Journal of Modern Physics, 2026, 17(9), 1170-1196
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

Ultra-Relativistic Particle Acceleration as an Observational Probe of Stationary Solutions of Einstein's Field Equations. I. Applications to the Taub-NUT Family of Metrics

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How to cite this paper: McGruder III, C.H. (2026) Ultra-Relativistic Particle Acceleration as an Observational Probe of Stationary Solutions of Einstein's Field Equations. I. Applications to the Taub-NUT Family of Metrics. *Journal of Modern Physics*, **17**, 1170-1196.
https://doi.org/10.4236/jmp.2026.179052

Received: June 23, 2026
Accepted: September 19, 2026
Published: September 22, 2026

Abstract

One of the central objectives of observational General Relativity is to determine which exact solution of Einstein's field equations most accurately describes the gravitational field associated with a specific physical system. Because Einstein's field equations admit many stationary solutions, the appropriate spacetime geometry cannot be assumed *a priori*; it must ultimately be established by comparing the observable predictions of candidate solutions with astronomical observations of the specific physical system under investigation. Here we develop an observational methodology for determining which stationary solution of Einstein's field equations is applicable to a specific compact astrophysical object and for constraining the associated metric parameters, using the radial coordinate acceleration of outward-moving relativistic particles as the theoretical basis. Beginning with a general expression for radial coordinate acceleration in a static radial metric sector, we apply the formalism to the Taub-NUT, dyonic Lorentzian Taub-NUT-AdS, Taub-NUT anti-de Sitter, and Kerr-Taub-NUT geometries. In the Kerr-Taub-NUT case the analysis is restricted to outward radial motion along the symmetry axis; no claim is made concerning general non-axial Kerr-Taub-NUT geodesics. For each spacetime, we derive the radial acceleration expression, determine the conditions for outward motion, $\mathrm{d}r/\mathrm{d}t > 0$, with positive radial coordinate acceleration, $\mathrm{d}^2r/\mathrm{d}t^2 > 0$, and evaluate the weak-field ultra-relativistic limit. The resulting acceleration expressions depend differently upon mass, NUT charge, electric charge, asymptotic curvature, and rotation. The radial acceleration is not itself directly measured by cosmic-ray or cosmic-ray neutrino observations; rather, the methodology relates metric-dependent propagation to observable particle quantities together with independently determined source properties. This publication, Paper I, establishes this mathematical and observational methodology. Quantitative applications to cosmic-ray neutrinos and cosmic-ray protons are undertaken in Papers II and III.

Keywords

General Relativity, Stationary Spacetimes, Gravitational Repulsion, Radial Coordinate Acceleration, Cosmic Rays, Neutrinos, Taub-NUT Metrics

1. Introduction

Newton's theory of gravitation is based upon a single equation, the universal law of gravitation. Einstein's theory of gravitation, General Relativity, is based upon Einstein's field equations, which admit many exact solutions. One of the central objectives of observational General Relativity is to determine, from astronomical observations, which exact solution of Einstein's field equations most accurately describes the gravitational field associated with a specific physical system. Mathematical consistency alone does not determine which solution provides the appropriate description of a particular astrophysical object. That determination must ultimately rest upon comparison between the predictions of candidate stationary solutions and observations of the physical system under investigation.

Throughout this investigation, *stationary* refers to a spacetime whose metric components are independent of the coordinate time. Familiar stationary solutions include the Schwarzschild, Kerr, Reissner-Nordström, and Kerr-Newman metrics. Other exact solutions introduce additional geometrical structure, including the NUT parameter in the Taub-NUT solution [1] [2], electromagnetic parameters and asymptotic curvature in extensions of the Taub-NUT geometry [3] [4], and rotation in the Kerr-Taub-NUT solution [5].

The present investigation builds upon the long history of gravitational repulsion in General Relativity. Droste [6] [7] and Hilbert [8] showed that sufficiently rapid outward radial motion in Schwarzschild coordinates is associated with positive radial coordinate acceleration. McGruder [9] subsequently investigated both radial and non-radial motion in the Schwarzschild field and concluded that gravitational repulsion occurs only for radial motion. Dickau, Kauffmann, and Robertson have recently examined related gravitational-repulsion ideas in connection with the metric of a static point mass [10] and cosmology [11].

Previous applications of the Schwarzschild gravitational-repulsion calculation to very-high-energy and ultra-high-energy cosmic-ray protons [12] [13] and cosmic-ray neutrinos [14] assumed that the gravitational field of the source was described by the Schwarzschild solution. That assumption need not hold for a particular compact astrophysical object. Occam's razor compels us to reduce the number of assumptions required in interpreting observations whenever this can be accomplished without sacrificing explanatory power. We therefore drop the assumption that the Schwarzschild solution is necessarily applicable and allow the observations themselves to determine the stationary spacetime geometry. The scientific problem considered here is consequently different: rather than assuming the spacetime geometry in advance, we investigate whether ultra-relativistic particle propagation can determine which stationary solution is applicable and whether it can be employed to determine the associated parameters of the metric.

The Taub-NUT family provides a useful first application of this approach. The ordinary Taub-NUT solution introduces the NUT parameter [1] [2]; the dyonic Lorentzian Taub-NUT-AdS solution introduces electromagnetic parameters together with an AdS curvature scale [3]; the neutral Taub-NUT-AdS solution isolates NUT and asymptotic-curvature effects [4]; and the Kerr-Taub-NUT solution introduces rotation [5].

The methodology developed in this paper is based upon the radial coordinate acceleration of outward-moving relativistic particles. The radial coordinate acceleration is not itself a directly observed cosmic-ray or cosmic-ray neutrino quantity. Rather, the metric determines the radial propagation of the particle, and the resulting theoretical expressions may subsequently be related to observable particle quantities together with independently determined properties of the astrophysical source. This publication, Paper I, establishes this mathematical and observational methodology. Section 8 considers, at a general level, why ultra-relativistic cosmic-ray protons and cosmic-ray neutrinos provide potentially useful astrophysical applications, but does not undertake metric-dependent quantitative determinations. Those quantitative applications are the subjects of Papers II and III: Paper II applies the methodology quantitatively to cosmic-ray neutrinos, whereas Paper III applies the methodology to very-high-energy and ultra-high-energy cosmic-ray protons.

For the Taub-NUT, dyonic Lorentzian Taub-NUT-AdS, and neutral Taub-NUT-AdS geometries, the analysis is restricted to radial sectors of the form adopted in Section 2. For both the Kerr and Kerr-Taub-NUT geometries, an additional restriction is required. McGruder [9] concluded that gravitational repulsion occurs only for radial motion. Because frame dragging prevents a general fixed-angle trajectory from being treated as a purely radial diagonal sector, the Kerr and Kerr-Taub-NUT analyses considered in connection with the present methodology are therefore restricted to outward radial motion along the symmetry axis. No claim is made concerning general non-axial motion in these fields.

The objective of this investigation is therefore to determine how the radial acceleration expressions for outward-moving relativistic particles depend upon the stationary spacetime geometry and its associated parameters. We derive a common radial acceleration formula and apply it to the ordinary Taub-NUT, dyonic Lorentzian Taub-NUT-AdS, neutral spherically symmetric Taub-NUT-AdS, and axial Kerr-Taub-NUT geometries.

The paper is organized as follows. Section 2 develops the general radial acceleration formula and specifies its domain of applicability. Sections 3-6 apply the formula to the four Taub-NUT-type metrics. Section 7 develops the observational interpretation and comparison among the stationary solutions. Section 8 discusses, at a general level, why ultra-relativistic cosmic-ray protons and cosmic-ray neutrinos provide potentially useful astrophysical applications, while the metric-dependent quantitative applications are reserved for Papers II and III. Section 9 discusses the physical interpretation, applicability, and limitations of the formalism. Section 10 presents the Conclusion.

2. General Radial Acceleration Formula

The central objective of the present investigation is to develop an observational methodology capable of distinguishing among stationary solutions of Einstein's field equations describing the gravitational fields associated with specific compact astrophysical objects. The methodology developed here is based upon the radial coordinate acceleration of outward-moving relativistic particles relative to a distant observer. For the radial metric sectors to which Equation (16) applies, this quantity can be calculated from the spacetime geometry and then incorporated into a quantitative propagation calculation relating the metric to observable particle quantities and independently determined source properties.

The approach adopted in this paper builds upon the work of Droste [6] [7], Hilbert [8], and McGruder [9], who demonstrated that positive radial coordinate acceleration may occur for sufficiently relativistic outward-moving particles in the Schwarzschild geometry. Here we generalize that approach to the Taub-NUT solutions of Einstein's field equations considered in this investigation. The resulting formalism provides a common mathematical framework that can be applied to each stationary metric considered in this investigation.

Throughout this paper we consider test particles moving along geodesics. Their motion is determined entirely by the spacetime geometry, thereby allowing the radial acceleration expression to probe, through its metric dependence, the underlying gravitational field—that is, the metric or specific solution of Einstein's field equations describing that field. Since the objective is to compare different stationary solutions under identical physical assumptions, the same observational quantity is derived for each spacetime.

We employ the coordinate time used by a distant observer as the independent time variable. The resulting radial coordinate acceleration is therefore formulated relative to a distant observer monitoring the motion of relativistic particles escaping from the compact object under study. The radial coordinate acceleration remains a theoretical, coordinate-dependent quantity; it is not itself assumed to be directly measured in cosmic-ray or cosmic-ray neutrino observations.

We begin with the general stationary metric

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\phi}\mathrm{d}t\mathrm{d}\phi + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2, \quad (1)$$

where the metric coefficients depend only upon the spatial coordinates. McGruder [9] concluded that gravitational repulsion occurs only for radial motion. In the following development we therefore specialize to radially outward-moving particles and derive a general expression for the radial coordinate acceleration, subject to the radial-sector restrictions stated below.

For convenience we denote the time-time component of the metric by

$$B(r) \equiv -g_{00}(r), \tag{2}$$

where the sign convention follows the metric signature adopted throughout this paper.

For the Schwarzschild spacetime,

$$B(r) = 1 - \frac{\alpha}{r},$$

so the notation adopted here is a direct generalization of the Schwarzschild metric function.

The motion of a freely moving test particle in the reciprocal radial sector is described by the geodesic Lagrangian,

$$L = -B(r)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{B(r)}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2. \tag{3}$$

Here τ is an affine parameter along the particle worldline. Because t does not appear explicitly in the radial Lagrangian, the t geodesic equation can be integrated once. We write the resulting integration constant as C:

$$C = B(r)\frac{\mathrm{d}t}{\mathrm{d}\tau}. \tag{4}$$

The normalization condition for a timelike geodesic is

$$-B(r)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{B(r)}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = -1. \tag{5}$$

Substituting Equation (4) into Equation (5) gives

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = C^2 - B(r). \tag{6}$$

We define the radial coordinate velocity by

$$\dot{r} \equiv \frac{\mathrm{d}r}{\mathrm{d}t}. \tag{7}$$

Since

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \dot{r}\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{8}$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{C}{B(r)}, \tag{9}$$

we obtain

$$\dot{r}^2 = B(r)^2 - \frac{B(r)^3}{C^2} = B(r)^2 + AB(r)^3, A \equiv -\frac{1}{C^2}. \tag{10}$$

Equation (10) is the direct generalization of the Schwarzschild radial-velocity relation used by McGruder [12]. The constant A is an integration constant; no energy interpretation is required in deriving the radial coordinate acceleration.

Differentiating Equation (10) with respect to the stationary coordinate time t gives

$$2\dot{r}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \left[2B(r)B'(r) + 3AB(r)^2 B'(r)\right]\dot{r}. \tag{11}$$

For outward motion with $\dot{r} \neq 0$,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = B(r)B'(r) + \left(\frac{3}{2}\right)AB(r)^2 B'(r). \tag{12}$$

From Equation (10),

$$A = \left[\dot{r}^2 - B(r)^2\right]/B(r)^3. \tag{13}$$

Substitution of Equation (13) into Equation (12) gives

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = B(r)B'(r) + \left[3B'(r)/2B(r)\right]\left[\dot{r}^2 - B(r)^2\right]. \tag{14}$$

Collecting terms,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \left[3B'(r)/2B(r)\right]\dot{r}^2 - \left(\frac{1}{2}\right)B(r)B'(r). \tag{15}$$

Therefore,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \left[\frac{B'(r)}{2}\right]\left[\frac{3\dot{r}^2}{B(r)} - B(r)\right]. \tag{16}$$

Equation (16) is the central result of this section. It provides a general expression for the radial coordinate acceleration of outward-moving particles in any stationary spacetime whose relevant radial sector has the reciprocal form adopted here. Throughout the remainder of this paper, this equation will be specialized to the Taub-NUT family of metrics by substituting the appropriate metric function B(r) and its radial derivative.

The condition for gravitational repulsion is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} > 0. \tag{17}$$

For metrics satisfying $B'(r) > 0$, Equation (16) immediately gives

$$\dot{r}^2 > \frac{B^2}{3}. \tag{18}$$

To illustrate the general formalism, it is useful to consider the Schwarzschild solution. For the Schwarzschild metric,

$$B(r) = 1 - \frac{\alpha}{r}, \quad \alpha = 2GM, \tag{19}$$

where α is the Schwarzschild radius (in units where $c = 1$). Substituting this metric function into the general radial acceleration formula derived above immediately reproduces the Schwarzschild acceleration expression obtained by McGruder [9]. Likewise, substitution into the general repulsion condition yields the Schwarzschild repulsion condition. The Taub-NUT family considered in the following sections is obtained by replacing the Schwarzschild metric function $B(r)$ with the corresponding metric function for each stationary solution.

Inequality 18 is the direct generalization of the Schwarzschild repulsion condition. It establishes the minimum outward radial velocity dr/dt, as assigned by the distant observer, required for positive radial acceleration d^2r/dt^2 assigned by that observer. Once the metric function $B(r)$ is specified, the corresponding repulsion condition follows immediately.

Having established this general formalism, we now apply it to the Taub-NUT family of stationary solutions, beginning with the ordinary Taub-NUT metric.

3. Application to the Taub-NUT Metric

We now apply the general radial acceleration formalism developed in Section 2 to the ordinary Taub-NUT metric. The Taub-NUT solution, originally derived by Taub [1] and subsequently generalized by Newman, Tamburino, and Unti [2], is one of the best-known stationary generalizations of the Schwarzschild spacetime. In addition to the Schwarzschild mass parameter, it contains the NUT parameter, introducing an additional geometric parameter that modifies the spacetime structure beyond the Schwarzschild solution. Consequently, it provides the simplest member of the Taub-NUT family for investigating how departures from the Schwarzschild geometry modify the radial acceleration of outward-moving relativistic particles.

$$ds^2 = -B(r)\left(dt + 2n\cos\theta\, d\varphi\right)^2 + \frac{dr^2}{B(r)} + \left(r^2+n^2\right)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \quad (20)$$

For radial motion, $d\theta = d\varphi = 0$, and the relevant metric function is

$$B(r) = g_{00}(r) = \frac{r^2 - \alpha r - n^2}{r^2 + n^2}. \quad (21)$$

Here

$$\alpha = 2GM, \quad (22)$$

is twice the gravitational mass parameter in units where $c = 1$, and n is the NUT parameter [1] [2] [15]. Differentiating Equation (21) with respect to the radial coordinate gives

$$B'(r) = \frac{\alpha\left(r^2 - n^2\right) + 4rn^2}{\left(r^2+n^2\right)^2}. \quad (23)$$

Thus both the metric function and its radial derivative are known explicitly. Substituting Equations (21) and (23) into the general radial acceleration formula, Equation (16), yields

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{\alpha\left(r^2-n^2\right)+4rn^2}{2\left(r^2+n^2\right)^2}\left[3\dot{r}^2\left(\frac{r^2+n^2}{r^2-\alpha r-n^2}\right) - \left(\frac{r^2-\alpha r-n^2}{r^2+n^2}\right)\right]. \quad (24)$$

Equation (24) is the expression for the radial coordinate acceleration of outward-moving relativistic particles in the Taub-NUT spacetime. It shows that the radial coordinate acceleration depends not only on the Schwarzschild mass parameter through α, but also on the NUT parameter through both the metric function and its radial derivative. Consequently, the Taub-NUT geometry generally predicts a different radial acceleration from that of the Schwarzschild spacetime. The following analysis investigates the physical and observational consequences of this modified acceleration.

Applying the general repulsion condition, Equation (18), to the Taub-NUT metric, and assuming $B'(r) > 0$, gives

$$\dot{r}^2 > \frac{1}{3}\left(\frac{r^2 - \alpha r - n^2}{r^2 + n^2}\right)^2. \tag{25}$$

For outward-moving particles, $\dot{r} > 0$, this condition leads to

$$\dot{r} > \frac{1}{\sqrt{3}}\left(\frac{r^2 - \alpha r - n^2}{r^2 + n^2}\right). \tag{26}$$

Equations (25) and (26) are the Taub-NUT generalizations of the Schwarzschild repulsion condition,

$$\dot{r} > \frac{1}{\sqrt{3}}\left(1 - \frac{\alpha}{r}\right), \tag{27}$$

The NUT parameter therefore modifies the minimum outward radial velocity dr/dt, as assigned by the distant observer, required for positive radial acceleration d^2r/dt^2 assigned by that observer, providing the first indication that departures from the Schwarzschild geometry may have observable consequences and may therefore be used to test the applicability of the Taub-NUT metric as a description of the gravitational field associated with a specific compact astrophysical object.

To determine the magnitude of these departures, we consider the weak-field region,

$$r \gg \alpha, \quad r \gg n. \tag{28}$$

Under these conditions,

$$B(r) \simeq 1 - \frac{\alpha}{r} - \frac{2n^2}{r^2} + O\left(\frac{\alpha n^2}{r^3}, \frac{n^4}{r^4}\right), \tag{29}$$

and

$$B'(r) \simeq \frac{\alpha}{r^2} + \frac{4n^2}{r^3} + O\left(\frac{\alpha n^2}{r^4}, \frac{n^4}{r^5}\right). \tag{30}$$

Equations (29) and (30) show that the leading Schwarzschild contribution is supplemented by terms involving the NUT parameter. We now determine how these additional geometric contributions modify the radial acceleration of ultra-relativistic particles.

For ultra-relativistic particles,

$$\dot{r}^2 \simeq 1. \tag{31}$$

Substituting the weak-field expansions for $B(r)$ and $B'(r)$, Equations (29) and (30), respectively, together with the ultra-relativistic condition, Equation (31), into Equation (24) gives, to leading order,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \simeq \frac{\alpha}{r^2} + \frac{4n^2}{r^3}. \tag{32}$$

Since the Newtonian gravitational acceleration is

$$g = \frac{GM}{r^2} = \frac{\alpha}{2r^2}, \tag{33}$$

the weak-field acceleration becomes

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \simeq 2g + \frac{4n^2}{r^3}. \tag{34}$$

Thus, in the weak-field ultra-relativistic limit, the Taub-NUT metric reproduces the leading Schwarzschild result, $2g$, while predicting an additional positive contribution proportional to n^2/r^3.

The natural way to compare the Taub-NUT and Schwarzschild accelerations is to normalize the radial acceleration by the Newtonian acceleration g. For the Schwarzschild metric,

$$\frac{1}{g}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \to 2, \tag{35}$$

whereas for the Taub-NUT metric,

$$\frac{1}{g}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \simeq 2 + \frac{8n^2}{\alpha r}. \tag{36}$$

Thus the departure from the Schwarzschild result is controlled by the dimensionless quantity

$$\frac{8n^2}{\alpha r} = \frac{8(n/\alpha)^2}{r/\alpha}. \tag{37}$$

Illustrative Magnitude of the Taub-NUT Correction

The weak-field expression permits a direct estimate of the magnitude of the departure from the Schwarzschild result. Defining

$$\delta_{\mathrm{TN}} \equiv \frac{a_{\mathrm{TN}} - a_{\mathrm{Schw}}}{a_{\mathrm{Schw}}}, \tag{38}$$

with $a_{\mathrm{Schw}} = \alpha/r^2$, the weak-field Taub-NUT expression gives

$$\delta_{\mathrm{TN}} = \frac{4n^2}{\alpha r} = 4\left(\frac{n}{\alpha}\right)^2\left(\frac{\alpha}{r}\right). \tag{39}$$

The numerical values used here are chosen for two different reasons. The NUT scale is observationally motivated. Ghasemi-Nodehi [16], using shadow observations of Sgr A* in the Kerr-Taub-NUT geometry, finds a dimensionless NUT charge of order $|n|/M \lesssim 0.5$ for Schwarzschild-like and very slowly rotating configurations. Since $\alpha = 2GM/c^2$, this limiting magnitude corresponds to

$n/\alpha = 0.25$ when the mass is expressed as its geometrized length. This value is employed only as an observationally motivated NUT scale; it is not asserted to be a determination of n for the ordinary Taub-NUT metric considered here.

The radial value is chosen from the domain of validity of the weak-field approximation. The weak-field expansion requires $r \gg \alpha$ and $r \gg n$. We therefore adopt $r/\alpha = 50$. For $n/\alpha = 0.25$, this also gives $r/n = 200$, so both weak-field conditions are well satisfied. The fractional departure is then

$$\delta_{\mathrm{TN}} = 4(0.25)^2\left(\frac{1}{50}\right) = 5\times10^{-3} = 0.5\%. \tag{40}$$

Thus an observationally motivated NUT scale can produce a sub-percent departure from the Schwarzschild radial acceleration expression in the weak-field region. This is an order-of-magnitude illustration, not a determination of the NUT parameter for a particular compact astrophysical object. Detailed metric-dependent applications are reserved for Papers II and III.

The physical interpretation of the NUT parameter remains an open question, with a variety of interpretations having been proposed in the literature. The acceleration derived here provides an operational interpretation of the NUT parameter in terms of its effect on the radial acceleration of outward-moving relativistic particles. This operational interpretation is developed in detail in Section 8, where it is compared with the principal interpretations proposed in the literature.

4. Dyonic Lorentzian Taub-NUT-AdS Metric

In our previous investigations, including the stationary-cosmology studies, we referred to this geometry as the Lorentzian Taub-NUT metric, following the nomenclature adopted there [15] [17]. Here (Paper I) we refine this nomenclature and refer to the metric as the *dyonic Lorentzian Taub-NUT-AdS metric*. No change in the metric itself is implied. The revised designation makes explicit the presence of both electric and magnetic parameters and the finite curvature length ℓ, for which $\Lambda = -3/\ell^2$, identifying the asymptotic geometry as anti-de Sitter [3].

The complete line element may be written as

$$\mathrm{d}s^2 = -B_{\mathrm{LTN}}(r)\left(\mathrm{d}t + 2n\cos\theta\,\mathrm{d}\phi\right)^2 + \frac{\mathrm{d}r^2}{B_{\mathrm{LTN}}(r)} + \left(r^2+n^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{41}$$

We next apply the general radial acceleration formalism developed in Section 2 to the dyonic Lorentzian Taub-NUT-AdS metric. This solution is commonly referred to in the literature as the dyonic Lorentzian Taub-NUT-AdS metric. In our previous cosmological investigation, however, the word *charged* was omitted to avoid suggesting that the universe itself possesses an electric charge [15]. The more explicit dyonic Lorentzian Taub-NUT-AdS terminology is adopted here.

The dyonic Lorentzian Taub-NUT-AdS metric represents the next level of generalization beyond the ordinary Taub-NUT spacetime. Whereas the ordinary Taub-NUT metric introduces the NUT parameter as an additional geometric parameter beyond the Schwarzschild mass parameter, the dyonic Lorentzian Taub-NUT-AdS metric also contains the parameters e, g, and l. It therefore pro-

vides an opportunity to investigate how these additional geometric ingredients modify the radial acceleration of outward-moving relativistic particles.
where, in the notation of the present manuscript,

$$B_{\mathrm{LTN}}(r)=\frac{r^2-\alpha r-n^2+4n^2g^2+e^2}{r^2+n^2}-\frac{3n^4-6n^2r^2-r^4}{l^2\left(r^2+n^2\right)}. \tag{42}$$

Equivalently,

$$B_{\mathrm{LTN}}(r)=\frac{r^2-\alpha r-n^2+4n^2g^2+e^2+\dfrac{r^4+6n^2r^2-3n^4}{l^2}}{r^2+n^2}. \tag{43}$$

Here $\alpha=2GM$ is twice the gravitational mass parameter in units where $c=1$, n is the NUT parameter, e and g are the additional parameters appearing in the Lorentzian Taub-NUT solution, and l is the additional length scale. Thus the metric depends upon the complete parameter set

$$\alpha, n, e, g, l. \tag{44}$$

For purely radial motion,

$$\mathrm{d}\theta=\mathrm{d}\phi=0, \tag{45}$$

and Equation (41) reduces to

$$\mathrm{d}s^2=-B_{\mathrm{LTN}}(r)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{B_{\mathrm{LTN}}(r)}. \tag{46}$$

The radial sector therefore has precisely the form required by the general radial acceleration formula, Equation (16).

Differentiating Equation (43) with respect to r gives

$$B'_{\mathrm{LTN}}(r)=\frac{\alpha l^2\left(r^2-n^2\right)+4l^2n^2r-2l^2e^2r-8l^2n^2g^2r+18n^4r+4n^2r^3+2r^5}{l^2\left(r^2+n^2\right)^2}. \tag{47}$$

Thus both $B_{\mathrm{LTN}}(r)$ and $B'_{\mathrm{LTN}}(r)$ are known explicitly. Substituting Equations (43) and (47) into the general radial acceleration formula, Equation (16), gives

$$\begin{aligned}\frac{\mathrm{d}^2r}{\mathrm{d}t^2}=&\frac{\alpha l^2\left(r^2-n^2\right)+4l^2n^2r-2l^2e^2r-8l^2n^2g^2r+18n^4r+4n^2r^3+2r^5}{2l^2\left(r^2+n^2\right)^2}\\&\times\left[\frac{3\dot{r}^2\left(r^2+n^2\right)}{r^2-\alpha r-n^2+4n^2g^2+e^2+\dfrac{r^4+6n^2r^2-3n^4}{l^2}}\right.\\&\left.-\frac{r^2-\alpha r-n^2+4n^2g^2+e^2+\dfrac{r^4+6n^2r^2-3n^4}{l^2}}{r^2+n^2}\right].\end{aligned} \tag{48}$$

Equation (48) shows that the radial coordinate acceleration depends upon all five parameters of the Lorentzian Taub-NUT spacetime. Unlike the ordinary Taub-NUT metric, where the departure from the Schwarzschild acceleration is governed solely by the NUT parameter, the dyonic Lorentzian Taub-NUT-AdS metric predicts departures arising from the combined influence of the parameters n, e, g, and l.

Applying the general repulsion condition, Equation (18), to the Lorentzian Taub-NUT metric, and assuming

$$B'_{\mathrm{LTN}}(r) > 0, \tag{49}$$

gives

$$\dot{r}^2 > \frac{1}{3} B^2_{\mathrm{LTN}}(r). \tag{50}$$

Substituting the explicit metric function, Equation (43), into Equation (50) gives

$$\dot{r}^2 > \frac{1}{3}\left[\frac{r^2 - \alpha r - n^2 + 4n^2 g^2 + e^2 + \dfrac{r^4 + 6n^2 r^2 - 3n^4}{l^2}}{r^2 + n^2}\right]^2. \tag{51}$$

For an outward-moving particle, $\dot{r} > 0$, the corresponding velocity condition may be written as

$$\dot{r} > \frac{1}{\sqrt{3}}\left[\frac{r^2 - \alpha r - n^2 + 4n^2 g^2 + e^2 + \dfrac{r^4 + 6n^2 r^2 - 3n^4}{l^2}}{r^2 + n^2}\right], \tag{52}$$

provided that the metric function appearing in square brackets is positive.

Equations (51) and (52) are the Lorentzian Taub-NUT forms of the general repulsion condition given by Equation (18). They show that the minimum outward radial velocity dr/dt, as assigned by the distant observer, required for positive radial acceleration d^2r/dt^2 assigned by that observer depends upon the complete parameter set (n, e, g, l). Consequently, the threshold for positive radial coordinate acceleration is not determined by the mass parameter alone.

The Lorentzian Taub-NUT result may be checked by taking the Schwarzschild limit,

$$n \to 0,\ e \to 0,\ g \to 0,\ l \to \infty. \tag{53}$$

Under the limiting conditions in Equation (53), the metric function becomes

$$B_{\mathrm{LTN}}(r) \leftrightarrow 1 - \frac{\alpha}{r}. \tag{54}$$

The repulsion condition, Equation (52), therefore reduces to

$$\dot{r} > \frac{1}{\sqrt{3}}\left(1 - \frac{\alpha}{r}\right), \tag{55}$$

which is the Schwarzschild form of the general repulsion condition, Equation (18). Likewise, Equation (48) reduces to the Schwarzschild expression for the radial coordinate acceleration. These reductions provide consistency checks on the Lorentzian Taub-NUT results.

We now consider the ultra-relativistic limit,

$$\dot{r}^2 \simeq 1. \tag{56}$$

Substituting Equation (56) into Equation (48) gives

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \simeq \frac{\alpha l^2\left(r^2-n^2\right)+4l^2n^2r-2l^2e^2r-8l^2n^2g^2r+18n^4r+4n^2r^3+2r^5}{2l^2\left(r^2+n^2\right)^2} \times\left[\frac{3\left(r^2+n^2\right)}{r^2-\alpha r-n^2+4n^2g^2+e^2+\frac{r^4+6n^2r^2-3n^4}{l^2}} - \frac{r^2-\alpha r-n^2+4n^2g^2+e^2+\frac{r^4+6n^2r^2-3n^4}{l^2}}{r^2+n^2}\right]. \tag{57}$$

Equation (57) is the expression for the radial coordinate acceleration of outward-moving ultra-relativistic particles in the dyonic Lorentzian Taub-NUT-AdS spacetime. Its fully expanded form makes the dependence upon the mass parameter α, the NUT parameter n, the parameters e and g, and the length scale l directly visible.

The parameters n, e, g, and l alter both factors in Equation (57). They modify the radial variation of the metric through $B'_{\mathrm{LTN}}\left(r\right)$ and also modify the velocity-dependent factor inherited from the general radial acceleration expression. Depending upon their values, these contributions may enhance, reduce, or partially cancel one another. The dyonic Lorentzian Taub-NUT-AdS metric therefore predicts a radial acceleration profile that may differ quantitatively from those predicted by the Schwarzschild and ordinary Taub-NUT metrics.

The ultra-relativistic limit is central to the observational methodology developed in this paper. For an outward-moving particle whose velocity is extremely close to the speed of light, Equation (57) gives the metric-specific radial coordinate acceleration entering the theoretical propagation calculation. Because the acceleration depends simultaneously upon several spacetime parameters, a quantitative application will generally constrain combinations of $\left(\alpha,n,e,g,l\right)$ rather than determine each parameter independently. The connection to observation must be made through observable particle quantities together with independently determined properties of the compact astrophysical source; the radial coordinate acceleration itself is not directly measured.

Applicability to Neutral and Charged Test Particles

The radial acceleration expression derived in this section describes geodesic motion. Because the dyonic Lorentzian Taub-NUT-AdS spacetime contains an electromagnetic field, a charged test particle is in general also subject to the Lorentz force and therefore does not follow a geodesic determined solely by the spacetime metric. Consequently, the expression derived here applies directly to neutral test particles, including neutrinos. Application to charged cosmic-ray particles such as protons requires inclusion of the Lorentz force in the equation of motion and is not considered here (Paper I). This distinction will be retained when the general relevance of neutrinos and cosmic-ray protons to the methodology is discussed in Section 8.

5. Taub-NUT Anti-de Sitter Metric

The geometry considered in this section is closely related to the dyonic Lorentzian Taub-NUT-AdS geometry investigated in Section 4. Setting the electric and magnetic parameters in Section 4 equal to zero gives its neutral Taub-NUT-AdS limit. The purpose of the present section is to examine that neutral AdS geometry independently and thereby isolate the combined effects of the NUT parameter and asymptotic AdS curvature from the electromagnetic contributions present in Section 4.

The Taub-NUT-AdS solution belongs to a broader topological family containing spherical ($\kappa=1$), planar ($\kappa=0$), and hyperbolic ($\kappa=-1$) spatial sections. Here (Paper I), we restrict the analysis to the spherically symmetric member, $\kappa=1$, because the methodology is being developed for compact astrophysical objects [4].

We next apply the general radial acceleration formula, Equation (16), to the Taub-NUT anti-de Sitter (AdS) metric. This solution extends the ordinary Taub-NUT geometry by introducing a negative cosmological constant and the associated AdS curvature length scale l. It therefore provides an opportunity to determine how the combined effects of the mass parameter, the NUT charge, and the asymptotic AdS curvature modify the radial coordinate acceleration of outward-moving relativistic particles.

The Taub-NUT AdS solution belongs to a broader topological family that permits spherical ($\kappa=1$), planar ($\kappa=0$), and hyperbolic ($\kappa=-1$) two-dimensional spatial sections. In the present investigation we restrict our attention to the spherically symmetric member of this family ($\kappa=1$). This specialization is appropriate because the objective of this paper is to investigate stationary metrics describing the gravitational fields associated with compact astrophysical objects, for which spherical symmetry provides the natural reference geometry. The planar and hyperbolic members of the Taub-NUT AdS family are therefore beyond the scope of the present investigation.

Following Mann, Pando Zayas, and Park [4], and adopting the notation introduced in our previous cosmological investigation [15], the spherically symmetric ($\kappa=1$) Taub-NUT AdS metric may be written as

$$\mathrm{d}s^2=-B_{\mathrm{TNAdS}}(r)\left(\mathrm{d}t+2s\cos\theta\,\mathrm{d}\phi\right)^2+\frac{\mathrm{d}r^2}{B_{\mathrm{TNAdS}}(r)}+\left(r^2+s^2\right)\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right), \quad (58)$$

where

$$B_{\text{TNAdS}}(r) = \frac{r^2 - \alpha r - s^2 + \dfrac{r^4 + 6s^2r^2 - 3s^4}{l^2}}{r^2 + s^2}. \tag{59}$$

Here $\alpha = 2GM$ is twice the gravitational mass parameter in units where $c = 1$, s is the NUT charge, and l is the anti-de Sitter curvature length. The corresponding cosmological constant is

$$\lambda = -\frac{3}{l^2}. \tag{60}$$

The metric therefore depends upon the three parameters

$$\alpha, s, l. \tag{61}$$

The parameter s characterizes the departure from the Schwarzschild geometry associated with the NUT charge, whereas the AdS length scale l determines the curvature of the asymptotic spacetime.

For purely radial motion,

$$\mathrm{d}\theta = \mathrm{d}\phi = 0, \tag{62}$$

and Equation (58) reduces to

$$\mathrm{d}s^2 = -B_{\text{TNAdS}}(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B_{\text{TNAdS}}(r)}. \tag{63}$$

The radial sector therefore has precisely the form required by the general radial acceleration formula, Equation (16).

Differentiating Equation (59) with respect to r gives

$$B'_{\text{TNAdS}}(r) = \frac{\alpha l^2\left(r^2 - s^2\right) + 4l^2s^2r + 18s^4r + 4s^2r^3 + 2r^5}{l^2\left(r^2 + s^2\right)^2}. \tag{64}$$

Thus both $B_{\text{TNAdS}}(r)$ and $B'_{\text{TNAdS}}(r)$ are known explicitly. Substituting Equations (59) and (64) into the general radial acceleration formula, Equation (16), gives

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{\alpha l^2\left(r^2 - s^2\right) + 4l^2s^2r + 18s^4r + 4s^2r^3 + 2r^5}{2l^2\left(r^2 + s^2\right)^2} \times\left[\frac{3\dot{r}^2\left(r^2 + s^2\right)}{r^2 - \alpha r - s^2 + \dfrac{r^4 + 6s^2r^2 - 3s^4}{l^2}} - \frac{r^2 - \alpha r - s^2 + \dfrac{r^4 + 6s^2r^2 - 3s^4}{l^2}}{r^2 + s^2}\right]. \tag{65}$$

Equation (65) is the expression for the radial coordinate acceleration of outward-moving relativistic particles in the spherically symmetric Taub-NUT AdS spacetime. Its fully expanded form shows explicitly that the acceleration depends simultaneously upon the mass parameter α, the NUT charge s, and the AdS curvature length l.

The NUT charge and the AdS curvature modify both the radial derivative of the metric function and the velocity-dependent factor inherited from the general radial acceleration formula. Consequently, the ultra-relativistic acceleration is not obtained by introducing a simple cosmological constant correction to the ordinary Taub-NUT result. Instead, it represents the combined influence of the complete spherically symmetric Taub-NUT AdS geometry. Depending upon their numerical values, the NUT charge and the AdS curvature may enhance, reduce, or partially cancel one another's contributions to the radial coordinate acceleration.

6. Kerr-Taub-NUT Metric: Axial Radial Sector

We next apply the general radial acceleration formula, Equation (16), to the Kerr-Taub-NUT metric. This metric is the most general stationary solution considered in the present investigation. It extends the previous metrics by incorporating rotation in addition to the mass parameter, the NUT parameter, and the electromagnetic parameter. Consequently, it provides the most general test of the observational methodology developed in this paper.

Following Miller [5], and adopting the notation introduced in our previous cosmological investigation [15], the complete Kerr-Taub-NUT metric may be written as

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta(r)}{\Sigma(r,\theta)}\left[\mathrm{d}t+\left(2n_K\cos\theta-a\sin^2\theta\right)\mathrm{d}\phi\right]^2 \\ &+\frac{\sin^2\theta}{\Sigma(r,\theta)}\left[a\,\mathrm{d}t-\left(r^2+a^2+n_K^2\right)\mathrm{d}\phi\right]^2 \\ &+\frac{\Sigma(r,\theta)}{\Delta(r)}\mathrm{d}r^2+\Sigma(r,\theta)\,\mathrm{d}\theta^2, \end{aligned} \tag{66}$$

where

$$\Sigma(r,\theta)=r^2+\left(n_K+a\cos\theta\right)^2, \tag{67}$$

and

$$\Delta(r)=r^2-\alpha r+a^2-n_K^2+q_e^2. \tag{68}$$

Here $\alpha = 2GM$ is twice the gravitational mass parameter in units where $c=1$, a is the rotation parameter, n_K is the Kerr-Taub-NUT parameter, and q_e is the electromagnetic parameter. The notation n_K is adopted to distinguish the Kerr-Taub-NUT parameter from the NUT parameters used in Sections 3-5.

The coefficient of $\mathrm{d}t^2$ obtained from Equation (66) is

$$g_{tt}(r,\theta)=\frac{a^2\sin^2\theta-\Delta(r)}{\Sigma(r,\theta)}, \tag{69}$$

which is the metric component employed in our previous cosmological investigation.

McGruder [9] concluded that gravitational repulsion occurs only for radial motion. In a rotating spacetime, frame dragging prevents a general fixed-angle trajectory from being represented by the reciprocal radial sector adopted in Section

2. We therefore restrict the Kerr-Taub-NUT calculation to outward radial motion along the symmetry axis.

For motion at an arbitrary fixed value of θ, setting

$$\mathrm{d}\theta = \mathrm{d}\phi = 0, \tag{70}$$

gives

$$\mathrm{d}s^2 = -\frac{\Delta(r) - a^2\sin^2\theta}{\Sigma(r,\theta)}\mathrm{d}t^2 + \frac{\Sigma(r,\theta)}{\Delta(r)}\mathrm{d}r^2. \tag{71}$$

At an arbitrary fixed angular direction the coefficients of $\mathrm{d}t^2$ and $\mathrm{d}r^2$ are not reciprocals. Consequently, the general radial acceleration formula, Equation (16), cannot be applied directly.

Because the general radial acceleration formula, Equation (16), derived in Section 2 assumes a reciprocal radial sector, we specialize to radial motion along the symmetry axis, where the Kerr-Taub-NUT metric possesses precisely this structure.

Accordingly, we consider

$$\theta = 0 \ \text{ or } \ \theta = \pi, \tag{72}$$

for which

$$\sin\theta = 0, \ \cos\theta = \epsilon, \ \epsilon = \pm 1. \tag{73}$$

The values $\epsilon = \pm 1$ correspond to the two directions along the rotation axis. Along either direction,

$$\Sigma_\epsilon(r) = r^2 + (n_K + \epsilon a)^2, \tag{74}$$

and the radial sector reduces to

$$\mathrm{d}s^2 = -B_{\mathrm{KTN},\epsilon}(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B_{\mathrm{KTN},\epsilon}(r)}, \tag{75}$$

where

$$B_{\mathrm{KTN},\epsilon}(r) = \frac{r^2 - \alpha r + a^2 - n_K^2 + q_e^2}{r^2 + (n_K + \epsilon a)^2}. \tag{76}$$

The axial radial sector therefore has precisely the form required by the general radial acceleration formula, Equation (16). This specialization applies specifically to outward-moving relativistic particles propagating along the symmetry axis. Along the axis, the frame-dragging cross terms do not contribute to the reduced radial line element, and the acceleration formalism developed in Section 2 applies directly.

Differentiating Equation (76) with respect to r gives

$$B'_{\mathrm{KTN},\epsilon}(r) = \frac{\alpha\left[r^2 - (n_K + \epsilon a)^2\right] + 2r\left[2n_K(n_K + \epsilon a) - q_e^2\right]}{\left[r^2 + (n_K + \epsilon a)^2\right]^2}. \tag{77}$$

Thus both $B_{\mathrm{KTN},\epsilon}(r)$ and $B'_{\mathrm{KTN},\epsilon}(r)$ are known explicitly. Substituting Equations (76) and (77) into the general radial acceleration formula, Equation

(16), gives

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}=\frac{\alpha\left[r^2-\left(n_K+\epsilon a\right)^2\right]+2r\left[2n_K\left(n_K+\epsilon a\right)-q_e^2\right]}{2\left[r^2+\left(n_K+\epsilon a\right)^2\right]^2}\times\left[\frac{3\dot{r}^2\left[r^2+\left(n_K+\epsilon a\right)^2\right]}{r^2-\alpha r+a^2-n_K^2+q_e^2}-\frac{r^2-\alpha r+a^2-n_K^2+q_e^2}{r^2+\left(n_K+\epsilon a\right)^2}\right]. \tag{78}$$

Equation (78) is the expression for the radial coordinate acceleration of outward-moving relativistic particles propagating along the symmetry axis of the Kerr-Taub-NUT spacetime. Its fully expanded form shows explicitly how the mass parameter α, the rotation parameter a, the Kerr-Taub-NUT parameter n_K, the electromagnetic parameter q_e, and the axial direction ϵ jointly determine the radial coordinate acceleration.

The appearance of the combination $n_K+\epsilon a$ shows that rotation and the Kerr-Taub-NUT parameter do not contribute independently to axial radial motion. Instead, they contribute through their combined influence, so that the predicted radial acceleration may differ along the two opposite directions of the symmetry axis. The Kerr-Taub-NUT geometry therefore predicts both parameter-dependent and direction-dependent departures from the Schwarzschild radial coordinate acceleration. The metric functions employed throughout the present investigation are summarized in **Table 1**.

Applying the general repulsion condition, Equation (18), to the axial Kerr-Taub-NUT metric, and assuming

$$B'_{\mathrm{KTN},\epsilon}\left(r\right)>0, \tag{79}$$

gives

$$\dot{r}^2>\frac{1}{3}B^2_{\mathrm{KTN},\epsilon}\left(r\right). \tag{80}$$

Substituting Equation (76) into Equation (80) gives

$$\dot{r}^2>\frac{1}{3}\left[\frac{r^2-\alpha r+a^2-n_K^2+q_e^2}{r^2+\left(n_K+\epsilon a\right)^2}\right]^2. \tag{81}$$

For an outward-moving relativistic particle, $\dot{r}>0$, the corresponding velocity condition is

$$\dot{r}>\frac{1}{\sqrt{3}}\left[\frac{r^2-\alpha r+a^2-n_K^2+q_e^2}{r^2+\left(n_K+\epsilon a\right)^2}\right]. \tag{82}$$

Equations (81) and (82) are the Kerr-Taub-NUT forms of the general repulsion condition, Equation (18). Unlike the previous metrics, the threshold for positive radial coordinate acceleration depends upon the combined influence of the mass parameter, the rotation parameter, the Kerr-Taub-NUT parameter, the electromagnetic parameter, and the direction of motion along the rotation axis.

As a consistency check, consider the Schwarzschild limit,

$$a \to 0,\ n_K \to 0,\ q_e \to 0. \tag{83}$$

Equation (76) then reduces to

$$B_{\mathrm{KTN},\epsilon}(r) \to 1 - \frac{\alpha}{r}, \tag{84}$$

while Equation (82) reduces to

$$\dot{r} > \frac{1}{\sqrt{3}}\left(1 - \frac{\alpha}{r}\right), \tag{85}$$

which is precisely the Schwarzschild form of the general repulsion condition, Equation (18). Likewise, Equation (78) reduces to the Schwarzschild radial coordinate acceleration. These reductions provide important consistency checks on the Kerr-Taub-NUT results.

We now consider the ultra-relativistic limit,

$$\dot{r}^2 \simeq 1. \tag{86}$$

Substituting Equation (86) into Equation (78) gives

$$\begin{aligned}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} \simeq\ & \frac{\alpha\left[r^2 - (n_K + \epsilon a)^2\right] + 2r\left[2n_K(n_K + \epsilon a) - q_e^2\right]}{2\left[r^2 + (n_K + \epsilon a)^2\right]^2} \\ & \times \left[\frac{3\left[r^2 + (n_K + \epsilon a)^2\right]}{r^2 - \alpha r + a^2 - n_K^2 + q_e^2} - \frac{r^2 - \alpha r + a^2 - n_K^2 + q_e^2}{r^2 + (n_K + \epsilon a)^2}\right].\end{aligned} \tag{87}$$

Equation (87) is the expression for the radial coordinate acceleration of outward-moving ultra-relativistic particles propagating along the symmetry axis of the Kerr-Taub-NUT spacetime. Its fully expanded form shows explicitly the combined influence of the mass parameter α, the rotation parameter a, the Kerr-Taub-NUT parameter n_K, the electromagnetic parameter q_e, and the axial direction ϵ.

Rotation and the Kerr-Taub-NUT parameter influence the acceleration through the combination $n_K + \epsilon a$. Consequently, depending upon their numerical values, rotation and the Kerr-Taub-NUT parameter may enhance, reduce, or partially cancel one another's contributions to the radial coordinate acceleration. In addition, the predicted acceleration may differ along the two opposite directions of the rotation axis.

The ultra-relativistic limit is central to the observational methodology developed in this paper. For outward-moving particles whose velocities are extremely close to the speed of light, Equation (87) gives the metric-specific radial coordinate acceleration entering the theoretical propagation calculation. Because the acceleration depends simultaneously upon α, a, n_K, and q_e, a quantitative application will generally constrain combinations of these parameters rather than determine each independently. The connection to observation must be made through observable particle quantities and independently determined source properties; the radial coordinate acceleration itself is not directly measured.

Applicability to Neutral and Charged Test Particles

The acceleration expression derived in this section describes geodesic motion. When $q_e \neq 0$, the Kerr-Taub-NUT spacetime also contains an electromagnetic field. A charged test particle is then generally subject to the Lorentz force in addition to the gravitational interaction. The expression derived here applies directly to neutral test particles, including neutrinos. Application to charged cosmic-ray particles such as protons in a Kerr-Taub-NUT spacetime with nonzero electromagnetic parameter requires inclusion of the Lorentz force and is not considered here (Paper I). For $q_e = 0$, this particular electromagnetic restriction is absent.

7. Observational Determination of Stationary Metrics

Sections 3-6 derive metric-dependent expressions for the radial coordinate acceleration of outward-moving relativistic particles for four stationary geometries: the Taub-NUT metric, the dyonic Lorentzian Taub-NUT-AdS metric, the neutral spherically symmetric Taub-NUT-AdS metric, and the Kerr-Taub-NUT metric restricted to axial radial motion. In every case, the expression follows from the general radial acceleration formula, Equation (16), providing a common theoretical framework for comparing the different stationary geometries.

The fundamental result is that different stationary solutions predict different expressions for the radial coordinate acceleration. Consequently, the propagation of an outward-moving relativistic particle depends upon which stationary solution of Einstein's field equations describes the gravitational field of the source.

7.1. Theoretical Quantities and Astronomical Observables

The radial coordinate acceleration derived in this investigation is a theoretical quantity, not a quantity directly measured in cosmic-ray or neutrino observations. Astronomical observations instead provide quantities such as particle energy, arrival direction, arrival time or relative timing when available, and an association with a possible astrophysical source. Independent astronomical observations may also provide information concerning the distance, mass, rotation, and other properties of that source.

The role of the radial acceleration expressions derived in Sections 3-6 is therefore to provide the metric-dependent theoretical description of particle propagation. To test a stationary metric observationally, that theoretical description must be related to the particle quantities measured at the observer and to independently determined properties of the source. The radial coordinate acceleration itself is not assumed to be measured directly.

7.2. Determination of the Stationary Metric and Its Parameters

The observational problem is an inverse problem. In the forward direction, a specified stationary metric and a specified set of spacetime parameters determine the corresponding radial acceleration expression through Equation (16). In the inverse direction, particle observations together with independently determined source properties may be used to constrain which metric and which parameter values are consistent with those observations.

A specific compact astrophysical object is first identified as a candidate source of an observed ultra-relativistic particle. Candidate stationary solutions are then considered for its gravitational field. For each candidate solution, the metric-dependent particle propagation is calculated using the corresponding radial acceleration expression and spacetime parameters. The resulting theoretical predictions are related to measured particle quantities and independently determined source properties. Candidate metrics and parameter combinations inconsistent with the observations can thereby be eliminated.

A single particle observation will not generally determine every spacetime parameter uniquely. The ordinary Taub-NUT metric introduces only one additional parameter beyond the mass and therefore provides the simplest illustration. The dyonic Lorentzian Taub-NUT-AdS and Kerr-Taub-NUT geometries contain several additional parameters, so an observation may constrain combinations or ranges of parameters rather than determine each one independently.

7.3. Comparison of the Stationary Geometries

Table 1 makes the comparative parameter dependence explicit while also displaying the domains in which the formulas apply. In particular, the ordinary Taub-NUT geometry has a transparent weak-field correction proportional to n^2/r^3, whereas the finite-l AdS geometries retain curvature-dependent ultra-relativistic expressions rather than an asymptotically Minkowskian weak-field limit. The Kerr-Taub-NUT result is restricted to axial radial motion.

7.4. Scope of Paper I

The purpose here (Paper I) is to establish this theoretical and observational methodology, not to claim that the radial coordinate acceleration itself is directly measured by present cosmic-ray or neutrino observations. The metric-dependent expressions derived here provide the theoretical input required for quantitative applications. Papers II and III apply this framework to cosmic-ray neutrinos and cosmic-ray protons, respectively.

Table 1. Side-by-Side comparison of the stationary geometries considered here (Paper I).

Geometry	Radial metric function $B(r)$	Repulsion condition	Weak-field/ ultra-relativistic behavior
Schwarzschild	$1-\alpha/r$	$\dot{r}^2 > B^2/3$	$\mathrm{d}^2r/\mathrm{d}t^2 \simeq \alpha/r^2 = 2g$
Taub-NUT	$\dfrac{r^2-\alpha r-n^2}{r^2+n^2}$	$\dot{r}^2 > B_{\mathrm{TN}}^2/3$	$\mathrm{d}^2r/\mathrm{d}t^2 \simeq \alpha/r^2 + 4n^2/r^3$

Continued

Dyonic LTN-AdS	$\dfrac{r^2-\alpha r-n^2+4n^2g^2+e^2+\left(r^4+6n^2r^2-3n^4\right)/l^2}{r^2+n^2}$	$\dot r^2 > B_{\mathrm{LTN}}^2/3$	Finite l: asymptotically AdS; Equation (57)
Neutral TN-AdS	$\dfrac{r^2-\alpha r-s^2+\left(r^4+6s^2r^2-3s^4\right)/l^2}{r^2+s^2}$	$\dot r^2 > B_{\mathrm{TNAdS}}^2/3$	Finite l: asymptotically AdS; set $\dot r^2 \simeq 1$ in Equation (65)
Kerr-Taub-NUT	$\dfrac{r^2-\alpha r+a^2-n_K^2+q_e^2}{r^2+\left(n_K+\epsilon a\right)^2}$	$\dot r^2 > B_{\mathrm{KTN},\epsilon}^2/3$	Axial only; Equation (87); Schwarzschild for $a, n_K, q_e \to 0$

Note. All repulsion inequalities are stated for connected stationary exterior regions with $B(r)>0$ and $B'(r)>0$. The Taub-NUT weak-field expression assumes $r \gg \alpha$ and $r \gg n$. For the two finite-l AdS geometries there is no asymptotically Minkowskian weak-field limit; the corresponding ultra-relativistic expressions are therefore referenced instead. The Kerr-Taub-NUT result applies only to outward radial motion along the symmetry axis. In the dyonic Lorentzian Taub-NUT-AdS case, and in Kerr-Taub-NUT when the electromagnetic parameter is nonzero, the geodesic formulas apply directly to neutral test particles; charged-particle motion additionally requires the Lorentz force.

8. General Relevance to Ultra-Relativistic Cosmic-Ray Protons and Cosmic-Ray Neutrinos

The methodology developed here (Paper I) concerns outward-moving ultra-relativistic particles whose propagation is influenced by the gravitational field associated with a stationary compact astrophysical object. Very-high-energy and ultra-high-energy cosmic-ray protons and high-energy and ultra-high-energy neutrinos provide natural candidates because their velocities are extremely close to the speed of light.

Previous investigations considered ultra-relativistic neutrinos [14] and cosmic-ray protons [13] under the assumption that the source gravitational field was described by the Schwarzschild solution. Here (Paper I), that assumption is dropped. The purpose of this section is not to perform the metric-dependent quantitative neutrino or proton calculations. Paper II undertakes the quantitative application to cosmic-ray neutrinos, whereas Paper III undertakes the quantitative cosmic-ray proton application.

8.1. Cosmic-Ray Neutrinos

Cosmic-ray neutrinos are particularly useful because they are electrically neutral. Their propagation is therefore not subject to the electromagnetic Lorentz-force complication discussed in Sections 4 and 6. Within the test-particle approximation adopted here, their motion can be treated geodesically in each stationary geometry considered in Paper I.

8.2. Cosmic-Ray Protons

Very-high-energy and ultra-high-energy cosmic-ray protons also provide potentially important applications because their velocities are extremely close to c. Their electric charge, however, requires greater care. For stationary geometries without a nonzero electromagnetic field, the geodesic expressions may be used when nongravitational forces are negligible. For the dyonic Lorentzian Taub-NUT-AdS geometry, and for Kerr-Taub-NUT when $q_e \neq 0$, a proton is subject to the Lorentz force in addition to gravitation; the charged-particle equation of motion must therefore include the electromagnetic contribution.

8.3. Observational Role

The observational significance of cosmic-ray protons and cosmic-ray neutrinos does not arise from direct measurement of their radial coordinate acceleration near the source. Observations provide particle quantities such as energy, arrival direction, timing information when available, and possible association with an astrophysical source. Independent astronomical observations provide additional information about the source. The metric-dependent radial acceleration expressions derived in Paper I supply the theoretical propagation framework needed to relate these observations to candidate stationary geometries.

9. Physical Interpretation and Applicability of the Present Formalism

9.1. Interpretation of the NUT Parameters

The Schwarzschild mass parameter has a well-established physical interpretation as the gravitational mass of the source. In contrast, the physical interpretation of the NUT parameter remains the subject of continuing investigation. Proposed interpretations include a gravitomagnetic monopole charge [18] [19], dual mass or gravitational dyon [20], a mass accompanied by a semi-infinite massless source of angular momentum [5] [21], and global, topological, or gravitational-instanton properties [22]-[24]. No general consensus has emerged concerning its fundamental physical origin.

The methodology developed here (Paper I) does not require that one of these interpretations be adopted in advance. Instead, NUT parameters are treated as spacetime parameters entering metric-dependent expressions for particle propagation. Quantitative applications can investigate whether observations, together with independently determined properties of a specific compact astrophysical object, constrain their allowed values. This provides an operational role for the NUT parameters without asserting that their fundamental physical meaning has thereby been established.

9.2. Applicability of the Present Formalism

The methodology provides a General Relativistic framework for investigating the applicability of stationary solutions to specific compact astrophysical objects. The expressions derived here provide the theoretical component of the observational methodology; they do not imply that radial coordinate acceleration itself is directly measured.

The methodology is applicable when the assumptions underlying the particular calculation are satisfied. The motion must lie within the radial sector for which Equation (16) is applicable. For Kerr-Taub-NUT this restricts the result to axial radial motion. In addition, the expressions describe geodesic motion. For electrically charged particles propagating in a spacetime with a nonzero electromagnetic field, the Lorentz force must also be included.

A metric containing several additional parameters will not generally permit all of them to be determined uniquely from a single particle observation. An observation may eliminate particular geometries, constrain combinations of parameters, or establish allowed parameter ranges. Quantitative applications are undertaken in Papers II and III for cosmic-ray neutrinos and cosmic-ray protons, respectively.

10. Conclusions

Newton's theory of gravitation is based upon a single gravitational equation, whereas Einstein's theory of gravitation, General Relativity, is based upon Einstein's field equations, which admit many exact solutions. This circumstance raises an observational question fundamental to General Relativity: which solution describes the gravitational field associated with a specific physical system? We have developed here (Paper I) a methodology for investigating this question using outward-moving ultra-relativistic particles.

Beginning with the general radial acceleration formula, Equation (16), we derived metric-dependent expressions for the radial coordinate acceleration and the conditions for positive radial coordinate acceleration for the Taub-NUT, dyonic Lorentzian Taub-NUT-AdS, neutral spherically symmetric Taub-NUT-AdS, and axial Kerr-Taub-NUT geometries. Different stationary solutions predict different expressions because their metric functions depend differently upon mass, NUT parameters, electromagnetic parameters, AdS curvature, and rotation.

The ordinary Taub-NUT geometry provides the simplest quantitative illustration. In the weak-field ultra-relativistic limit, the fractional departure from Schwarzschild is

$$\delta_{\mathrm{TN}} = 4\left(\frac{n}{\alpha}\right)^2\left(\frac{\alpha}{r}\right). \tag{88}$$

Using the observationally motivated illustrative NUT scale $n/\alpha = 0.25$ and the representative weak-field radius $r/\alpha = 50$, Section 3 obtains $\delta_{\mathrm{TN}} = 5\times10^{-3} = 0.5\%$. This is an order-of-magnitude illustration and not a determination of the NUT parameter for a particular compact astrophysical object.

The methodology has important limitations. The radial coordinate acceleration is a theoretical, coordinate-dependent quantity and is not directly measured in cosmic-ray or cosmic-ray neutrino observations. The Kerr-Taub-NUT results are restricted to outward radial motion along the symmetry axis. In geometries containing a nonzero electromagnetic field, charged particles such as protons are generally also subject to the Lorentz force. Finally, metrics containing several additional parameters will not generally permit every parameter to be determined from a single particle observation.

The principal contribution of Paper I is therefore the establishment of the methodology rather than a quantitative determination of the spacetime geometry of a particular compact astrophysical object. Paper II and Paper III undertake the metric-dependent quantitative applications to cosmic-ray neutrinos and cosmic-ray protons, respectively. The stationary spacetime geometry and its associated parameters are to be constrained by confronting metric-dependent particle propagation with observations rather than assumed in advance.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix. Independent Verification of the Coordinate-Time Acceleration

For completeness, this Appendix records the coordinate-time geodesic derivation in a compact form. For the reciprocal radial metric sector adopted in Section 2, and with τ chosen as an affine parameter, the temporal and radial geodesic equations can be written explicitly as

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = -\frac{B'}{B}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}, \tag{89}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = -\frac{1}{2}BB'\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{B'}{2B}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2. \tag{90}$$

Using

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{1}{\left(\mathrm{d}t/\mathrm{d}\tau\right)^2}\left[\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - \frac{\mathrm{d}r/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau}\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2}\right] \tag{91}$$

and substituting the two affine equations gives

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \left[\frac{B'(r)}{2B(r)}\right]\left[3\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - B(r)^2\right], \tag{92}$$

which is identical to Equation (16). This also shows precisely where the factor of three arises: one contribution comes from the radial Christoffel symbol and two contributions arise from the non-affine use of coordinate time.