent filter rather than as a simple constant attenuation factor.

8.3. Energy-Domain Interpretation

For a stationary component of definite energy, the complete state in each spatial region has the form

$$\Psi_j(x,z)=\psi_j(x)\chi_z(E),\quad j=1,2,3. \tag{147}$$

The same energy-domain phase multiplies the incident, reflected, evanescent, and transmitted spatial components. It therefore cancels from the continuity conditions at the two boundaries and from the incident and transmitted probability currents. Consequently, the stationary tunneling coefficient is unchanged.

For a localized incident packet, the transmitted packet is

$$\Psi_{\mathrm{tr}}(x,z)=\int A(E)t(E)\mathrm{e}^{\mathrm{i}k(E)x}\chi_z(E)\mathrm{d}E. \tag{148}$$

Writing $t(E)=\left|t(E)\right|\exp\left[\mathrm{i}\delta_t(E)\right]$, this becomes

$$\Psi_{\mathrm{tr}}(x,z)=\int A(E)\left|t(E)\right|\exp\left[\mathrm{i}k(E)x+\mathrm{i}\delta_t(E)\right]\chi_z(E)\mathrm{d}E. \tag{149}$$

The barrier modifies the packet in two ways. The magnitude $\left|t(E)\right|$ changes the relative weight of each energy component, while the phase $\delta_t(E)$ changes the interference between those components. The energy-domain phase supplies the additional relative phase associated with the parameter z.

8.4. Probability versus Traversal Time

The fixed-energy transmission coefficient answers whether a detector can register

the particle beyond a finite classically forbidden region. It follows from the evanescent spatial solution, boundary matching, and probability-current ratios. No traversal duration is required for this calculation.

A different question asks how long the process takes. Wigner-Smith delay, dwell time, phase time, detector response, and attosecond tunneling delays are operational time observables or clock-dependent constructs [20] [21]. They cannot be identified with the auxiliary parameter z without an additional physical relation between z and a clock. The present framework therefore reproduces the stationary tunneling probability but does not claim to determine a tunneling time.

The contribution of the energy-domain formulation is this explicit separation of the existence and probability of tunneling from the duration assigned to it. The first belongs to the stationary spatial boundary-value problem. The second requires an operational clock, a time-dependent measurement protocol, or a relational degree of freedom beyond the present model.

9. z-Ordering, Reference Covariance, and Dynamical Scope

9.1. Generator of the Ordering Parameter

The complete state is ordered by

$$\hat{U}_z\left(z,z_0\right)=\exp\left[-\frac{\mathrm{i}\left(\hat{H}_x-E_{\mathrm{ref}}\hat{I}\right)^2\Delta z}{\hbar}\right], \tag{150}$$

so that

$$\mathrm{i}\hbar\frac{\partial}{\partial z}\left|\Psi\left(z\right)\right\rangle=\left(\hat{H}_x-E_{\mathrm{ref}}\hat{I}\right)^2\left|\Psi\left(z\right)\right\rangle. \tag{151}$$

The spectral differential identity and the operator evolution equation have distinct roles. Equation (10) acts on the spectral phase, whereas Equation (151) governs the complete state.

For self-adjoint $\hat{H}_x$, the generator in Equation (151) is self-adjoint and the ordering is unitary. For an observable $\hat{A}$ with no explicit z-dependence,

$$\frac{\mathrm{d}\left\langle\hat{A}\right\rangle}{\mathrm{d}z}=\frac{\mathrm{i}}{\hbar}\left\langle\left[\left(\hat{H}_x-E_{\mathrm{ref}}\hat{I}\right)^2,\hat{A}\right]\right\rangle. \tag{152}$$

This differs from the standard Heisenberg equation generated by $\hat{H}_x$ and makes clear that the z-ordered theory is not a simple replacement $t\mapsto z$.

9.2. Reference-Energy Covariance

The zero-of-energy issue is central because a bare $\hat{H}_x^2$ generator is not invariant under $\hat{H}_x\mapsto\hat{H}_x+C\hat{I}$. The referenced generator resolves this representational ambiguity by transforming the pair

$$\left(\hat{H}_x,E_{\mathrm{ref}}\right)\to\left(\hat{H}_x+C\hat{I},E_{\mathrm{ref}}+C\right). \tag{153}$$

Then

$$\left(\hat{H}_x + C\hat{I} - \left(E_{\text{ref}} + C\right)\hat{I}\right)^2 = \left(\hat{H}_x - E_{\text{ref}}\hat{I}\right)^2. \tag{154}$$

The physical prescription is therefore not an arbitrary shift of the Hamiltonian while keeping the reference fixed. A single declared threshold or baseline belongs to the definition of each problem and shifts covariantly with the Hamiltonian.

9.3. Narrow-Band Correspondence

Let the referenced energies be concentrated near a non-zero value ε_c,

$$\varepsilon_n = \varepsilon_c + \delta E_n. \tag{155}$$

Then

$$\varepsilon_n^2 = \varepsilon_c^2 + 2\varepsilon_c \delta E_n + \left(\delta E_n\right)^2. \tag{156}$$

If

$$\sigma_E^2 \left|\Delta z\right| / \hbar \ll 1, \tag{157}$$

the last term contributes a negligible relative phase over the occupied spectral width. The remaining relative phase has the standard form with

$$t_{\text{eff}} = 2\varepsilon_c \Delta z. \tag{158}$$

This is a local approximation tied to a selected spectral band. It is not a universal definition of time and does not apply automatically to broad distributions or to states centred on $\varepsilon_c = 0$.

9.4. Explicitly Time-Dependent Hamiltonians and Attosecond Physics

The present construction assumes a time-independent Hamiltonian. If the physical Hamiltonian is explicitly time dependent, $\hat{H} = \hat{H}(t)$, its definition already contains an external clock parameter and the stationary spectral decomposition used above is generally unavailable. Replacing that problem by a z-ordered state would require an enlarged relational theory in which the driving field, clock, or measurement apparatus is included as a quantum subsystem. Such a construction is not supplied here.

This limitation is relevant to attosecond science, strong-field ionization, and tunneling-delay measurements. These experiments compare clock-resolved or phase-resolved signals and are commonly analysed with time-dependent wave packets, Wigner-Smith-type delays, or operational dwell-time constructions [20] [21]. The present fixed-energy tunneling coefficient can be used as a stationary input, but the measured delay cannot be inferred from z alone.

9.5. Remaining Limitations

Three limitations remain. First, no direct measurement protocol for z has been established. Second, the reference energy must be selected by a physical threshold, baseline, or enlarged closed-system description; different non-covariant choices

would define different orderings. Third, broad superpositions evolve under a generator quadratic in the referenced Hamiltonian and generally differ from standard Schrödinger evolution. These limitations define the current boundary of the proposal rather than secondary qualifications.

10. Discussion

10.1. Principal Results

The analysis establishes four principal results. First, fixed-energy spatial eigenfunctions, probability currents, reflection and transmission coefficients, barrier-penetration probabilities, matrix elements, and selection rules are unchanged because the spectral phase is global within one component. Second, a complete bound-plus-continuum state can be written with generalized normalization, explicit channel labels, and the appropriate spectral measure. Third, the z-ordering is unitary and is generated by the square of a referenced Hamiltonian. Fourth, the Gaussian example gives explicit mathematical consequences for packet displacement and spreading.

These results do not establish complete equivalence with standard time-dependent quantum mechanics. In particular, the exact broadband generator is quadratic in the referenced Hamiltonian. Transition rates, lifetimes, linewidths, time delays, tunneling times, and explicitly driven dynamics remain outside the present framework.

10.2. Spectral Structure and Dynamical Content

The fixed-energy coefficients derived in Sections 5-8 follow from the stationary spatial problem. The energy-domain structure enters through the separation of the spatial basis from the spectral phase, the channel-resolved continuum state, the reference-covariant generator, and the distinction between stationary observables and the ordering of a superposition.

This distinction also explains why the stationary results are recovered almost automatically while the broadband dynamics raises new questions. The fixed-energy calculations establish compatibility with the spatial boundary-value problem. The Gaussian packet and Equation (152) describe the distinct consequences of the proposed generator for superpositions.

10.3. Relation to Timeless and Relational Approaches

Page-Wootters and related relational-clock approaches recover change through correlations between subsystems [1] [6]. Quantum cosmology encounters a related problem because the Wheeler-DeWitt equation contains no external Schrödinger time [7]-[11]. Bohmian and wave-function-based cosmological models provide examples in which histories or semiclassical evolution are extracted from a timeless constraint [12] [13].

The present proposal is not equivalent to these frameworks. It does not introduce a clock subsystem, and it is not derived from a Hamiltonian constraint of

general relativity. Its narrower purpose is to examine whether a unitary ordering based on referenced energy can preserve stationary quantum predictions and generate a mathematically consistent ordering of superpositions. A future relational completion would need to explain how z is correlated with a physical clock and why the selected energy reference is operationally preferred.

10.4. Future Work

The next steps are numerical comparison of exact z-ordered and standard time-ordered packets, extension to multi-channel and higher-dimensional scattering, and the construction of a relational clock model that can connect z with an experimental duration. Further work should also determine whether the referenced quadratic generator can be derived from a larger constrained system rather than postulated, and whether any experimentally distinguishable prediction survives once the clock and measurement interaction are included.

11. Conclusions

This paper develops an energy-domain formulation for time-independent non-relativistic quantum mechanics. The energy E is treated as a spectral label, and the energy derivative acts on the spectral phase. The spatial eigenfunction $\psi_E(x)$ is retained as an element of the spectral basis. Continuum states are represented as generalized eigenstates with channel labels and the appropriate spectral measure.

A reference energy E_{ref} has been incorporated directly into the phase and the generator. The complete state is ordered by

$$\hat{U}_z(z,z_0)=\exp\left[-\frac{\mathrm{i}\left(\hat{H}_x-E_{\mathrm{ref}}\hat{I}\right)^2(z-z_0)}{\hbar}\right], \tag{159}$$

which is invariant when the Hamiltonian and reference energy are shifted together. This resolves the representational ambiguity associated with the arbitrary zero of energy, while making the reference convention an explicit part of the physical problem.

For one fixed-energy component, the additional phase is global. The standard spatial equations, probability currents, reflection and transmission coefficients, tunneling probabilities, stationary transition strengths, and selection rules are therefore preserved. These results are exact but should be interpreted as consistency checks of the stationary sector.

For superpositions, the relative phases are generated by the square of the referenced Hamiltonian. The explicit Gaussian example gives

$$x_c(z)=x_0+\frac{\hbar^3k_0^3}{m^2}(z-z_0), \tag{160}$$

and

$$\sigma_x^2(z)=\frac{1}{4\sigma_k^2}+\left(\frac{3\hbar^3k_0^2}{m^2}\right)^2\sigma_k^2(z-z_0)^2 \tag{161}$$

in the narrow-k approximation. These expressions show both the local correspondence with standard group motion and the different spreading produced by the exact quadratic-energy phase.

The transition construction defines Born-rule probabilities conditioned on a specified interaction event. It does not define a rate per unit time. Likewise, the stationary transmission coefficient answers whether tunneling occurs and with what probability, but it does not determine traversal time, dwell time, Wigner-Smith delay, or an attosecond detector response.

The framework provides a unitary, reference-covariant spectral ordering and preserves fixed-energy stationary observables for time-independent Hamiltonians. It does not replace standard time-dependent quantum mechanics for broadband states or explicitly driven systems. An operational meaning for z and an embedding in a relational clock theory remain necessary for a complete time-free dynamics.

Author Contributions

M.L. conceived the energy-domain framework, developed the principal mathematical derivations, and wrote the main manuscript text. N.M.K. contributed to the Gaussian wave-packet analysis and to the interpretation of the fixed-energy and superposition sectors. J.R.W. contributed to the mathematical analysis, notation consistency, continuum-state formulation, and interpretation of the physical claims. All authors reviewed and approved the final manuscript.

Conflicts of Interest

The authors declare that they have no competing interests and that no external financial support was required for this work.

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