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Temporal Mechanics: Phase-Locked Internal Holonomy

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Abstract

By identifying Schrödinger's phase, proper time, and internal holonomy as related aspects of a single temporal structure, Temporal Mechanics proposes a common geometric origin for the distinct notions of time appearing in quantum theory and relativity. In this framework, clocks unwind a hidden periodic structure whose Abelian holonomy is shared by quantum phase and proper time. The internal sector is modeled as a compact three-torus $\mathcal{M}_\tau \cong \mathbb{T}^3$ equipped with a $U(3)$ bundle whose determinant line is identified, by a chosen phase lock, with the auxiliary $U(1)$ line of a $\mathrm{Spin}^c(1,3)$ structure on an emergent Lorentzian spacetime $\mathcal{M}_4$. The traceless sector carries non-Abelian geometric structures allowing Higgs and flavor degrees of freedom, which are interpreted as modes of the internal connection, while the induced internal Dirac spectrum provides rest masses. This framework proposes a geometric mechanism by which quantum phase, proper time, internal gauge holonomy, and spectral masses may be realized as different projections of a single phase-locked temporal bundle that is compatible with the Standard Model.

Keywords

Schrödinger Phase, Proper Time, Holonomy

1. Introduction

Modern field theory reconciles quantum mechanics with special relativity through relativistic quantum field theory, and general relativity gives a precise geometric account of proper time and gravitation. Temporal Mechanics (TM) attempts to address the coexistence of several distinct temporal structures: external coordinate time in quantum evolution, proper time along Lorentzian worldlines, compact phase in $U(1)$ quantum evolution, and holonomy phases in gauge theory. TM proposes that these structures are not independent by introducing a compact internal temporal sector and a phase-lock that identifies the determinant line of an internal $U(3)$ bundle with the auxiliary $U(1)$ line of an external $\mathrm{Spin}^c(1,3)$ structure. TM hypothesizes that quantum phase, proper time, hypercharge, and masses are different realizations of one locked temporal holonomy.

The central distinction from a conventional scalar-potential or symmetry-breaking construction is that TM is kinetic and geometric at the fundamental level. The primitive objects are the phase-locked product geometry, the internal connection, the curvature of the temporal bundle, and the corresponding Dirac operators. Potentials, Higgs-sector parameters, and Yukawa matrices are not introduced as independent inputs. When such objects appear in the four-dimensional effective description, they are derived response functionals of the internal kinetic geometry, determined by curvature, holonomy, compact temporal volume, and overlap integrals of internal eigenmodes. Likewise, the reduction to the observed gauge structure is not treated as a free spontaneous breaking of an unrelated larger gauge group by an externally chosen potential. It is a stabilizer and phase-lock selection inside a dual temporal holonomy.

Temporal Mechanics incorporates Kaluza-Klein and string-theoretic compactification ideas by using compact internal structure with Wilson lines and gauge-Higgs unification mechanisms, in which scalar Higgs degrees of freedom can arise from internal components of a gauge connection [1]-[5]. The present construction can be viewed as a geometric extension of the Hosotani mechanism. In the Hosotani mechanism, Wilson-line phases associated with compact extra dimensions become physical degrees of freedom, and extra-dimensional components of a gauge connection can play the role of Higgs fields whose vacuum configuration dynamically selects the low-energy gauge symmetry. TM adopts this Wilson-line/gauge-Higgs mechanism as a structural ansatz, but generalizes it.

First, the compact sector is interpreted as an internal temporal manifold $\mathcal{M}_\tau \simeq \mathbb{T}^3$, rather than as ordinary spatial extra dimensions or the orbifold $\mathbb{T}^2/\mathbb{Z}_2$. Second, the gauge connection is promoted from the illustrative $SU(2)$ setting to a $U(3)$ internal bundle whose traceless sector carries non-Abelian holonomy with a determinant line that carries the Abelian temporal phase. Third, this determinant $U(1)$ is phase-locked to the auxiliary $U(1)$ line of the external $\mathrm{Spin}^c(1,3)$ structure, so that quantum phase, proper time, and internal gauge structure are treated as related projections of the same holonomy. In this phase-locked temporal $U(3)$ geometry, Higgs degrees of freedom, flavor mixing, and spectral rest masses arise from internal connection modes and the internal Dirac spectrum. TM therefore seeks a structural origin for generation, charge, and mass patterns in a spirit related to preon, composite, and permutation-symmetric approaches [6]-[8], while proposing a constrained geometric framework and benchmark construction based on an extended Hosotani-type ansatz.

Definition: The determinant line of a rank-3 Hermitian bundle E is the complex line bundle whose transition functions are the determinants of the $U(3)$-

valued transition functions of E, where the line bundle carries the determinant $U(1)$ phase of the internal $U(3)$ bundle. A $\mathrm{Spin}^c(1,3)$ structure is a spin-Lorentz structure with an auxiliary $U(1)$ line. The phase lock is a chosen isomorphism between the pullback of $\det(E)$ and the pullback of the Spin^c auxiliary line on the product space. The Lorentzian metric is treated as the external chronometric response to the locked temporal sector, making Lorentzian spacetime emergent rather than as an independently postulated primitive.

$\mathcal{M}_4$ denotes the emergent Lorentzian spacetime with metric $g_{\mu\nu}$ and signature $+---$, and $\mathcal{M}_\tau$ denotes the compact internal temporal manifold specialized to $\mathbb{T}^3$. Let $\mathcal{M}_7 := \mathcal{M}_4 \times \mathcal{M}_\tau$ be the global product with projections $\pi_4 : \mathcal{M}_7 \to \mathcal{M}_4$ and $\pi_\tau : \mathcal{M}_7 \to \mathcal{M}_\tau$. Greek indices $\mu, \nu = 0,1,2,3$ refer to $\mathcal{M}_4$ with coordinate x^μ on $\mathcal{M}_4$, Latin indices $a,b = 1,2,3$ to $\mathcal{M}_\tau$, and capital indices $A,B = 0,\cdots,6$ to $\mathcal{M}_7$. The internal rank-3 Hermitian bundle is $E \to \mathcal{M}_\tau$. Its determinant line is $L := \det E$, and $L_{\mathrm{Spin}^c} \to \mathcal{M}_4$ is the auxiliary line of the external $\mathrm{Spin}^c(1,3)$ structure. Let s be the worldline parameter, $t = x^0$ be coordinate time, τ be proper time, and $\tilde{\theta}$ be the lifted phase of the locked temporal $U(1)$ line. Finally, let $\hbar = c = 1$.

"The theory of relativity is concerned with the connection between the descriptions of phenomena as viewed from two different coordinate systems which are in motion relatively to each other" [9]. In TM, the compact manifold is temporal, and its determinant $U(1)$ is locked to proper time, so that spacetime geometry, gauge charges, and rest masses are all geometric manifestations of internal time. Relativity is then a statement of connections to the temporal manifold. Einstein's original postulates of special and general relativity [9]-[12] can be reinterpreted with temporal geometry.

Postulate 1: Time oscillation is universal, where all physical systems possess intrinsic motion within the three-dimensional temporal manifold $\mathcal{M}_\tau$ with coordinates θ^a, defining the system's rest mass and inertial identity.

Postulate 2: All physical trajectories are dictated by curvature within the total pseudo-Riemannian manifold $\mathcal{M}_7$, with temporal structure encoded in the internal geometry of $\mathcal{M}_\tau$ and associated bundle.

Postulate 3: The generalized temporal bundle carries the natural split-pairing

$$\mathcal{E}_\tau := T\mathcal{M}_\tau \oplus T^*\mathcal{M}_\tau. \tag{1}$$

After choosing a compatible generalized metric and generalized complex structure, its complexification selects a conjugate pair of rank-three unitary sectors, $(\mathcal{E}_\tau)_{\mathbb{C}} = E \oplus \overline{E}$. Their determinant lines are Hermitian duals, $\det \overline{E} \simeq (\det E)^*$, providing the corresponding Abelian holonomies as conjugate inverse phases. TM identifies these determinant phases as the two orientations of a single temporal $U(1)$.

The phase lock then identifies this shared temporal determinant line with the auxiliary $U(1)$ line of the external $\mathrm{Spin}^c(1,3)$ structure, producing

$$\mathcal{G} = U(3) \times_{U(1)} \mathrm{Spin}^c(1,3), \tag{2}$$

where $\mathcal{G}$ denotes the fiber product over the common $U(1)$ phase selected by the lock. The allowed pair consists of an internal $U(3)$ transformation and an external $\mathrm{Spin}^c(1,3)$ transformation whose determinant/auxiliary $U(1)$ phases agree under the chosen line-bundle isomorphism. This removes an independent Abelian duplication and leaves one locked temporal $U(1)$.

Postulate 4: Temporal coherence and gauge information propagate at a universal limiting speed. When projected onto $\mathcal{M}_4$, this limit is the speed of light, preserving standard relativistic causality.

Postulate 5: The curvature of the external spacetime manifold $\mathcal{M}_4$ is sourced by the energy-momentum associated with the internal geometry of the temporal manifold $\mathcal{M}_\tau$ through the phase-locked determinant line and the induced external Lorentzian geometry.

The defining geometric assumptions of the model are the product geometry $\mathcal{M}_7 = \mathcal{M}_4 \times \mathcal{M}_\tau$, with a compact internal temporal sector $\mathcal{M}_\tau \simeq \mathbb{T}^3$, an internal $U(3)$ bundle, and a phase lock between the internal determinant $U(1)$ and the auxiliary $U(1)$ line of the external $\mathrm{Spin}^c(1,3)$ structure.

From this structure, TM constructs a locked temporal holonomy sector, an SM-compatible gauge and charge assignment, a geometric interpretation of chirality in terms of conjugate-dual fiber circulation, and an effective mass-generating spectral structure. The fundamental variational principle is the Einstein-Hilbert action on the phase-locked temporal geometry. SM-like mass and Yukawa terms are interpreted as effective sectors arising from rigid and flexible perturbations of that geometry, rather than as an unrelated Lagrangian appended by hand.

The construction is organized perturbatively. At zeroth order, $\mathcal{M}_\tau$ is rigid over $\mathcal{M}_4$, the product metric separates, and the internal temporal Dirac operator supplies spectral rest masses. At first order, x-dependent perturbations of the internal connection and metric generate Higgs-coset dynamics, geometric overlap matrices, and flavor mixing. Thus, the benchmark mass law is a rigid temporal spectral construction, while the Yukawa-like structures arise as perturbative overlap operators rather than as independent fundamental rest masses.

The benchmark mass construction is not an independent phenomenological fit detached from the geometry, nor is it a uniqueness proof for the SM. It is a compatibility benchmark showing that the observed charged-fermion and neutrino mass patterns can be represented within the spectral structure of the phase-locked $U(3)$ temporal bundle. The constructed consequences are the locked Abelian sector, the SM-compatible bundle assignments, the Higgs-coset interpretation, and the spectral form of the zeroth-order mass operator. The numerical mass tables are benchmark reproductions using fitted holonomy.

Section 2 defines time from first principles, showing how periodicity and phase evolution define temporal dynamics. Section 3 provides the global $U(1)$ symmetry that encodes time evolution. Section 4 generalizes this to $U(3)$ to account for flavor and charge. Section 5 introduces the extended symmetry group $SO(3,3)$ and the SM. Section 6 demonstrates how both spacetime and gauge symmetries arise through phase locking. Section 7 formalizes the product manifold and its perturbation. Section 8 provides the particles of the SM. Section 9 addresses how mass and energy emerge from the spectrum of the Dirac operator. Section 10 provides rest masses, and Section 11 discusses mixing and Higgs dynamics using holonomy and curvature. The geometric benchmark fit is provided in the **Appendix**.

2. Temporal Phase and Internal Time

This section defines the temporal terminology used in TM. Its purpose is to explain why the compact internal manifold $\mathcal{M}_\tau$ is interpreted as temporal rather than spatial. The standard spatial interpretation of compact internal directions is mathematically equivalent at the level of periodic coordinates and holonomy. TM makes a different physical identification, where the compact coordinate is read as clock phase, Schrödinger phase, and proper-time phase rather than as an unobserved spatial displacement. This interpretation motivates the phase-lock between the internal determinant $U(1)$, the external $\mathrm{Spin}^c(1,3)$ auxiliary line, and the quantum phase of matter fields.

Definition. A clock is any physical system whose state evolves through stable, periodic trajectories on the temporal manifold $\mathcal{M}_\tau$, such that each completed cycle defines measured intervals of external time in the emergent spacetime $\mathcal{M}_4$.

Each oscillation of a physical system, like an atomic transition, maps onto periodic trajectories in the temporal manifold, establishing a direct correspondence between internal geometry and experimental time intervals. These periodicities trace closed loops in $\mathcal{M}_\tau$, linking observable time intervals in $\mathcal{M}_4$ to geometric properties of the temporal manifold. This defines the external time parameter $t \in \mathbb{R}$, which labels points on an observer's worldline, from the internal phase $\theta \in S^1$ of a clock. The phase lives on a circle, and its evolution is described by a $U(1)$ action, while t is the lift of that circular motion to the universal cover $\mathbb{R}$.

A periodic clock with angular frequency ω has phase $\theta(t) = \omega t$, $\mathrm{e}^{-i\theta(t)} = \mathrm{e}^{-i\omega t} \in U(1)$. Thus time translations act on the clock state by the homomorphism $t \in (\mathbb{R},+) \mapsto \mathrm{e}^{-i\omega t} \in U(1)$, where addition of time intervals corresponds to multiplication of phases, $\mathrm{e}^{-i\omega t_1}\mathrm{e}^{-i\omega t_2} = \mathrm{e}^{-i\omega(t_1+t_2)}$. For a clock with period $T = 2\pi/\omega$, physical states are identified under $t \sim t + nT$, $n \in \mathbb{Z}$, so the periodic state space is $\mathbb{R}/T\mathbb{Z} \simeq \mathbb{S}^1$, the group manifold of $U(1)$. External time is therefore the unwrapped parameter of a compact internal clock phase.

This is the minimal reason that $U(1)$ appears before the full $U(3)$ construction. A clock does not merely label time. It realizes time operationally as an accumulated phase. The phase lives on a compact circle, while the measured time t is the lift of that circular motion to the universal cover $\mathbb{R}$.

A pendulum gives the elementary prototype. In the harmonic approximation its angular coordinate φ satisfies $\mathcal{L} \approx \frac{1}{2}m\ell^2\dot{\varphi}^2 - \frac{1}{2}mg\ell\varphi^2 = \frac{\beta}{2}\dot{\varphi}^2 - \frac{\beta\omega^2}{2}\varphi^2$, where $\beta = m\ell^2$, $\omega^2 = \frac{g}{\ell}$. The Euler-Lagrange equation gives $\ddot{\varphi} + \omega^2\varphi = 0$,

$\varphi(t) = A\cos(\omega t + \theta_0)$. The physically relevant clock variable is the accumulated phase of this periodic trajectory. The configuration may be represented on a compact circle, and one completed oscillation corresponds to one completed cycle of phase.

The conjugate momentum to the phase is $p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \beta\dot{\varphi}$. The parameter β is the phase inertia. It measures resistance to changes in the rate of phase winding. In the free-rotor limit, $\mathcal{L}_{\text{free}} = \frac{\beta}{2}\dot{\varphi}^2$, the shift symmetry $\varphi \mapsto \varphi + \epsilon$ gives a conserved phase momentum by Noether's theorem [13]. Upon quantization, the compactness of the phase circle gives discrete winding numbers. Thus, a periodic clock already contains the essential geometric ingredients used in TM. It has a compact phase, a $U(1)$ action, a conserved winding momentum, and an external time parameter obtained by lifting the phase to its universal cover.

TM promotes this elementary clock phase to an internal field-theoretic structure. The single angular coordinate $\varphi \in \mathbb{S}^1$ is generalized to local angular coordinates $\varphi^a \in \mathbb{T}^3$, $a = 1,2,3$, on the compact temporal manifold $\mathcal{M}_\tau \simeq \mathbb{T}^3$. The phase inertia becomes an internal tensor $\beta_{ab} \propto g_{ab}$, and the corresponding momenta $p_a = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^a} = \beta_{ab}\dot{\varphi}^b$ measure winding in the compact temporal directions. After quantization, these momenta become discrete internal windings, analogous in form to Kaluza-Klein momenta, but interpreted here as temporal rather than spatial phase windings.

The same $U(1)$ structure here appears in quantum time evolution. The time-dependent Schrödinger equation is $i\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t) = \hat{H}\Psi(\boldsymbol{r},t)$. For an energy eigenstate, $\hat{H}\psi_n(\boldsymbol{r}) = E_n\psi_n(\boldsymbol{r})$, separation of variables gives $\Psi(\boldsymbol{r},t) = \psi_n(\boldsymbol{r})\chi(t)$, $\chi(t) = \mathrm{e}^{-iE_n t}\chi(0)$. The time dependence of an energy eigenstate is therefore a $U(1)$-valued phase evolution. For oscillatory systems, including pendula, atomic transitions, and spinor precession, the measured clock rate is equivalently a phase rate [14]. The Hamiltonian evolution in external time is interpreted in TM as the external shadow of winding in internal time.

The temporal evolution operator for a single energy eigenstate is $\hat{U}(t) = \mathrm{e}^{-iE_n t} \in U(1)$, with phase defined modulo 2π. Thus, the Schrödinger phase, the clock phase, and the proper-time phase all have the same Abelian structure. TM identifies them as associated realizations of one temporal holonomy rather than as unrelated phase conventions.

The phase field may also be written in spacetime form. For a scalar clock phase φ on $(\mathcal{M}_4, g_{\mu\nu})$, take $\mathcal{L} = \frac{\beta}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\beta\omega^2}{2}\varphi^2$, with action $S_\varphi = \int_{\mathcal{M}_4}\mathrm{d}^4x\sqrt{-g_4}\mathcal{L}$. Stationarity gives $\frac{1}{\sqrt{-g_4}}\partial_\mu\left(\sqrt{-g_4}\beta g^{\mu\nu}\partial_\nu\varphi\right) + \beta\omega^2\varphi = 0$, which reduces to $\ddot{\varphi} + \omega^2\varphi = 0$ in flat space and in the homogeneous limit. This form connects the elementary clock model to field dynamics on the emergent external spacetime.

At the level of differential geometry, the same compact circle can be described using the standard language of a compact spatial coordinate. The difference is the physical identification of the compact holonomy. In the standard spatial view, the compact coordinate is an unobserved spatial position. In the temporal view, the compact coordinate is the internal clock phase, which is the periodic structure through which physical systems accumulate proper time and quantum phase. The mathematics of $\mathbb{S}^1$, $U(1)$, winding and holonomy is the same. TM changes the physical interpretation of what the compact direction measures.

3. *U*(1)

The $U(1)$ group is the minimal symmetry that encodes the phase evolution of a quantum system [14]. TM assumes a single underlying temporal holonomy line $L_\tau \to \mathcal{M}_7$ with a structure group $U(1)$. The Schrödinger phase of quantum evolution, the determinant line of the internal $U(3)$ bundle, the auxiliary $U(1)$ line of the external $\mathrm{Spin}^c(1,3)$ structure, and the line-bundle twists carried by matter fields are taken to be associated realizations of this temporal $U(1)$, not independent, Abelian sectors. The phase lock is the geometric identification that ties these realizations together.

A clock modeled by free motion on $\mathbb{S}^1$ with angular coordinate $\theta \sim \theta + 2\pi$ and free-rotor Lagrangian has a shift symmetry $\theta \to \theta + \epsilon$ yielding conserved momentum $p_\theta = \beta\dot{\theta}$. It is useful to distinguish the signed lifted phase, the signed winding count, and the operationally accumulated clock time:

$$\tilde{\theta}(s) = \int_0^s \dot{\theta}(u)\,\mathrm{d}u, N_\theta(s) = \frac{\tilde{\theta}(s)}{2\pi},$$

$$t_{\mathrm{clk}}(s) = \frac{1}{2\pi}\int_0^s \left|\dot{\theta}(u)\right|\mathrm{d}u. \tag{3}$$

Here $\tilde{\theta}$ is the lifted phase angle in radians, N_θ is the signed winding count, and t_{clk} is the positively accumulated clock time. Reverse winding changes the sign of N_θ but not of t_{clk}.

$U(1)$ is the group of complex numbers under multiplication,

$$U(1) = \left\{ \mathrm{e}^{-i\theta} \mid \theta \in [0, 2\pi) \right\}. \tag{4}$$

For any continuous path $\mathcal{C}:[0,T] \to \mathbb{S}^1$, the winding number is

$$w[\mathcal{C}] = \frac{1}{2\pi}\int_0^T \frac{\mathrm{d}\theta}{\mathrm{d}s}(s)\,\mathrm{d}s = N_\theta(T) - N_\theta(0) = \frac{\Delta\tilde{\theta}}{2\pi}. \tag{5}$$

The real line $t \in \mathbb{R}$ parameterizes the covering space of these windings, and the Hamiltonian determines how the wavefunction winds around the circle fiber as time progresses and is the generator of the Lie group $U(1)$. The total Hamiltonian $\hat{H}$ is self-adjoint and $-i\hat{H}$ lies in the Lie algebra $\mathfrak{u}(n)$, generating a one-parameter subgroup $\hat{U}(t) = \mathrm{e}^{-i\hat{H}t} \in U(n)$ acting on a finite-dimensional internal Hilbert space. Its central $U(1)$ factor corresponds to the overall phase generated by the trace part of $\hat{H}$, while the traceless part lives in $\mathfrak{su}(n)$ with

$$-i\hat{H} \in \mathfrak{u}(\mathcal{H}), \text{ with } \det \hat{U}(t) = \mathrm{e}^{-it\mathrm{tr}\hat{H}t} \in U(1). \tag{6}$$

The global phase of $\hat{U}(t)$ given by $\frac{\mathrm{tr}\hat{H}}{n}$ defines a central $U(1) \subset U(n)$ that encodes the total energy of the system. When $\hat{H}$ acts as $E_\tau \mathbf{1}$ on the subspace $\mathcal{H}_\tau$, the dynamics reduce to a pure $U(1)$ rotation generated by $-i\hat{H}_\tau \in \mathfrak{u}(1)$.

Let $P \xrightarrow{\pi} \mathcal{M}_\tau$ be a principal $U(1)$ bundle with global connection one-form $\mathcal{A} \in \Omega^1(P, \mathfrak{u}(1))$. In a local trivialization $s_\alpha : U_\alpha \to P$, the local gauge potential is

$$A_\alpha = s_\alpha^* \mathcal{A} \in \Omega^1(U_\alpha, \mathbb{R}), \tag{7}$$

and on overlaps $U_\alpha \cap U_\beta$ the local potentials differ by a gauge transformation. The curvature is the globally defined closed two-form $F \in \Omega^2(\mathcal{M}_\tau, \mathbb{R})$ characterized locally by

$$F\big|_{U_\alpha} = dA_\alpha. \tag{8}$$

The topology of the $U(1)$ bundle is encoded by the first Chern class

$$c_1(P) = \left[\frac{1}{2\pi} F\right] \in H^2(\mathcal{M}_\tau, \mathbb{Z}), \tag{9}$$

whose pairing with a closed 2-cycle $\Sigma \in H_2(\mathcal{M}_\tau, \mathbb{Z})$ gives the quantized flux condition

$$\frac{1}{2\pi} \int_\Sigma F \in \mathbb{Z}. \tag{10}$$

For a loop $\mathcal{C}$ contained in a local chart with $\mathcal{C} = \partial\Sigma$, Stokes' theorem gives

$$\oint_{\mathcal{C}} A_\alpha = \int_\Sigma F,$$

and the associated Wilson loop is

$$W(\mathcal{C}) = \exp\left(-i\oint_{\mathcal{C}} A_\alpha\right) = \exp\left(-i\int_\Sigma F\right). \tag{11}$$

For a flat connection, holonomy depends only on the homotopy class of the loop. In general, it depends on the connection along the loop, and Stokes' theorem relates it locally to the curvature on a spanning surface when one exists, while the integral cohomology class $[F/2\pi]$ records the global topological sector [15]-[19]. Equivalently, a spinor transported around $\mathcal{C}$ picks up the geometric phase $\psi \mapsto W(\mathcal{C})\psi$, and the resulting charge quantization is the bundle content of the Dirac-Wu-Yang condition on a nontrivial 2-cycle [15] [20]-[23].

This follows the Aharonov-Bohm effect, where the quantum phase can depend on gauge holonomy even when the local field strength vanishes along the particle trajectory [24]. It is also the bundle-theoretic language underlying Berry's geometric phase, in which cyclic evolution produces a phase determined by the geometry of the parameter-space path [25]. TM applies this holonomy principle to the compact temporal line, so that clock phase, the Schrödinger phase, and determinant-line phase are treated as associated realizations of one temporal $U(1)$.

For the $U(1)$ group, the global phase symmetry is $\psi \to e^{iq_1\alpha}\psi$. The covariant derivative is

$$D_\mu = \partial_\mu + iq_1 A_\mu \tag{12}$$

with the gauge field

$$A_\mu \to A_\mu - \partial_\mu \alpha \tag{13}$$

and has the local symmetry $\psi \to e^{iq_1\alpha}\psi$. The conserved current is

$$J^\mu = q_1 \bar{\psi}\gamma^\mu \psi \tag{14}$$

and the field strength is

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{15}$$

The commutator of the covariant derivative is

$$\left[D_\mu, D_\nu\right] = iq_1 \mathcal{F}_{\mu\nu}, \tag{16}$$

where the generators T^A are Hermitian, the gauge potential A_μ has real components A_μ^A, and the field $\mathcal{F}_{\mu\nu}$ is Hermitian.

The unitary group $U(1)$ appears in multiple foundational contexts of modern physics as the gauge group of electromagnetism and hypercharge [26]-[28]. $U(1)$ emerges as the residual symmetry of temporal geometry, and acts as the generator of phase evolution. This geometrizes both quantum time evolution and gauge interactions under a single temporal symmetry.

The global $U(1)$ symmetry defines a principal fiber bundle over the temporal manifold $\mathcal{M}_\tau$, with group $U(1)$, total space P, and projection map $\pi: P \to \mathcal{M}_\tau$,

$$U(1) \hookrightarrow P \xrightarrow{\pi} \mathcal{M}_\tau. \tag{17}$$

This bundle admits local trivializations over charts $(U_\alpha, \varphi_\alpha)$ covering $\mathcal{M}_\tau$, with transition functions valued in $U(1)$. The dual interpretation of $U(1)$ as both a temporal generator and a gauge symmetry provides the foundational mechanism by which internal temporal structure encodes observable particle properties through the traceless internal sector, providing the geometry that underlies the gauge theory of electromagnetism and hypercharge.

TM interprets $U(1)$ as a symmetry fundamentally arising from temporal geometry. Kaluza-Klein (KK) theory [1] [2] interprets charge as momentum along spatial compactified dimensions, whereas TM locates this momentum along compact temporal directions, providing a winding in internal time. This unifies quantum evolution and gauge interaction under a shared temporal symmetry [3], where geometric quantization formalism identifies such phase evolution with holonomy in fiber bundles and physical observables emerge from winding numbers [23]. The locked $U(1)$ holonomy is the temporal analog of the Berry phase, while the non-Abelian $U(3)$ holonomies acting on internal temporal sectors are the analog of Wilczek-Zee holonomy.

4. $U(3)$

The internal temporal sector is modeled by a rank-3 Hermitian bundle $E \to \mathcal{M}_\tau$, whose unitary automorphisms define a $U(3)$ structure. Its determinant line carries the Abelian phase, while its traceless sector carries non-Abelian internal structure. A temporal manifold with higher-rank unitary symmetry extends the geometric description of gauge symmetries.

The Lie group $U(n)$ is central to gauge theories and field interactions in quantum mechanics and particle physics [18] [19], where n reflects the number of coupled internal phase modes providing for the emergence of mass, flavor, and gauge.

Throughout the local gauge formulas, the matrices T^α are Hermitian generators normalized by $\mathrm{Tr}\left(T^\alpha T^\beta\right) = \frac{1}{2}\delta^{\alpha\beta}$. The local gauge potentials $A_A = A_A^\alpha T^\alpha$ are therefore Hermitian matrix-valued one-forms. The corresponding principal-bundle connection is the anti-Hermitian form iqA_A. With matter fields transforming as $\psi \mapsto U\psi$, the associated covariant derivative is written $\nabla_A = \partial_A + \omega_A + iqA_A$, and gauge covariance requires $A_A \mapsto A'_A = UA_AU^{-1} + \frac{i}{q}(\partial_A U)U^{-1}$. For $U = \mathrm{e}^{iq\alpha}$ in the Abelian case, this reduces to $A_A \mapsto A_A - \partial_A \alpha$, matching the $U(1)$ convention.

Defining the Lie group as

$$U(n) = \left\{U \in \mathbb{C}^{n\times n} \mid U^\dagger U = I\right\}, \tag{18}$$

and choosing a basis of Hermitian matrices $T^A = \left(T^A\right)^\dagger$ ($A = 0,\cdots,n^2-1$), the corresponding Lie algebra elements are anti-Hermitian and spanned by iT^A, in the physics convention. Any $X \in \mathfrak{u}(n)$ can be written as $X = i\alpha_A T^A$ with real coefficients $\alpha_A \in \mathbb{R}$. The Lie algebra is

$$\mathfrak{u}(n) = \left\{X \in M_n(\mathbb{C}) \mid X^\dagger = -X\right\}. \tag{19}$$

For the real vector space of Hermitian generators under gauge transformation with Lie bracket induced by the commutator $[X,Y] = XY - YX$, an n component complex matter field transforms as

$$\psi(x) \mapsto U(x)\psi(x),\ U(x) \in U(n). \tag{20}$$

$\psi(x)$ is a complex vector-valued field and transforms under the fundamental representation of $U(n)$.

The covariant derivative is

$$D_\mu = \partial_\mu + iq_n A_\mu, \tag{21}$$

where

$$A_\mu = A_\mu^\alpha T^\alpha \tag{22}$$

and the gauge symmetry is

$$A_\mu \to UA_\mu U^{-1} + \frac{i}{q_n}\left(\partial_\mu U\right)U^{-1}. \tag{23}$$

The conserved current is

$$J^{\mu\alpha} = q_n \bar{\psi}\gamma^{\mu}T^{\alpha}\psi \tag{24}$$

and the field strength is

$$\mathcal{F}_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + iq_n\left[A_{\mu}, A_{\nu}\right]. \tag{25}$$

The commutator of the covariant derivative is

$$\left[D_{\mu}, D_{\nu}\right] = iq_n\mathcal{F}_{\mu\nu}. \tag{26}$$

The group $U(n)$ has n^2 real parameters, and the determinant of a unitary matrix lies on the circle with the map, $\det : U(n) \to U(1)$ and is a group homomorphism. The higher-order unitary groups $U(n)$ are isomorphic to

$$U(n) \simeq \frac{SU(n)\times U(1)}{\mathbb{Z}_n}, \tag{27}$$

where $\mathbb{Z}_n = \left\{\left(\mathrm{e}^{2\pi ik/n}\mathbf{1}_n, \mathrm{e}^{-2\pi ik/n}\right) \mid k = 0,\cdots,n-1\right\}$. This decomposition of the unitary group is standard in gauge theory [29]. When $n = 3$, $\mathbb{Z}_3 = \left\{\left(\omega^{-k}\mathbf{1}_3, \omega^{k}\right) \mid k = 0,1,2\right\}$, and $\omega = \mathrm{e}^{2\pi i/3}$.

$SU(n)$ denotes the special unitary group of unitary matrices with determinant 1. $\mathbb{Z}_n$ is the discrete central subgroup of $SU(n)\times U(1)$ that is identified in the quotient to produce $U(n)$, which always includes a global phase with a real cover map and the gauge connection to $SU(n)$.

In the decomposition, $\mathfrak{u}(n) = \mathfrak{su}(n)\oplus\mathfrak{u}(1)$ is the central $U(1)$. The generators T^{α} with $\alpha = 0,1,\cdots,n^2-1$ have $T^0 \propto 1$. For the $SU(n)$ sector, the generators T^{a} are traceless. The normalization is

$$\mathrm{Tr}\left(T^{\alpha}T^{\beta}\right) = \frac{1}{2}\delta^{\alpha\beta}. \tag{28}$$

Using the Lie algebra structure constants $f^{\alpha\beta\gamma}$, the general generator commutator is

$$\left[T^{\alpha}, T^{\beta}\right] = if^{\alpha\beta\gamma}T^{\gamma}. \tag{29}$$

The $U(1)$ sector is $f^{0\beta\gamma} = 0$, $\left[T^0, T^{\alpha}\right] = 0$. In the $SU(n)$ sector, $\left\{T^{\alpha}, T^{\beta}\right\} = \frac{1}{n}\delta^{\alpha\beta}\mathbf{1} + d^{\alpha\beta\gamma}T^{\gamma}$. The field is then

$$\mathcal{F}_{\mu\nu} = \left(\partial_{\mu}A_{\nu}^{\alpha} - \partial_{\nu}A_{\mu}^{\alpha} - q_n f^{\alpha\beta\gamma}A_{\mu}^{\beta}A_{\nu}^{\gamma}\right)T^{\alpha}. \tag{30}$$

Take $T^0 = \mathbf{1}_3/\sqrt{6}$ for the central $U(1)$, and the Gell-Mann matrices are $T^{\alpha} = \lambda^{\alpha}/2$ for $\alpha = 1,\cdots,8$. In the $SU(3)$ group $T^{\alpha} = \lambda^{\alpha}/2$ and $f^{123} = 1$, $f^{147} = f^{246} = f^{257} = f^{345} = \frac{1}{2}$, $f^{156} = f^{367} = -\frac{1}{2}$, $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$. In the $SU(2)$ group, $T^{\alpha} = \sigma^{\alpha}/2$ and $f^{\alpha\beta\gamma} = \varepsilon^{\alpha\beta\gamma}$ [30].

$U(1)$ governs the global phase and $SU(n)$ introduces internal degrees of freedom, including gauge interactions. As with $U(1)$, the group $U(n)$ defines a principal fiber bundle over $\mathcal{M}_{\tau}$, where curvature forms encode generalized gauge field strengths,

$$F = dA + iqA \wedge A \in \Omega^2\left(\mathcal{M}_\tau, \mathfrak{u}(n)\right). \tag{31}$$

When the representation charge or coupling is absorbed into A, this is written simply as $F = dA + iA \wedge A$.

The higher gauge groups of $U(n)$ on $\mathcal{M}_\tau$ would admit additional topological invariants through higher Chern characters. These structures are defined by Chern-Weil theory, where characteristic classes arise as topological invariants associated with the curvature form of a principal bundle [31]-[33].

Let $\mathcal{C}:[0,1] \to \mathcal{M}_\tau$ be a closed loop, and A connection on the $U(3)$ bundle, where the Wilson line and the internal spinors transform under flavor mixing with

$$W(\mathcal{C}) = \mathcal{P}\exp\left(-i\oint_{\mathcal{C}} A\right), \quad \chi_n \mapsto \hat{W}_{nk}\chi_k. \tag{32}$$

$\mathcal{P}$ denotes path ordering, and the $U(n)$ valued connection A does not commute at different points along the loop. The Wilson line $W(\mathcal{C})$ is the path ordered exponential solving $\frac{\mathrm{d}}{\mathrm{ds}}W(s) = -iA(s)W(s)$, $W(0) = \mathbf{1}$, $W(\mathcal{C}) = W(1)$. χ transforms in the fundamental representation of $U(n)$ and $W(\mathcal{C})$ is an $n \times n$ unitary matrix obtained by parallel transport along $\mathcal{C}$.

This is the non-Abelian analog of geometric phase. In the Wilczek-Zee extension of Berry phase, adiabatic evolution in a degenerate subspace produces matrix-valued holonomy rather than a single $U(1)$ phase [34]. In TM, the noncommuting $U(3)$ temporal Wilson lines play the corresponding role for internal temporal eigenmodes, where their failure to be simultaneously diagonalized becomes, after projection, the relative left-handed basis mismatch observed as CKM and PMNS mixing.

5. Conjugate Temporal Windings

The natural temporal geometry contains a duality between phase-flow directions and phase-gradient directions. The dual temporal pair is the organizing bridge between temporal holonomy and the SM structure. It arises from the traceless parts of two conjugate rank-three temporal sectors after their determinant phases have been locked to a single temporal $U(1)$.

The temporal manifold in TM is then not modeled only by its tangent directions, but by the generalized bundle $\mathcal{E}_\tau := T\mathcal{M}_\tau \oplus T^*\mathcal{M}_\tau$, whose fibers carry the canonical split-signature pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}\left(\xi(Y) + \eta(X)\right), \quad X, Y \in T\mathcal{M}_\tau, \ \xi, \eta \in T^*\mathcal{M}_\tau. \tag{33}$$

For $\dim \mathcal{M}_\tau = 3$, this pairing has signature $(3,3)$, so the natural structure group of $\mathcal{E}_\tau$ is $O(3,3)$, or $SO(3,3)$ after fixing orientation.

This dual temporal structure also supplies the bundle-theoretic origin of charge conjugation. The conjugate pair constructed this way is therefore not an auxiliary doubling and not merely a notation for opposite windings. It is the dual rank-three temporal architecture from which TM obtains the color/electroweak split, the locked determinant time direction, and the Higgs coset.

The use of $SO(3,3)$ reflects a basic feature of TM where temporal geometry comes with two mutually paired directions at each point, from the phase-flow directions in $T\mathcal{M}_\tau$ and conjugate phase-gradient directions in $T^*\mathcal{M}_\tau$. In this sense, the temporal sector already contains a dual winding that uses a geometric origin for the two global windings.

To pass from the real split-signature bundle to internal unitary structure, TM chooses a generalized metric together with a compatible generalized complex structure on $\mathcal{E}_\tau$ [35]-[37]. After complexification, this produces a decomposition

$$(\mathcal{E}_\tau)_{\mathbb{C}} = E \oplus \bar{E}, \tag{34}$$

where E is the $+i$ eigenbundle and $\bar{E}$ is its complex conjugate $-i$ eigenbundle. Each has a complex rank 3, and together they define a $U(3)$ reduction of the internal generalized structure.

The pair $(E,\bar{E})$ is the mathematical expression of the two global temporal windings in TM, which are conjugate realizations of the same underlying temporal holonomy. For any closed loop $\mathcal{C} \subset \mathcal{M}_\tau$, parallel transport gives unitary holonomies

$$W_E(\mathcal{C}) \in U(3),\ W_{\bar{E}}(\mathcal{C}) = \overline{W_E(\mathcal{C})}. \tag{35}$$

Passing to determinant lines,

$$L_E := \det E,\ L_{\bar{E}} := \det \bar{E}, \tag{36}$$

the Hermitian structure identifies

$$L_{\bar{E}} \cong \overline{L_E} \cong L_E^*. \tag{37}$$

Hence, the Abelian holonomies are conjugate inverse phases,

$$\det W_E(\mathcal{C}) = \mathrm{e}^{-i\vartheta(\mathcal{C})},\ \det W_{\bar{E}}(\mathcal{C}) = \mathrm{e}^{+i\vartheta(\mathcal{C})}. \tag{38}$$

This is the precise sense in which TM has two global windings. One winding carries the phase $\mathrm{e}^{-i\vartheta}$ and the other carries the conjugate winding $\mathrm{e}^{+i\vartheta}$. They are opposite orientations of one and the same temporal circle, and equivalently, two conjugate lifts of a single global temporal $U(1)$.

Since $L_{\bar{E}} \simeq L_E^*$, this dualization reverses the orientation of the locked temporal holonomy. The spectral mass norm is unchanged, while the internal temporal charges are reversed. Thus, matter and antimatter are opposite associated-bundle orientations of the same phase-locked temporal geometry and not two independent temporal sectors.

The two $U(3)$ holonomies are interpreted as conjugate flow directions on the same compact temporal manifold, not as two independent orthogonal internal spaces. The conjugate-dual fibers are therefore correlated by the phase lock, where the same circulation mode cannot simultaneously exist as an identical flow in both conjugate-dual fibers. Antiparticles are represented by conjugate or swapped fiber circulation on the same internal temporal manifold rather than by a freely independent mirror sector.

The phase lock identifies this common determinant phase with the auxiliary $U(1)$ line of the external $\mathrm{Spin}^c(1,3)$ structure. Thus, the two rank-three sectors do not carry independent Abelian gauge factors. Their determinant $U(1)$ phases are conjugate realizations of one locked temporal $U(1)$, while their traceless unitary parts remain available for non-Abelian gauge structure.

Assigning one member of the conjugate pair to the color sector

$$C := \bar{E}, \tag{39}$$

and retaining the full traceless unitary structure on this sector,

$$SU(3)_C \subset SU(3)_{\bar{E}}. \tag{40}$$

The conjugate partner E cannot be assigned the same physical gauge role without producing a mirror copy of color rather than the observed SM gauge structure. Once one rank-three temporal sector is used for unbroken color, the SM compatible use of the remaining rank-three sector must therefore be asymmetric.

The minimal asymmetric possibility is to select a holonomy-preserved rank-two subbundle and a complementary line bundle,

$$E = V \oplus L_0,\ \operatorname{rank}(V) = 2,\ \operatorname{rank}(L_0) = 1. \tag{41}$$

This splitting is the geometry that distinguishes weak-doublet directions from a singlet direction. The stabilizer of the splitting $E = V \oplus L_0$ inside $U(3)_E$ is

$$U(V) \times U(L_0) \simeq U(2) \times U(1). \tag{42}$$

Restricting to the traceless sector $SU(3)_W$ imposes the determinant-one condition,

$$S\big(U(2)\times U(1)\big) = \big\{(A,z) \in U(2)\times U(1) \mid \det(A)\, z = 1\big\}. \tag{43}$$

Here $A \in U(V) \simeq U(2)$ acts on the rank-two subbundle V, while $z \in U(L_0) \simeq U(1)$ acts on the line bundle L_0. This group is locally isomorphic to $SU(2)\times U(1)$ with

$$S\big(U(2)\times U(1)\big) \simeq \frac{SU(2)\times U(1)}{\mathbb{Z}_2}. \tag{44}$$

At the Lie-algebra level,

$$\mathfrak{su}(3)_E \supset \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_Y. \tag{45}$$

Thus, the holonomy-preserved splitting of the conjugate partner of color selects the electroweak gauge structure as the stabilizer of a rank-(2 + 1) decomposition.

Because the color and electroweak actions are supported on conjugate temporal sectors, their projected four-dimensional gauge actions commute. The resulting low-energy gauge algebra is therefore the Standard Model product algebra,

$$G_{\mathrm{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{46}$$

However, this product should not be interpreted in TM as three unrelated orthogonal internal spaces. The two non-Abelian sectors are complementary realizations of one phase-locked dual temporal geometry. Color occupies one rank-three conjugate flow, while the opposite conjugate flow cannot carry an independent, identical color copy without producing a mirror color sector. The minimal allowed use of the conjugate partner is therefore the holonomy-preserved rank-(2 + 1) splitting whose stabilizer is the electroweak $SU(2)_L \times U(1)_Y$ subgroup.

Thus, the Standard Model gauge product appears after projection, but its factors are correlated by the underlying dual-fiber geometry. The determinant $U(1)$ of the two sectors is the same temporal phase seen with opposite orientation in the conjugate windings, and this common determinant line is phase-locked to the external $\mathrm{Spin}^c(1,3)$ line. The role of $SO(3,3)$ is therefore more than kinematic. It supplies the generalized bundle whose compatible unitary reduction produces the conjugate temporal flows. One flow carries unbroken color, and the other carries the electroweak stabilizer with the Higgs coset. This gives a selection rule where a physical low-energy sector cannot contain both an unbroken color copy and its identical conjugate-dual mirror as independent gauge structures.

Chirality has a geometric meaning in this construction. Right-handed fermions correspond to circulation on one fiber of the conjugate-dual internal structure, whereas left-handed fermions correspond to circulation on both fibers. Thus, chirality is not introduced only as an external representation label. It is encoded in the allowed circulation structure of the internal temporal bundle. TM therefore shows that the observed chiral assignments of the SM are compatible with the phase-locked $U(3)$ temporal geometry.

6. Phase Locking

Definition: The phase lock is a chosen product-manifold structure on $\mathcal{M}_7 = \mathcal{M}_4 \times \mathcal{M}_\tau$. It selects a distinguished Hermitian $U(1)$ line bundle with connection

$$(L_{\mathrm{lock}}, A_{\mathrm{lock}}) \to \mathcal{M}_7 \tag{47}$$

together with connection-preserving identifications

$$(L_{\mathrm{lock}}, A_{\mathrm{lock}}) \simeq (\pi_\tau^* L, \pi_\tau^* \mathcal{A}) \simeq (\pi_4^* L_{\mathrm{Spin}^c}, \pi_4^* A_{\mathrm{Spin}^c}). \tag{48}$$

This can be written in the locked sector as

$$\Lambda : (\pi_\tau^* L, \pi_\tau^* \mathcal{A}) \xrightarrow{\sim} (\pi_4^* L_{\mathrm{Spin}^c}, \pi_4^* A_{\mathrm{Spin}^c}). \tag{49}$$

This is not an automatic identity between arbitrary line bundles on the two factors. It is the defining compatibility condition of the phase-locked product sector. The product manifold is therefore restricted to configurations for which the internal determinant line and the external Spin^c auxiliary line are represented by the same locked temporal $U(1)$ line after pullback to $\mathcal{M}_7$.

The lock is not meant as a generic equality between arbitrary pullback classes from the two factors of the product. On a direct product, $\pi_4^* H^2(\mathcal{M}_4)$ and $\pi_\tau^* H^2(\mathcal{M}_\tau)$ generally occupy different Künneth summands. TM therefore treats L_{lock} as the primary temporal line on $\mathcal{M}_7$. The internal determinant line and the external Spin^c auxiliary line are its associated realizations or restrictions in the locked sector. In the minimal benchmark used, the lock may be taken at the level of flat connection holonomy, with trivial or compatible Chern class. More general nontrivial locks require a diagonal, restricted, or mixed class on $\mathcal{M}_7$ rather than an unconstrained equality of arbitrary product pullbacks.

Within a topologically compatible locked sector, the associated Chern data and curvatures are represented by the same locked temporal line,

$$c_1\left(\pi_\tau^* L\right) = c_1\left(L_{\text{lock}}\right) = c_1\left(\pi_4^* L_{\text{Spin}^c}\right),$$
$$\pi_\tau^* F_{\text{det}} = F_{\text{lock}} = \pi_4^* F_{\text{Spin}^c}, \tag{50}$$

and for the conjugate determinant line

$$c_1\left(\pi_\tau^* L^*\right) = -c_1\left(L_{\text{lock}}\right),$$
$$\pi_\tau^* F_{\text{det}}^* = -F_{\text{lock}}. \tag{51}$$

The lifted phase of the locked line then provides the external proper-time parameter from a single global temporal $U(1)$. The phase lock is the chosen identification of the determinant line of the internal $U(3)$ bundle with the auxiliary line of the external $\text{Spin}^c(1,3)$ structure.

Let $L_{\text{lock}} \to \mathcal{M}_7$ denote the distinguished principal $U(1)$ bundle carrying the universal time phase. Its holonomies are generated by $W(\mathcal{C})$. The lifted phase $\tilde{\theta} \in \mathbb{R}$ defines a covering map $\text{Cov}: \mathbb{R} \to U(1)$ with $\text{Cov}(\tilde{\theta}) = \mathrm{e}^{-i\tilde{\theta}}$ and winding number $w = \frac{1}{2\pi}\Delta\tilde{\theta}$. The unwrapped temporal parameter t in the external Lorentz sector is identified with $\tilde{\theta}$ and the phase lock.

The diagonal phase lock identifies the universal cover of the locked $U(1)$ phase with the external proper-time coordinate. Let A_{lock} denote a local representative of the locked $U(1)$ connection on L_{lock}. In adapted gauges, the lock may be represented by

$$A_{\text{lock}} \sim \pi_\tau^* \mathcal{A} \sim \pi_4^* A_{\text{Spin}^c}, \tag{52}$$

with locally defined curvature

$$F_{\text{lock}} = dA_{\text{lock}}, \tag{53}$$

while globally the corresponding curvature is the common 2-form determined by the locked Chern class. The external time direction is not an additional independent coordinate. It is the lifted determinant phase of the internal $U(3)$ structure, and $e^m{}_\mu$ and $g_{\mu\nu}(\phi)$ respond to internal energy through the locked Spin^c line.

In the $U(3)$ reduction, $U(3)$ acts on the rank-3 Hermitian bundle $\mathcal{E} \to \mathcal{M}_\tau$ with connection $A \in \Omega^1(\mathcal{M}_\tau, \mathfrak{u}(3))$. The standard isogeny of

$U(3)=\dfrac{SU(3)\times U(1)}{\mathbb{Z}_3}$ is

$$A=A^{traceless}+\frac{1}{3}\operatorname{tr}(A)\mathbf{1}. \tag{54}$$

The internal decomposition

$$1\to SU(3)\to U(3)\xrightarrow{\det}U(1)\to 1 \tag{55}$$

identifies a traceless $SU(3)$ sector as the $SU(3)$ fiber base and a determinant $U(1)$ sector attachment, which is phase-locked to the external Lorentzian $\mathrm{Spin}^c(1,3)$ line.

This identifies the total internal temporal phase, which is the sum of internal eigenphases in $U(3)$ with the external proper time $U(1)$ phase governing Spin^c transport on $\mathcal{M}_4$. There is a single global temporal $U(1)$. The temporal evolution on the physical subspace $\mathcal{H}_\tau$ is a pure fiber rotation,

$$\hat{U}_\tau(t)=\mathrm{e}^{-iE_\tau t}\mathbf{1}_{\mathcal{H}_\tau},\ -i\hat{H}_\tau\in\mathfrak{u}(1), \tag{56}$$

while in general $-i\hat{H}\in\mathfrak{u}(n)$ with $\hat{U}(t)=\mathrm{e}^{-i\hat{H}t}\in U(n)$.

The phase lock organizes the external Lorentzian sector through the chronometric field, $\phi:T\mathcal{M}_4\to\mathbb{R}$, homogeneous quadratic function on the tangent space $T_p\mathcal{M}_4$, such that $\phi_p(v):=g_p(v,v)$ has a Lorentzian signature $(1,3)$. The metric is reconstructed by polarization,

$$g_p(u,v)=-\frac{1}{2}\left[\phi_p(u+v)-\phi_p(u)-\phi_p(v)\right]. \tag{57}$$

Thus, $(\mathcal{M}_4,g(\phi))$ carries the emergent orthonormal structure group $SO(1,3)$. The external Lorentzian metric is reconstructed from the chronometric field ϕ, so that $(\mathcal{M}_4,g(\phi))$ carries proper time along a clock's worldline [9] [11] [38],

$$d\tau^2=g_{\mu\nu}dx^\mu dx^\nu, \tag{58}$$

where operational time measured by clocks in the emergent external spacetime is the proper time induced by the phase-locked metric.

The chronometric field is not an additional matter field independent of the metric. It is the quadratic norm functional of the external Lorentzian block of the phase-locked product geometry. Thus, specifying $\phi_p(v)=-g_p(v,v)$ on each tangent space is equivalent, by polarization, to specifying $g_{\mu\nu}$. Variations may therefore be taken with respect to $g_{\mu\nu}$, or equivalently with respect to ϕ restricted to the open cone of Lorentzian quadratic forms. The Einstein-Hilbert term on the phase-locked product manifold supplies the gravitational dynamics, while the internal temporal sector enters through the effective stress-energy obtained after projection.

The phase lock is a geometric identification rather than an additional phenomenological field. In the present construction, the internal determinant line $L_\tau=\det(E_\tau)$ of the internal $U(3)$ bundle is identified with the auxiliary $U(1)$ line associated with the external $\mathrm{Spin}^c(1,3)$ structure. The minimal consistency condition is therefore the existence of a globally well-defined locked $U(1)$ connection on $\mathcal{M}_7 = \mathcal{M}_4 \times \mathcal{M}_\tau$. Equivalently, the pullbacks of the internal determinant line and the external Spin^c line to $\mathcal{M}_7$ must be isomorphic, with compatible first Chern data and with holonomies that are single-valued on the compact temporal cycles.

This condition is not imposed as an independent anomaly-cancellation postulate. Rather, the locked $U(1)$ sector is part of the defining geometric structure of the theory. The SM-compatible charge assignments are then tested within this locked bundle structure, rather than appended as an arbitrary list of chiral fields. Thus, the relevant consistency requirement for TM is the global existence of the phase-locked determinant connection and its compatibility with the observed representation structure.

7. Temporal Manifold

Working in a locally adapted product chart $\mathcal{M}_7 = \mathcal{M}_4 \times \mathcal{M}_\tau$, the associated locked line bundle is the common temporal $U(1)$ line $L_{\mathrm{lock}} \to \mathcal{M}_7$. The phase lock identifies the determinant $U(1)$ phase of the internal $U(3)$ structure with the auxiliary $U(1)$ phase of the external $\mathrm{Spin}^c(1,3)$ structure. Thus, the external time parameter is not introduced as an independent Abelian degree of freedom, but it is the lifted determinant phase of the locked temporal line.

$$\begin{array}{ccc} \mathcal{G} = U(3) \times_{U(1)} \mathrm{Spin}^c(1,3) & \to & U(3) \\ \downarrow & & \downarrow \det \\ \mathrm{Spin}^c(1,3) & \xrightarrow{\zeta} & U(1) \end{array} \tag{59}$$

Definition: Let $\mathcal{M}_7$ be a seven-dimensional product manifold equipped with the locked structure group $\mathcal{G} = U(3) \times_{U(1)} \mathrm{Spin}^c(1,3)$. Using locally adapted coordinates (x^μ, θ^a), $\mu, \nu = 0,1,2,3$, and $a, b = 1,2,3$, where x^μ are coordinates on the emergent Lorentzian spacetime $\mathcal{M}_4$, and θ^a are coordinates on the compact internal temporal manifold $\mathcal{M}_\tau$, the temporal manifold carries an atlas $\{(U_\alpha, \varphi_\alpha)\}$, with smooth transition functions on overlaps. These local charts allow local trivializations of the internal bundle and therefore support the definition of internal connections, spinor sections, Wilson lines, and curvature forms.

The internal temporal bundle is represented by a principal $U(3)$ bundle

$$U(3) \hookrightarrow P \xrightarrow{\pi} \mathcal{M}_\tau, \quad A \in \Omega^1(P, \mathfrak{u}(3)), \tag{60}$$

where P is the total space, π is the projection, and A is the connection one-form. The corresponding matrix-valued gauge potential is written using Hermitian generators, while the principal connection is represented by its anti-Hermitian form. The curvature encodes how the internal temporal geometry twists across local trivializations, and the Chern classes distinguish topologically inequivalent internal sectors.

Topologically, $\mathcal{M}_\tau$ is compact and orientable. In the benchmark model, it is specialized to $\mathcal{M}_\tau \cong \mathbb{T}^3$, whose nontrivial cohomology supports quantized holonomy and Wilson lines. Nothing in the local construction requires the internal bundle to be globally trivial. Local trivializations exist over chart domains, while transition functions encode the global bundle.

The internal connection (A_C, A_L, A_Y) and their curvatures (F_C, F_L, F_Y) fix the internal temporal energy density and the internal spectrum. Through the locked determinant line, this internal temporal structure sources the external Lorentzian chronometric geometry. The external metric $g_{\mu\nu}(\phi)$ is therefore interpreted as the chronometric response to the phase-locked internal temporal sector.

At zeroth order, TM imposes a rigid product ansatz. This is not a constraint on the full perturbed theory. It is the background about which the coupled theory is expanded. The zeroth-order metric is

$$g^{(7,0)} = g^{(4)}(x) \oplus g^{(\tau)}(\theta), \tag{61}$$

or, in adapted coordinates,

$$ds_0^2 = g^{(4)}_{\mu\nu}(x)\,dx^\mu dx^\nu + g^{(\tau)}_{ab}(\theta)\,d\theta^a d\theta^b. \tag{62}$$

The corresponding zeroth-order connection separates as

$$A^{(0)} = A^{(0)}_\mu(x)\,dx^\mu + A^{(0)}_a(\theta)\,d\theta^a. \tag{63}$$

Equivalently, in the rigid product sector,

$$g^{(0)}_{\mu a} = 0,\ \partial_a g^{(0)}_{\mu\nu} = 0,\ \partial_\mu g^{(0)}_{ab} = 0,$$

$$\partial_a A^{(0)}_\mu = 0,\ \partial_\mu A^{(0)}_a = 0. \tag{64}$$

For this zeroth-order background, the volume form and scalar curvature separate as

$$\sqrt{\left|g^{(7,0)}\right|} = \sqrt{-g_4}\sqrt{g_\tau},$$

$$R\left[g^{(7,0)}\right] = R\left[g^{(4)}\right] + R\left[g^{(\tau)}\right], \tag{65}$$

where $g_4 := \det\left(g^{(4)}_{\mu\nu}\right)$, and $g_\tau := \det\left(g^{(\tau)}_{ab}\right)$. The internal metric $g^{(\tau)}$ is rigid over $\mathcal{M}_4$ at this order, and the internal temporal connection $A^{(0)}_a(\theta)$ determines the zeroth-order internal Dirac spectrum. The determinant component is phase-locked to the external Spin^c line, while the traceless $\mathfrak{su}(3)$ directions carry the non-Abelian internal holonomy.

External spinors are sections of the $\mathrm{Spin}^c(1,3)$ bundle over $\mathcal{M}_4$. Internal temporal spinors are sections over $\mathcal{M}_\tau$. Physical four-dimensional fermion fields arise by expanding product spinors into external fields multiplied by internal temporal eigenmodes. Because $\mathcal{M}_\tau$ is compact, its volume is finite,

$$\kappa_\tau := \int_{\mathcal{M}_\tau} \mathrm{d}^3\theta \sqrt{g_\tau}. \tag{66}$$

In the rigid product sector, the zeroth-order Lagrangian separates as

$$\mathcal{L}^{(0)}(x,\theta) = \mathcal{L}^{(0)}_4(x) + \mathcal{L}^{(0)}_\tau(\theta), \tag{67}$$

with

$$\mathcal{L}_4^{(0)} = \frac{1}{16\pi G_4} R_4 + \mathcal{L}_{S,4}^{(0)}, \tag{68}$$

and

$$\mathcal{L}_\tau^{(0)} = \frac{1}{16\pi G_\tau} R_\tau + \mathcal{L}_{S,\tau}^{(0)}, \tag{69}$$

where $R_4 := R\left[g^{(4)}\right]$, and $R_\tau := R\left[g^{(\tau)}\right]$. The internal curvature term R_τ is retained because the internal geometry and covariant derivative on $\mathcal{M}_\tau$ are part of the temporal spectral mechanism. In the minimal decoupled benchmark $\mathcal{L}_{S,4}^{(0)} = 0$, so that the zeroth-order sources are internal temporal circulation.

The zeroth-order action is

$$\begin{aligned} S^{(0)} &= \int_{\mathcal{M}_4} \mathrm{d}^4x \sqrt{-g_4} \int_{\mathcal{M}_\tau} \mathrm{d}^3\theta \sqrt{g_\tau} \left(\mathcal{L}_4^{(0)}(x) + \mathcal{L}_\tau^{(0)}(\theta) \right) \\ &= \kappa_\tau \int_{\mathcal{M}_4} \mathrm{d}^4x \sqrt{-g_4} \mathcal{L}_4^{(0)}(x) + \int_{\mathcal{M}_4} \mathrm{d}^4x \sqrt{-g_4} \mathcal{L}_{\tau,\mathrm{eff}}^{(0)}. \end{aligned} \tag{70}$$

The internal temporal contribution appears in the four-dimensional effective theory as the source density

$$\mathcal{L}_{\tau,\mathrm{eff}}^{(0)} := \int_{\mathcal{M}_\tau} \mathrm{d}^3\theta \sqrt{g_\tau} \mathcal{L}_\tau^{(0)}(\theta). \tag{71}$$

If the rigid internal background is exactly independent of x, then $\mathcal{L}_{\tau,\mathrm{eff}}^{(0)}$ is constant and behaves as a vacuum-energy-like effective source. When internal temporal modes are projected to four-dimensional fermion fields, the external mode coefficients carry x-dependence, and the effective source inherits this dependence through the projected fields.

With Einstein-Hilbert normalization, the zeroth-order external gravitational action is

$$S_{\mathrm{grav}}^{(0)} = \frac{\kappa_\tau}{16\pi G_4} \int_{\mathcal{M}_4} \mathrm{d}^4x \sqrt{-g_4} R_4. \tag{72}$$

The zeroth-order effective source action becomes

$$S_{\mathrm{src}}^{(0)} = \kappa_\tau \int_{\mathcal{M}_4} \mathrm{d}^4x \sqrt{-g_4} \mathcal{L}_{S,4}^{(0)} + \int_{\mathcal{M}_4} \mathrm{d}^4x \sqrt{-g_4} \mathcal{L}_{\tau,\mathrm{eff}}^{(0)}. \tag{73}$$

Define the effective four-dimensional stress-energy tensor by the local functional derivative

$$T_{\mu\nu}^{(0)}(x) := -\frac{2}{\sqrt{-g_4}} \frac{\delta S_{\mathrm{src}}^{(0)}}{\delta g^{\mu\nu}(x)}. \tag{74}$$

The local metric variation of the zeroth-order effective action gives

$$\frac{1}{\sqrt{-g_4}} \frac{\delta S^{(0)}}{\delta g^{\mu\nu}(x)} = \frac{\kappa_\tau}{16\pi G_4} G_{\mu\nu} - \frac{1}{2} T_{\mu\nu}^{(0)}. \tag{75}$$

Stationarity with respect to $g^{\mu\nu}$ therefore yields

$$G_{\mu\nu} = \frac{8\pi G_4}{\kappa_\tau} T_{\mu\nu}^{(0)}. \tag{76}$$

In the minimal temporal-source sector, where $\mathcal{L}_{S,4}^{(0)} = 0$, this becomes

$$G_{\mu\nu}\left[g(\phi)\right] = \frac{8\pi G_4}{\kappa_\tau} T_{\mu\nu}^{(\tau,0)}\left[F_C, F_L, F_Y, g_\tau, \chi\right], \tag{77}$$

where $T_{\mu\nu}^{(\tau,0)}$ is the effective stress-energy induced by the rigid internal temporal sector after projection through the phase-locked product geometry. Equivalently, the finite internal volume can be absorbed into an effective four-dimensional Newton constant $G_{\text{eff}} := G_4/\kappa_\tau$.

The full coupled theory is organized as a perturbation around the rigid product background with the action,

$$S = S^{(0)} + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + \cdots. \tag{78}$$

The metric and connection are expanded as

$$\begin{aligned}
g_{\mu a}(x,\theta) &= \epsilon h_{\mu a}(x,\theta) + \mathcal{O}\left(\epsilon^2\right), \\
g_{ab}(x,\theta) &= g_{ab}^{(\tau)}(\theta) + \epsilon h_{ab}(x,\theta) + \mathcal{O}\left(\epsilon^2\right), \\
g_{\mu\nu}(x,\theta) &= g_{\mu\nu}^{(4)}(x) + \epsilon h_{\mu\nu}^{\text{mix}}(x,\theta) + \mathcal{O}\left(\epsilon^2\right), \\
A_a(x,\theta) &= A_a^{(0)}(\theta) + \epsilon \delta A_a(x,\theta) + \mathcal{O}\left(\epsilon^2\right), \\
A_\mu(x,\theta) &= A_\mu^{(0)}(x) + \epsilon \delta A_\mu(x,\theta) + \mathcal{O}\left(\epsilon^2\right).
\end{aligned} \tag{79}$$

At zeroth order, the product factorization gives separate external and internal sectors. At first order, the mixed metric components, x-dependent internal metric fluctuations, and x-dependent internal connection fluctuations break the strict separation. Consequently,

$$\begin{aligned}
\sqrt{\left|g^{(7)}\right|} &= \sqrt{-g_4}\sqrt{g_\tau}\left[1 + \frac{\epsilon}{2}\left(g_4^{\mu\nu} h_{\mu\nu}^{\text{mix}} + g_\tau^{ab} h_{ab}\right)\right] + \mathcal{O}\left(\epsilon^2\right), \\
R\left[g^{(7)}\right] &= R\left[g^{(4)}\right] + R\left[g^{(\tau)}\right] + \epsilon \delta R_{\text{mix}} + \mathcal{O}\left(\epsilon^2\right).
\end{aligned} \tag{80}$$

The first-order correction δR_{mix} collects the terms generated by $h_{\mu a}$, $h_{\mu\nu}^{\text{mix}}$, h_{ab}, and their mixed derivatives. Similarly, the gauge curvature develops mixed components

$$F_{\mu a} = \partial_\mu A_a - \partial_a A_\mu + i\left[A_\mu, A_a\right], \tag{81}$$

which vanish in the rigid product background only under the zeroth-order constraints in Equation (64). At first order, these components are generally nonzero and encode the coupling between external propagation and internal temporal geometry.

Using the Lagrangian density with

$$\mathcal{L} = \mathcal{L}^{(0)} + \epsilon \mathcal{L}^{(1)} + \epsilon^2 \mathcal{L}^{(2)} + \cdots, \tag{82}$$

and with a convenient first-order decomposition

$$\mathcal{L}^{(1)} = \mathcal{L}_{S,4}^{(1)} + \delta \mathcal{L}_\tau^{(1)} + \mathcal{L}_{\text{mix}}^{(1)}. \tag{83}$$

Here, $\mathcal{L}^{(1)}_{S,4}$ restores external four-dimensional source structure, $\delta\mathcal{L}^{(1)}_{\tau}$ contains first-order changes in the internal temporal geometry and connection, and $\mathcal{L}^{(1)}_{\text{mix}}$ contains terms that cannot be written as a pure x-term plus a pure θ-term. These mixed terms include contributions from $F_{\mu a}$, $h_{\mu a}$, mixed curvature terms, and x-dependent internal connection fluctuations.

The corresponding first-order correction to the action is

$$S^{(1)} = \kappa_\tau \int_{\mathcal{M}_4} \mathrm{d}^4 x \sqrt{-g_4}\mathcal{L}^{(1)}_{S,4}(x) + \int_{\mathcal{M}_4} \mathrm{d}^4 x \sqrt{-g_4} \int_{\mathcal{M}_\tau} \mathrm{d}^3\theta \sqrt{g_\tau}\left[\delta\mathcal{L}^{(1)}_{\tau}(x,\theta) + \mathcal{L}^{(1)}_{\text{mix}}(x,\theta)\right]. \tag{84}$$

The external spacetime remains noncompact, and the field equations are obtained locally by varying the action density on $\mathcal{M}_4$. The x-dependent fluctuations $\delta A_a(x,\theta)$ in the traceless coset directions contain the Higgs-like connection modes. Their overlaps with internal temporal spinor eigenmodes produce the effective Yukawa-like matrices in the four-dimensional theory. Thus, the zeroth-order rigid temporal sector supplies the rest masses, while first-order perturbations of the temporal connection and metric generate mixing and geometric overlap corrections.

First-principles construction of the action uses external diffeomorphism invariance to select the Einstein-Hilbert term on $\mathcal{M}_4$, internal $U(3)$ gauge invariance to select curvature terms such as $\mathrm{Tr}\left(F_{AB}F^{AB}\right)$, and spin geometry to select Dirac actions with covariant derivatives built from the corresponding external and internal connections. The phase lock fixes the universal temporal $U(1)$, eliminating an independent Abelian redundancy. The rigid ansatz defines the stable zeroth-order compact temporal sector with finite κ_τ, while the perturbations of g_{ab}, A_a, and the mixed fields generate the Higgs-coset dynamics, Yukawa-like overlap operators, and flavor mixing.

8. The Standard Model

In the same way that SO(10) grand unifying theories arrange quarks and leptons into a single 16-dimensional spinor representation [6] [7], TM arranges Lorentz spin and flavor into a single $U(3)\times_{U(1)}\mathrm{Spin}^c(1,3)$ spinor bundle, with internal quantum numbers arising from holonomy in the temporal manifold. The temporal $U(1)$ appearing after locking is the central phase of the full unitary evolution promoted to a geometric holonomy and identified with the internal determinant phase. The internal geometry contains two conjugate rank-3 sectors, identifying one with the color bundle C and the other with the Higgs-electroweak bundle E. The electroweak subgroup is embedded in the traceless part of $U(3)$ acting on E, while the full traceless part of the conjugate sector furnishes $SU(3)_C$.

In TM the electroweak embedding is taken inside the traceless sector of $U(3)$ acting on $\mathcal{E}$. After locking, the trace $U(1)$ is fixed, where the dynamical internal gauge directions lie in $\mathfrak{su}(3)$. From Equation (27) the residual gauge group is $SU(3)$. The chosen holonomy-preserved rank-2 subbundle $V \subset E$, where

$E = V \oplus L_0$ and $\det E = \det V \otimes L_0$. This embeds $SU(2)_L$ as endomorphisms of V and picks a traceless diagonal $U(1)_Y$ orthogonal to both T_3 and the locked trace.

In the $SU(3)$ embedding $SU(2)_L \times U(1)_Y \subset SU(3)$, $SU(2)_L$ acts on the first two components generated by the Gell-Mann matrices $\lambda_1, \lambda_2, \lambda_3$ in the chosen upper-left 2 × 2 block as $T_L^a = \frac{\lambda_a}{2}$, $a = 1,2,3$. Taking the hypercharge as the diagonal generator in $SU(3)$ as $Y = y\,\mathrm{diag}(1,1,-2)$, where y is a normalization constant. Then picking one $SU(2)$, the first two components form an $SU(2)$ doublet and the third component is an $SU(2)$ singlet with hypercharge eigenvalues $Y_1 = Y_2 = y$ and $Y_3 = -2y$. Then, the coset

$$\Pi = \frac{SU(3)}{SU(2)_L \times U(1)_Y} \tag{85}$$

is represented at the Lie algebra level by the generators that are not in $\mathfrak{su}(2)_L$ or Y. The adjoint of $SU(3)$ has 8 generators and 3 of them are $SU(2)_L(\lambda_1, \lambda_2, \lambda_3)$. The one $U(1)_Y$ direction is identified as λ_8. The remaining generators that span the coset are $\lambda_4, \lambda_5, \lambda_6, \lambda_7$. The complex doublet can then be written as

$$\begin{aligned} E_{13} &\sim \lambda_4 + i\lambda_5, \\ E_{23} &\sim \lambda_6 + i\lambda_7, \end{aligned} \tag{86}$$

which rotates under $SU(2)_L(E_{13}, E_{23})$.

In order to compute the hypercharge, consider E_{13}, which mixes $(1,3)$ and E_{23} which mixes $(2,3)$. Acting on the basis $\{e_1, e_2, e_3\}$, E_{13} takes $e_3 \mapsto e_1$ under Y, the phase of e_i is Y_i where $Y(e_1) = y$, $Y(e_2) = y$, $Y(e_3) = -2y$. Then the charge of the raising operator E_{13} is
$Y(E_{13}) = Y(e_1) - Y(e_3) = y - (-2y) = 3y$ and
$Y(E_{23}) = Y(e_2) - Y(e_3) = y - (-2y) = 3y$, where the coset doublet (E_{13}, E_{23}) has the hypercharge $Y = 3y$.

Identifying the Higgs doublet's quantum number as $Y_H = +\frac{1}{2}$ then
$3y = \frac{1}{2} \Rightarrow y = \frac{1}{6}$ giving the expected hypercharge. The Higgs doublet field $\xi(x)$ arises from fluctuations of the internal connection along the coset directions spanned by (E_{13}, E_{23}). This fixes the embedding to $Y = \frac{1}{6}\mathrm{diag}(1,1,-2)$, demonstrating that TM produces the correct hypercharge normalization with $y = 1/6$. The fundamental 3 of $SU(3)$ decomposes under $SU(2)_L \times U(1)_Y$ as the two components with $Y = +1/6$ and a component with $Y = -1/3$, where
$\mathbf{3} \to \mathbf{2}_{+1/6} \oplus \mathbf{1}_{-1/3}$.

After locking, the holonomy-preserved splitting of the rank-3 internal bundle

$$E = V \oplus L_0, \tag{87}$$

where $\mathrm{rank}(V) = 2$ and $L_0 := V^{\perp}$ is a line bundle. Under the Hermitian traceless diagonal generator

$$Y_0 = \frac{1}{6}\operatorname{diag}(1,1,-2),\ Y_{\text{eff}} = Y_0 + \frac{q}{3}. \tag{88}$$

the two summands transform as

$$V \cong \mathbf{2}_{+1/6},\ L_0 \cong \mathbf{1}_{-1/3}. \tag{89}$$

The coset directions are the off-diagonal maps between these summands,

$$H \cong \operatorname{Hom}(L_0, V) \cong V \otimes L_0^*, \tag{90}$$

so the Higgs bundle carries

$$Y(H) = Y(V) - Y(L_0) = \frac{1}{6} - \left(-\frac{1}{3}\right) = +\frac{1}{2}. \tag{91}$$

Thus, the coset mode is an $SU(2)_L$ doublet with the SM Higgs hypercharge.

The lepton doublet and the right-handed charged lepton arise from the same structure once the unique locked temporal line $L = \det(E)$ is used to twist the base $SU(3)$ weights. Normalizing the temporal line so that tensoring by L^q shifts the effective hypercharge by $q/3$, where $Y_{\text{eff}} = Y_0 + \frac{q}{3}$. Since $V^* \cong \mathbf{2}_{-1/6}$, one obtains

$$L_L \cong V^* \otimes L^{-1} \cong \mathbf{2}_{-1/2},\ e_R \cong L_0 \otimes L^{-2} \cong \mathbf{1}_{-1}, \tag{92}$$

and similarly

$$Q_L \cong C \otimes V,\ d_R \cong C \otimes L_0,$$
$$u_R \cong C \otimes L_0^* \otimes L,\ \nu_R \cong L_0 \otimes L. \tag{93}$$

Here C denotes the color triplet bundle. In this way, the full one-generation hypercharge pattern is produced by the traceless $SU(3)$ embedding together with integer powers of the same locked temporal line.

This construction reproduces the one-generation SM hypercharges listed in Table 1. The three generations are then introduced by assigning the three low-lying temporal sectors to three copies of these bundle types. It provides a common geometric origin for the electroweak embedding, the Higgs doublet, and the associated Abelian charge assignments.

The $U(3)$ symmetry of $\mathcal{E}$ carries both the electroweak subgroup and the coset directions that generate the Higgs doublet in the gauge-Higgs unification interpretation. Reducing the unitary structure to $U(2)$ the gauge-Higgs mode disappears, and only the electroweak group can exist. Temporal manifolds larger than 3 result in more Higgs doublets. Generalizing the coset to $SU(2)_L \times U(1)_Y \times SU(n-2) \subset SU(n)$, where $\dim SU(n) = n^2 - 1$, $\dim SU(2)_L = 3$, $\dim SU(n-2) = (n-2)^2 - 1$, and $\dim U(1)_Y = 1$. The subgroup $\Pi_n = SU(2)_L \times U(1)_Y \times SU(n-2)$ has $\dim \Pi_n = 3 + 1 + (n-2)^2 - 1 = (n-2)^2 + 3 = n^2 - 4n + 7$. The coset dimension is then

$$\dim \frac{SU(n)}{SU(2)_L \times U(1)_Y \times SU(n-2)} = (n^2 - 1) - (n^2 - 4n + 7) = 4n - 8 = 4(n-2),$$

where each complex $SU(2)$ doublet has 4 degrees of freedom and $n-2$ Higgs doublets.

Table 1. One-generation bundle assignment in TM. The rank-2 bundle V and line bundle L_0 arise from the decomposition $E = V \oplus L_0$ of the internal temporal bundle, $L = \det(E)$ is the locked temporal line, and C denotes the color triplet bundle. Each SM fermion f transforms in a representation of the internal temporal group $U(3)$, whose Cartan generators T^3 and Y act on the temporal fiber with eigenvalues (t^3, y). Phase locking and the $SU(3)$ embedding identify these internal weights with the external weak isospin and hypercharge, $(T^3, Y) \equiv (t^3, y)$, so that the electric charge is $Q = T^3 + Y$.

Field f	Bundle	$\left(SU(3)_C, SU(2)_L\right)_Y^{T^3}$	Charge Q
(u_L, d_L)	$C \otimes V$	$(\mathbf{3},\mathbf{2})_{+1/6}^{\pm1/2}$	$\left(+\frac{2}{3}, -\frac{1}{3}\right)$
u_R	$C \otimes L_0^* \otimes L$	$(\mathbf{3},\mathbf{1})_{+2/3}^{0}$	$+\frac{2}{3}$
d_R	$C \otimes L_0$	$(\mathbf{3},\mathbf{1})_{-1/3}^{0}$	$-\frac{1}{3}$
(ν_L, e_L)	$V^* \otimes L^{-1}$	$(\mathbf{1},\mathbf{2})_{-1/2}^{\pm1/2}$	$(0, -1)$
e_R	$L_0 \otimes L^{-2}$	$(\mathbf{1},\mathbf{1})_{-1}^{0}$	-1
ν_R	$L_0 \otimes L$	$(\mathbf{1},\mathbf{1})_{0}^{0}$	0

The discovery of a 125 GeV scalar with SM-like couplings by ATLAS and CMS, together with combined measurements of Higgs signal strengths in multiple channels, disfavors large admixtures of additional Higgs doublets or other light scalar multiplets that mix significantly with the observed 125 GeV state. Extended Higgs sectors such as two-Higgs-doublet models remain phenomenologically allowed where extra states are heavy and weakly mixed, but there is at present no positive evidence for additional doublets [39]-[42].

Defining each SM Weyl multiplet [43] with the field, f, enumerates the SM of particles with

$$f \in \{Q_L, u_R, d_R, L_L, e_R, \nu_R\}, \tag{94}$$

where L_L is the other doublet (ν_l, e_l).

After the phase-locked stabilizer selection $U(3) \rightsquigarrow SU(2)_L \times U(1)_Y$ on the electroweak conjugate sector, these internal representations generate the visible fermion multiplets. This operation is not a conventional potential-driven spontaneous symmetry breaking. It is the restriction to the holonomy-preserved rank-(2+1) splitting of the conjugate temporal bundle. The phase lock identifies the external $\mathrm{Spin}^c\, U(1)$ with the internal determinant $U(1)$, so there is a single Abelian connection. The internal electroweak group is therefore the stabilizer of the selected temporal splitting, not an independent gauge factor appended to color.

Table 1 enumerates the SM Weyl multiplets, where ν_R is a true singlet identified as a right sterile neutrino [44]. The SM field content is represented in this construction through bundle assignments derived from the temporal $U(3)$ structure. Since Table 1 reproduces exactly one SM generation of chiral hypercharges, with ν_R neutral, the anomaly cancellations are the usual generation-by-generation SM cancellations. The nontrivial TM assignment is the temporal-bundle origin of these assignments and not a new anomaly algebra.

The electroweak gauge group is the chiral stabilizer acting on the holonomy $V \oplus L_0$ splitting of the temporal bundle. An $SU(2)_L \times U(1)_Y$ subgroup acts on the left-handed doublet directions, while right-handed modes are singlets under $SU(2)_L$. The Higgs field is not a fundamental scalar added to break this group. It is the lowest connection fluctuation in the coset directions of the same temporal splitting, and its effective vacuum behavior is a statement about the selected holonomy background. Because the phase lock identifies the determinant $U(1)$ with the external Spin^c line, TM contains no independent additional Abelian factor. The construction, therefore, yields the observed gauge group

$SU(3)_C \times SU(2)_L \times U(1)_Y$ in the effective theory, but with no separate light Z', no mirror color copy, and no freely adjustable scalar potential in the fundamental description.

The SM discussion should not append the SM as an external quantum field theory, nor should it claim a conventional bottom-up reconstruction of the SM from an unrelated set of assumptions. The fundamental variational structure in TM is kinetic and geometric. It is built from the Einstein-Hilbert term on the phase-locked temporal geometry, internal curvature terms, and Dirac operators coupled to the corresponding connections. SM-like mass, Higgs, and Yukawa terms are effective sectors arising from rigid and flexible perturbations of the internal temporal geometry. In this sense, the observed SM structure is tested for compatibility with the geometric perturbation theory of the locked internal bundle.

Conventional symmetry-breaking terminology must therefore be translated with care. TM does not introduce an arbitrary scalar potential whose minimum breaks a fundamental gauge symmetry. A four-dimensional effective potential may be written after projection or after integrating out temporal fluctuations, but it is a derived functional of the kinetic geometry composed of curvature, Wilson-line holonomy, radiative corrections, compact temporal volume, and internal-mode overlaps. The Higgs-like sector is therefore closer in spirit to gauge-Higgs/Hosotani constructions than to a theory in which a scalar potential is freely added by hand, but TM further constrains the construction by the phase lock and by the conjugate-dual fiber selection rules.

9. Temporal Mechanics

The $\mathcal{M}_4$ covariant derivative acts on external fields induced after projection of temporal modes, whereas the M_τ covariant derivative acts on internal temporal sections and defines the spectral problem whose eigenvalues supply the rest masses.

After locking, the internal gauge dynamics reside in the traceless $\mathfrak{su}(3)$ sector, while the shared $U(1)$ governs universal time transport. Let γ^μ be a four-dimensional Clifford representation on $\mathcal{M}_4$ obeying

$$\left\{\gamma^\mu, \gamma^\nu\right\} = 2g_4^{\mu\nu}, \tag{95}$$

and define the external chirality operator

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3,\ \left(\gamma^5\right)^2 = \mathbf{1},\ \left\{\gamma^5, \gamma^\mu\right\} = 0. \tag{96}$$

Let γ^a be a three-dimensional Euclidean Clifford representation on $\mathcal{M}_\tau$,

$$\left\{\gamma^a, \gamma^b\right\} = 2g_\tau^{ab}. \tag{97}$$

The graded product Clifford generators

$$\Gamma^\mu = \gamma^\mu \otimes \mathbf{1},\ \Gamma^a = \gamma^5 \otimes \gamma^a, \tag{98}$$

so that $\left\{\Gamma^\mu, \Gamma^\nu\right\} = 2g_4^{\mu\nu}$, $\left\{\Gamma^a, \Gamma^b\right\} = 2g_\tau^{ab}$, and $\left\{\Gamma^\mu, \Gamma^a\right\} = 0$. The appearance of γ^5 in Γ^a reflects the Lorentzian grading on $L^2\left(\mathcal{S}_{(1,3)}\right)$.

Defining the total Dirac operator algebraically by

$$D_7 = i\Gamma^A\nabla_A = i\Gamma^\mu\nabla_\mu + i\Gamma^a\nabla_a, \tag{99}$$

then the Dirac operator on the product of the external Lorentz spin geometry with the internal temporal spectral geometry, with the covariant derivatives split as

$$\nabla_\mu = \partial_\mu + \omega_\mu + i\sum_{k\in R_f} q_k A_\mu^{(k)}(x) T_k, \tag{100}$$

$$\nabla_a = \partial_a + \omega_a + iA_a(\theta), \tag{101}$$

where $A_\mu^{(k)}$ are the four-dimensional gauge fields obtained by gauging the appropriate internal symmetries over $\mathcal{M}_4$, and $A_a(\theta)$ is the rigid internal temporal connection. The field produced by the causal response to the fermion is then summed over its representation R_f. After locking, the determinant component of $A_a(\theta)$ is fixed by the shared $U(1)$ time phase, so the dynamical internal directions lie in $\mathfrak{su}(3)$.

The internal connection $A_a(\theta)$ is Hermitian in the local physics convention. Thus $\partial_a + \omega_a + iA_a$ is the covariant derivative on the internal temporal spinor bundle, while $D_\tau = -i\gamma^a\left(\partial_a + \omega_a + iA_a\right)$ is the corresponding self-adjoint internal Dirac operator on the compact Riemannian temporal sector, subject to the chosen spin structure and boundary conditions. This Hermiticity is internal to the spectral problem on $\mathcal{M}_\tau$. The external Lorentzian causal structure is instead controlled by the $\mathrm{Spin}^c(1,3)$ operator on $\mathcal{M}_4$ and by the phase-locked chronometric field.

With the representation in Equation (98), the operator splits as

$$D_7 = D_4 \otimes \mathbf{1} - \gamma^5 \otimes D_\tau, \tag{102}$$

where

$$D_4 = i\gamma^\mu\left(\partial_\mu + \omega_\mu + i\sum_{k\in R_f} q_k A_\mu^{(k)}(x) T_k\right),$$

$$D_\tau = -i\gamma^a \left(\partial_a + \omega_a + iA_a(\theta)\right). \tag{103}$$

Lorentz chirality is governed by γ^5 and the projectors are

$$P_{L,R} = \frac{1}{2}\left(1 \mp \gamma^5\right) \tag{104}$$

on the external spinor bundle.

Assume a separated expansion

$$\Psi(x,\theta) = \sum_n \psi_n(x) \otimes \chi_n(\theta), \tag{105}$$

where the internal modes satisfy

$$D_\tau \chi_n = E_n \chi_n, \ \langle \chi_n, \chi_m \rangle_\tau = \delta_{nm}. \tag{106}$$

Substitution into the Dirac equation $D_7 \Psi = 0$ and projection onto χ_n yields the effective four-dimensional equation

$$\left(D_4 - E_n \gamma^5\right)\psi_n(x) = 0. \tag{107}$$

E_n is the spectral mass set by the internal temporal geometry. More invariantly, E_n is the singular value of the internal temporal mass operator after restriction to an allowed paired temporal sector. TM does not introduce this mass through an external Yukawa parameter. The internal temporal Dirac operator defines a bundle morphism between the conjugate left- and right-handed temporal sectors selected by $E \oplus \bar{E}$, the locked determinant line, and the bundle assignments in Table 1.

In the four-dimensional chiral notation, this morphism is represented as a mass operator acting between left- and right-handed fields, but its magnitude is fixed by the spectrum of D_τ. The Higgs-coset and Yukawa-like overlap operators describe the gauge-covariant orientation, mixing, and perturbative deformation of this spectral operator; they are not the fundamental source of the zeroth-order rest mass. The factor γ^5 in Equation (107) is the grading operator required by the Clifford product $\Gamma^a = \gamma^5 \otimes \gamma^a$. The zeroth-order masses are spectral and are fixed by the eigenvalue of the internal temporal Dirac operator D_τ.

The conjugate temporal decomposition $E \oplus \bar{E}$ supplies mirrored left- and right-handed Lorentz projections after locking. The Hermitian-dual pairing of the conjugate temporal sectors gives the same internal singular spectrum for the paired modes. Thus, the four-dimensional rest-mass parameter associated with the paired temporal eigenmode is

$$m_n = |E_n|. \tag{108}$$

The internal eigenvalues can be expressed as expectation values of the internal Dirac operator. With χ normalized on $\mathcal{M}_\tau$ with a flat internal frame,

$$E_\tau = \int_{\mathcal{M}_\tau} \mathrm{d}^3\theta \sqrt{g_\tau}\, \chi^{?}(\theta)\left(-i\gamma^a\left(\partial_a + iA_a\right)\right)\chi(\theta). \tag{109}$$

After pairing the conjugate temporal modes and projecting onto the positive mass branch of the internal singular spectrum, the four-dimensional effective action takes the usual massive Dirac form,

$$S_4 = \int_{\mathcal{M}_4} \mathrm{d}^4 x \sqrt{-g_4}\,\bar{\psi}(x)\left[i\gamma^\mu\left(\partial_\mu + i\sum_{k\in R_f} q_k A_\mu^{(k)} T_k\right) - m_n\right]\psi(x). \tag{110}$$

The origin of this mass is the internal temporal spectral geometry, including the internal connection, spin structure, holonomy, and curvature of $\mathcal{M}_\tau$. $m_n = |E_n|$ is not a Yukawa-generated parameter. It is the singular value of the temporal Dirac mass operator obtained from the rigid internal temporal geometry. The Yukawa structures enter at the next order, when the rigid temporal background is perturbed. They do not replace the spectral origin of mass. Instead, the geometric overlap integrals generated by fluctuations of A_a deform the zeroth-order spectral mass operator.

This spectral use of the internal Dirac operator is related in spirit to noncommutative spectral geometry, where a Dirac operator encodes metric, gauge, Higgs, and fermions. In the Chamseddine-Connes spectral action, the SM action coupled to gravity is recovered from spectral data of an almost-commutative geometry [45]. TM differs from the internal spectral sector by an ordinary compact temporal geometry rather than a finite noncommutative space, and the zeroth-order rest masses are identified with the spectrum of the temporal Dirac operator, while Yukawa-like structures arise from perturbative overlap operators.

This provides a geometric mass generation mechanism in which rest masses arise as the spectrum of the internal temporal Dirac operator. The geometry of $\mathcal{M}_\tau$, its spin structure, the locked determinant line, and the traceless $SU(3)$ holonomy together control the allowed internal modes and therefore the observable four-dimensional mass spectrum. In this way, gauge interactions, mass generation, and the Lorentzian chirality are unified by a single phase-locked temporal geometry.

10. Mass

TM does not prove the uniqueness of $\mathbb{T}^3$, dynamically derive the generation-dependent matrices B_n, compute the full Wilson-line effective potential, prove quantum consistency of the locked background, or establish stability against all metric and connection perturbations. These are dynamical problems not presented here as part of this geometric construction.

For a compact three-dimensional temporal manifold, $\mathcal{M}_\tau = \mathbb{T}^3$, the second cohomology $H^2(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z}^3$ has three independent generators represented by the three coordinate $\mathbb{T}^2$ subtori. These three primitive 2-cycles motivate a three-sector organization of the low-lying temporal holonomies. The three distinguished low-lying sectors are identified with the three observed generations. After phase locking removes the trace direction, the remaining traceless Cartan data of the internal $U(3)$ holonomy distinguish these sectors and set their zeroth-order spectrum.

The choice $\mathcal{M}_\tau = \mathbb{T}^3$ is the minimal toroidal ansatz. Once the temporal sector is taken to have a split rank-three structure or equivalent $U(3,3)$ temporal structure with conjugate rank-three unitary sectors, then the SM compatibility requires rank-three internal holonomy with one rank-three sector carrying color and one rank-three sector whose reduction contains $SU(2)_L \times U(1)_Y$ and the Higgs coset. The compact temporal base must therefore support three independent commuting holonomy directions in the Cartan limit. Among smooth compact three-manifolds, $\mathbb{T}^3$ is the minimal choice with $\pi_1(\mathbb{T}^3) = \mathbb{Z}^3$, $H^1(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z}^3$, and $H^2(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z}^3$. This supplies three independent Wilson-line phases and three primitive dual two-cycle classes.

The choice $\mathcal{M}_\tau \simeq \mathbb{T}^3$ is not asserted here as a proven uniqueness theorem. It is the minimal compact temporal choice used in the present construction. A three-torus provides three independent compact temporal cycles, supports Wilson-line holonomies, and matches the three-generation structure without introducing additional internal dimensions, curvature, or fixed-point structure. A single circle $\mathbb{S}^1$ or a two-torus $\mathbb{T}^2$ would not supply the three independent compact cycles used in this construction, whereas $\mathbb{T}^n$ with $n > 3$, orbifolds, curved compact three-manifolds, or higher-dimensional compactification spaces would introduce additional geometric structure requiring separate physical justification.

The three observed generations are then identified with the three lowest temporal holonomy sectors of this rank-three compact geometry. This identification is not an arbitrary enumeration of generations, but it is the low-energy SM assignment selected after imposing the $U(3,3)$ temporal structure, the phase lock, and the observed gauge representation.

Higher temporal sectors are not excluded mathematically. They correspond to heavier internal Dirac modes or projected sectors and must either decouple, lie above present experimental scales, or be removed by additional boundary/orbifold selection rules. TM uses the $\mathbb{T}^3$ model as the minimal smooth benchmark. Throughout this section, $n = 1, 2, 3$ labels the observed generation sector, whereas $n_a \in \mathbb{Z}$ labels the integer winding component around the a^{th} fundamental cycle of $\mathbb{T}^3$.

TM assigns an internal Hilbert space $\mathbb{H}_\tau^{(f)}$ of spinor sections on $\mathcal{M}_\tau$ transforming in a representation R_f of the internal $U(3)$ bundle, where f is the fermion index. The internal Dirac operator in this sector is

$$D_\tau^{(f)} = -i\gamma^a \left(\partial_a + \omega_a + iA_a^{(f)}(\theta) \right),$$

$$A_a^{(f)} = A_a^A(\theta) T_A^{(R_f)}, \tag{111}$$

where $T_A^{(R_f)}$ are the generators of $\mathfrak{u}(3)$ in the representation R_f. Internal eigenmodes satisfy

$$D_\tau^{(f)} \chi_{n,i}^{(f)}(\theta) = E_{n,i}^{(f)} \chi_{n,i}^{(f)}(\theta),$$

$$\left\langle \chi_{n,i}^{(f)}, \chi_{m,j}^{(f)} \right\rangle_\tau = \delta_{nm}\delta_{ij}, \tag{112}$$

and the corresponding four-dimensional fermions have rest masses

$$m_{n,i}^{(f)} = \left| E_{n,i}^{(f)} \right|. \tag{113}$$

Here, n is the index labeling the three generations, and i labels the species within a generation, where the spectrum of $D_\tau^{(f)}$ with different f and i sample different eigenvalues. The eigenvalues are determined by the internal geometry and the $U(3)$ representation R_f, where Table 1 shows the allowed temporal structures of the SM. It is convenient to arrange the charged fermions against (n,i) as

$$f = \begin{pmatrix} e & u & d \\ \mu & c & s \\ \tau & t & b \end{pmatrix}. \tag{114}$$

Charged fermions exist as the lowest twisted plane waves [46] compatible with a twisted boundary condition having stable eigenmodes of $D_\tau^{(f)}$ in each sector, which are aligned along inequivalent Cartan directions of $U(3)$.

Specializing to the semi-flat torus polarization $\mathcal{M}_\tau = \mathbb{T}^3$ that supplies the holonomy and choosing a lattice basis $\{\ell_a\}_{a=1}^3 \subset \mathbb{R}^3$ for Λ, which represents points on $\mathbb{T}^3$ by $\theta \in \mathbb{R}^3$ with the identification

$$\theta \sim \theta + 2\pi\ell_a. \tag{115}$$

Let $\{\ell^a\}$ denote the dual basis, $\ell^a \cdot \ell_b = \delta^a{}_b$. Define the 2π-periodic scalar coordinates along the cycles by

$$\begin{gathered} \vartheta^a(\theta) = \ell^a \cdot \theta \\ \Downarrow \\ \vartheta^a(\theta + 2\pi\ell_b) = \vartheta^a(\theta) + 2\pi\delta^a{}_b. \end{gathered} \tag{116}$$

The rest masses are universal over $\mathcal{M}_4$ and the metric on $\mathbb{T}^3$ is taken to be

$$g_{ab} = g(\ell_a, \ell_b), \ (g^{ab}) = (g_{ab})^{-1}. \tag{117}$$

For each fermion type f, the background of commuting Wilson lines are in the internal $U(3)$ bundle. In a commuting Cartan background, the connection components are taken to be constant and diagonalizable, where

$$\begin{gathered} A^{(f)} = \frac{1}{2\pi}\Theta_a^{(f)} d\vartheta^a, \\ \Theta_a^{(f)} = \mathrm{diag}\left(\Theta_a^{(f,1)}, \Theta_a^{(f,2)}, \Theta_a^{(f,3)}\right), \end{gathered} \tag{118}$$

and the phases $\Theta_a^{(f,i)}$ are the eigenphases of the holonomy in the representation R_f around the a^{th} fundamental cycle. The Wilson line around that cycle is

$$W_a^{(f)} = \exp\left(-i\Theta_a^{(f)}\right) = \mathrm{diag}\left(\mathrm{e}^{-i\Theta_a^{(f,1)}}, \mathrm{e}^{-i\Theta_a^{(f,2)}}, \mathrm{e}^{-i\Theta_a^{(f,3)}}\right). \tag{119}$$

The constant Cartan connection and the twisted boundary condition are two gauge-equivalent descriptions of the same flat temporal bundle. In a periodic trivialization, the wavefunctions are periodic, and the flat connection appears explicitly in the covariant derivative. In a twisted-boundary trivialization, the flat connection is gauged locally to zero, and the same holonomy appears in the transition functions of the wavefunctions. The spectrum is independent of which representative is used.

In the periodic trivialization, one writes

$$D_{\tau}^{(f)} = -i\gamma^{a}\left(\partial_{a} + iA_{a}^{(f)}\right), \ \chi^{(f,i)}\left(\theta + 2\pi\ell_{a}\right) = \chi^{(f,i)}\left(\theta\right), \tag{120}$$

where, in the commuting Cartan background,

$$A^{(f)} = \frac{1}{2\pi}\Theta_{a}^{(f)} d\vartheta^{a}. \tag{121}$$

Equivalently, after the local gauge transformation that removes the flat connection from the derivative, one uses the twisted-boundary trivialization

$$D_{\tau}^{(f)} = -i\gamma^{a}\partial_{a}, \ \chi^{(f,i)}\left(\theta + 2\pi\ell_{a}\right) = \mathrm{e}^{-i\Theta_{a}^{(f,i)}}\chi^{(f,i)}\left(\theta\right), \tag{122}$$

with $\omega_{a} = 0$ in this flat background. The following plane-wave calculation is performed in this twisted-boundary representative.

For a plane-wave mode on the covering space,

$$\chi^{(f,i)}\left(\theta\right) = \exp\left(ik_{a}^{(f,i)}\vartheta^{a}\left(\theta\right)\right)u^{(f,i)}, \tag{123}$$

the boundary condition gives

$$\mathrm{e}^{i2\pi k_{a}^{(f,i)}} = \mathrm{e}^{-i\Theta_{a}^{(f,i)}} \Leftrightarrow k_{a}^{(f,i)} = n_{a} - \frac{\Theta_{a}^{(f,i)}}{2\pi}, \ n_{a} \in \mathbb{Z}. \tag{124}$$

Thus, the holonomy enters the spectrum once, either through the explicit flat connection in the periodic gauge or through the quasi-periodic boundary condition in the twisted gauge. In the twisted-boundary representative used, the corresponding internal Dirac eigenvalues satisfy

$$E^{2} = g^{ab}k_{a}k_{b}. \tag{125}$$

This is the temporal analog of a Scherk-Schwarz mass shift. In the usual compact spatial setting, a boundary twist $\Phi\left(y + 2\pi R\right) = \mathrm{e}^{2\pi i\alpha}\Phi\left(y\right)$ shifts the KK momentum by $n + \alpha$. In TM, the twist is temporal holonomy, $\chi\left(\theta + 2\pi\ell_{a}\right) = \mathrm{e}^{-i\Theta_{a}}\chi\left(\theta\right)$, so the internal temporal momentum is shifted to $k_{a} = n_{a} - \frac{\Theta_{a}}{2\pi}$, and the corresponding rest mass is the internal spectral norm $m^{2} = g^{ab}k_{a}k_{b}$.

This use of quasi-periodic boundary conditions is structurally analogous to Scherk-Schwarz compactification, where fields acquire phases under transport around compact directions and the corresponding shifted momenta appear as mass splittings in the lower-dimensional theory [47] [48]. In TM, the same mathematical mechanism is interpreted temporally, where the twist is a temporal Wilson-line phase, and the shifted compact momentum is an internal temporal Dirac eigenvalue.

Choosing a Cartan basis $\left\{\Phi_{I}\right\}_{I=1}^{3}$ for $\mathrm{u}(3)$ and then the constant Cartan connection can be written as

$$A_{a} = \frac{1}{2\pi}\Upsilon_{a}^{I}\Phi_{I}, \ \left[\Phi_{I}, \Phi_{J}\right] = 0, \tag{126}$$

The Wilson loop around ℓ_{a} is then

$$W_a = \mathcal{P}\exp\left(-i\oint_{\ell_a} A\right) = \exp\left(-i\Upsilon^I_a\Phi_I\right) \in U(3). \tag{127}$$

Let R_f be a representation of $U(3)$ and $|w\rangle$ a simultaneous eigenstate of the Cartan generators,

$$\Phi_I |w\rangle = w_I |w\rangle. \tag{128}$$

Then $|w\rangle$ is an eigenvector of each Wilson loop with eigenvalue

$$W_a |w\rangle = \mathrm{e}^{-i\Upsilon^I_a w_I} |w\rangle,$$

$$\Theta_a(w) = \Upsilon^I_a w_I \,(\mathrm{mod}\,2\pi). \tag{129}$$

For a fixed set of type covectors $k_i \in \mathbb{R}^3$, the three observed families correspond to three distinguished Cartan-weight vectors $w_{n,i} \in \mathbb{R}^3$. Any linear rule $w_{n,i} = S_n k_i$ with $S_n \in \mathrm{GL}(3)$ induces the factorized holonomy shifts

$$\frac{\Theta^{(n,i)}}{2\pi} = B_n k_i \,(\mathrm{mod}\,1), \quad B_n = \frac{1}{2\pi}\Upsilon S_n \in \mathbb{R}^{3\times 3}. \tag{130}$$

For the charged fermions $n_a = 0$, where the allowed internal momenta are

$$k_a^{(n,i)} = -\left(B_n k_i\right)_a, \tag{131}$$

and the corresponding masses are therefore

$$m_{n,i}^2 = g^{ab} k_a^{(n,i)} k_b^{(n,i)} = \left(B_n k_i\right)^{\mathrm{T}} g^{-1}\left(B_n k_i\right) = k_i^{\mathrm{T}} g_n k_i, \tag{132}$$

where $g_n = B_n^{\mathrm{T}} g^{-1} B_n$.

Koide's charged-lepton relation [49] admits a geometric interpretation as a fixed angle between the square-root mass vector and the trace direction, along with a discussion of the relation and its radiative-stability problem [50] [51]. The internal metric g^{-1} then defines the species-dependent generational Gram matrix

$$\mathcal{G}^{(i)}_{nm} := \left(B_n k_i\right)^{\mathrm{T}} g^{-1}\left(B_m k_i\right). \tag{133}$$

Its diagonal entries are precisely the squared rigid spectral masses,

$$\mathcal{G}^{(i)}_{nn} = m_{ni}^2. \tag{134}$$

Thus $\mathcal{G}^{(i)}$ contains both the mass norms and the relative generation geometry sampled by the charged species i. The normalized Gram matrix

$$\mathcal{C}^{(i)}_{nm} := \frac{\mathcal{G}^{(i)}_{nm}}{\sqrt{\mathcal{G}^{(i)}_{nn}\mathcal{G}^{(i)}_{mm}}} \tag{135}$$

gives the cosine of the internal temporal angle between the n^{th} and m^{th} generation vectors for that species.

The Koide relation is not imposed on the primitive vectors directly. It is a trace functional of the square-root diagonal spectrum induced by the Gram matrix. Define

$$R_i := \begin{pmatrix} \left(\mathcal{G}^{(i)}_{11}\right)^{1/4} \\ \left(\mathcal{G}^{(i)}_{22}\right)^{1/4} \\ \left(\mathcal{G}^{(i)}_{33}\right)^{1/4} \end{pmatrix} = \begin{pmatrix} \sqrt{m_{1i}} \\ \sqrt{m_{2i}} \\ \sqrt{m_{3i}} \end{pmatrix}. \tag{136}$$

In a rank-three temporal $U(3)$ phase basis, the determinant line is generated by the trace direction proportional to $(1,1,1)$. Since the phase lock identifies this determinant $U(1)$ with the external Spin^c temporal line, the normalized locked trace direction in generation-amplitude space is $d_{\det} := \frac{1}{\sqrt{3}}(1,1,1)^{\mathrm{T}}$. The corresponding Koide trace functional is therefore

$$K_i := \frac{\sum_{n=1}^{3}\left(\mathcal{G}_{nn}^{(i)}\right)^{1/2}}{\left[\sum_{n=1}^{3}\left(\mathcal{G}_{nn}^{(i)}\right)^{1/4}\right]^2}. \tag{137}$$

Equivalently, if θ_i is the angle between R_i and the locked trace direction $d_{\det}$, then $K_i^{-1} = 3\cos^2\theta_i$, and the charged-lepton Koide condition is the specialization $i = e$, $K_e = \frac{2}{3}$.

In TM, this is interpreted as a trace-balance condition on the charged-lepton Gram spectrum, where the square-root spectral vector R_e lies at $\theta_e = \pi/4$ from the locked determinant direction. Equivalently, writing $R_e = R_{e,\parallel} + R_{e,\perp}$, $R_{e,\parallel} := (R_e \cdot d_{\det})d_{\det}$, and $R_{e,\perp} \cdot d_{\det} = 0$, the Koide value $K_e = 2/3$ is equivalent to $\|R_{e,\parallel}\| = \|R_{e,\perp}\|$. Thus, the phase lock supplies the meaning of the conventional democratic direction, where it is the normalized determinant/trace direction of the rank-three temporal $U(3)$ structure. Koide is then a minimal charged-lepton constraint on the diagonal spectrum of the TM generational Gram matrix, not an independent Yukawa postulate.

In the commuting Cartan background, the internal $U(3)$ connection has gauge-invariant Wilson loops around the three non-contractible cycles. These Hosotani-Wilson [4] phases may be equivalently implemented as quasi-periodic boundary conditions for the internal wavefunctions. The holonomy generation-dependent linear map $B_n \in \mathrm{GL}(3)$ acting on type covectors $k_i \in \mathbb{R}^3$ defines the boundary twist phases, which are a generation-dependent momentum $k^{(n,i)} = -B_n k_i$ on a rigid manifold $(\mathbb{T}^3, g)$, or as a family of generation-dressed, symmetric, positive-definite forms $\{g_n\}$ acting on common type covectors $\{k_i\}$.

This packaging of the Hosotani holonomy phases for charged fermions by restricting to the ground state winding provides an effective mass law, and compactification on a rigid $\mathbb{T}^3$ with commuting $U(3)$ Wilson loops in which rest masses are internal Dirac eigenvalues, and the species phase vector is dressed by a generation-dependent mixing.

Table 2 and **Table 3** summarize one benchmark realization of the spectral ansatz. The target masses in **Table 2** are phenomenological mass inputs, not predictions of the SM mass. The fitted matrices and vectors used to obtain the benchmark are given in the Appendix. **Table 3** shows a benchmark normal-hierarchy fit using the same internal metric that generated **Table 2**. After fixing the observed splittings and choosing a discrete set of integer core vectors $\{\boldsymbol{k}_n\}$, TM yields mass eigenvalues near $m_1 = 0.026420\ \mathrm{eV}$, $m_2 = 0.027785\ \mathrm{eV}$, and $m_3 = 0.056551\ \mathrm{eV}$, with total mass $\sum_i m_i \simeq 0.111\,\text{eV}$.

In particular, the effective generation forms g_n are decomposed there as $g_n = \lambda_n^2 \bar{g}_n$, with $\det \bar{g}_n = 1$. This decomposition makes explicit that the benchmark contains both a generation-scale hierarchy and anisotropic holonomy-dressing. The present fit should therefore be read as a reproducible benchmark of the spectral mechanism, with no dynamical principle that fixes g, B_n, and the integer neutrino cores.

Table 2. Benchmark fit to charged-fermion input masses. The second column lists measured or running mass inputs at the stated renormalization convention. The third column gives the TM benchmark values obtained from the internal metric, type covectors, and generation matrices B_n. They are phenomenological inputs used to test whether the temporal holonomy mass law can reproduce the observed hierarchy with a controlled parameter structure.

Fermion	Input (GeV)	TM (GeV)
e	0.000510999	0.000510987
u	0.002160000	0.002160110
d	0.004700000	0.004700050
μ	0.105658000	0.105657000
c	1.273000000	1.272990000
s	0.093500000	0.093501400
τ	1.776930000	1.776860000
t	172.5700000	172.5680000
b	4.183000000	4.183070000

The neutrino sector is distinguished in TM by allowing a nontrivial integer winding sector that cores the dressed lepton covector down to a small residual. Introduce an integer core vector $\boldsymbol{k}_n \in \mathbb{Z}^3$ for each generation and define the residual neutrino momentum by

$$\boldsymbol{k}_{\nu,n} = B_n \boldsymbol{k}_i - \boldsymbol{k}_n. \tag{138}$$

Because the charged-fermion fit fixes the dimensionless temporal manifold $\left(g^{-1}, B_n, \boldsymbol{k}_i\right)$ only up to an overall scale, the neutrino sector introduces a physical calibration constant α_ν converting the internal residual norm into an observed neutrino mass,

$$m_{\nu,n} = \alpha_\nu \sqrt{\boldsymbol{k}_{\nu,n}^{\mathrm{T}} g^{-1} \boldsymbol{k}_{\nu,n}} = \alpha_\nu \left\| \boldsymbol{k}_{\nu,n} \right\|_{g^{-1}}. \tag{139}$$

The observed splittings fix α_ν once a discrete choice of integer cores $\{\boldsymbol{k}_n\}$ is made

$$\Delta m_{21}^2 = \alpha_\nu^2 \left(\left\| \boldsymbol{k}_{\nu,2} \right\|_{g^{-1}}^2 - \left\| \boldsymbol{k}_{\nu,1} \right\|_{g^{-1}}^2 \right),$$

$$\Delta m_{31}^2 = \alpha_\nu^2 \left(\left\| \boldsymbol{k}_{\nu,3} \right\|_{g^{-1}}^2 - \left\| \boldsymbol{k}_{\nu,1} \right\|_{g^{-1}}^2 \right). \tag{140}$$

Table 3. Fit to the known neutrino mass splitting.

Neutrino	Input (eV^2)	TM (eV^2)
Δm^2_{21}	0.000074	0.000074
Δm^2_{31}	0.002500	0.002500
NH	n	TM (eV)
ν_1	(0, −1, 0)	0.026420
ν_2	(1, −1, 1)	0.027785
ν_3	(−2, 1, 0)	0.056551

Table 3 shows the fit obtained using the same metric as in Table 2. For the discrete winding choice and calibration fixed by the measured normal-hierarchy splittings, TM yields a lightest neutrino mass of approximately 0.026 eV. In the Cartan quantization, the lowest-lying mode is $(0,-1,0)$, while the two heavier neutrinos correspond to higher winding sectors. The total neutrino mass is approximately 0.111 eV.

Neutrinos are constructed as spectral modes of the internal temporal Dirac operator with independent left- and right-handed components paired by the same phase-locked temporal geometry with $\nu_R \sim L_0 \otimes L$. The fermion bundle assignments in Table 1 include a sterile right-handed neutrino mode, and the mass term is therefore Dirac in character. A Majorana mass term would require an invariant pairing of a neutrino mode with its charge-conjugate mode, along with a violation of the locked temporal $U(1)$ charge structure. Therefore, in this ansatz, neutrinos are Dirac fermions and lepton number is conserved at the level of the spectral mass construction.

This is a deliberate contrast with the usual Majorana mechanisms for small neutrino mass. The seesaw idea introduces heavy right-handed neutrino degrees of freedom or equivalent high-scale lepton-number violation to suppress the observed light-neutrino masses [52]-[54]. In effective field theory, the corresponding low-energy Majorana mass is represented by the dimension-five Weinberg operator [55]. The minimal TM ansatz does not include such a self-conjugate temporal pairing, where a Majorana term would require an additional lepton-number-violating identification of a neutrino mode with its charge-conjugate temporal sector, or a controlled breaking of the locked temporal $U(1)$.

This gives a geometric selection rule. Processes that require a state to be identical to its own conjugate-dual flow are absent or non-generic in the minimal phase-locked theory. The most direct implication is that neutrinos are naturally Dirac rather than Majorana particles in the minimal construction. A conventional Majorana identification would require a neutrino flow to be equivalent to its own conjugate-dual circulation, which is not generic when identical flow occupancy of the dual conjugate fibers is excluded. Neutrinoless double-beta decay, therefore, provides a direct phenomenological test of the minimal dual-holonomy ansatz.

In the four-dimensional effective language, the photon and Z arise as the usual

Cartan mixtures after the electroweak stabilizer and Higgs-connection background have been selected. In the fundamental TM description, this is not driven by an independently postulated scalar potential. The Higgs doublet is the lowest scalar excitation of the internal connection along the coset directions Π, with its effective mass, self-coupling, and vacuum response controlled by the internal gauge kinetic term, holonomy, radiative corrections, and the volume of $\mathbb{T}^3$.

Gauge-inert spinor modes in trivial representations of the internal temporal bundle are dark-matter candidates only if their masses, lifetimes, production mechanism, free-streaming length, and self-annihilation or decay channels satisfy cosmological and halo-structure constraints. This is distinct from dark energy, which would require an approximately homogeneous vacuum contribution with an equation of state close to -1. TM identifies possible temporal sectors whose cosmological viability must be tested by relic-abundance, CMB, BBN, structure-formation, and halo calculations.

11. Mixing and Higgs Dynamics

The previous section treated a commuting Cartan background on $\mathbb{T}^3$, which is sufficient to extract a zeroth-order spectral mass law. The present section relaxes that restriction and allows generic non-Abelian holonomy. Once the Wilson loops around independent cycles fail to commute, the internal Dirac operators in different sectors are no longer simultaneously diagonalizable, and the resulting misalignment appears in four dimensions as CKM and PMNS mixing. Thus, the commuting background controls leading-order masses, while noncommuting holonomy and coset fluctuations control flavor mixing and Higgs-induced off-diagonal structure.

The same distinction applies to Higgs and flavor dynamics. The effective four-dimensional theory may be written using familiar language of Higgs potentials, vacuum expectation values, and Yukawa matrices, but in TM, these are not independent primitive structures. They are the projected description of a connection on the phase-locked temporal bundle. A Higgs vacuum corresponds to a stable holonomy/stabilizer background of the internal connection. A Yukawa matrix is an overlap operator induced by fluctuations of that connection. Flavor mixing is therefore not an arbitrary set of independent unitary rotations. It is constrained by the same noncommuting internal temporal holonomy that controls the mass-generating spectral operator.

In the Cartan plane-wave limit, the Yukawa-like overlap integral reduces to a temporal momentum matching condition. The Higgs-coset connection mode must carry the internal temporal momentum difference between the left and right eigenmodes. Thus, Yukawa entries are kinetic overlap selection rules of the phase-locked temporal connection, not arbitrary fundamental constants. This same phase-locked dual-holonomy structure should also constrain particle-antiparticle phase relations. CP-violating phases are not expected to be arbitrary independent parameters, but should arise from the mismatch between allowed circulation sectors under the global phase lock. Once the internal connection is fixed, the construction implies correlations among the CKM phase, the PMNS phase, and the mass-generating holonomies. Failure of these correlations would falsify the minimal dual-holonomy realization.

Let $\mathcal{C} \subset \mathbb{T}^3$ be a closed loop and define the temporal Wilson line $W^{(f)}(\mathcal{C}) = \mathcal{P}\exp\left(-i\oint_{\mathcal{C}} A^{(f)}\right) \in U(3)$, where $A \in \Omega^1\left(\mathcal{M}_\tau, \mathfrak{u}(3)\right)$ is the fixed internal connection. Each SM fermion sector f carries a representation $R_f : U(3) \to U\left(\mathbb{H}_\tau^{(f)}\right)$, with the induced sector connection $A^{(f)} = R_{f*}(A)$, and holonomy $W^{(f)}(\mathcal{C}) = R_f\left(\mathcal{P}\mathrm{e}^{-i\oint_{\mathcal{C}} A}\right) \in U\left(\mathbb{H}_\tau^{(f)}\right)$. For flavor, a generic $U(3)$ background on $\mathbb{T}^3$ admits noncommuting holonomies for loops $\mathcal{C}_1, \mathcal{C}_2$. The matrices $W(\mathcal{C}_1)$ and $W(\mathcal{C}_2)$ need not commute, so there is, in general, no global basis that diagonalizes the temporal parallel transport.

In TM, this non-Abelian holonomy is processed through the sector representations R_f and through the sector's temporal Dirac operators $D_\tau^{(f)}$, whose eigenmodes define the internal wavefunctions $\chi^{(f)}(\theta)$ that enter the effective four-dimensional theory. In the rigid Cartan limit, the fermion mass operator is already present as the internal temporal spectral operator. For each sector f, write the zeroth-order mass matrix in the temporal eigenbasis as

$$M_f^{(0)} = \mathrm{diag}\left(\left|E_1^{(f)}\right|, \left|E_2^{(f)}\right|, \left|E_3^{(f)}\right|\right). \tag{141}$$

When the rigid temporal background is perturbed by x-dependent coset fluctuations of the internal connection, the mass operator receives geometric overlap corrections,

$$M_f^{\mathrm{eff}} = M_f^{(0)} + \epsilon \delta M_f + \mathcal{O}\left(\epsilon^2\right). \tag{142}$$

In Standard-Model notation, the perturbative correction may be parameterized as

$$\delta M_f = \frac{v}{\sqrt{2}} Y_f, \tag{143}$$

but in TM Y_f is not a fundamental source of mass. It is the overlap matrix induced by perturbations of the internal temporal connection.

The physical mass eigenstates are obtained by biunitary diagonalization

$$U_L^{(f)\dagger} M_f U_R^{(f)} = \mathrm{diag}\left(m_{f_1}, m_{f_2}, m_{f_3}\right), \tag{144}$$

where the neutrinos are just appended to Equation (114) as $f = \{u, d, e, \nu\}$, and only the left-handed matrices $U_L^{(f)}$ enter charged currents. Therefore, the observable mixing matrices are the relative left-handed basis mismatches between paired sectors,

$$U_{\mathrm{PMNS}} = U_L^{(e)\dagger} U_L^{(\nu)}, \; V_{\mathrm{CKM}} = U_L^{(u)\dagger} U_L^{(d)}. \tag{145}$$

In TM, the matrices $U_L^{(f)}$ are not free inputs. They are determined by the same underlying temporal holonomy of A evaluated in the representation R_f and restricted to the relevant left-handed subbundle selected by the $SU(2)_L$

embedding together with the sector-dependent spectrum and eigenmodes of $D_\tau^{(f)}$. Because the background holonomies are noncommuting, the induced effective Yukawa matrices need not be simultaneously diagonalizable across sectors, and CKM-PMNS [56]-[58] mixing emerges as a geometric consequence of a single internal $U(3)$ connection.

Within TM, the Higgs doublet is interpreted as the lowest scalar excitation associated with fluctuations of the internal $U(3)$ connection along the coset directions of the electroweak embedding described at the end of Section 6. Concretely, the internal gauge field is expanded about the locked background as

$$A_a(x,\theta) = A_a^{(bg)}(\theta) + \delta A_a(x,\theta),$$
$$\delta A_a(x,\theta) = \delta A_a^{(\text{unbroken})}(x,\theta) + \xi(x)\xi_a(\theta) + \xi^\dagger(x)\xi_a^\dagger(\theta) + \cdots, \tag{146}$$

where $\xi_a(\theta)$ takes values in the coset generators, so that $\xi(x)$ transforms as an $SU(2)_L$ doublet with hypercharge +1/2. The locking ansatz is imposed on the background fields, while Higgs dynamics arise from small x-dependent fluctuations $\delta A_a(x,\theta)$ in the traceless coset directions. These fluctuations generate mixed field strength components of linear order,

$$F_{\mu a} = \partial_\mu A_a - \partial_a A_\mu + i[A_\mu, A_a], \tag{147}$$

and their four-dimensional kinetic term descends from the higher-dimensional Yang-Mills [59] action

$$S = -\frac{1}{4g_\tau^2}\int \mathrm{d}^4x\mathrm{d}^3\theta\sqrt{-g_4 g_\tau}\,\mathrm{Tr}\,\mathcal{F}_{AB}\mathcal{F}^{AB}. \tag{148}$$

Separating background and fluctuations as

$$A_\mu = A_\mu^{(\mathrm{bg})}(x) + \delta A_\mu(x,\theta),$$
$$A_a = A_a^{(\mathrm{bg})}(\theta) + \delta A_a(x,\theta), \tag{149}$$

the mixed curvature expands to first order as

$$\mathcal{F}_{\mu a} = \partial_\mu \delta A_a - D_a^{(\mathrm{bg})}\delta A_\mu + \cdots, \tag{150}$$

and inserting the KK-style mode decomposition of Equation (146) into $\int \mathrm{d}^3\theta\,\mathrm{Tr}\,\mathcal{F}_{\mu a}\mathcal{F}^{\mu a}$ yields the Higgs kinetic term

$$\left|D_\mu\xi\right|^2, D_\mu\xi = \left(\partial_\mu + i\sum_{k\in EW} q_k A_\mu^{(k)}(x)T_k^{(H)}\right)\xi. \tag{151}$$

providing the geometric origin of $(D_\mu\xi)^\dagger(D^\mu\xi)$, and in the standard way with an $SU(2)_L\times U(1)_Y$ basis, $A_\mu^{(k)}T_k^{(H)} = gW_\mu^a T_a^{(H)} + g'B_\mu Y_H$, with $Y_H = +1/2$.

In gauge-Higgs unification, the quartic interaction is inherited from the non-Abelian curvature in the purely internal sector, $\mathrm{Tr}\,F_{ab}F^{ab} \supset \mathrm{Tr}[A_a, A_b]^2$. Once the coset mode is projected onto its normalized internal profile $\xi_a(\theta)$, and its effective coupling is set by g_τ. The Higgs mass term is controlled by the same geometry as in standard gauge-Higgs unification, where "the Higgs boson is identified with the fifth component of a gauge field" [5], which arise from the holonomy and radiative effects consistent with the higher-dimensional gauge symmetry. TM treats the Higgs/electroweak as an internal component of the $U(3)$ temporal connection with its dynamics emerging from the time-bundle curvature rather than from an independent scalar sector.

The same identification also ties Yukawa couplings directly to geometry. Starting from the fermion action

$$S = \int \mathrm{d}^4 x \mathrm{d}^3\theta \sqrt{-g_4 g_\tau}\, \bar{\Psi}\left(i\Gamma^\mu D_\mu + i\Gamma^a D_a\right)\Psi, \tag{152}$$

with

$$\begin{aligned} D_\mu &= \partial_\mu + \omega_\mu + iA_\mu(x,\theta), \\ D_a &= \partial_a + \omega_a + iA_a(x,\theta), \end{aligned} \tag{153}$$

and the product representation $\Gamma^\mu = \gamma^\mu \otimes \mathbf{1}$ and $\Gamma^a = \gamma^5 \otimes \gamma^a$ with the internal gauge interaction contains

$$S_{\text{int}} \supset \int \mathrm{d}^4 x \sqrt{-g_4} \int \mathrm{d}^3\theta \sqrt{g_\tau}\, \bar{\Psi}(x,\theta)\, i\Gamma^a A_a(x,\theta)\Psi(x,\theta). \tag{154}$$

Expanding the fermions into four-dimensional fields times temporal eigenmodes,

$$\Psi(x,\theta) \sim \sum_i \psi_{L,i}^{(f)}(x)\chi_{L,i}^{(f)}(\theta) + \sum_j \psi_{R,j}^{(f)}(x)\chi_{R,j}^{(f)}(\theta) + \cdots, \tag{155}$$

and inserting the coset fluctuation $\delta A_a(x,\theta) \supset \xi(x)\xi_a(\theta) + \xi^\dagger(x)\xi_a^\dagger(\theta)$ gives

$$\begin{aligned} S_{\text{int}} = \int \mathrm{d}^4 x \sqrt{-g_4} \sum_{i,j} \bar{\psi}_{L,i}^{(f)}(x)\xi(x)\psi_{R,j}^{(f)}(x) \\ \left[\int \mathrm{d}^3\theta \sqrt{g_\tau}\, \chi_{L,i}^{(f)\dagger}(\theta)\left(i\Gamma^a\right)\xi_a(\theta)\chi_{R,j}^{(f)}(\theta)\right] + \cdots. \end{aligned} \tag{156}$$

which reduces to the four-dimensional Yukawa interaction

$$S_{\text{int}} = \int \mathrm{d}^4 x \sqrt{-g_4} \sum_{i,j} \bar{\psi}_{L,i}^{(f)}(x)\xi(x)\psi_{R,j}^{(f)}(x)\left(Y_f\right)_{ij} + \cdots, \tag{157}$$

with the geometric Yukawa matrix given by the overlap integral

$$\left(Y_f\right)_{ij} = i\int_{\mathcal{M}_\tau} \mathrm{d}^3\theta \sqrt{g_\tau}\, \chi_{L,i}^{(f)\dagger}(\theta)\Gamma^a \xi_a(\theta)\chi_{R,j}^{(f)}(\theta). \tag{158}$$

This recovers the Yukawa Lagrangian,

$$\mathcal{L}_Y^{(f)} = -\left(Y_f\right)_{ij} \bar{\psi}_{L,i}^{(f)}(x)\xi(x)\psi_{R,j}^{(f)}(x) - \left(Y_f\right)_{ij}^* \bar{\psi}_{R,j}^{(f)}(x)\xi^\dagger(x)\psi_{L,i}^{(f)}(x), \tag{159}$$

and makes explicit why flavor structure is not independent in TM. This object is called a Yukawa matrix only after projection to the four-dimensional effective description. A Yukawa entry exists only if the Higgs-coset connection carries the internal temporal momentum needed to connect the left and right temporal modes.

In TM, the first-order geometric overlap operator is produced by perturbing the rigid temporal connection. The dominant rest masses are supplied by the zeroth-order internal Dirac spectrum, while the overlap matrix $\left(Y_f\right)_{ij}$ controls the mixing of the spectral mass operator. Thus, flavor hierarchies arise from the combined data of the rigid temporal spectrum and the perturbative overlap geometry, rather than from arbitrary external Yukawa parameters.

12. Conclusions

TM is not an attempt to quantize spacetime directly. It is a proposal that quantum phase, proper time, gauge charge, and rest mass arise from a common internal temporal geometry whose determinant phase is locked to the external Spin^c time structure. In this sense, TM changes what is taken as primitive. Rather than beginning with fields on a fixed or dynamical spacetime and then adding internal quantum numbers, it begins with phase-locked temporal holonomy and treats spacetime geometry, gauge structure, and mass spectra as external manifestations of that internal temporal organization.

The central construction is the locked temporal line. The determinant line of the internal $U(3)$ temporal bundle, the auxiliary $U(1)$ line of the external $\mathrm{Spin}^c(1,3)$ structure, the Schrödinger phase, and the proper-time phase are not treated as independent Abelian structures. They are identified as realizations of one underlying temporal holonomy. This phase lock removes an otherwise redundant Abelian sector and gives a geometric meaning to the relation between internal phase evolution and external clock time. Proper time is then the external Lorentzian expression of the same compact temporal phase that appears internally as a quantum phase.

The generalized temporal bundle $\mathcal{E}_\tau = TM_\tau \oplus T^*M_\tau$ supplies the split-signature structure from which the conjugate rank-three complex sectors $E \oplus \bar{E}$ arise. Their determinant lines are Hermitian duals, and their Abelian holonomies are opposite orientations of the same locked temporal circle. The two sectors do not introduce two independent $U(1)$ gauge factors, and their traceless parts remain available for non-Abelian gauge structure.

TM assigns one member of this conjugate pair to the color sector and retains its full traceless $SU(3)$ structure. The conjugate partner is then reduced by a holonomy-preserved splitting $E = V \oplus L_0$, $\mathrm{rank}(V) = 2$, and $\mathrm{rank}(L_0) = 1$. The stabilizer of this rank-2 + 1 decomposition inside the traceless sector is locally $SU(2)_L \times U(1)_Y$, while the off-diagonal coset directions transform as an electroweak doublet with $Y_H = \frac{1}{6} - \left(-\frac{1}{3}\right) = \frac{1}{2}$. Thus, the SM gauge structure is not introduced as three unrelated gauge groups. It is obtained from the conjugate temporal sectors after the common determinant $U(1)$ has been locked, where one sector carries unbroken color, and the other carries the electroweak stabilizer with the Higgs coset.

In this framework, the Higgs field is not an independent scalar appended to the theory. It is the lowest connection fluctuation along the broken coset directions of the rank-2 + 1 temporal splitting. The four-dimensional Higgs kinetic term descends from the mixed curvature components of the higher-dimensional gauge action, while the quartic and mass terms are controlled by the internal gauge curvature, holonomy, radiative corrections, and the compact temporal volume. The same internal connection also produces Yukawa structures, where effective Yukawa matrices arise as overlap integrals of internal temporal eigenmodes with the

Higgs-profile fluctuation. Flavor hierarchies and complex phases are therefore interpreted as geometry of the temporal connection rather than as arbitrary external parameters.

Rest mass is described spectrally. After separation of variables on $\mathcal{M}_4 \times \mathcal{M}_\tau$, the internal temporal Dirac operator produces eigenvalues that enter the effective four-dimensional Dirac equation as masses. In the commuting Cartan limit on $\mathbb{T}^3$, these eigenvalues can be written in terms of Wilson-line phases, temporal momenta, and the internal metric. The benchmark charged-fermion and neutrino fits presented should therefore be read as evidence of compatibility and a parametrization of the spectral mechanism, not as a final derivation of the observed mass spectrum. The physical content of the construction is that masses are controlled by internal holonomy and the temporal Dirac spectrum.

The three observed generations are associated with distinguished low-lying temporal sectors of the compact internal geometry. On $\mathbb{T}^3$, the existence of three primitive two-cycles motivates a natural three-sector organization of the lowest holonomy classes. It is a structural identification that the three observed generations are assigned to the three lowest stable temporal sectors. Higher sectors, if present, would correspond to heavier or otherwise inaccessible excitations and must be dynamically suppressed or lifted by the internal geometry.

Neutrinos arise from the same internal temporal Dirac construction as the other fermions. The bundle assignment includes a sterile right-handed neutrino mode, so the minimal TM mass term is Dirac in character. A Majorana mass term would require an invariant self-conjugate pairing of a neutrino temporal mode with its charge-conjugate mode, or an additional lepton-number-violating temporal identification. Such a structure is not part of the minimal phase-locked TM ansatz. Therefore, within the present construction, neutrinos are Dirac fermions, and lepton number is conserved at the level of the spectral mass mechanism.

The external Lorentzian metric is interpreted as the macroscopic geometric response to internal temporal energy and holonomy. The chronometric field reconstructs the external metric, and the resulting spacetime geometry obeys the usual diffeomorphism-governed description in the effective four-dimensional limit. Gravitational dynamics are therefore not discarded or replaced. Rather, TM proposes that the source of external geometry is the internal temporal organization whose spectral and gauge structures also give matter its observed quantum numbers.

The falsifiable content of the present framework is not merely the absence of extra compactification states. TM imposes several linked selection rules. First, particle-antiparticle conjugation is a constrained fiber-swap operation inside a phase-locked dual $U(3)$ holonomy, not an independent mirror copy of the particle spectrum. Second, color and electroweak structure coexist as conjugate flows on the same compact temporal manifold. The observed product gauge group is their projected commuting action, not evidence for unrelated orthogonal gauge spaces. Third, the Higgs and Yukawa sectors must be derivable from the temporal connection rather than from freely chosen scalar potentials and arbitrary flavor matrices. Fourth, Majorana self-conjugate neutrino flow is absent in the minimal locked ansatz unless an additional lepton-number-violating temporal identification is introduced.

These constraints give concrete ways to challenge the minimal realization. It would be disfavored or ruled out by an unavoidable independent mirror-family spectrum, an extra light Abelian gauge boson associated with a second unlocked $U(1)$, a confirmed minimal Majorana neutrino mechanism such as neutrinoless double-beta decay, or flavor and CP data that cannot be embedded in any common noncommuting locked temporal connection. Conversely, the minimal TM pattern is supported if neutrinos remain Dirac, no light mirror color or extra Z' sector appears, Higgs self-interactions remain compatible with a derived connection/holonomy origin, and future predictive versions of the internal connection correlate charged-fermion masses, CKM mixing, PMNS mixing, and CP phases rather than treating them as unrelated inputs.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix: Benchmark Fit Parameters

This appendix gives the numerical data used in the benchmark charged-fermion and neutrino fits discussed in Section 10. The purpose of the appendix is reproducibility and diagnostic transparency. It shows that the phase-locked spectral ansatz has sufficient structure to reproduce the observed charged-fermion hierarchy and a normal-ordering neutrino spectrum once the holonomy dressings are specified.

Operationally, the charged-sector fit minimizes a logarithmic residual over the nine charged-fermion masses, while the neutrino stage fixes the residual winding scale and integer cores against the two measured normal-ordering mass-squared splittings. Thus, the charged-fermion masses and neutrino splittings are inputs to the benchmark. The displayed TM masses, the lightest-neutrino mass, and the total neutrino mass are outputs of this fitted realization. A dynamical selection process is not presented.

The benchmark should not be read as a uniqueness proof for the observed mass hierarchy. The observed charged-fermion masses and neutrino mass-squared splittings are treated as target data for this benchmark. The fitted quantities are the species covectors, the generation-dependent dressed quadratic forms, and the residual integer cores used in the neutrino sector. The predictive content of the larger framework would arise from additional restrictions on the allowed holonomy dressings, the dual-fiber phase lock, and the admissible flavor-mixing structure. The data below are rounded for readability. The determinants, scale factors, and determinant-normalized matrices are computed from the full-precision benchmark values.

The benchmark is organized as a two-stage fit. In Stage 1, the input observables are the charged-fermion masses listed in **Table 2**, m_{ni}^{obs}, $n=1,2,3$, $i \in \ell,u,d$, where n labels generation and i labels charged-lepton, up-type-quark, and down-type-quark species. The fitted charged-sector quantities are the three shared species covectors $k_\ell, k_u, k_d \in \mathbb{R}^3$, and the three generation-dependent dressed quadratic forms g_1, g_2, g_3. Using the charged-sector mass law of Equation (132), the logarithmic charged-sector objective is

$$\mathcal{O}_{\text{ch}} = \sum_{n=1}^{3} \sum_{i \in \ell,u,d} \left[\log\left(\frac{m_{ni}^{\text{TM}}}{m_{ni}^{\text{obs}}} \right) \right]^2 . \tag{160}$$

A logarithmic objective is used because the charged-fermion masses span many orders of magnitude. The Stage 1 charged-only objective value for the benchmark is $\mathcal{O}_{\text{ch}} = 2.08335 \times 10^{-5}$.

In the representative convention used for the benchmark, choose $g^{-1} = I$ and take B_n to be the symmetric positive square-root representative satisfying $B_n^{\text{T}} B_n = g_n$. The effective neutrino residual is the integer-cored winding vector defined in Equation (138), evaluated here with the lepton covector k_ℓ and the displayed representative B_n. The physical neutrino masses are then obtained from the residual norm using the scale relation in Equation (139).

For each candidate integer-core triplet, the benchmark chooses the scale Λ_ν by least squares against the two normal-ordering mass-squared splittings. In the code implementation, this is done by minimizing

$$\mathcal{O}_\nu = \left(\Delta m_{21}^{2,\mathrm{TM}} - \Delta m_{21}^{2,\mathrm{obs}}\right)^2 + \left(\Delta m_{31}^{2,\mathrm{TM}} - \Delta m_{31}^{2,\mathrm{obs}}\right)^2, \tag{161}$$

with the splittings evaluated in common physical units. The charged masses and neutrino mass-squared splittings are reproduced from the target data in this appendix. They should not be counted as independent predictions of the benchmark.

The three shared species covectors are

$$k_\ell = (-0.08072, -0.34339, 0.15021),$$

$$k_u = (-6.56131, 30.75623, -0.80207),$$

$$k_d = (-0.74011, -0.77700, 1.42029).$$

The three effective generation quadratic forms are

$$g_1 = \begin{pmatrix} 3.1776\times10^{-7} & 3.6731\times10^{-8} & -1.0783\times10^{-6} \\ 3.6731\times10^{-8} & 4.2902\times10^{-9} & -1.4083\times10^{-7} \\ -1.0783\times10^{-6} & -1.4083\times10^{-7} & 9.5646\times10^{-6} \end{pmatrix},$$

$$\det(g_1) = 3.27422\times10^{-30},$$

$$g_2 = \begin{pmatrix} 1.91852 & 0.49779 & 1.19601 \\ 0.49779 & 0.12916 & 0.31032 \\ 1.19601 & 0.31032 & 0.74691 \end{pmatrix},$$

$$\det(g_2) = 4.19907\times10^{-17},$$

and

$$g_3 = \begin{pmatrix} 10.85846 & -8.91886 & 4.25944 \\ -8.91886 & 26.90018 & -4.45881 \\ 4.25944 & -4.45881 & 2.32062 \end{pmatrix},$$

$$\det(g_3) = 128.09740.$$

The matrices B_n are therefore not unique, since $B_n \mapsto Q_n B_n$ with $Q_n^{\mathrm{T}} Q_n = I$ leaves g_n unchanged. For reproducibility, we choose the canonical symmetric positive square-root representative $B_n = g_n^{1/2}$. For the benchmark matrices used above, this gives

$$B_1 = \begin{pmatrix} 4.72496665\times10^{-4} & 5.10518952\times10^{-5} & -3.0314644\times10^{-4} \\ 5.10518952\times10^{-5} & 5.53901879\times10^{-6} & -4.0660249\times10^{-5} \\ -3.0314644\times10^{-4} & -4.0660249\times10^{-5} & 3.07751192\times10^{-3} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1.15547853 & 0.29980349 & 0.70250045 \\ 0.29980349 & 0.07778793 & 0.18227721 \\ 0.70250045 & 0.18227721 & 0.46922943 \end{pmatrix},$$

and

$$B_3 = \begin{pmatrix} 2.99384132 & -1.0451595 & 0.89611224 \\ -1.0451595 & 5.04762262 & -0.5738670 \\ 0.89611224 & -0.5738670 & 1.09008315 \end{pmatrix}.$$

These matrices satisfy $B_n^{\mathrm{T}} B_n = g_n$ to numerical precision. Thus, the reported g_n matrices are the invariant data of the benchmark fit, while the displayed B_n matrices provide one explicit admissible representative frame.

Only the dressed form g_n enters the charged-sector mass law in Equation (132). The factorization implicit in that equation is not unique without an additional choice of internal frame, where transformations of the internal frame leave the invariant dressed forms g_n unchanged. For this reason, the benchmark is specified primarily by the invariant dressed quadratic forms g_n, together with the shared species covectors k_i.

If a representative convention is desired, one may choose $g^{-1} = I$ and take any matrix square root B_n satisfying $B_n^{\mathrm{T}} B_n = g_n$. This representative choice fixes a convenient gauge for displaying B_n, but it does not change the charged-sector masses or the physical content of the benchmark. In this representative gauge, the quantities may be taken as $B_n = g_n^{1/2}$, where $g_n^{1/2}$ is the positive symmetric square root computed from the full-precision dressed form. Any other displayed B_n related by an internal frame rotation gives the same invariant matrices g_n and the same mass table. For this reason, TM reports the frame-invariant g_n as the primary reproducible data and gives an explicit representative convention for reconstructing one admissible set of B_n.

The raw determinants of the dressed quadratic forms span many orders of magnitude,

$$\det(g_1) = 3.27422 \times 10^{-30},$$

$$\det(g_2) = 4.19907 \times 10^{-17},$$

$$\det(g_3) = 1.28097 \times 10^{2}.$$

g_n contains both scale and shape as

$$g_n = \lambda_n^2 \bar{g}_n,\ \det(\bar{g}_n) = 1. \tag{162}$$

For a 3 × 3 matrix, this gives

$$\lambda_n = \det(g_n)^{1/6},\ \bar{g}_n = \frac{g_n}{\det(g_n)^{1/3}}. \tag{163}$$

The benchmark scale factors are therefore

$$\lambda_1 = 1.21857 \times 10^{-5},$$

$$\lambda_2 = 1.86434 \times 10^{-3},$$

$$\lambda_3 = 2.24521.$$

With this decomposition, the charged-sector mass law becomes

$$m_{ni} = \lambda_n \sqrt{k_i^{\mathrm{T}} \bar{g}_n k_i}, \tag{164}$$

where λ_n carries the gross generation scale and $\overline{g}_n$ carries the determinant-normalized anisotropic shape of the holonomy dressing.

Using the determinants above, the determinant-normalized matrices are

$$\overline{g}_1 \simeq \begin{pmatrix} 2.1399\times10^{3} & 2.4736\times10^{2} & -7.2613\times10^{3} \\ 2.4736\times10^{2} & 2.8892\times10^{1} & -9.4843\times10^{2} \\ -7.2613\times10^{3} & -9.4843\times10^{2} & 6.4412\times10^{4} \end{pmatrix},$$

$$\overline{g}_2 \simeq \begin{pmatrix} 5.51970\times10^{5} & 1.43216\times10^{5} & 3.44098\times10^{5} \\ 1.43216\times10^{5} & 3.71596\times10^{4} & 8.92812\times10^{4} \\ 3.44098\times10^{5} & 8.92812\times10^{4} & 2.14890\times10^{5} \end{pmatrix},$$

$$\overline{g}_3 \simeq \begin{pmatrix} 2.15405 & -1.76928 & 0.844965 \\ -1.76928 & 5.33632 & -0.884515 \\ 0.844965 & -0.884515 & 0.460353 \end{pmatrix}.$$

The large entries in $\overline{g}_1$ and $\overline{g}_2$ show that the benchmark does not contain only an overall generation scale. It also contains strong anisotropies in the first two generations of dressing forms. Thus, the observed charged-fermion hierarchy is represented by both the scale hierarchy in Equation (164) and the anisotropic shapes of the determinant-normalized forms.

The Stage 2 neutrino benchmark uses the same charged-sector structure but allows a residual integer-cored winding vector for each neutrino generation, as defined in Equation (138). The neutrino mass is then obtained from the residual norm using Equation (139). In the following, the physical neutrino scale is

$$\Lambda_\nu = 2.64298\times10^{-2}\ \text{eV}. \tag{165}$$

The small value of this scale in GeV units is not a small dimensionless coupling. It is the conversion from a dimensionless internal residual norm to the physical neutrino mass scale.

The Stage 2 integer cores and residual vectors are

$$k_{\text{core},1} = (0,-1,0),$$

$$k_{\nu,1}^{\text{eff}} = \left(-0.000355, 0.999633, 7.235378\times10^{-8}\right),$$

$$k_{\text{core},2} = (1,-1,1),$$

$$k_{\nu,2}^{\text{eff}} = (-0.142431,-0.551845,-0.883386),$$

$$k_{\text{core},3} = (-2,1,0),$$

$$k_{\nu,3}^{\text{eff}} = (1.89448,-0.99455,0.00018).$$

The integer cores are fitted residual windings selected from a bounded integer search around the nearest metric-whitened lattice vectors. They are only low-residual integer-core representatives for this benchmark. The benchmark contains two visible hierarchies. First, the generation scale factors in Equation (164) satisfy $\lambda_1 \ll \lambda_2 \ll \lambda_3$. Second, the determinant-normalized matrices $\overline{g}_1$ and $\overline{g}_2$ are highly anisotropic. These features are not numerical artifacts. They are the way this benchmark encodes the observed charged-fermion hierarchy within the spectral ansatz.

Several layers of the construction should therefore be distinguished. The phase lock, the rigid product ansatz, and the choice $\mathcal{M}_\tau \simeq \mathbb{T}^3$ are structural assumptions of the present model. The SM-compatible representation assignments follow from the selected $U(3)$ splitting and locked determinant line. The rest-mass formula follows from the internal Dirac spectrum in the rigid Cartan limit. The numerical charged-fermion and neutrino data in this appendix are benchmark fits demonstrating compatibility of the spectral ansatz with the observed mass pattern. They do not by themselves prove the uniqueness of the scale hierarchy, the anisotropic shape of the dressed quadratic forms, or the integer cores.

The benchmark fit above should be understood as the semi-flat Cartan limit of a more general variational principle. In that limit, the relevant gauge-invariant background data are the conjugacy classes of the Wilson loops around the three fundamental cycles of $\mathbb{T}^3$, as in Equation (127). For a field in representation R with Cartan weight w, the internal temporal momenta are shifted by the corresponding holonomy phase according to the same shifted-momentum rule as Equation (124). The fluctuation spectrum around the background depends on Θ through

$$\lambda_{n,w}^2(\Theta) = g^{ab}\left(n_a - \frac{\Theta_a(w)}{2\pi}\right)\left(n_b - \frac{\Theta_b(w)}{2\pi}\right) + \cdots. \tag{166}$$

Integrating out the temporal gauge, Higgs-coset, and matter fluctuations gives a Wilson-line effective potential of the schematic form

$$V_{\mathrm{eff}}(\Theta) = V_{\mathrm{cl}}(\Theta) + \frac{1}{2}\sum_{\mathrm{bosons}} d_b \operatorname{Tr}\log\Delta_b(\Theta) - \frac{1}{2}\sum_{\mathrm{fermions}} d_f Tr\log\Delta_f(\Theta), \tag{167}$$

with the Θ-independent divergent pieces absorbed into local counterterms. The remaining finite part is a periodic class function of the Wilson-loop phases and is the temporal analog of the Hosotani effective potential. The physical Cartan background is selected by $\frac{\partial V_{\mathrm{eff}}}{\partial \Theta_a^I} = 0$, modulo large gauge transformations and Weyl identifications. The stationary holonomy phases $\Theta_\star$ determine the twisted temporal momenta by the shifted-momentum rule of Equation (124), evaluated at $\Theta = \Theta_\star$.

In the present appendix, the full effective-potential calculation is not performed. Instead, the fitted dressed forms g_n, species covectors k_i, and residual integer cores are used as benchmark data for the semi-flat Cartan limit. This is sufficient to test whether the spectral mass law can represent the observed charged and neutrino hierarchies.