Journal of Modern Physics, 2026, 17(8), 929-939
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

Solid Angle and the Fine-Structure Constant: A Geometric View of Hydrogen Length Scales

Zaki Harari

Novelant Scientific Research Inc., Toronto, Canada
Email: zaki.harari@novelant.com

How to cite this paper: Harari, Z. (2026) Solid Angle and the Fine-Structure Constant: A Geometric View of Hydrogen Length Scales. *Journal of Modern Physics*, **17**, 929-939.
https://doi.org/10.4236/jmp.2026.178041

Received: June 5, 2026
Accepted: August 14, 2026
Published: August 17, 2026

Abstract

The Lyman-limit wavelength $\lambda_{\mathrm{Ly}}^{(\infty)} \equiv 1/R_\infty$ (defined for infinite nuclear mass) and the Bohr radius a_0 are two of the most basic length scales in atomic physics. We show that their ratio satisfies the exact identity

$$\frac{\lambda_{\mathrm{Ly}}^{(\infty)}}{a_0} = \frac{4\pi}{\alpha} = \frac{\Omega_{\mathrm{sph}}}{\alpha},$$

where $\alpha \approx 1/137$ is the fine-structure constant and $\Omega_{\mathrm{sph}} = 4\pi$ sr is the solid angle of a complete sphere. We write λ_{Ly} for $\lambda_{\mathrm{Ly}}^{(\infty)}$ throughout, and $\lambda_{\mathrm{Ly}}^{(\mathrm{real})}$ where the finite-proton-mass correction is discussed. The identity follows from CODATA 2022 definitions in four lines of algebra. The constant α is the ratio of the electron's ground-state speed to c; a factor 2π arises from $h = 2\pi\hbar$; a factor 2 arises from the Coulomb virial theorem, $\langle T \rangle = -E_{\mathrm{total}}$. These two factors are *structurally* identical to the azimuthal and polar integrals whose product gives the solid angle of a sphere. We stress that this is a *formal, convention-dependent analogy*: it does not imply angular integration over the hydrogen wavefunction or the photon's emission pattern, and it is tied to expressing photon energy as $h\nu$ rather than $\hbar\omega$; in the latter convention the factor 2π disappears entirely while the physics is unchanged. The result is verified numerically to better than one part in 10^8 using CODATA 2022 data; because λ_{Ly}, a_0, R_∞, and α are themselves algebraically linked through their CODATA definitions, this is best read as a consistency check on those definitions, not an independent test. We also present the hierarchy of the three electromagnetic length scales of hydrogen (r_e, a_0, λ_{Ly}), separated by powers of α and 4π, and discuss the leading finite-proton-mass correction (+0.054%) and extensions to hydrogenic ions and atomic units. No new physics is proposed. The paper packages standard textbook definitions into a single exact, precision-testable identity with a transparent (if convention-dependent) geo-

metric reading, offered as a pedagogical and diagnostic tool.

Keywords

Fine-Structure Constant, Hydrogen Atom, Lyman Limit, Bohr Radius, Solid Angle, Dimensional Analysis, Atomic Units, Hydrogenic Ions

1. Introduction

When the ground-state electron of hydrogen is ionised, the emitted photon lies at the Lyman limit. For infinite nuclear mass, this threshold wavelength is $\lambda_{\mathrm{Ly}} \equiv \lambda_{\mathrm{Ly}}^{(\infty)} = 1/R_\infty = 91.1267$ nm [1], where R_∞ is the Rydberg constant in the infinite-nuclear-mass limit; the finite-proton-mass (real hydrogen) threshold is denoted $\lambda_{\mathrm{Ly}}^{(\mathrm{real})}$ and treated separately in Section 6. The size of the atom is set by the Bohr radius $a_0 = 52.9177$ pm [1]. Their ratio is

$$\frac{\lambda_{\mathrm{Ly}}}{a_0} = 1722.05\ldots \tag{1}$$

Standard treatments note only that this is "of order $1/\alpha^2$" without making its exact value or geometric content explicit. Here we show that the ratio is not approximately $1/\alpha^2$ but *exactly* $4\pi/\alpha$, with every factor traceable to a distinct physical origin. This exactness holds within the idealised, infinite-nuclear-mass, non-relativistic Coulomb model in which R_∞ and a_0 are themselves defined; the *measured* hydrogen threshold additionally contains reduced-mass, relativistic, recoil, radiative, and finite-nuclear-size corrections not captured by $4\pi/\alpha$ alone (Section 6 treats the leading, reduced-mass term quantitatively).

The fine-structure constant, introduced by Sommerfeld [2], is

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}, \quad \alpha^{-1} = 137.035999177 \quad [1], \tag{2}$$

determined most precisely by atom-recoil interferometry with rubidium [3] and cesium [4], both agreeing with the CODATA 2022 adjustment to within 0.3 ppb—negligible next to the 10^{-8} precision of the identity below.

The relation $\lambda_{\mathrm{Ly}}/a_0 = 4\pi/\alpha$ follows directly from the standard definitions $a_0 = \hbar/(m_e c\alpha)$ and $R_\infty = m_e c\alpha^2/(2h)$: substituting into $R_\infty a_0$ gives $\alpha/(4\pi)$ immediately, so the identity is implicit in every standard reference [5] [6], even though it is not stated explicitly there. Our contribution is 1) the observation that $4\pi = 2\pi_{\mathrm{phase}} \times 2_{\mathrm{virial}}$, where each factor has a distinct physical origin; 2) the recognition that, in the h-based convention for photon energy, these two factors map onto the azimuthal and polar integrals of the sphere, so that $\lambda_{\mathrm{Ly}}/a_0 = \Omega_{\mathrm{sph}}/\alpha$—an interpretive, not derivational, correspondence made precise in Section 4; 3) a numerical verification to 10^{-8} including the reduced-mass correction; and 4) the hierarchy of hydrogen's three electromagnetic length scales and its extension to hydrogenic ions and atomic units. We derive no new physics and make no claim to obtain α from first principles.

2. Notation and Definitions

Table 1 lists the symbols used, with CODATA 2022 values [1] where applicable; SI units throughout.

Table 1. Notation, definitions, and numerical values. Measured constants use CODATA 2022 [1]. †Mathematical constant; not a measured quantity.

Symbol	Definition	Value
	Measured physical constants (CODATA 2022)	
α	Fine-structure constant $e^2/(4\pi\varepsilon_0\hbar c)$	7.2974×10^{-3}
α^{-1}	Inverse fine-structure constant	137.035 999 177
a_0	Bohr radius $\hbar/(m_e c\alpha)$	5.2918×10^{-11} m
r_e	Classical electron radius $\alpha^2 a_0$	2.8179×10^{-15} m
$\bar{\lambda}_C$	Reduced Compton wavelength $\hbar/(m_e c)$	3.8616×10^{-13} m
R_∞	Rydberg constant $m_e c\alpha^2/(2h)$	1.09737×10^{7} m^{-1}
E_R	Rydberg energy $\frac{1}{2}m_e c^2\alpha^2$	13.6057 eV
λ_{Ly}	Lyman-limit wavelength (infinite nuclear mass) $1/R_\infty = hc/E_R$	91.1267 nm
h	Planck constant $2\pi\hbar$	6.6261×10^{-34} J s
$\hbar$	Reduced Planck constant	1.0546×10^{-34} J s
m_e	Electron rest mass	9.1094×10^{-31} kg
m_p	Proton rest mass	1.6726×10^{-27} kg
c	Speed of light in vacuum	2.9979×10^{8} m·s^{-1}
	Mathematical constant	
Ω_{sph}	Solid angle of a complete sphere†	$4\pi = 12.566\cdots$ sr

3. Algebraic Derivation

This section derives the exact identity between λ_{Ly}, a_0, α, and the number 4π; its geometric reading as a solid angle (Section 4) is an interpretation added afterward and does not affect the derivation.

Bohr radius. From Equation (2) and $a_0 = \bar{\lambda}_C/\alpha$:

$$a_0 = \frac{\hbar}{m_e c\alpha}. \tag{3}$$

Rydberg energy. The ground-state ionisation energy (infinite nuclear mass) is

$$E_R = \frac{1}{2}m_e c^2\alpha^2, \tag{4}$$

where $\frac{1}{2}$ is fixed by the virial theorem for the Coulomb potential, $\langle T\rangle = -E_{\text{total}}$ (Section 4).

Lyman-limit wavelength. The photon emitted for $n=1\to\infty$ has energy $E_\gamma = E_R$, so

$$\lambda_{\mathrm{Ly}} = \frac{hc}{E_R} = \frac{2\pi\hbar c}{\frac{1}{2}m_e c^2\alpha^2} = \frac{4\pi\hbar}{m_e c\alpha^2}. \tag{5}$$

The 4π here has two independent origins: $h = 2\pi\hbar$ contributes a factor 2π, and the $\frac{1}{2}$ in the denominator of E_R, moved to the numerator, contributes a factor 2.

Main identity. Forming $\lambda_{\mathrm{Ly}}/a_0$ from Equations (3) and (5):

$$\frac{\lambda_{\mathrm{Ly}}}{a_0} = \frac{4\pi\hbar}{m_e c\alpha^2} \times \frac{m_e c\alpha}{\hbar} = \frac{4\pi}{\alpha}. \tag{6}$$

m_e, c, and $\hbar$ cancel exactly: the ratio of a macroscopic photon wavelength to a sub-angstrom radius depends only on α and the pure number 4π. Equivalently, using $R_\infty = 1/\lambda_{\mathrm{Ly}}$,

$$R_\infty a_0 = \frac{\alpha}{4\pi}. \tag{7}$$

Semiclassical cross-check. The hydrogen 1*s* state is spherically symmetric and has no definite orbit or speed; the following is Bohr-model shorthand, retained only because it is pedagogically standard. With the characteristic speed scale $v_1 = \alpha c$ set by $\langle T\rangle = \frac{1}{2}m_e v_1^2 = E_R$, the semiclassical de Broglie wavelength is

$$\lambda_{\mathrm{dB}} = \frac{h}{m_e v_1} = 2\pi a_0, \tag{8}$$

exactly one Bohr-orbit circumference. This is simply the Bohr quantization condition ($n\lambda_{\mathrm{dB}} = 2\pi a_0$ at $n=1$) from which a_0 is itself derived, so it is a self-consistency check on a_0's definition rather than an independent verification of Equation (6).

4. Geometric Origin of the Factor 4π

The correspondence developed here—between the 2π and 2 of Equation (5) and the azimuthal and polar integrals of the sphere—is an *interpretive analogy*, not a new derivation: it illuminates the origin of the prefactor 4π but does not alter Section 3.

4.1. Two Independent Factors

As shown above, $4\pi = \underbrace{2\pi}_{h=2\pi\hbar} \times \underbrace{2}_{\text{virial theorem}}$. Using h rather than $\hbar$ in Equation (5) introduces 2π: the emitted photon carries the energy of one complete oscillation cycle, and a full cycle sweeps 2π radians of phase. This factor is fixed by the definition of linear frequency ν relative to angular frequency ω; had the photon energy been written as $\hbar\omega$ throughout, as in most modern formulations, this 2π would not appear (Section 4.3).

The factor 2 instead comes from $E_R = \frac{1}{2}m_e c^2\alpha^2$: by the virial theorem for the $1/r$ Coulomb potential, $\langle V\rangle = -2\langle T\rangle$, so the total energy is exactly $-\langle T\rangle$, giving the $\frac{1}{2}$ that (once inverted into the numerator of λ_{Ly}) supplies the factor 2. This factor is a quantum-mechanical consequence of the Coulomb potential, not a free parameter.

4.2. Match to the Solid Angle of a Sphere

The solid angle of a complete sphere is

$$\Omega_{\text{sph}} = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta \mathrm{d}\theta = \underbrace{\int_0^{2\pi} \mathrm{d}\varphi}_{=2\pi} \times \underbrace{\int_0^{\pi} \sin\theta \mathrm{d}\theta}_{=2} = 4\pi \text{ sr}. \tag{9}$$

As shown in **Figure 1**, this decomposes into exactly the same two factors identified algebraically above: the azimuthal integral matches the phase factor from $h = 2\pi\hbar$, and the polar integral matches the virial-theorem factor of 2. We stress that this match is *formal and convention-dependent*: it does not imply angular integration over the hydrogen wavefunction or the photon's emission pattern, and it is tied specifically to writing photon energy as $h\nu$ rather than $\hbar\omega$. In a $\hbar$-based convention the ratio $\lambda_{\text{Ly}}/a_0 = 4\pi/\alpha$ is unchanged—it remains exact—but the decomposition becomes $4\pi = 1\times(4\pi)$ and no longer matches the sphere integrals term by term. The identification itself is therefore exact and convention-*independent*; only the sphere-integral picture that makes the $2\pi\times 2$ structure visible is convention-dependent. We write the identity in its most transparent form:

$$\boxed{\frac{\lambda_{\text{Ly}}}{a_0} = \frac{\Omega_{\text{sph}}}{\alpha},} \tag{10}$$

where $\Omega_{\text{sph}} = 4\pi$ sr is the solid angle of a complete sphere.

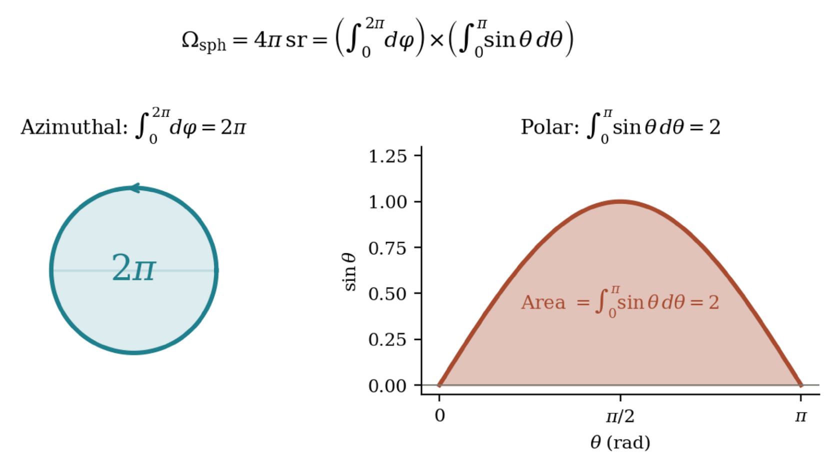

Figure 1. The solid angle of the sphere, $\Omega_{\text{sph}} = 4\pi$ sr, is expressed as the product of two integrals. *Left*: azimuthal integral $\int_0^{2\pi} \mathrm{d}\varphi = 2\pi$, corresponding to the phase factor from $h = 2\pi\hbar$. *Right*: polar integral $\int_0^{\pi} \sin\theta \mathrm{d}\theta = 2$, corresponding to the virial-theorem factor of 2 contributed to the numerator of λ_{Ly} by $E_R = \frac{1}{2} m_e c^2 \alpha^2$. The correspondence is formal and convention-dependent; it does not imply angular integration over the hydrogen wavefunction or the photon emission pattern.

4.3. Why Not 2π?

If E_R had no $\frac{1}{2}$ (kinetic-energy scale $m_e c^2\alpha^2$ instead), $\lambda_{\text{Ly}} \to \lambda_{\text{Ly}}/2$ and the ratio would be $2\pi/\alpha$, a hemisphere ($\Omega_{\text{sph}}/2$). Working throughout in $\hbar$-based natural units instead— $E=\hbar\omega$, reduced wavelength $\bar{\lambda}=\hbar c/E$ —gives $\bar{\lambda}_{\text{Ly}}/a_0\big|_{\hbar\text{-conv.}}=2/\alpha$, matching the polar integral alone. So the match to the full sphere is a feature of the h-convention, not an intrinsic property of the physics: $4\pi=\Omega_{\text{sph}}$ arises specifically when photon energy is measured against linear frequency ν *and* the virial theorem applies to the Coulomb potential, the latter being the only piece that is a universal (not conventional) property of the $1/r$ potential.

5. The Three Length Scales of Hydrogen

Three electromagnetic length scales appear naturally in hydrogen physics, each expressible via $\bar{\lambda}_C=\hbar/(m_e c)$:

$$r_e=\alpha\bar{\lambda}_C=\alpha^2 a_0, \tag{11}$$

$$a_0=\bar{\lambda}_C/\alpha, \tag{12}$$

$$\lambda_{\text{Ly}}=\frac{4\pi}{\alpha}a_0=\frac{4\pi}{\alpha^2}\bar{\lambda}_C. \tag{13}$$

From r_e to a_0 the scale grows by $1/\alpha^2\approx 1.88\times 10^4$; from a_0 to λ_{Ly} by a further $4\pi/\alpha\approx 1722$. The three scales span about seven and a half orders of magnitude ($\lambda_{\text{Ly}}/r_e=4\pi/\alpha^3\approx 3.2\times 10^7$), yet every ratio is a pure number built from α and 4π alone (Table 2, Figure 2—bar lengths in the figure are illustrative only and do not reproduce this true separation).

Table 2. The three characteristic electromagnetic length scales of hydrogen. CODATA 2022 values [1].

Scale	Symbol	Value	Ratio to a_0
Classical electron radius	$r_e=\alpha\bar{\lambda}_C$	2.817 94 fm	$\alpha^2=5.325\times 10^{-5}$
Bohr radius	$a_0=\bar{\lambda}_C/\alpha$	52.917 7 pm	1 (reference)
Lyman limit	$\lambda_{\text{Ly}}=(4\pi/\alpha)a_0$	91.126 7 nm	$4\pi/\alpha=1722.05$

Physically: $r_e=\alpha^2 a_0$ is where the electron's electrostatic potential energy equals its rest-mass energy, $e^2/(4\pi\varepsilon_0 r_e)=m_e c^2$, setting the scale of Thomson scattering [7]. $a_0=\bar{\lambda}_C/\alpha$ is where potential and kinetic energy balance via the virial theorem, fixing the ground-state radius [5]. $\lambda_{\text{Ly}}=(4\pi/\alpha)a_0$ is the wavelength of the photon that frees the ground-state electron; its ratio to a_0 is exactly $\Omega_{\text{sph}}/\alpha$.

6. Numerical Verification

6.1. CODATA 2022 Consistency Check

Using CODATA 2022 [1], three routes give λ_{Ly}/a_0: 1) direct ratio, (91.126705 nm)/(52.917721 pm) = 1722.045153; 2) via R_∞, $1/(R_\infty a_0) = 1722.045153$; 3) algebraic, 4π × 137.035999177 = 1722.045154. Because λ_{Ly}, a_0, R_∞, and α are themselves algebraically linked through their CODATA definitions, agreement of these three routes to better than 10^{-8} (Table 3) confirms the *internal consistency* of the CODATA 2022 adjustment, not an independent experimental test of the identity.

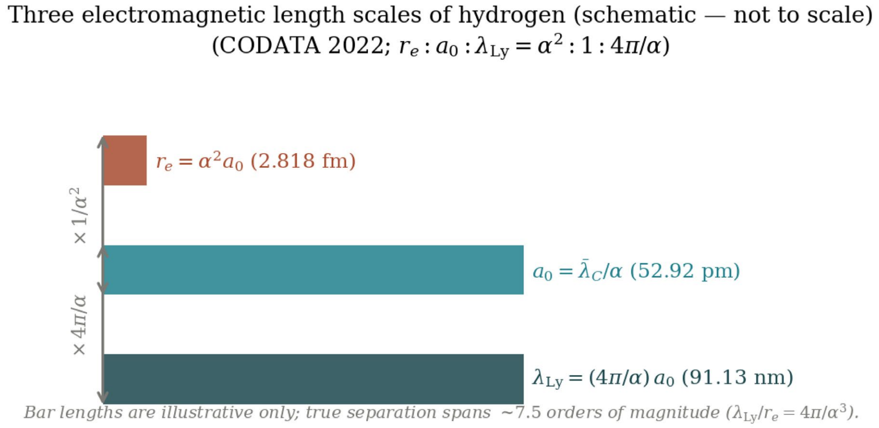

Figure 2. Schematic (not to scale). The three nested electromagnetic length scales of hydrogen. Each scale is related to its neighbour by a factor built from α and 4π alone; m_e, c, $\hbar$ cancel in the ratios. Bar lengths are illustrative and do not represent the true logarithmic spacing, which spans about seven and a half orders of magnitude from r_e to λ_{Ly} (Table 2). CODATA 2022 values [1].

Table 3. Numerical consistency check of $\lambda_{Ly}/a_0 = 4\pi/\alpha$. CODATA 2022 [1]. The three routes are algebraically linked and serve as mutual consistency checks, not independent measurements.

Method	λ_{Ly}/a_0
Direct ratio (λ_{Ly}/a_0)	1722.045153
From $R_\infty a_0$	1722.045153
Algebraic: $4\pi/\alpha$	1722.045154
Finite proton mass (below)	1722.983008

6.2. Finite Proton Mass

R_∞ is defined for infinite nuclear mass; for real hydrogen the appropriate Rydberg is $R_H = R_\infty \mu/m_e$ with reduced mass $\mu = m_e m_p/(m_e + m_p)$, giving

$$\frac{\mu}{m_e} = \frac{m_p}{m_e + m_p} = \frac{1836.152}{1837.152} = 0.99945568 \tag{14}$$

using $m_p/m_e = 1836.15267343$ [1]. Since $\mu < m_e$, $\lambda_{Ly}^{(\text{real})} > \lambda_{Ly}^{(\infty)}$: real hydrogen's Lyman limit sits at a longer wavelength,

$$\lambda_{Ly}^{(\text{real})} = \frac{\lambda_{Ly}}{\mu/m_e} = 91.1763\ \text{nm}, \quad \left.\frac{\lambda_{Ly}}{a_0}\right|_{\text{real}} = \frac{4\pi}{\alpha} \times \frac{m_e}{\mu} = 1722.983, \tag{15}$$

consistent with the measured laboratory value (91.18 nm [8]) within experimental resolution. The exact identity $4\pi/\alpha$ describes only the infinite-nuclear-mass model used to define R_∞ and a_0; the physical value carries the multiplicative correction $m_e/\mu \approx 1 + 5.446\times10^{-4}$. Because $\lambda_{\mathrm{Ly}}/a_0 = 4\pi/\alpha$ is linear in $1/\alpha$, both points in **Table 3** lie exactly on this line by construction, so no separate plot is needed.

7. Discussion

Since $\alpha = v_1/c$ is the characteristic ground-state electron speed relative to c (semiclassically, Section 3; [5]), the identity reads $\lambda_{\mathrm{Ly}} = 4\pi(c/v_1)a_0$: the photon's wavelength exceeds the Bohr radius by 4π times the ratio of characteristic speeds, with 4π the geometric factor converting a speed ratio into a length ratio.

Standard textbooks [5] [6] introduce α, a_0, R_∞, E_R separately and compute λ_{Ly} numerically, so $\lambda_{\mathrm{Ly}}/a_0 = 4\pi/\alpha$ is implicit but never stated. It operates at $O(\alpha^2)$ (the order of E_R and a_0 themselves), well below the $O(\alpha^4)$ fine-structure splitting [7]; relativistic corrections to E_R, of order $\alpha^2 = 5.3\times10^{-5}$, are negligible here. By the Buckingham Π theorem, any dimensionless $\lambda_{\mathrm{Ly}}/a_0$ in non-relativistic hydrogen must be a function of α alone (the small m_e/m_p enters only as the separate reduced-mass correction above); Equation (6) shows that function to be precisely $4\pi/\alpha$, verifiable to ten significant figures given CODATA's $\delta\alpha/\alpha = 1.5\times10^{-10}$ [1].

8. Implications and Extensions

The applications below follow from $\lambda_{\mathrm{Ly}}/a_0 = 4\pi/\alpha$; any solid-angle language remains the interpretive shorthand of Section 4.

Atomic units. $R_\infty a_0 = \alpha/(4\pi)$ (Equation (7)) bridges Hartree units ($\hbar = m_e = e = 1$) and Rydberg units ($2m_e = \hbar = e^2/(2\cdot 4\pi\varepsilon_0) = 1$) directly, with 4π acting as a universal bridge constant between length and energy scales whenever α is the coupling.

Hydrogenic ions. For nuclear charge Z, $E_1(Z) = Z^2 E_R$ and $a_0(Z) = a_0/Z$ [5], giving

$$\frac{\lambda_{\mathrm{Ly}}(Z)}{a_0(Z)} = \frac{hc}{Z^2 E_R}\times\frac{Z}{a_0} = \frac{4\pi}{Z\alpha} = \frac{\Omega_{\mathrm{sph}}}{Z\alpha}. \tag{16}$$

This is exact within the same non-relativistic, point-nucleus model used throughout, but should *not* be read as exact across the full isoelectronic sequence: as Z grows, $(Z\alpha)^2$ relativistic corrections and finite-nuclear-size effects become essential (e.g. for He^+ through U^{91+}), as treated in the standard reference on relativistic hydrogenic ions [9]. Equation (16) is best understood as the leading-order term of an expansion in $Z\alpha$, valid for low- to moderate-Z systems.

Two-photon spectroscopy. The $1S$-$2S$ transition, the most precisely measured atomic transition (4×10^{-15} [1]), has $1S$ ionisation energy E_R and $1S$-$2S$ interval $\frac{3}{4}E_R$; both scale as $E_R = \frac{1}{2}m_e c^2\alpha^2$, so any future revision of α propagates directly into $\lambda_{\mathrm{Ly}}/a_0$ via $4\pi/\alpha$ with no free parameters, making Equation (10) a useful diagnostic in global CODATA adjustments.

Multi-electron atoms (*speculative*). An effective-charge analogue, $\lambda_{\mathrm{ion}}/a_0^* \approx \Omega_{\mathrm{sph}}/(Z_{\mathrm{eff}}\alpha)$ with $Z_{\mathrm{eff}} \approx 1.34$ for helium, is suggested only as a possible direction for future work; no calculation establishing its validity is given here.

Pedagogy. $\lambda_{\mathrm{Ly}}/a_0 = \Omega_{\mathrm{sph}}/\alpha$ packages four separate textbook definitions (α, a_0, R_∞, E_R) into one relation, making explicit that a single coupling α and the geometry of three-dimensional space ($\Omega_{\mathrm{sph}} = 4\pi$) suffice to connect hydrogen's two most fundamental length scales—a convenient entry point for discussions of naturalness and dimensional analysis [6].

9. Conclusions

The ratio of the hydrogen Lyman-limit wavelength (infinite nuclear mass) to the Bohr radius satisfies the exact identity

$$\frac{\lambda_{\mathrm{Ly}}}{a_0} = \frac{4\pi}{\alpha} = \frac{\Omega_{\mathrm{sph}}}{\alpha}, \tag{17}$$

where $\Omega_{\mathrm{sph}} = 4\pi$ sr is the solid angle of a sphere. In summary:

1) The identity follows from four lines of algebra using only the standard definitions of α, a_0, R_∞; no model beyond textbook quantum mechanics is needed.

2) $4\pi = 2\pi \times 2$, with 2π from $h = 2\pi\hbar$ (one full phase cycle in the h-convention) and 2 from the virial theorem, $\langle T \rangle = -E_{\mathrm{total}}$.

3) These two factors are structurally identical to the sphere's azimuthal and polar integrals *when* photon energy is written as $h\nu$. This correspondence is interpretive and convention-dependent—it does not imply angular integration over atomic wavefunctions or emission patterns—whereas the identity $\lambda_{\mathrm{Ly}}/a_0 = \Omega_{\mathrm{sph}}/\alpha$ itself is exact and convention-independent.

4) r_e, a_0, λ_{Ly} form a hierarchy governed by specific powers of α and 4π.

5) The identity extends to leading order in $Z\alpha$, to hydrogenic ions as $\lambda_{\mathrm{Ly}}(Z)/a_0(Z) = \Omega_{\mathrm{sph}}/(Z\alpha)$.

6) The finite proton mass shifts the ratio by +0.054%, from 1722.045 to 1722.983, matching the measured Lyman limit.

7) The three numerical routes in Section 6 are consistency checks among algebraically linked CODATA constants, confirming internal consistency to 10^{-8}, not an independent experimental test.

The exact ratio of hydrogen's two most fundamental length scales reduces to the fine-structure constant and the pure number 4π. We offer this as a bridge constant linking Hartree and Rydberg atomic units, an exact leading-order scaling law across the hydrogenic isoelectronic sequence, and a zero-free-parameter diagnostic for tracking future revisions of α. The solid-angle correspondence behind its 4π prefactor is a formal, convention-dependent one, not evidence of a physical angular integration—but within that stated scope it gives hydrogen's length-scale hierarchy a geometric reading left implicit in standard textbook treatments.

10. Python Code for Numerical Verification

All numerical results and figures were produced with the Python script plots.py (**Supplemental Material**). The key check:

```
import numpy as np
alpha  = 7.2973525643e-3    # CODATA 2022
a0     = 5.29177210544e-11  # m
h      = 6.62607015e-34     # J.s
c      = 2.99792458e8       # m/s
me     = 9.1093837139e-31   # kg
mp     = 1.67262192595e-27  # kg
Ry_eV  = 13.605693122994    # eV
eV     = 1.602176634e-19    # J
E_R    = Ry_eV * eV
lam_Ly = h * c / E_R
ratio  = lam_Ly / a0        # 1722.045153
algebraic = 4*np.pi / alpha # 1722.045154
mu     = me * mp / (me + mp)
ratio_real = ratio * me / mu   # 1722.983008
```

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Mohr, P.J., Newell, D.B., Taylor, B.N. and Tiesinga, E. (2025) CODATA Recommended Values of the Fundamental Physical Constants: 2022. *Reviews of Modern Physics*, **97**, Article ID: 025002. https://doi.org/10.1103/revmodphys.97.025002

[2] Sommerfeld, A. (1916) Zur Quantentheorie der Spektrallinien. *Annalen der Physik*, **356**, 1-94. https://doi.org/10.1002/andp.19163561702

[3] Morel, L., Yao, Z., Cladé, P. and Guellati-Khélifa, S. (2020) Determination of the Fine-Structure Constant with an Accuracy of 81 Parts per Trillion. *Nature*, **588**, 61-65. https://doi.org/10.1038/s41586-020-2964-7

[4] Parker, R.H., Yu, C., Zhong, W., Estey, B. and Müller, H. (2018) Measurement of the Fine-Structure Constant as a Test of the Standard Model. *Science*, **360**, 191-195. https://doi.org/10.1126/science.aap7706

[5] Griffiths, D.J. (2018) Introduction to Quantum Mechanics, 3rd Edition, Cambridge University Press.

[6] Landau, L.D. and Lifshitz, E.M. (1977) Quantum Mechanics: Non-Relativistic Theory. 3rd Edition, Butterworth-Heinemann.

[7] Berestetskii, V.B., Lifshitz, E.M. and Pitaevskii, L.P. (1982) Quantum Electrodynamics. 2nd Edition, Butterworth-Heinemann.

[8] Morton, D.C. (2003) Atomic Data for Resonance Absorption Lines. III. Wavelengths Longward of the Lyman Limit for the Elements Hydrogen to Gallium. *The Astrophysical Journal Supplement Series*, **149**, 205-238. https://doi.org/10.1086/377639

[9] Bethe, H.A. and Salpeter, E.E. (1957) Quantum Mechanics of One- and Two-Electron Atoms. Springer.

Supplemental Material

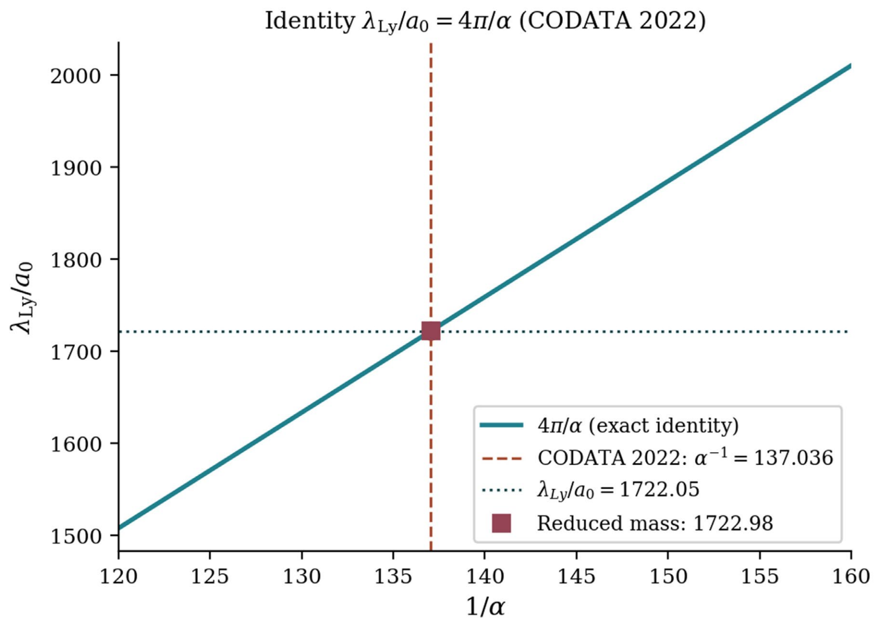

Figure S1. Ratio Vs Alpha.

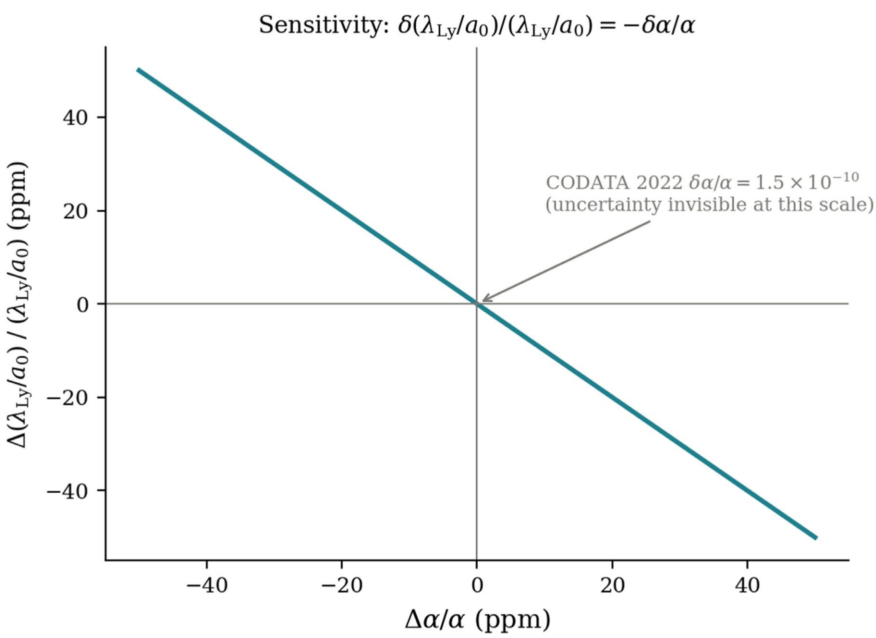

Figure S2. Sensitivity.