

Born's Rule from Reversible Evolution and Record Formation

Oskar Axelsson 

Independent Researcher, Höör, Sweden

Email: oskar@oascitech.se

How to cite this paper: Axelsson, O.

(2026) Born's Rule from Reversible Evolution and Record Formation. *Journal of Modern Physics*, 17, 1282-1294.

<https://doi.org/10.4236/jmp.2026.1710056>

Received: June 20, 2026

Accepted: October 6, 2026

Published: October 9, 2026

Copyright © 2026 by author(s) and

Scientific Research Publishing Inc.

This work is licensed under the Creative

Commons Attribution International

License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

We show that the quadratic measure associated with the Born rule need not be introduced as an independent probabilistic postulate, but follows, under the stated structural and regularity assumptions, from the compatibility between reversible linear evolution and the formation of stable records. Physical processes alternate between reversible dynamics and episodes of record formation. Quantities that remain well defined throughout such processes must therefore admit a representation that remains consistent across both regimes. Prior to record formation, alternatives combine additively at the level of complex amplitudes, whereas successive record-forming distinctions admit a multiplicative representation through refinement. Requiring compatibility between these compositional structures constrains the admissible weight assignment to the family $|\alpha|^p$. The finite-dimensional specialization of Lamperti's isometry theorem then singles out $p = 2$ as the unique value compatible with continuous reversible linear mixing while preserving total weight. The Born rule therefore emerges as the unique weight assignment compatible with reversible linear composition, multiplicative record refinement, phase invariance, and continuous reversible invariance, without invoking probabilistic assumptions or a specific interpretation of quantum measurement.

Keywords

Born's Rule, Quantum Measurement, Reversible Dynamics, Irreversible Record Formation, Quantum Foundations

1. Introduction

The Born rule plays a central role in quantum theory, connecting the mathematical formalism of amplitudes with experimentally observed outcome frequencies. In the standard formulation it is introduced as a postulate, and considerable effort

has been devoted to deriving it from more primitive physical principles.

Several approaches have been proposed. Decision-theoretic arguments in the Everett interpretation attempt to recover the rule from rational consistency requirements [1]. Envariance-based derivations relate the rule to symmetry properties of entangled states [2]. Other approaches derive the quadratic measure from structural features of Hilbert space or Gleason-type theorems [3]. Derivations based on time-reversal symmetry have also been proposed [4]. The present approach differs in deriving a general weight assignment from the compatibility between reversible evolution and irreversible record formation, without assuming a probabilistic interpretation from the outset.

As emphasized in several reviews, derivations of the Born rule generally require assumptions beyond the standard dynamical postulates of quantum mechanics [5]. At the same time, the role of these assumptions remains a matter of debate. Recurring criticisms include the implicit introduction of probabilistic concepts through auxiliary assumptions and the absence of a clear account of physical outcomes within the formalism. A recent review likewise highlights the variety of existing formulations of the Born rule, their differing domains of validity, and the continuing lack of a universally accepted account of the emergence of definite physical outcomes [6].

The present derivation differs from representative existing approaches primarily in the assumptions from which it begins. It does not invoke decision-theoretic rationality, probabilistic symmetry arguments, expectation values, or a pre-existing quadratic measure. Instead, it starts from reversible linear composition, persistent record formation, and consistency of the weight assignment across these two regimes. A broader comparison of existing Born-rule derivations and their assumptions is provided in Refs. [5] [6].

The present work addresses these concerns from a different perspective. Probabilities are not introduced as primitive quantities, and outcomes are represented by stable records, understood as persistent physical outcomes that can be identified after their formation. Recent analyses have also emphasized the indispensable role of the superposition principle, that is, the additive composition of amplitudes, in many proposed derivations of the Born rule [7]. In the present work, additive composition is not derived independently but taken directly from the linear structure of reversible quantum evolution.

We therefore pursue a structural route based on the compatibility between reversible evolution and irreversible record formation. Reversible evolution is described by the Schrödinger equation and possesses continuous symmetries together with the associated conservation laws. Physical processes, however, are not composed exclusively of reversible evolution, but repeatedly alternate between reversible dynamics and episodes of record formation.

Both ingredients are independently motivated by established physical practice. Reversible evolution is a central element of the standard quantum formalism, while stable records are the operational basis by which outcomes are identified

and compared in experiment. The physical starting points of the present derivation are therefore structures that already play a fundamental role in the description of physical processes, rather than on additional assumptions concerning observers, decision-making, or subjective probabilities. We do not attempt to explain the origin of irreversibility or the arrow of time. Rather, we start from the empirical coexistence of reversible quantum evolution and persistent records, and investigate the constraints imposed by requiring that quantities associated with possible outcomes remain consistently represented across both regimes (**Figure 1**).

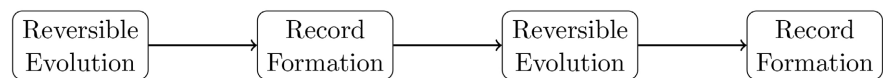


Figure 1. Physical evolution alternates between reversible dynamics and the formation of stable records. Reversible evolution is linear and time symmetric, while record formation produces effectively irreversible outcomes. Each record distinguishes a single realized outcome, after which reversible evolution continues from the recorded configuration.

If quantities such as energy remain well defined throughout a physical process, their representation cannot be determined independently in these two regimes. Reversible and irreversible processes must therefore admit a common structural description. The present work leverages the constraints imposed by this requirement on admissible outcome weights.

In this setting, reversible evolution combines alternatives additively at the amplitude level, whereas record formation composes multiplicatively through successive refinement, whereby each new record further specifies the realized outcome. Any quantity that remains consistently represented across both regimes must therefore be compatible with these distinct compositional structures.

The distinction between reversible evolution and irreversible record-forming processes has been explored in various forms in the literature. In particular, related ideas appear in the work of Horodecki, where reversible and irreversible aspects of quantum processes are discussed in a broader informational context [8].

The derivation presented here is formulated within the standard framework of quantum mechanics, assuming reversible linear evolution with complex amplitudes together with the occurrence of irreversible record formation. No specific interpretation or mechanism for this process is assumed. Nor does the derivation assume probabilities or expectation values as primitive concepts. Instead, the admissible weight assignment is derived from the compatibility of reversible evolution and irreversible record formation. A probabilistic interpretation is considered only after the quadratic form has been derived.

We show that additive reversible composition and multiplicative record refinement, together with the requirement that outcome weights remain consistently represented across both regimes, uniquely select a quadratic assignment of outcome weights. The Born rule therefore emerges as the unique weighting compatible with reversible linear evolution and multiplicative irreversible refinement. To

make this argument precise, we first identify the structural features of the reversible and irreversible regimes separately before examining the constraints imposed by their compatibility.

2. Two Operational Regimes

Measurement interactions are commonly described as involving two operationally distinct regimes: reversible linear evolution prior to record formation and effectively irreversible formation of persistent records associated with measurement outcomes [9]. This operational distinction is widely used in both textbook and research treatments of quantum measurement.

1) Reversible regime. Prior to formation of a stable record, evolution is reversible and linear at the amplitude level. States associated with different potential outcomes remain operationally interconvertible.

2) Irreversible regime. Once a stable record forms, configurations corresponding to distinct outcomes become operationally distinguishable and are no longer reversibly interconvertible.

Here irreversibility refers to operational irreversibility: the recorded configurations cannot be returned to their pre-record state by physically accessible reversible transformations, even though the underlying microscopic dynamics may remain time symmetric.

For clarity, the derivation below uses the following physical (**P**) and mathematical (**M**) assumptions.

P1. Reversible linear composition. Prior to record formation, alternatives are represented by complex amplitudes and combine linearly.

P2. Persistent record formation. Physical processes may produce stable records that render distinct outcomes operationally distinguishable and whose recorded distinction remains available under subsequent evolution and is not erased by the reversible transformations under consideration.

P3. Cross-regime consistency. A non-negative weight assigned to a possible record before its formation retains the same physical meaning when that record becomes established.

P4. Phase invariance. States differing only by an overall complex phase receive the same weight.

M1. Regularity. The weight assignment and the composition laws introduced below are taken to be continuous on their stated domains. Where stronger regularity properties are required, they are stated explicitly at the point of use.

No probabilistic interpretation of the weight is assumed at this stage. Additional structural requirements associated with sequential record refinement and reversible invariance are introduced explicitly where they enter the derivation.

2.1. Reversible Composition

At the microscopic level, the dynamical laws governing physical systems are reversible. If a quantum state evolves according to a deterministic dynamical law,

the same law also determines how the earlier state can be reconstructed from its later state.

A familiar example is the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi, \quad (1)$$

which generates unitary evolution. Given a solution $\psi(t)$, the state at an earlier time can be obtained by evolving with the inverse unitary operator $U^{-1}(t) = U(-t)$. The equation itself therefore contains no intrinsic direction of time.

This microscopic reversibility contrasts with everyday experience. Macroscopic processes such as measurement or record formation appear irreversible because they involve the creation of persistent records that are impossible to erase in practice through local physical interactions. Such irreversibility reflects operational limitations rather than a fundamental asymmetry of the underlying dynamical laws.

The coexistence of reversible evolution and irreversible record formation raises the question of how quantities that remain well defined throughout a physical process can be represented consistently across both regimes. This compatibility requirement imposes strong constraints on how non-negative weights, rather than probabilities, may be assigned to possible outcomes. The term *weight* is used because no probabilistic interpretation is assumed a priori; the relation to probability emerges only after the quadratic form has been established.

The present work does not attempt to derive the additive composition of amplitudes, which is taken directly from the linear structure of reversible quantum evolution [7].

If two alternatives lead to the same subsequent state without leaving a record of their distinction, they cannot be operationally distinguished and must be represented by a single effective alternative. Since reversible evolution is linear, the corresponding amplitudes combine additively,

$$\alpha_{\text{combined}} = \alpha_1 + \alpha_2. \quad (2)$$

Addition is therefore the fundamental composition law of the reversible regime.

Reversible evolution admits a continuous symmetry corresponding to global phase transformations,

$$\alpha \mapsto e^{i\theta} \alpha. \quad (3)$$

A *weight* is a non-negative quantity assigned to a possible persistent record prior to its realization. The realized record retains this same assignment upon record formation. Throughout the present work, the term *weight* refers to this single assignment both before and after record formation.

States differing only by a global phase are operationally indistinguishable prior to record formation and must therefore be assigned the same weight. Outcome weights are therefore invariant under global phase transformations.

It follows that weights cannot depend on the overall phase of α , and can therefore be written as a function of its magnitude,

$$\mu = f(|\alpha|). \quad (4)$$

2.2. Irreversible Regime

The preceding discussion identifies the composition law and symmetry properties of the reversible regime. We now turn to the corresponding structure associated with irreversible record formation.

Physical processes consist of alternating stages of reversible evolution and irreversible record formation. At discrete stages, irreversible record formation occurs, selecting a single outcome from a set of alternatives. No specific mechanism for this selection is assumed.

Once a stable record forms, configurations become operationally distinguishable. Subsequent evolution proceeds within the configuration consistent with the recorded outcome until a further irreversible record forms. Upon record formation, the realized record inherits the corresponding weight μ .

The present derivation does not assume a particular microscopic mechanism by which persistent records are established. In modern quantum theory, effective record formation is commonly described in decoherence-based approaches through environmental interaction, where the global evolution remains reversible while local subsystems acquire effectively irreversible records [2] [10]. The present argument is compatible with such descriptions. It requires only that a regime exists in which persistent records can be treated as operationally established. The derivation therefore concerns the structural consequences of record formation rather than its microscopic origin.

2.3. Consistency of Sequential Refinement

Successive record formation progressively specifies the realized outcome: each new record introduces information not contained in previous records. When multiple record-formation events occur in sequence, the final record encodes all distinctions introduced at each stage.

Let $\mu(R)$ denote the non-negative weight assigned to a record R . We require that this assignment respect the compositional structure of successive refinement. In particular, the weight assigned to a final record must not depend on how the refinement leading to that record is partitioned into intermediate stages.

Suppose the composition law for successive refinements is represented by an operation $\circ$,

$$\mu(R_{12}) = \mu(R_1) \circ \mu(R_2). \quad (5)$$

For positive weights, we require the composition law $\circ$ to be continuous, associative, and strictly increasing in each argument. Associativity expresses independence of the grouping of successive refinements,

$$(\mu_1 \circ \mu_2) \circ \mu_3 = \mu_1 \circ (\mu_2 \circ \mu_3), \quad (6)$$

while strict monotonicity expresses that increasing the contribution of either re-

finement increases the weight of the cumulative record.

Proposition. Let $\circ$ be a binary operation on a real interval of positive weights. If $\circ$ is continuous, associative, and strictly increasing in each argument, then, up to a continuous strictly monotonic reparameterization, it admits an additive representation.

By Aczél's representation theorem for continuous, associative, strictly monotonic operations on a real interval [11], there exists a continuous strictly monotonic function h such that

$$h(\mu_1 \circ \mu_2) = h(\mu_1) + h(\mu_2). \quad (7)$$

Defining the reparameterized weight

$$\tilde{\mu} = \exp[h(\mu)], \quad (8)$$

therefore gives

$$\tilde{\mu}(R_{12}) = \tilde{\mu}(R_1) \tilde{\mu}(R_2). \quad (9)$$

We henceforth choose this multiplicative representation and suppress the tilde, writing

$$\mu(R_{12}) = \mu(R_1) \mu(R_2). \quad (10)$$

The distinguished zero-weight case is treated separately as the boundary corresponding to absence of an alternative. Multiplication extends continuously to this boundary through

$$0 \cdot \mu = \mu \cdot 0 = 0. \quad (11)$$

Thus multiplication is not assumed as the primitive refinement law. It is the canonical representation, up to continuous monotonic reparameterization, of any continuous, associative, strictly monotonic composition law for positive refinement weights.

This relation expresses the cumulative character of record formation: each stage contributes an additional distinction while preserving those already encoded in previous records.

3. Compatibility between Regimes

We now examine how the two regimes must connect. The central question is how quantities that remain well defined throughout a physical process can survive transitions between reversible evolution and record formation.

In the reversible regime, alternatives combine additively at the amplitude level. In the irreversible regime, distinctions accumulate through multiplicative refinement. The compatibility problem is therefore to identify a weight assignment that remains consistently defined across regimes whose fundamental composition laws are addition and multiplication, respectively.

The refinement associated with record formation changes the relative prominence of an existing alternative without altering its identity. For a nonzero amplitude magnitude, we characterize a particular refinement by its relative change

$$\lambda = \frac{|\alpha'|}{|\alpha|}, \quad \lambda > 0, \quad (12)$$

so that, equivalently,

$$|\alpha'| = \lambda |\alpha|. \quad (13)$$

This relation defines the refinement factor λ ; it does not assert that every continuous transformation fixing zero is intrinsically linear.

Successive refinement factors compose multiplicatively. If

$$|\alpha'| = \lambda_1 |\alpha|, \quad |\alpha''| = \lambda_2 |\alpha'|, \quad (14)$$

then

$$|\alpha''| = \lambda_2 \lambda_1 |\alpha|. \quad (15)$$

To represent the same refinement consistently at different initial amplitude magnitudes, we further require that its effect on the weight depend only on the refinement factor λ , rather than separately on the initial magnitude. Thus

$$f(\lambda |\alpha|) = g(\lambda) f(|\alpha|). \quad (16)$$

Successive rescalings do not represent independent physical dynamics. Rather, they provide a mathematical representation of successive stages of record formation. Each stage further suppresses competing alternatives while preserving the realized alternative. If the description is to remain independent of how the refinement is partitioned into intermediate stages, successive rescalings must compose consistently in the same manner as the corresponding sequence of record refinements.

Since weights represent physically meaningful quantities associated with continuously varying amplitudes, we assume that the scaling function $g(\lambda)$ is positive and satisfies the mild regularity condition of continuity (or, equivalently for the present purpose, measurability). These assumptions exclude pathological solutions without restricting physically admissible behavior.

Consistency therefore requires

$$g(\lambda_1 \lambda_2) = g(\lambda_1) g(\lambda_2). \quad (17)$$

This is the standard multiplicative functional equation, whose positive continuous (or measurable) solutions are

$$g(\lambda) = \lambda^p, \quad (18)$$

for some real $p > 0$.

Substituting this result into Equation (16) gives

$$f(\lambda |\alpha|) = \lambda^p f(|\alpha|). \quad (19)$$

Since this relation holds for every positive scaling parameter λ , for $|\alpha| > 0$ we may choose

$$\lambda = \frac{1}{|\alpha|}. \quad (20)$$

This choice specifies the scaling transformation; it does not impose a condition on the original amplitude. Under this particular transformation,

$$|\alpha'| = \lambda |\alpha| = 1. \quad (21)$$

Consequently,

$$f(1) = |\alpha|^{-p} f(|\alpha|), \quad (22)$$

or equivalently,

$$f(|\alpha|) = f(1) |\alpha|^p. \quad (23)$$

Absorbing the constant factor $f(1)$ into the normalization yields

$$\mu = f(|\alpha|) = |\alpha|^p. \quad (24)$$

Compatibility between reversible evolution and record formation therefore restricts the admissible weight assignments to the family $\mu = |\alpha|^p$. To determine which member of this family is physically admissible, we must now impose the additional requirement that continuous reversible evolution preserve the total weight assigned to mutually exclusive alternatives.

4. Reversible Invariance

Consider a finite set of n mutually exclusive alternatives, represented by the complex amplitude vector

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n. \quad (25)$$

Prior to record formation, states evolve according to reversible linear dynamics. Such transformations are represented by invertible complex-linear maps

$$\alpha' = U\alpha, \quad (26)$$

where U is an invertible $n \times n$ complex matrix.

By continuous reversible evolution we mean continuous families of such invertible linear transformations, including transformations that smoothly mix distinct amplitude components.

The weight assignment must be compatible with this reversible evolution.

A reversible transformation changes the amplitudes describing a state but does not itself create or remove possible outcomes. We therefore require that, if the same weight assignment is to remain consistently defined during reversible evolution, reversible transformations may redistribute weight among alternatives but may not change the total weight associated with the complete set. This is a separate cross-regime consistency requirement and does not follow from multiplicative record refinement alone. At this stage neither the exponent p nor any normalization is assumed.

If the weight associated with an amplitude is

$$\mu = |\alpha|^p, \quad (27)$$

then the total weight assigned to a complete set of mutually exclusive records is

$$\sum_{i=1}^n |\alpha_i|^p. \quad (28)$$

The reversible-invariance requirement is therefore

$$\sum_{i=1}^n |(U\alpha)_i|^p = \sum_{i=1}^n |\alpha_i|^p \quad (29)$$

for every amplitude vector α and every admissible reversible transformation U .

The relevant finite-dimensional specialization of Lamperti's isometry theorem is the following [12]. For $0 < p < \infty$, $p \neq 2$, an invertible complex-linear transformation preserving

$$\sum_i |\alpha_i|^p \quad (30)$$

for every amplitude vector has the form of a generalized permutation: it permutes the components and multiplies each component by a complex factor of unit modulus. Such transformations cannot continuously mix distinct amplitude components.

For $p = 2$, by contrast, the invariant quantity

$$\sum_i |\alpha_i|^2 \quad (31)$$

is the squared Hermitian norm, whose linear isometry group is the unitary group. This group contains continuous transformations that smoothly mix distinct components.

A useful geometric intuition can be given for the special role of $p = 2$. The constant-weight surfaces $\sum_i |\alpha_i|^2 = \text{const.}$ are hyperspheres, which admit continuous rotations mixing their components while preserving the total weight. For $p \neq 2$, the corresponding linear isometries reduce to generalized permutations and componentwise phase transformations.

Consequently, $p = 2$ is uniquely singled out by compatibility with continuous reversible linear mixing.

5. Result

The preceding analysis first restricts admissible weight assignments to the family $|\alpha|^p$ and then uses reversible invariance to select the unique value $p = 2$.

Under the stated structural and regularity assumptions, the unique weight assignment compatible with:

- Reversible linear composition,
- Multiplicative representation of record refinement,
- Phase invariance,
- Cross-regime consistency, and
- Continuous reversible linear invariance,

Is

$$\mu = |\alpha|^2. \quad (32)$$

The uniquely selected weight coincides with the standard quadratic measure of quantum theory. Normalization gives these non-negative weights, with the additive aggregation used above for mutually exclusive alternatives, the mathematical form of a finite probability distribution. The further physical interpretation of this measure as probabilities of individual outcomes or as relative frequencies in repeated experiments depends on the interpretation of quantum measurement and is not assumed in the present derivation.

The derivation shows that the quadratic form arises from the requirement that outcome weights remain consistently represented across reversible evolution and irreversible record formation, without invoking probabilities or expectation values as primitive quantities.

The emergence of the same quadratic structure from both Hilbert-space measure theory and compatibility between reversible and irreversible processes suggests that the Born rule reflects a deeper structural constraint rather than a feature of any particular formalism.

6. Conclusions

The goal of this work was to determine whether the quadratic measure associated with the Born rule can be derived from structural properties of physical processes rather than introduced as an independent postulate.

We have shown that, under the stated structural and regularity assumptions, the quadratic measure of Born's rule need not be introduced as an independent probabilistic postulate, but follows from the compatibility between reversible linear evolution and persistent record formation.

The central motivation for this compatibility requirement is that physical processes are not composed exclusively of reversible evolution, but alternate between reversible dynamics and episodes of record formation. Quantities that remain well defined throughout a physical process must therefore admit a representation that remains consistent across both regimes. The present derivation leverages the constraints imposed by this requirement on admissible outcome weights.

The argument takes reversible linear evolution and operationally irreversible record formation as its physical starting point and imposes the structural and regularity conditions stated explicitly in Section 2. Weights are associated with possible persistent records and are retained after record formation. Weights are assigned only to physically realized records, which include the full conditions of their formation. Different interpretations may describe the emergence of such records differently—for example as single-outcome selection or as branching into multiple outcomes—but the underlying structural distinction between reversible evolution and irreversible record formation remains the same.

A recurring concern in derivations of the Born rule is the possibility of circularity, whereby probabilistic concepts enter implicitly through auxiliary assumptions and subsequently reappear in the final result. The present derivation avoids this route by introducing neither probabilities nor expectation values as primitive

quantities. Instead, the argument begins with structural assumptions concerning reversible evolution, record formation, and the compatibility of their respective composition laws. The quadratic measure emerges from these structural requirements rather than from probabilistic postulates.

The present formulation characterizes measurement through the formation of stable records, understood as physically persistent outcomes that remain identifiable after their formation. This shifts the discussion from a concept whose precise role varies across interpretations to a physical process characterized by the formation and persistence of recorded outcomes.

In this way, the Born rule emerges as a structural consequence of the requirement that outcome weights remain consistently represented across reversible evolution and irreversible record formation. The quadratic measure is therefore not introduced as an additional dynamical law, but appears as the unique weight assignment compatible with additive reversible composition, multiplicative record refinement, and continuous reversible invariance.

Use of Large Language Models, AI and Machine Learning Tools

ChatGPT was used to improve language, checking consistency, literature search, and translation.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Deutsch, D. (1999) Quantum Theory of Probability and Decisions. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, **455**, 3129-3137. <https://doi.org/10.1098/rspa.1999.0443>
- [2] Zurek, W.H. (2005) Probabilities from Entanglement, Born's Rule $p_k = |\psi_k|^2$ from Envariance. *Physical Review A*, **71**, Article ID: 052105. <https://doi.org/10.1103/physreva.71.052105>
- [3] Gleason, A. (1957) Measures on the Closed Subspaces of a Hilbert Space. *Indiana University Mathematics Journal*, **6**, 885-893. <https://doi.org/10.1512/iumj.1957.6.56050>
- [4] Ilyin, A.V. (2016) The Born Rule and Time-Reversal Symmetry of Quantum Equations of Motion. *Foundations of Physics*, **46**, 845-851. <https://doi.org/10.1007/s10701-016-0006-y>
- [5] Vaidman, L. (2020) Derivations of the Born Rule. In: Hemmo, M. and Shenker, O., Eds., *Quantum, Probability, Logic*, Springer, 567-584.
- [6] Neumaier, A. (2025) The Born Rule—100 Years Ago and Today. *Entropy*, **27**, Article 415. <https://doi.org/10.3390/e27040415>
- [7] Zhang, J.X. (2026) Summing to Uncertainty: On the Necessity of Additivity in Deriving the Born Rule. *Physical Review A*. <https://doi.org/10.1103/mqy7-s2db>
- [8] Horodecki, R. (2025) Does Quantum Information Require Additional Structure? arXiv: 2408.11183.

- [9] Schlosshauer, M. (2005) Decoherence, the Measurement Problem, and Interpretations of Quantum Mechanics. *Reviews of Modern Physics*, **76**, 1267-1305. <https://doi.org/10.1103/revmodphys.76.1267>
- [10] Zurek, W.H. (2003) Decoherence, Einselection, and the Quantum Origins of the Classical. *Reviews of Modern Physics*, **75**, 715-775. <https://doi.org/10.1103/revmodphys.75.715>
- [11] Aczel, J. (1948) Sur les opérations définies pour nombres réels. *Bulletin de la Société mathématique de France*, **2**, 59-64. <https://doi.org/10.24033/bsmf.1396>
- [12] Lamperti, J. (1958) On the Isometries of Certain Function-Spaces. *Pacific Journal of Mathematics*, **8**, 459-466. <https://doi.org/10.2140/pjm.1958.8.459>