

# Kruskal Coordinates, Semi-Complex Extension of Spacetime

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## Abstract

In this article, we re-examine the geometric nature of the extension of the Schwarzschild solution into the domain  $0 < r < \alpha$ . A preliminary study of the sphere, the torus, and Flamm's surface demonstrates that real geometries can admit semi-complex extensions beyond the limits of their real embedding domains; in such cases, certain coordinates become purely imaginary, while their projections onto real subspaces remain observable as *ghost images*. Kruskal's standard construction is then repeated line by line, separately in the outer  $r > \alpha$  and the inner region  $0 < r < \alpha$ . It appears that the usual actual definition of the tortoise coordinate,  $r^* = r + \alpha \ln \left| \frac{r}{\alpha} - 1 \right|$  encompasses two distinct changes of variables on either side of  $r = \alpha$ , supplemented by different sign choices for the exponential coordinates that allow for the construction of the four sectors of the Kruskal diagram. We then consider another continuation, obtained by maintaining a single analytic change of variables across  $r = \alpha$ , without introducing the absolute value. In the domain  $0 < r < \alpha$ , the Kruskal coordinates thus extended naturally become purely imaginary. The standard real representation can be recovered via an imaginary translation of the time coordinate, which simultaneously produces the swapping of hyperbolic functions characteristic of the interior regions. A three-dimensional representation  $r = H(u, v)$  allows for the visualization of the connection between the real and semi-complex sectors. This analysis leads to an alternative interpretation of the Schwarzschild extension: rather than attributing swapped causal roles to the coordinates  $r$  and  $t$  in the interior region, the latter can be viewed as belonging to a semi-complex extension of spacetime geometry. Finally, the physical consequences of this interpretation are discussed, along with the possibility of alternative compact states, such as *plugstars*, that could potentially lead to discriminating observational tests.

## Keywords

Semi-Complex Extension, Kruskal Coordinates, Kruskal Diagram, Schwarzschild Geometry, Autoparallel Curves, Ghost Images, Ghost Geometry, Ghost Symmetry, Pseudo-Real Coordinates

## 1. Real and Semi-Complex Geometric Extension: The Ghost Phenomenon

Let us consider the sphere  $S^2$ , defined by its metric, expressed in terms of the two angular coordinates  $\theta$  and  $\varphi$ :

$$ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (1)$$

Introduce the change of coordinate:

$$\theta = \arcsin \frac{r}{R} \quad (2)$$

The line element becomes:

$$ds^2 = r^2 d\varphi^2 + \frac{R^2 dr^2}{R^2 - r^2} \quad (3)$$

We thus have a Riemannian metric, in the sense that the signs of the signature (+ +) are identical. One can construct the geodesics of this object as length-minimizing curves:

$$J = \int ds = \int \sqrt{r^2 d\varphi^2 + \frac{R^2 dr^2}{R^2 - r^2}} \quad (4)$$

By locating the position of the point using a parameter  $\lambda$ :

$$J = \int \sqrt{r^2 \left( \frac{d\varphi}{d\lambda} \right)^2 + \frac{R^2}{R^2 - r^2} \left( \frac{dr}{d\lambda} \right)^2} \quad (5)$$

By posing:

$$\dot{\varphi} = \frac{d\varphi}{d\lambda} \quad \dot{r} = \frac{dr}{d\lambda} \quad (6)$$

We get:

$$J = \int \sqrt{r^2 \dot{\varphi}^2 + \frac{R^2}{R^2 - r^2} \dot{r}^2} d\lambda = \int F(r, \varphi, \dot{r}, \dot{\varphi}) d\lambda \quad (7)$$

The quantity  $F(r, \varphi, \dot{r}, \dot{\varphi})$  ...being the corresponding Lagrange function. Minimizing the length would lead us to the Lagrange equations:

$$\frac{d}{d\lambda} \left( \frac{\partial F}{\partial \dot{r}} \right) = \frac{\partial F}{\partial r} \quad (8)$$

$$\frac{d}{d\lambda} \left( \frac{\partial F}{\partial \dot{\varphi}} \right) = \frac{\partial F}{\partial \varphi} \quad (9)$$

What happens if we replace the Lagrange function  $F$  by its square  $F^2$ ?

$$J = \int \left( r^2 \dot{\varphi}^2 + \frac{R^2}{R^2 - r^2} \dot{r}^2 \right) d\lambda = \int F^2(r, \varphi, \dot{r}, \dot{\varphi}) d\lambda = \int \Phi d\lambda \quad (10)$$

The Lagrange equations are:

$$\frac{dF}{d\lambda} \left( \frac{\partial F}{\partial \dot{r}} \right) + F \left( \frac{\partial F}{\partial \dot{r}} \right) = F \frac{\partial F}{\partial r} \quad (11)$$

$$\frac{dF}{d\lambda} \left( \frac{\partial F}{\partial \dot{\varphi}} \right) + F \left( \frac{\partial F}{\partial \dot{\varphi}} \right) = F \frac{\partial F}{\partial \varphi} \quad (12)$$

To obtain the same equations, a specific parameterization is required, such as:

$$\frac{dF}{d\lambda} = 0 \quad (13)$$

But we have:

$$\left( \frac{ds}{d\lambda} \right)^2 = r^2 \dot{\varphi}^2 + \frac{R^2}{R^2 - r^2} \dot{r}^2 = F^2 = \Phi \quad (14)$$

The condition  $\frac{dF}{d\lambda} = 0$  means that  $\lambda$  is an affine parameter along the geodesic. Since  $F = \frac{ds}{d\lambda}$ , this condition imposes a constant rate of variation of arc length with respect to  $\lambda$ . Choosing  $F = 1$  simply amounts to taking  $\lambda = s$ .

We now choose an affine parameterization, and more precisely the normalization  $F = 1$ , which, in the Riemannian sector, amounts to choosing  $\lambda = s$ . Thus:

$$1 = r^2 \dot{\varphi}^2 + \frac{R^2}{R^2 - r^2} \dot{r}^2 \quad (15)$$

Since we have a cyclical nature:

$$\frac{\partial \Phi}{\partial \varphi} = 0 \quad (16)$$

Choosing  $\Phi/2$  as Lagrangian:

$$r^2 \dot{\varphi}^2 = \text{Cst} \rightarrow \dot{\varphi} = \frac{h}{r} \quad (17)$$

By combining these two equations, one immediately obtains the differential equation yielding the solution curves:

$$d\varphi = \frac{hR}{r} \frac{dr}{\sqrt{(R^2 - r^2)(r^2 - h^2)}} \quad (18)$$

From which we can construct two families of curves:

- For  $r < R$  with  $|h| < R$
- For  $r > R$  with  $|h| > R$

For  $r < R$ , these are ellipses—planar projections of the geodesics of the sphere  $\mathcal{S}^2$ , which are its great circles. For  $r < R$ , these elliptical projections can be viewed as the planar projections of the great circles of the sphere  $\mathcal{S}^2$  embedded in  $\mathbb{R}^3$ .

But what about the curves corresponding to  $r > R$ , and what meaning should be assigned to them? Let us first show that they are ellipses. Let us set:

$$r = \frac{1}{u} \quad (19)$$

$$\frac{d\varphi}{du} = -\frac{hRu}{\sqrt{(R^2u^2-1)(1-h^2u^2)}} \quad (20)$$

$$v = u^2 = \frac{1}{r^2} \quad (21)$$

$$\left(\frac{dv}{du}\right)^2 = \frac{4}{h^2 R^2} (R^2v-1)(1-h^2v) \quad (22)$$

$$\frac{d^2v}{d\varphi^2} + 4v = 2\left(\frac{1}{h^2} + \frac{1}{R^2}\right) \quad (23)$$

$$v = \frac{1}{2}\left(\frac{1}{h^2} + \frac{1}{R^2}\right) + A \cos 2(\varphi - \varphi_0) \quad (24)$$

The turning points are

$$r = R \text{ et } r = |h| \quad (25)$$

What fixes the origin and provides, after the angular origin has been chosen:

$$\frac{1}{r^2} = \frac{\cos^2(\varphi - \varphi_0)}{h^2} + \frac{\sin^2(\varphi - \varphi_0)}{R^2} \quad (26)$$

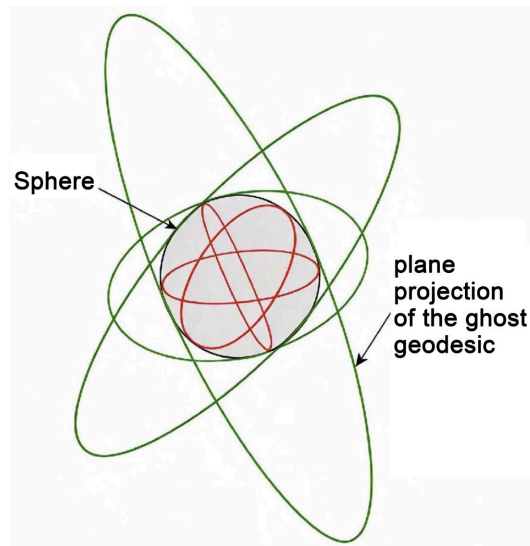
With:

$$x = r \cos(\varphi - \varphi_0) \quad (27)$$

$$y = r \sin(\varphi - \varphi_0) \quad (28)$$

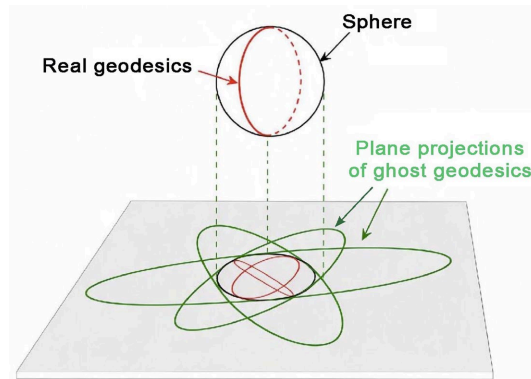
$$\frac{x^2}{h^2} + \frac{y^2}{R^2} = 1 \quad (29)$$

All these curves are ellipses, as shown in **Figure 1**.



**Figure 1.** Real plane projections of geodesics and ghost images of antiparallels inscribed on the semi-complex extension of the sphere for  $r > \alpha$ .

The curves lying outside the projection of the sphere's apparent contour are therefore also ellipses. However, these real curves are 2D projections; they cannot be regarded as the projections of actual curves capable of being embedded in  $\mathbb{R}^3$ . (See **Figure 2**.)



**Figure 2.** Sphere and projection of the sphere, plus geodesics, and ghost images of antiparallel curves that belong to the semi-complex geometric extension.

Let us show that they can be embedded in a three-dimensional semi-complex space with coordinates  $(r, \varphi, \eta)$ , where

$$(r, \varphi) \in \mathbb{R}^2 \quad \eta \in \mathbb{C} \quad (30)$$

That is to say, where the first two are real and the third is purely imaginary.

Regarding the region  $r < R$  the sphere  $\mathcal{S}^2$ , embedded in the space described by the real coordinates  $(r, \varphi, z)$  and projected onto the subspace—exhibits an apparent contour in the form of the circle:

$$r^2 + z^2 = R^2 \quad (31)$$

For  $r > R$ , if we attempt to embed the geometric object defined by its metric into  $\mathbb{R}^3$ , using coordinates  $(r, \varphi, z)$ , we obtain

$$dz = \frac{rdr}{\sqrt{R^2 - r^2}} \quad (32)$$

$z$  becomes purely imaginary. If we then move to an embedding in a semi-complex space described by the coordinates  $(r, \varphi, \eta)$  where

$$(r, \varphi) \in \mathbb{R}^2 \quad \eta \in \mathbb{C} \quad (33)$$

We would have an apparent contour corresponding to the equation:

$$r^2 + \eta^2 = R^2 \quad (34)$$

The curves we have obtained thus lie within a space formed by joining, along the sphere  $\mathcal{S}^2$ , which serves as a common boundary—the portion of the space  $(r, \varphi, z) \in \mathbb{R}^3$  contained inside ( $r < R$ ) that sphere and the portion of a semi-complex space  $(r, \varphi, \eta)$  corresponding to ( $r > R$ ). The embeddings are merely systems of representation. Let us set:

$$\eta = i\zeta \quad \zeta \in \mathbb{R} \quad (35)$$

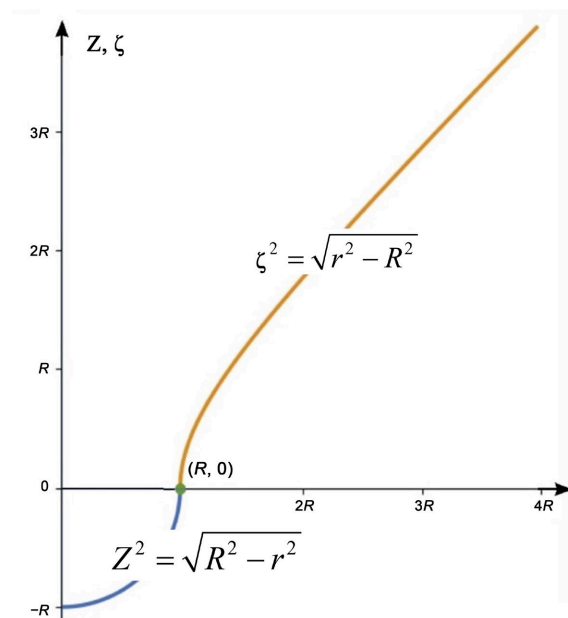
The semi-complex geometric object upon which the curves corresponding to  $r > R$  are traced then becomes a hyperboloid of one sheet with the equation:

$$r^2 - \zeta^2 = R^2 \quad (36)$$

For  $r < R$ , the metric of the sphere  $\mathcal{S}^2$  is Riemannian, with signature  $(+, +)$ . For  $r > R$ , in this representation  $(r, \varphi, \zeta)$ , the (induced) metric becomes Lorentzian, with signature  $(+, -)$ .

$$ds^2 = r^2 d\varphi^2 - \frac{R^2}{r^2 - R^2} dr^2 \quad (37)$$

In the figure below, we have the extension of the meridian constructed in this way. In **Figure 3**, this extension corresponds to the blue curve:



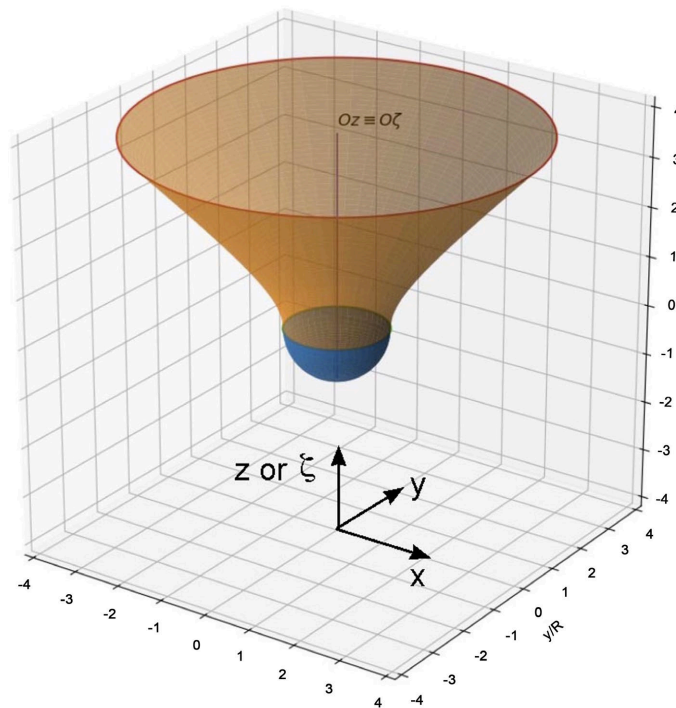
**Figure 3.** Extended apparent outline.

And its representation as a double 3D embedding. In **Figure 4** below, we have only the half part of the global semi-complex extension.

Of course, this needs to be completed by the reflection of this figure, including the symmetrical part:

$$z \rightarrow -z \quad \zeta \rightarrow -\zeta \quad (38)$$

Note the analogy with the same operation applied later to Flamm's surface. These figures serve merely to illustrate the extension of a real geodesic arc on the sphere  $\mathcal{S}^2$  by a branch lying within the semi-complex spatial region. We shall not pursue this concept of extending real geodesics from the sphere  $\mathcal{S}^2$  into a semi-complex extension of space any further. Nevertheless, this sheds light on the nature of the curves shown in the figure, which arise from Lagrange's equations. These curves fall into two families.



**Figure 4.** Part of the semi-complex extension of the  $S^2$  sphere.

- Real curves
- Curves embedded in a semi-complex space

We have immediate access to the projection of these paths onto the real component of these possible projections. For  $r < R$ , these curves are geodesics of the sphere. But what name should be given to the semi-complex extensions corresponding to  $r > R$ ? We might consider classical geodesics as belonging to the “here and now,” whereas those lying within the semi-complex extension inhabit a sort of “beyond.” Yet, we have access to their projection via the real coordinates of the semi-complex extension; they thus behave like ghosts. Our existence in the “here and now” unfolds within a real four-dimensional space  $(t, x, y, z)$ . Let us suppose that death transports us to a semi-complex space  $(\tau, x, y, z)$ , where the “beyond” is simply a space in which the time coordinate becomes purely imaginary (which would, of course, make communication problematic).

## 2. The Ghost Symmetry

However, if an observer in the “here-and-now” were to glimpse a ghost, they would gain access to the projection—onto the real  $(x, y, z)$  component of the semi-complex space—of the realm where that ghost exists. By analogy, we propose terming these curves (where  $r > R$ ) “ghost images.”

Conversely, if the observer were situated within the semi-complex world, the great circles of the sphere would appear to them as ghosts. Under this hypothesis, death would correspond to a shift from the real world to a semi-complex world—a phenomenon comparable to a  $90^\circ$  rotation of the time vector. This brings to

mind individuals in extreme situations who report seeing their entire life's events flash before them in rapid succession; this could be likened to the onset of a time-vector rotation that was subsequently aborted.

### 3. Autoparallel Curves of the Torus

Let us leave aside these digressions. What do Lagrange's equations represent? They yield autoparallel trajectories, the existence of which in no way requires length and coordinates to be real. We could repeat this exercise with the torus, defined by its metric:

$$ds^2 = r_0^2 d\theta^2 + (R + r_0 \cos \theta)^2 d\varphi^2 \quad (39)$$

The topology  $S^1 \oplus S^1$  then appears. The corresponding signature is  $(+ +)$ .

By considering the paths at  $\theta = \text{Cst}$  and then at  $\varphi = \text{cst}$ , we immediately highlight the non-contractility of this surface. Let's now introduce the change of variable:

$$\theta = \arccos \frac{r - R}{r_0} \quad (40)$$

The line element becomes:

$$ds^2 = r^2 d\varphi^2 + \frac{dr^2}{-R^2 + r_0^2 - r^2 + 2rR} \quad (41)$$

It is immediately apparent that the signature  $(+ +)$  will only be maintained if:

$$R - r \leq r \leq R + r_0 \quad (42)$$

Otherwise, we are outside the torus. The length  $s$  is expressed as follows:

$$s = \int \sqrt{r^2 \dot{\varphi}^2 + \frac{\dot{r}^2}{-R^2 + r_0^2 - r^2 + 2rR}} ds \quad (43)$$

But we know that we will obtain the same system of Lagrange equations by doing:

$$\delta \int \left( r^2 \dot{\varphi}^2 + \frac{\dot{r}^2}{-R^2 + r_0^2 - r^2 + 2rR} \right) ds = 0 \quad (44)$$

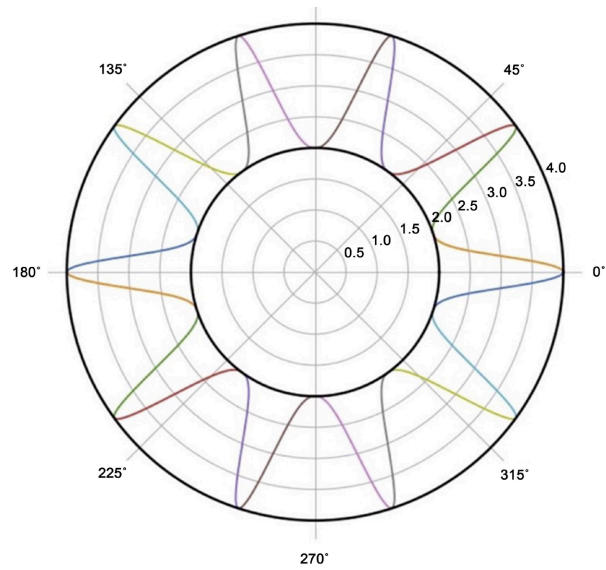
Lagrange's equations give us the projection of the curves resulting from this variation calculation onto the plane  $(r, \varphi)$ :

$$d\varphi = \frac{\pm h dr}{r^2 \sqrt{(-R^2 + r_0^2 - r^2 + 2rR) \left( 1 - \frac{h}{r^2} \right)}} \quad (45)$$

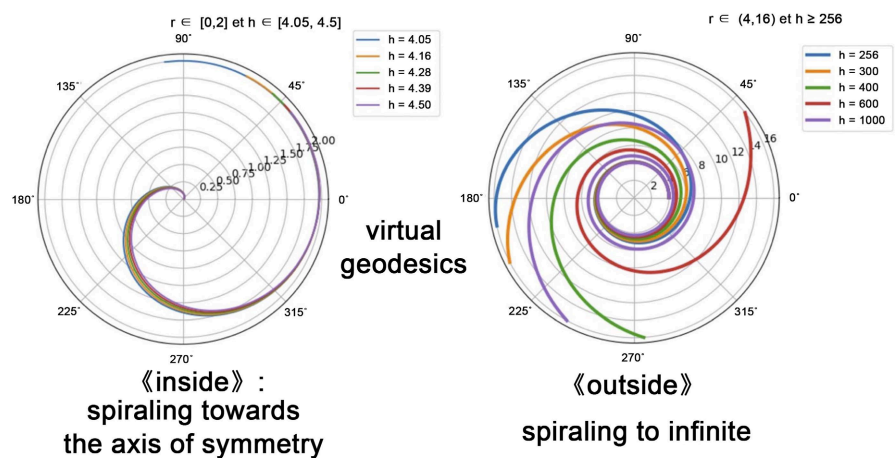
This allows us to plot the projection of a geodesic of the torus. See **Figure 5**.

But this solution also generates real curves that are the projection of curves equipped with an imaginary length. Below are the planar projections of these virtual geodesics, "inside and outside the torus", as shown in **Figure 6**.

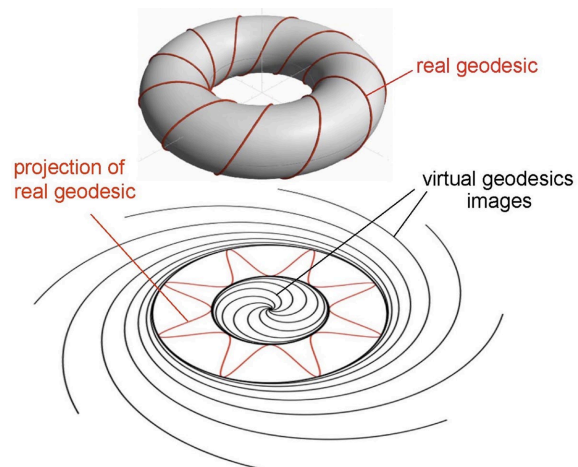
Things become clearer when we group these results into a perspective view. See **Figure 7**.



**Figure 5.** Plane projection of the real geodesics of the torus.



**Figure 6.** Projections of ghost geodesics of the torus.



**Figure 7.** The torus. Projections of real and ghost images of autoparallel curves inscribed on the semi-complex extension of the torus.

By extending the meridian curves, one could visualize the double semi-complex extension of the torus—“inside its throat circle” and “outside its rim circle.”

We shall not dwell on this, moving on instead to another, far more interesting 2D geometric object: the surface presented by L. Flamm as early as 1916 [1].

#### 4. Semi-Complex Extension of the Flamm Surface

Its line elements is:

$$ds^2 = \frac{dr^2}{1 - \frac{\alpha}{r}} + r^2 d\varphi^2 \quad (46)$$

For  $r > \alpha$  the two coefficients are positive. The metric is Riemannian and the signature is  $(++)$ . Let us embed this geometric object in 3D Euclidean space. In the 3D Euclidean space  $(x, y, z)$ :

$$x = r \cos \varphi \quad (47)$$

$$y = r \sin \varphi \quad (48)$$

The corresponding line element is:

$$dl^2 = dr^2 + r^2 d\varphi^2 + dz^2 \quad (49)$$

We have:

$$1 + \left( \frac{dz}{dr} \right)^2 = \frac{r}{r - \alpha} \quad (50)$$

$$\frac{dz}{dr} = \pm \sqrt{\frac{\alpha}{r - \alpha}} \quad (51)$$

$$z = \pm 2\sqrt{\alpha(r - \alpha)} \quad (52)$$

It is the horizontal parabola:

$$r = \alpha - \frac{z^2}{4\alpha} \quad (53)$$

We therefore have two layers that meet at the throat circle in

$$r = \alpha, z = 0 \quad (54)$$

For  $r < \alpha$ , we can consider an embedding into the semi-complex space  $(r, \varphi, \eta)$  with:

$$(r, \varphi) \in \mathbb{R}^2 \quad \eta \in \mathbb{C} \quad (55)$$

Posing:

$$\eta = i\zeta \quad (56)$$

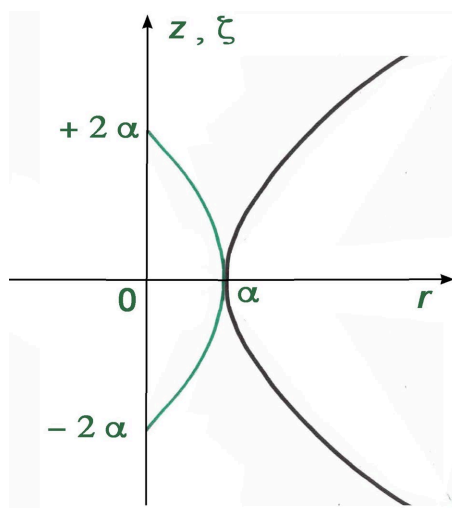
$$ds^2 = r^2 d\varphi^2 - \frac{r}{r - \alpha} dr^2 \quad (57)$$

The induced metric becomes Lorentzian, with signature:

$$(+ -) \quad (58)$$

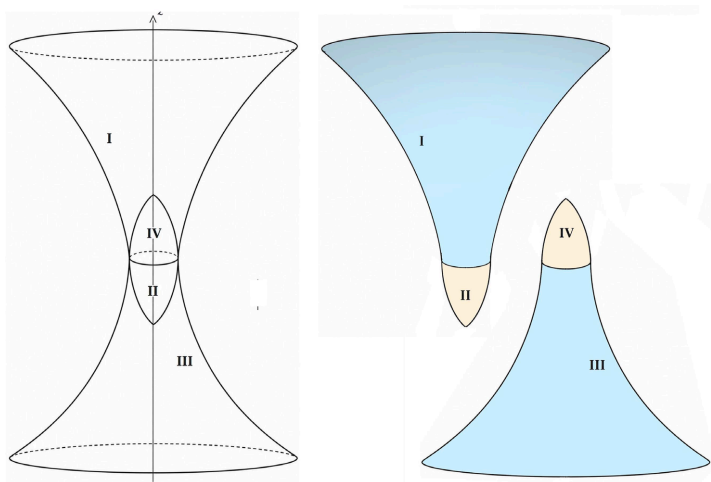
Two embeddings can be considered. The autoparallel curves for  $r > \alpha$ , which

lie on Flamm's surface—and thus serve as its geodesics—can be embedded in a 3D Euclidean space  $(r, \varphi, z)$ , and the meridian of this surface of revolution is a sideways-oriented parabola. The surface generated by rotating this sideways parabola consists of two sheets located on either side of the throat circle; we will designate these as regions I and III. The autoparallel curves corresponding to  $r < \alpha$  lie on a 2D object that constitutes the semi-complex extension of Flamm's surface. This object can be embedded in a 3D Euclidean space  $(r, \varphi, \eta = -\zeta)$ . A representation can be created by identifying  $\zeta$  with  $z$ . This yields two surface elements—portions of sideways parabolas—that meet at the throat circle, each featuring a conical point on the Oz axis. The real meridian and the meridian of the semi-complex extension are shown in **Figure 8** below.



**Figure 8.** Meridians. In black, from the Flamm surface. In green, of its semi-complex extension.

In **Figure 9**, 3D-views.



**Figure 9.** Semi complex extension of the Flamm's surface.

Let's deal with the calculation of the autoparallel curves.

$$L = \frac{1}{2} \left[ \frac{r}{r-\alpha} \dot{r}^2 + r^2 \dot{\phi}^2 \right] \quad (59)$$

$$\frac{\partial L}{\partial \dot{\phi}} = r^2 \dot{\phi} = h \quad (60)$$

$$\dot{\phi} = \frac{h}{r^2} \quad (61)$$

$$1 = \frac{r}{r-\alpha} \dot{r}^2 + r^2 \dot{\phi}^2 \quad (62)$$

$$r > \alpha \rightarrow r > |h| \quad (63)$$

$$r < \alpha \rightarrow r < |h| \quad (64)$$

In the semi-complex part:

$$|h| > \alpha \quad (65)$$

In the vicinity of the throat, when  $r$  tends to  $\alpha$  :

$$\frac{d\phi}{dr} \sim \frac{\text{cst}}{\sqrt{\alpha-r}} \quad (66)$$

$$\frac{dr}{d\phi} \rightarrow 0 \quad (67)$$

In its embedding, the parallel curve is tangent to the circle of striction, the junction between the two surfaces.

Behaviour in  $r = 0$

$$\frac{d\phi}{dr} \sim \frac{1}{\sqrt{\alpha r}} \quad (68)$$

$$\Delta\phi \cong 2\sqrt{\frac{r}{\alpha}} \quad (69)$$

The curve does not spiral. In its semi-complex part, the meridian is the section of a horizontal parabola:

$$\zeta^2 = 4\alpha(\alpha - r) \quad (70)$$

In  $r = 0$ , we have  $\zeta = \pm 2\alpha$  and:

$$g_{\phi\phi} = r^2 \rightarrow 0 \quad (71)$$

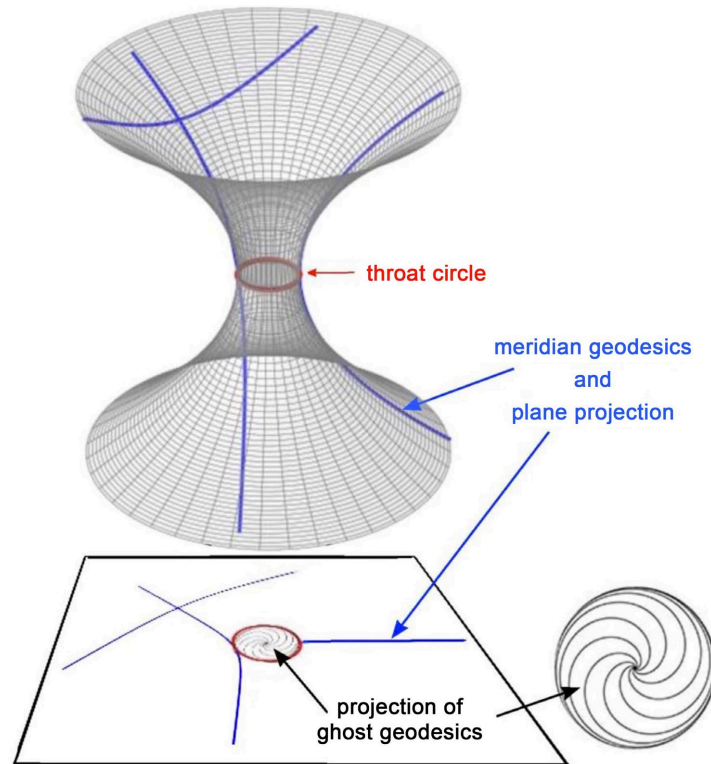
$$g_{rr} = -\frac{r}{\alpha-r} \rightarrow 0 \quad (72)$$

The induced metric is degenerate. In **Figure 10**, below, a perspective view of the Flamm's surface, imbedded in a 3D Euclidean space. Joined, the plane projections of real and ghost geodesics.

## 5. Where Is the Physical World?

In 1909, H. Minkowski invented the concept of spacetime [2] [3] and proposed an initial set of rules of the game. Hereafter, in **Figure 11**, the English translation

of his axiom, presented [3] in 1909.



**Figure 10.** Flamm's surface and the ghost images of the autoparallels of its semi-complex extension.

The axiom means that at every point in the universe the expression

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

is always positive, or, equivalently, that the speed  $v$  is always less than  $c$ . Thus,  $c$  is the upper limit for the speed of any substantial point, and this is precisely where its true meaning lies.

$$d\tau = \frac{1}{c} \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}$$

The integral  $\int d\tau$  of this quantity taken on the world line will be called the proper time (Eingeinzeit) of the substantial point.

**Figure 11.** Minkowski's axiomatic definition [3] of spacetime (1909).

According to its prescriptions, material particles follow geodesics defined by a length equated to a real proper time. Photons follow geodesics of zero length within the manifold. This approach would subsequently be adopted—either explicitly or implicitly—by Einstein [4] and Schwarzschild [5] [6]. In **Figure 12**, the English translation of the very beginning of his paper [5].

Droste [7], Flamm [1], and Weyl [8] adopt the same view of physics. In **Figure 13**, the English translation of Weyl's article.

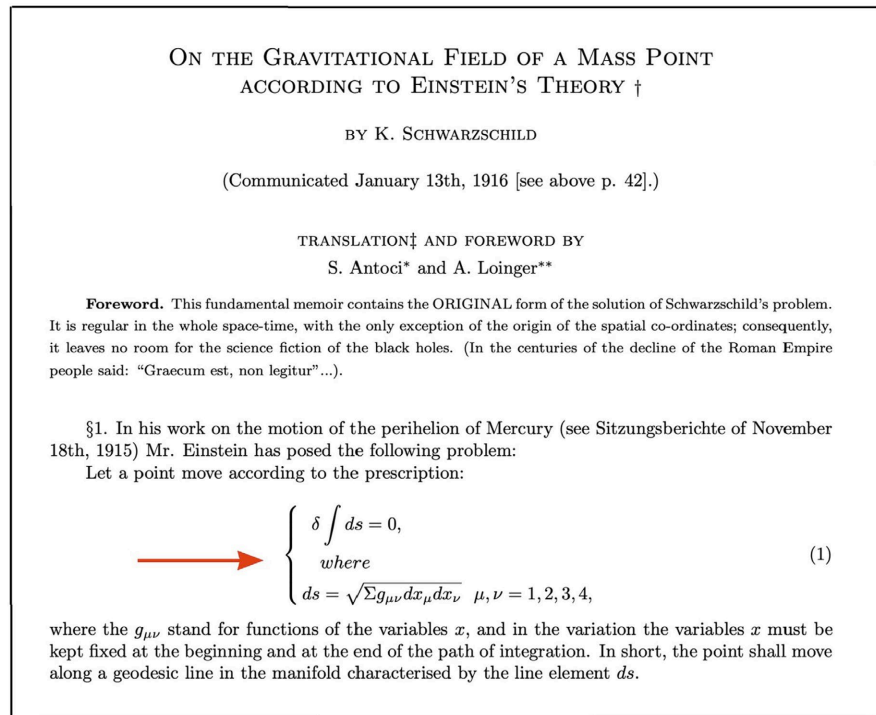


Figure 12. Schwarzschild foreword [5].

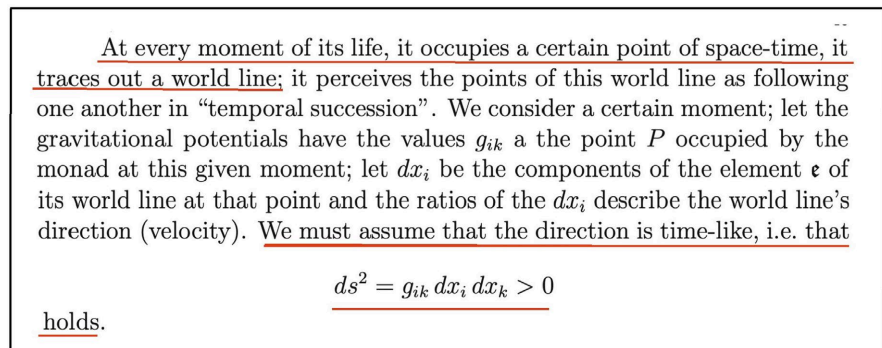


Figure 13. Weyl restricts himself to real proper time [8].

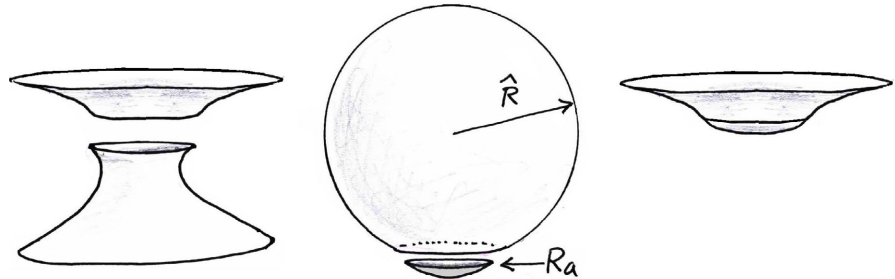
For H. Weyl, the path of a material particle lies along a trajectory endowed with a necessarily real length [8]. In this context, any solution curve possessing an imaginary length element does not belong to the physical world and lies outside the real hypersurface. All this is consistent with a Lorentzian signature  $(+---)$ .

## 6. In Search of the Geometry of the Hypersurface

In 1916, L. Flamm provided a complete geometric interpretation of the two solutions presented by Schwarzschild. In the exterior region, where  $r > R_a$ , the cross-section of the solution hypersurface at constant  $t$  and  $\theta$  is a portion of one of the two sheets of the surface generated by rotating a sideways-oriented parabola. In the interior region, where  $0 < r < R_a$ , the cross-section of the solution hyper-

surface at constant  $t$  and  $\theta$  is a portion of a sphere of radius. In **Figure 14**, building the final cut of the Schwarzschild global surface at  $t$  and  $\theta$  constants.

$$\hat{R} = \sqrt{\frac{3c^2}{8\pi G\rho}} \quad (73)$$



**Figure 14.** The Flamm's construction [9].

However, in January 1916, Karl Schwarzschild did not present the true expression of his exterior metric solution [5], expressed in terms of his coordinates  $(t, r, \theta, \varphi)$  which would not be published in that form until 2026 [9].

$$ds^2 = \frac{(r^3 + \alpha^3)^{1/3} - \alpha}{(r^3 + \alpha^3)^{1/3}} c^2 dt^2 - \frac{r^4 dr^2}{(r^3 + \alpha^3) \left[ (r^3 + \alpha^3)^{1/3} - \alpha \right]} - (r^3 + \alpha^3)^{2/3} (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (74)$$

With:

$$r = \sqrt{x^2 + y^2 + z^2} \geq 0 \quad x, y, z, r \in \mathbb{R} \quad (75)$$

In this context (with real variables  $x, y, z$ , and  $r$ ), the topology of the solution hypersurface is clearly revealed. The variable  $r$ —a simple spatial marker—is not a “radius” in the Euclidean sense. The surface can be foliated by a family of  $\mathcal{S}^2$  spheres with area:

$$A = 4\pi (r^3 + \alpha^3)^{2/3} \quad (76)$$

With a minimum value in  $r = 0$

$$A_{\min} = 4\pi\alpha^2 \quad (77)$$

The hypersurface associated with the exterior metric solution is therefore non-contractible, a fact confirmed by its cross-sections at constant  $t$  and  $\theta$  (Flamm's paraboloid). However, Schwarzschild published his result in this form:

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) c^2 dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad R = (r^3 + \alpha^3)^{1/3} \quad (78)$$

By explicitly stating, in a letter sent to Einstein [10], that this quantity  $R$  is merely an “intermediate quantity” (*Hilfsgroße*).

David Hilbert modified his ambitious 1915 article, “The Foundations of Physics”

[11], and set out to incorporate Schwarzschild's exterior metric solution into a new paper published in December 1916 [12]—a paper in which he made no mention whatsoever of the existence of an interior metric solution. He defined geometry as deriving not from the expression of an elementary length (the term “ $ds$ ” is absent from his article) but from a bilinear form that could take on either sign; depending on that sign, this approach led to the definition of two lengths associated with two sets of solution curves, described either as timelike trajectories or as “segments” (spacelike). He made no mention of the interior metric. As explained in detail in [9], in his reconstruction of the solution to the Einstein equation, he “de facto” conflated Schwarzschild's coordinates “ $r$ ” and “ $R$ ” while making what appeared to him to be a logical simplification to reach the solution more quickly—a confusion first pointed out in 1989, 73 years later, by the Canadian mathematician Abrams [13].

This confusion is implicitly reflected in the works published by A. Eddington in 1918 [14] and R. C. Tolman in 1934 [15]—extensively documented works that subsequently popularized general relativity worldwide.

In 1939, Albert Einstein published a paper [16] containing a few lines in which—without relying on substantive arguments—he expressed skepticism regarding the description of geometry within masses as presented by Schwarzschild in his second paper of 1916. This may well have deterred many scientists from focusing on that work, particularly given that Schwarzschild's paper was difficult to read, was written in German, and was not translated into English until 1999.

These two factors focus the scientific community's interest on the exterior metric. Many propose changes of variables aimed at eliminating the singularity at  $r = \alpha$ . The challenge is, in particular, to ensure that the metric coincides with the Lorentz metric at infinity. In 1935, A. Einstein and N. Rosen proposed [17] the change of variable:

$$u^2 = r - 2\alpha \quad (79)$$

And get:

$$ds^2 = \frac{u^2}{u^2 + \alpha} dt^2 - 4(u^2 + \alpha) du^2 - (u^2 + \alpha)^2 (d\theta^2 + \sin^2 \theta) \quad (80)$$

Their goal was to propose a geometric description of masses. They introduce the concept of wormhole, writing:

*We call such a connection between two sheets a “bridge”.*

However, this metric is not Lorentzian at infinity. In 1924, however, A. Eddington [18] proposed a change of variable involving the time coordinate that resolves the issue. Denoting “Schwarzschild time” by  $t_s$  and Eddington time by  $t_E$ , we have:

$$t_s = t_E + \varepsilon \frac{\alpha}{c} \ln \left( \frac{r}{\alpha} - 1 \right) \quad \varepsilon = \pm 1 \quad (81)$$

$$ds^2 = \left( 1 - \frac{\alpha}{r} \right) c^2 dt_E^2 + \varepsilon c \alpha \frac{dt_E dr}{r} - \left( 1 + \frac{\alpha}{r} \right) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (82)$$

The coordinate singularity is eliminated, but a cross-term in  $drdt_E$  appears.

In 1923, G.D. Birkhoff [19] stated a theorem:

Any spherically symmetric solution to Einstein's vacuum field equations is locally isometric to the Schwarzschild metric.

This result does not rule out the presence of a cross-term in the metric, provided that it arises from a simple coordinate change and the new formulation remains isometric to the Schwarzschild solution. This is precisely the case for the Eddington-Finkelstein metric, which is derived from the Schwarzschild metric via a transformation of the time coordinate and introduces a  $dt dr$  cross-term without altering the underlying geometry. A metric lacking a cross-term and admitting a hypersurface-orthogonal timelike Killing field is termed static. The Eddington-Finkelstein solution, by contrast, is stationary but not static, while remaining isometric to Schwarzschild.

In 1939, Oppenheimer and Snyder [20]—drawing on the fact that, in the Schwarzschild solution to the vacuum Einstein field equations, an external observer assigns an infinite Schwarzschild time to the crossing of the surface  $r = \alpha$  (an infinite value for the free-fall time of a test particle, corresponding to an infinite escape time)—proposed [19] that this solution describes the highly non-stationary phenomenon of the implosion of a massive star at the end of its life.

D. Finkelstein [21] in 1958 and H. A. Buchdahl [22] in 1959—recalling that non-static solutions containing a cross-term in  $drdt$  are solutions to the vacuum Einstein equations—begin by presenting the most general form of the exterior metric, as given by Tolman [15]:

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) c^2 dt^2 + q \frac{dt dr}{r} - \frac{dr^2}{1 - \frac{\alpha}{Rr}} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (83)$$

However, they do not associate this form with a physical solution and hasten to point out that the cross term can be eliminated by a change of variable involving the time coordinate (the inverse of the Eddington coordinate transformation [18]). Nevertheless, Finkelstein, focusing on the idea of a past-future asymmetry, writes:

*The surface is thus a true unidirectional membrane: causal influences can pass through it only in one sens. This again demonstrates the asymmetry between past and future. Moreover it demonstrates the existence of two distinct completions of the Schwarzschild solution is asserted. One has the structure.*

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) c^2 dt^2 + \frac{2\alpha c dt dr}{r} - \frac{dr^2}{1 - \frac{\alpha}{Rr}} - r^2 \Omega^2 \quad (84)$$

*and the other, obtained by time-reversal, has a negative coefficient for  $drdt$ . The completions must be distinct even if the resulting manifolds are isomorphic under  $t$ , because a particular geodesic segment reaches the center and in the other completion does not. A transformation of the variable  $t$  that cancels the cross term*

shows that the “core” of the manifold is that of the interior solution<sup>1</sup>. The only information, in this work, accordingly, is the connection between the interior and exterior solutions to form a single manifold.

The community is thus still seeking a geometric interpretation of this Schwarzschild solution. It is worth noting that Finkelstein suggests a two-sheeted, T-symmetric extension.

Incidentally, in the post-war period—and without any supporting paper—the cosmology community gradually opted for a change of signature:

$$(+---) \rightarrow (-+++)$$
 (85)

Thus, what is henceforth considered the standard form of the Schwarzschild metric becomes:

$$ds^2 = -\left(1 - \frac{\alpha}{r}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{\alpha}{Rr}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$
 (86)

In 1959, during a private meeting, Martin Kruskal presented to John Archibald Wheeler a coordinate transformation he had devised to eliminate the coordinate singularity at  $r = \alpha$ . Wheeler immediately grasped its significance and presented the construction a few weeks later at a colloquium in Royaumont, France. He was so enthusiastic and eager to share this with the scientific community that he wrote the paper [23] himself and proposed that Kruskal put his name to it—a proposal Kruskal accepted<sup>2</sup>. It should be noted that the change of variable is presented without details regarding its construction.

We now arrive at the core of this article’s subject. We must therefore present a version that can be considered standard, and we opt for the one found in Wald’s work [24].

## 7. The Standard Construction of the Kruskal Representation [24]

| Région I                                      | Région II                                                       |
|-----------------------------------------------|-----------------------------------------------------------------|
| $r > \alpha \quad A(r) > 0$ (87.I)            | $r < \alpha \quad A(r) < 0$ (87.II)                             |
| $ds^2 = A(r)dt^2 - \frac{1}{A(r)}dr^2$ (88.I) | $ds^2 = \frac{1}{ A(r) }dr^2 -  A(r) dt^2$ (88.II)              |
| $t$ is timelike. $r$ is spacelike.            | $\rightarrow r$ becomes timelike and $t$ is spacelike, in other |

<sup>1</sup>Finkelstein thus discusses Schwarzschild’s second solution—describing the geometry inside the mass—which he implicitly cites by referring to Tolman’s writings.

<sup>2</sup>It would be of considerable historical interest to determine to what extent the global four-region interpretation already existed in Kruskal’s original work or whether it emerged through Wheeler’s physical interpretation of the new coordinate system. The proceedings of the Royaumont conference (June 1959), where Wheeler first presented Kruskal’s construction before its publication, could shed light on this question.

We perform the change of variable:

$$r^* = r + \alpha \ln\left(\frac{r}{\alpha} - 1\right) \quad (89.I)$$

$$dr^* = dr \left(1 + \frac{1}{\frac{r}{\alpha} - 1}\right) = dr \left(\frac{1}{1 - \frac{r}{\alpha}}\right) \quad (90.I)$$

$$dr^* = \frac{dr}{A(r)} \quad (91.I)$$

The line element becomes:

$$ds^2 = A(r)(dt^2 - dr^{*2}) \quad (92.I)$$

We introduce null coordinates:

$$x^- = t - r^* \quad (93a.I)$$

$$x^+ = t + r^* \quad (93b.I)$$

Whence:

$$dt = \frac{1}{2}(dx^+ + dx^-) \quad (94.I)$$

$$dr^* = \frac{1}{2}(dx^+ - dx^-) \quad (95.I)$$

$$x^- = t - r^* \quad (96.I)$$

$$x^+ = t + r^* \quad (97.I)$$

$$dt = \frac{1}{2}(dx^+ + dx^-) \quad (98.I)$$

$$dr^* = \frac{1}{2}(dx^+ - dx^-) \quad (99.I)$$

$$ds^2 = A(r)(dt^2 - dr^{*2}) = A(r)dx^+dx^- \quad (100.I)$$

In the outer region  $r > \alpha$  we set up:

$$U = -\exp\left(-\frac{x^-}{2\alpha}\right) \quad (101.I)$$

$$V = \exp\left(\frac{x^+}{2\alpha}\right) \quad (102.I)$$

$$dU = \frac{1}{2\alpha} \exp\left(-\frac{x^-}{2\alpha}\right) dx^- - \frac{U}{2\alpha} dx^- \quad (103.I)$$

$$dV = \frac{V}{2\alpha} dx^+ \quad (104.I)$$

words, the causal and temporal nature of the directions  $\frac{\partial}{\partial t}$  et  $\frac{\partial}{\partial r}$  are swapped.

→ We use a *different* change of variable:

$$r^* = r + \alpha \ln\left(1 - \frac{r}{\alpha}\right) \quad (89.II)$$

$$dr^* = dr \left(1 - \frac{1}{1 - \frac{r}{\alpha}}\right) = dr \left(-\frac{\alpha}{\alpha - r}\right) \quad (90.II)$$

$$dr^* = \frac{dr}{A(r)} \quad (91.II)$$

The line element becomes:

$$ds^2 = A(r)(dt^2 - dr^{*2}) \quad (92.II)$$

We introduce null coordinates:

$$x^- = t - r^* \quad (93a.II)$$

$$x^+ = t + r^* \quad (93b.II)$$

Whence:

$$dt = \frac{1}{2}(dx^+ + dx^-) \quad (94.II)$$

$$dr^* = \frac{1}{2}(dx^+ - dx^-) \quad (95.II)$$

$$x^- = t - r^* \quad (96.II)$$

$$x^+ = t + r^* \quad (97.II)$$

$$dt = \frac{1}{2}(dx^+ + dx^-) \quad (98.II)$$

$$dr^* = \frac{1}{2}(dx^+ - dx^-) \quad (99.II)$$

$$ds^2 = A(r)(dt^2 - dr^{*2}) = A(r)dx^+dx^- \quad (100.II)$$

→ In the inner region  $r < \alpha$  we set up:

(This is the second act of this mathematical artifice).

$$U = +\exp\left(-\frac{x^-}{2\alpha}\right) \quad (101.II)$$

$$V = \exp\left(\frac{x^+}{2\alpha}\right) \quad (102.II)$$

$$dU = \frac{1}{2\alpha} \exp\left(-\frac{x^-}{2\alpha}\right) dx^- - \frac{U}{2\alpha} dx^- \quad (103.II)$$

$$dV = \frac{V}{2\alpha} dx^+ \quad (104.II)$$

$$dUdV = -\frac{UV}{4\alpha^2} dx^+ dx^- \quad (105.I)$$

$$dx^+ dx^- = -\frac{4\alpha^2}{UV} dUdV \quad (106.I)$$

Moreover:

$$UV = -\exp\left(\frac{x^+ - x^-}{2\alpha}\right) \quad (107.I)$$

$$x^+ - x^- = 2r^* \quad (108.I)$$

$$\frac{r^*}{\alpha} = \frac{r}{\alpha} + \ln\left(\frac{r}{\alpha} - 1\right) \quad (109.I)$$

$$e^{r^*/\alpha} = \left(\frac{r}{\alpha} - 1\right) e^{r/\alpha} \quad (110.I)$$

$$UV = -\left(\frac{r}{\alpha} - 1\right) e^{r/\alpha} \quad (111.I)$$

The line element becomes:

$$ds^2 = -4\alpha^2 \frac{A(r)}{UV} dUdV \quad (112.I)$$

That's to say:

$$ds^2 = \frac{4\alpha^3}{r} e^{-r/\alpha} dUdV \quad (113.I)$$

In four dimensions:

$$ds^2 = \frac{4\alpha^3}{r} e^{-r/\alpha} dUdV - r^2 d\Omega^2 \quad (114.I)$$

$$U = v - u \quad V = v + u \quad (115.I)$$

$$dUdV = dv^2 - du^2 \quad (116.I)$$

Whence:

$$ds^2 = \frac{4\alpha^3}{r} e^{-2\alpha} (dv^2 - du^2) - r^2 d\Omega^2 \quad (117.I)$$

$$UV = (v - u)(v + u) = v^2 - u^2 \quad (118.I)$$

In this region I:  $U < 0$   $V > 0$  we set up:

$$q_{ext} = e^{r/2\alpha} \sqrt{\frac{r}{\alpha} - 1} \quad (119.I)$$

$$U = -q_{ext} e^{-t/2\alpha} \quad (120.I)$$

$$V = q_{ext} e^{t/2\alpha} \quad (121.I)$$

$$u = \frac{V - U}{2} q_{ext} \operatorname{ch}\left(\frac{t}{2\alpha}\right) \quad (122.I)$$

$$dUdV = -\frac{UV}{4\alpha^2} dx^+ dx^- \quad (105.II)$$

$$dx^+ dx^- = -\frac{4\alpha^2}{UV} dUdV \quad (106.II)$$

Moreover:

$$UV = -\exp\left(\frac{x^+ - x^-}{2\alpha}\right) \quad (107.II)$$

$$x^+ - x^- = 2r^* \quad (108.II)$$

$$\frac{r^*}{\alpha} = \frac{r}{\alpha} + \ln\left(1 - \frac{r}{\alpha}\right) \quad (109.II)$$

$$e^{r^*/\alpha} = \left(1 - \frac{r}{\alpha}\right) e^{r/\alpha} \quad (110.II)$$

$$UV = +\left(1 - \frac{r}{\alpha}\right) e^{r/\alpha} \quad (111.II)$$

The line element becomes:

$$ds^2 = -4\alpha^2 \frac{A(r)}{UV} dUdV \quad (112.II)$$

That's to say:

$$ds^2 = \frac{4\alpha^3}{r} e^{-r/\alpha} dUdV \quad (113.II)$$

In four dimensions:

$$ds^2 = \frac{4\alpha^3}{r} e^{-r/\alpha} dUdV - r^2 d\Omega^2 \quad (114.II)$$

$$U = v - u \quad V = v + u \quad (115.II)$$

$$dUdV = dv^2 - du^2 \quad (116.II)$$

→ We get the *same* line element:

$$ds^2 = \frac{4\alpha^3}{r} e^{-2\alpha} (dv^2 - du^2) - r^2 d\Omega^2 \quad (117.II)$$

$$UV = (v - u)(v + u) = v^2 - u^2 \quad (118.II)$$

In this region II:  $U > 0$   $V > 0$  we set up:

$$q_{int} = e^{r/2\alpha} \sqrt{1 - \frac{r}{\alpha}} \quad (119.II)$$

$$U = +q_{int} e^{-t/2\alpha} \quad (120.II)$$

$$V = q_{int} e^{t/2\alpha} \quad (121.II)$$

→ And that is where the *permutation*  $\operatorname{sh} \leftrightarrow \operatorname{ch}$  appears.

$$u = \frac{V - U}{2} q_{int} \operatorname{sh}\left(\frac{t}{2\alpha}\right) \quad (122.II)$$

$$v = \frac{V+U}{2} q_{ext} \operatorname{sh}\left(\frac{t}{2\alpha}\right) \quad (123.I)$$

Hence the change of variables:

$$u = e^{r/2\alpha} \sqrt{\frac{r}{\alpha} - 1} \operatorname{ch}\left(\frac{t}{2\alpha}\right) \quad (124.I)$$

$$v = e^{r/2\alpha} \sqrt{\frac{r}{\alpha} - 1} \operatorname{sh}\left(\frac{t}{2\alpha}\right) \quad (125.I)$$

$$u^2 - v^2 = \left(\frac{r}{\alpha} - 1\right) > 0. \quad (126.I)$$

$$v = \frac{V+U}{2} q_{int} \operatorname{ch}\left(\frac{t}{2\alpha}\right) \quad (123.II)$$

Hence the change of variables:

$$u = e^{r/2\alpha} \sqrt{1 - \frac{r}{\alpha}} \operatorname{sh}\left(\frac{t}{2\alpha}\right) \quad (124.II)$$

$$v = e^{r/2\alpha} \sqrt{1 - \frac{r}{\alpha}} \operatorname{ch}\left(\frac{t}{2\alpha}\right) \quad (125.II)$$

$$u^2 - v^2 = -\left(1 - \frac{r}{\alpha}\right) < 0 \quad (126.II)$$

We thus see that this result is obtained through the use of two computational devices:

A—We have two changes of variables that differ for  $r > \alpha$  and for  $r < \alpha$ .

$r > \alpha$ :

$$r^* = r + \alpha \ln\left(\frac{r}{\alpha} - 1\right) \quad (127.I)$$

$r < \alpha$

$$r^* = r + \alpha \ln\left(1 - \frac{r}{\alpha}\right) \quad (127.II)$$

B—Different sectors are created through different choices of signs in the definition of the exponential variables  $U$  and  $V$ .

Sector I

$$U = -\exp\left(-\frac{x^-}{2\alpha}\right) \quad (128.I)$$

$$V = +\exp\left(-\frac{x^+}{2\alpha}\right) \quad (129.I)$$

Sector II

$$U = +\exp\left(-\frac{x^-}{2\alpha}\right) \quad (128.II)$$

$$V = +\exp\left(-\frac{x^+}{2\alpha}\right) \quad (129.II)$$

Additional calculations could be performed based on the choices:

Sector III

$$U = +\exp\left(-\frac{x^-}{2\alpha}\right) \quad (130.I)$$

$$V = -\exp\left(-\frac{x^+}{2\alpha}\right) \quad (131.I)$$

Sector IV

$$U = -\exp\left(-\frac{x^-}{2\alpha}\right) \quad (130.II)$$

$$V = -\exp\left(-\frac{x^+}{2\alpha}\right) \quad (131.II)$$

It is thanks to this that the permutation of hyperbolic lines appears in the various changes of variable.  $\boxed{\operatorname{sh} \rightleftharpoons \operatorname{ch}}$

These calculations gave rise to the Kruskal diagram. See **Figure 15** below.

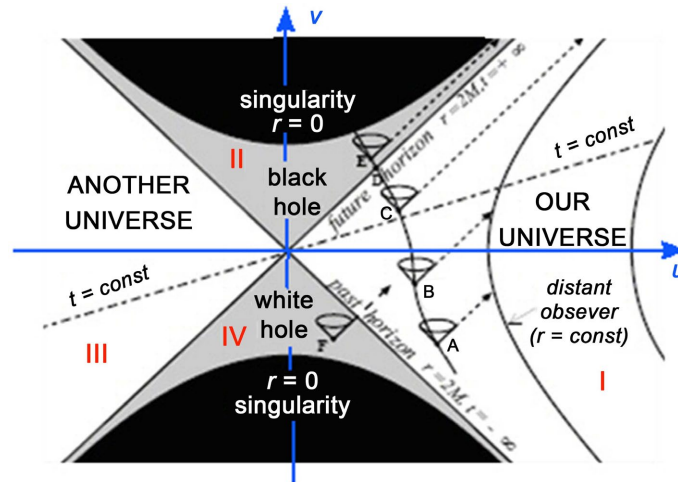


Figure 15. The Kruskal diagram.

## 8. Interpretation of the Region $r < \alpha$ as a Semi-Complex Extension of the Manifold

We start again from the form of the metric with constant  $\theta$  and  $\varphi$  :

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} \quad (132)$$

The 1960 article specifies that the aim is to express it in the form:

$$ds^2 = F^2(r)(dv^2 - du^2) \quad (133)$$

The curves with  $ds = 0$  correspond to:

$$\frac{du}{dv} = \pm 1 \quad (134)$$

The coordinates  $u$  and  $v$  form a conformally Minkowskian pair; their combinations  $v-u$  and  $v+u$  are null coordinates in the strict sense.

Let's introduce a change of variable of the form:

$$u = f(r)\text{ch}(\eta t) \quad (135)$$

$$v = f(r)\text{sh}(\eta t) \quad (136)$$

$\eta$  being a constant to be determined.

$$du = f'(r)\text{ch}(\eta t)dr + \eta f(r)\text{sh}(\eta t)dt \quad (137)$$

$$dv = f'(r)\text{sh}(\eta t)dr + \eta f(r)\text{ch}(\eta t)dt \quad (138)$$

This allows the cross-terms to be eliminated by forming:

$$dv^2 - du^2 = \eta^2 f^2 dt^2 - f'^2 dr^2 \quad (139)$$

By identifying:

$$F^2 \eta^2 f^2 = A \quad (140)$$

$$F^2 f'^2 = A^{-1} \quad (141)$$

Whence:

$$\frac{f'}{f} = \pm \frac{\eta}{A} \quad (142)$$

We choose the sign +

$$\frac{df}{f} = \eta \left( 1 + \frac{\alpha}{r - \alpha} \right) dr \quad (143)$$

By integrating:

$$\ln f = \eta r + \alpha \ln(r - \alpha) + C \quad (144)$$

As long as  $r > \alpha$ , we have the logarithm of a positive quantity—that's fine, let's continue. So:

$$f(r) = C e^{\eta r} (r - \alpha)^{\alpha \eta} \quad (145)$$

The preceding relations yield:

$$f^2 = C^2 e^{2\eta r} (r - \alpha)^{2\alpha \eta} \quad (146)$$

Whence:

$$F^2 = \frac{1}{\eta^2 C^2 r} e^{-2\eta r} \left( \frac{r}{\alpha} - 1 \right)^{1-2\alpha \eta} \quad (147)$$

Which gives:

$$ds^2 = \frac{1}{\eta^2 C^2 r} e^{-2\eta r} \left( \frac{r}{\alpha} - 1 \right)^{1-2\alpha \eta} (dv^2 - du^2) \quad (148)$$

If  $\eta < \frac{1}{2\alpha}$  then, when:

$$r \rightarrow \alpha \Rightarrow F^2 \rightarrow 0 \quad (149)$$

If  $\eta > \frac{1}{2\alpha}$  then, when:

$$r \rightarrow \alpha \Rightarrow F^2 \rightarrow +\infty \quad (150)$$

To avoid this, we choose:

$$\eta = \frac{1}{2\alpha} \quad \text{et} \quad C = 1 \quad (151)$$

Which gives:

$$ds^2 = \frac{4\alpha^3}{r} e^{-r/\alpha} \left( \frac{r}{\alpha} - 1 \right)^0 (dv^2 - du^2) \quad (152)$$

For  $r > \alpha$ , this factor is identically equal to unity. Its value at  $r = \alpha$  is defined by the continuous extension of the conformal factor, rather than by an independent evaluation of the formal expression  $0^0$ . We thus obtain:

$$ds^2 = \frac{4\alpha^3}{r} e^{-r/\alpha} (dv^2 - du^2) \quad (153)$$

And, in four dimensions:

$$ds^2 = \frac{4\alpha^3}{r} e^{-r/\alpha} (dv^2 - du^2) - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (154)$$

And:

$$u = e^{r/2\alpha} \sqrt{r-\alpha} \operatorname{ch}\left(\frac{t}{2\alpha}\right) \quad (155)$$

$$v = e^{r/2\alpha} \sqrt{r-\alpha} \operatorname{sh}\left(\frac{t}{2\alpha}\right) \quad (156)$$

Let us revisit this construction for  $0 < r < \alpha$ . We obtain:

$$u = i e^{r/2\alpha} \sqrt{\alpha-r} \operatorname{ch}\left(\frac{t}{2\alpha}\right) \quad (157)$$

$$v = i e^{r/2\alpha} \sqrt{\alpha-r} \operatorname{sh}\left(\frac{t}{2\alpha}\right) \quad (158)$$

If  $t$  is real, then the coordinates  $u$  and  $v$  become purely imaginary. Let us introduce a translation by a purely imaginary amount  $-i\frac{\pi}{2}$  and make use of the relation:

$$\operatorname{ch}\left(\tau - i\frac{\pi}{2}\right) = -i \operatorname{sh}(\tau) \quad (159)$$

$$\operatorname{sh}\left(\tau - i\frac{\pi}{2}\right) = -i \operatorname{ch}(\tau) \quad (160)$$

When applied to the time variable  $t$ , a purely imaginary translation suffices:

$$t \rightarrow t - i\pi\alpha \quad (161)$$

Then, for  $0 < r < \alpha$  we get:

$$u = e^{r/2\alpha} \sqrt{r-\alpha} \operatorname{sh}\left(\frac{t}{2\alpha}\right) \quad (162)$$

$$v = e^{r/2\alpha} \sqrt{r-\alpha} \operatorname{ch}\left(\frac{t}{2\alpha}\right) \quad (163)$$

This refers to Kruskal's change of variables. However, for a purely imaginary translation to be performed on a variable, that variable must be complex.

*We are thus dealing with a genuine sleight of hand, enabling the description of a complex geometric extension using coordinates that can be termed pseudo-real.*

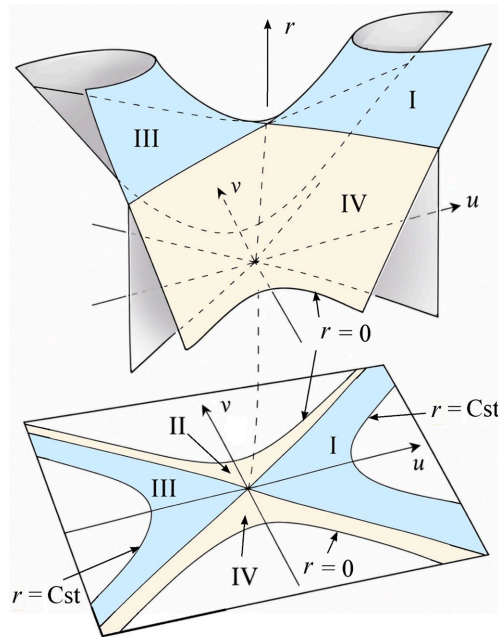
An alternative construction thus emerges, leading to a semi-complex geometric object. Just as we did earlier with the 2D objects—the sphere and Flamm's surface—we obtain, in **Figure 16**, the drawing below.

The blue folds correspond to the real values of the coordinates  $u$  and  $v$ ; the yellow surfaces, to imaginary folds.

Thus, the four-dimensional solution hypersurface in  $(t, r, \theta, \varphi)$  lends itself to two representations based on the form of two intersections:

- At  $t$  and  $\theta$  constants, this gives us a semi-complex representation in  $(r, \varphi)$

- At  $\theta$  and  $t$   $\varphi$  constants this gives us a semi-complex representation in  $(t, r)$



**Figure 16.** Kruskal complex extension (yellow).

In both cases, one obtains a complex hypersurface comprising four sectors that correspond to one another in pairs. This reflects a fundamental concept in differential geometry: that the nature of a geometric object cannot depend on the choice of coordinates used to describe it.

At the bottom, the Kruskal diagram. At the top, the surface  $r(u, v)$ . It indeed possesses four sheets. Sheets II and IV correspond to purely imaginary values of the coordinates  $u$  and  $v$  which a sleight-of-hand maneuver allows to be converted into real quantities. There is thus a one-to-one correspondence with the four sheets of the complex extension of the Flamm surface. This demonstrates that the geometric properties of this object—existing in a complex space—are intrinsic and independent of the chosen mode of representation, a fact that aligns with the principles of differential geometry.

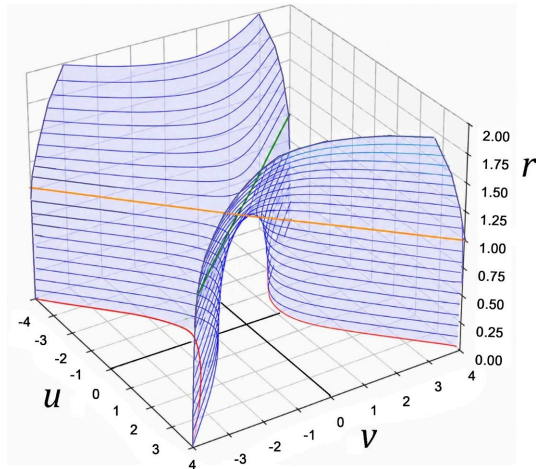
This surface can be constructed using a foliation of hyperbolas defined by the equations:

$$u^2 - v^2 = \frac{e^{r/\alpha}}{r - \alpha} \quad (164)$$

Depending on the values of  $r > 0$ , four branches are obtained. The real branches connect to the complex branches along lines  $u = \pm v$  situated in the plane  $r = \alpha$ . The point  $(0, 0, \alpha)$  lies on a saddle point of the surface. The surface is bounded by hyperbolas located at  $r = 0$ , which are the image of the central singularity. It has been arbitrarily bounded by the plane  $r = 2\alpha$ , but these elements I and III extend to infinity.

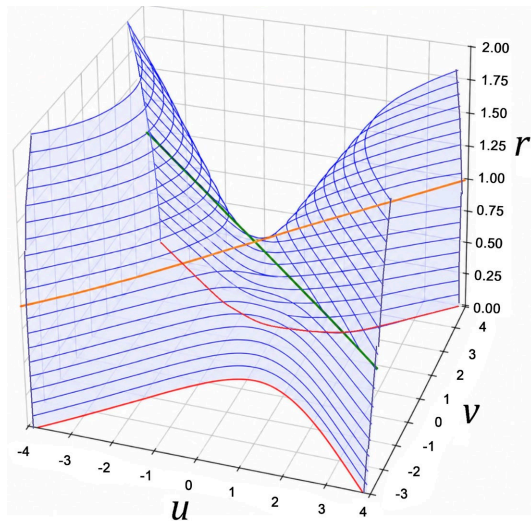
To the best of our knowledge, the literature generally depicts the Kruskal diagram in the  $(u, v)$  plane. We have found no study interpreting the implicit relation  $r = H(u, v)$  as a three-dimensional surface foliated by hyperbolas of constant  $r$ .

Here are some other images—this time, computer-generated. See **Figure 17**.



**Figure 17.** Kruskal complex extension.  $r > 1$  corresponds to real world.

And in **Figure 18**, another view.



**Figure 18.** Kruskal complex extension.  $0 < r < 1$  is the semi-complex world.

We will now consider another way of proceeding.

## 9. Two Different Representations of the Schwarzschild Geometry

On the left, the Kruskal representation. On the right, construction using a single change of variables.

**The Kruskal representation:**

It is arbitrarily decided that the following expression will henceforth be called the standard form of the Schwarzschild solution to the source-free Einstein equation.

The line element becomes:

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2 d\Omega^2 \quad (169)$$

In the real field, coordinate singularity for  $g_{RR}$  in  $r = \alpha$  and  $r = 0$ .

At this stage, the signature is still  $(+---)$

But one suddenly ceases to regard the reality of length as a criterion for belonging to physics.

So, arbitrarily and without any argument, they decide to change their signature.

$$( + --- ) \rightarrow ( - + + + ) \quad (171)$$

It is decided that this spherically symmetric stationary solution describes a physical object and that the “standard Schwarzschild metric” is:

$$ds^2 = - \left(1 - \frac{\alpha}{r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{\alpha}{r}} + r^2 d\Omega^2 \quad (173)$$

It is decided that if we obtain a representation of the geometric object using real coordinates  $(u, v, \theta, \varphi)$  this corresponds to a real physical object, known as a black hole. This representation is obtained from the coordinates.

$(t, r, \theta, \varphi)$ . But they are the coordinates  $(t, R, \theta, \varphi)$  where the intermediate Schwarzschild's quantity  $R$  is assimilated to a radial  $r$  coordinate.

It is set out to eliminate the singularity at  $r = \alpha$  and to obtain a representation using two variables,  $u$  and  $v$  such that light rays correspond to:

$$\frac{du}{dv} = \pm 1 \quad (177)$$

We consider a cross-section at constant  $\theta$  and  $\varphi$ . Then, two different changes of variable

**Construction using a single change of variables:**

We stick to the coordinates chosen by Schwarzschild:

$$(t, x, y, z) \in \mathbb{R}^4 \quad (165)$$

With its polar variable:

$$r = \sqrt{x^2 + y^2 + z^2} \geq 0 \quad (166)$$

He gets:

$$ds^2 = \frac{(r^3 + \alpha^3)^{1/3} - \alpha}{(r^3 + \alpha^3)^{1/3}} c^2 dt^2 - \frac{r^4 dr^2}{(r^3 + \alpha^3) \left[ (r^3 + \alpha^3)^{1/3} - \alpha \right]} \quad (167)$$

It is noted that this representation is free of singularities and that this portion of spacetime is non-contractible: within a foliation at constant  $r$ , the area of this sphere has a minimum value of  $4\pi\alpha^2$

Schwarzschild opts for a simpler presentation by introducing an intermediate variable  $R$

$$R = (r^3 + \alpha^3)^{1/3} \Rightarrow R \geq \alpha \quad (168)$$

The line element becomes:

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) c^2 dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2 d\Omega^2 \quad (170)$$

In the real field, coordinate singularity for  $g_{RR}$  in  $R = \alpha$

The signature is  $(+---)$

Quel que soit l'usage qui soit fait de cette métrique on ne lui attribue un lien avec la physique que si l'élément  $ds$  (le temps propre) est réel.

Schwarzschild n'attribue de sens physique à cette métrique que si celle-ci se trouve raccordée à une métrique intérieure, de mêmes symétries.

We keep the signature:

$$( + --- ) \quad (172)$$

We're staying in shape:

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) c^2 dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2 d\Omega^2 \quad (174)$$

We use Eddington's time change of variable [14] to eliminate the coordinate singularity in  $r = \alpha$

$$t = t_E + \varepsilon \frac{\alpha}{c} \ln \left( \frac{R}{\alpha} - 1 \right) \quad \varepsilon = \pm 1 \quad (175)$$

are implemented:

For  $r > \alpha$

$$r^* = r + \alpha \ln \left( \frac{r}{\alpha} - 1 \right) \quad (178)$$

For  $r < \alpha$

$$r^* = r + \alpha \ln \left( 1 - \frac{r}{\alpha} \right) \quad (179)$$

Exponential coordinates  $U$  and  $V$  are then introduced:

$$U = \varepsilon_1 \exp \left( \frac{r^* - t}{2\alpha} \right) \quad (180)$$

$$V = \varepsilon_2 \exp \left( \frac{r^* + t}{2\alpha} \right) \quad (181)$$

A description is then obtained using a single metric:

$$ds^2 = \frac{4\alpha^3}{r} e^{-2\alpha} (dv^2 - du^2) - r^2 d\Omega^2 \quad (184)$$

We then introduce four possible choices regarding the pairs ( $\varepsilon_1 = \pm 1$ ;  $\varepsilon_2 = \pm 1$ ) that arbitrarily define four sectors.

|            | $\varepsilon_1$ | $\varepsilon_2$ |
|------------|-----------------|-----------------|
| Sector I   | -1              | +1              |
| Sector II  | +1              | +1              |
| Sector III | +1              | -1              |
| Sector IV  | -1              | -1              |

Sectors I and III are then treated as two external regions, situated in two different spacetimes.

Sector II is likened to the interior of a black hole, organized around a central singularity.

Sector IV is identified with the interior of a white fountain, also organized around a central singularity.

From a physical standpoint, one is led to consider that, within these objects, the roles of the time and space coordinates are interchanged.

With  $R > \alpha$  still holding in the real domain. The choice  $\varepsilon = \pm 1$  introduces two possible solutions (two sheets).

$$ds^2 = \left( 1 - \frac{\alpha}{R} \right) c^2 dt_E^2 + \varepsilon c \alpha \frac{dt_E dR}{R} - \left( 1 + \frac{\alpha}{R} \right) dR^2 - R^2 d\Omega^2 \quad (176)$$

We merge these two sheets into a single one by choosing  $\varepsilon = -1$  and introducing the change of variable:

$$R = \alpha (1 + \ln \text{ch } \rho) \quad (182)$$

We get:

$$ds^2 = \frac{\ln \text{ch } \rho}{1 + \ln \text{ch } \rho} c^2 dt_E^2 - \frac{c \text{th } \rho dt_E d\rho}{1 + \ln \text{ch } \rho} - \alpha^2 \left[ \frac{1 + \ln \text{ch } \rho}{\ln \text{ch } \rho} \text{th}^2 \rho d\rho^2 + (1 + \ln \text{ch } \rho)^2 d\Omega^2 \right] \quad (183)$$

This representation depicts a single *non-contractile* sheet with two leaves,  $\rho > 0$  and  $\rho < 0$ , connected at a throat sphere  $\rho = 0$  with an area of  $4\pi\alpha^2$

This object is a bridge between two sheets of Lorentzian spacetime. This metric coincides with the Lorentz metric at infinity on both sheets.

$$\rho \simeq \frac{R}{\alpha} \quad (185)$$

$$g_{t_E t_E} \rightarrow c^2 \quad (186)$$

$$g_{\rho t_E} \rightarrow 0 \quad (187)$$

$$g_{\rho\rho} \rightarrow -\alpha^2 (g_{RR} \rightarrow -1) \quad (188)$$

$$g_{\theta\theta} \rightarrow -\alpha^2 \rho^2 = -R^2 \quad (189)$$

$$g_{\varphi\varphi} \rightarrow -\alpha^2 \rho^2 \sin^2 \theta = -R^2 \sin^2 \theta \quad (190)$$

All of this (upon grouping terms) is achieved using a single change of variables applied to the initial Schwarzschild variables:

$$r = \alpha \left[ 1 - (1 + \ln \text{ch } \rho)^3 \right]^{1/3} \quad (191)$$

$$t = t_E - \frac{\alpha}{c} \ln (\ln \rho) \quad (192)$$

This representation and its topology are consistent with the analysis of sections at constant  $t$  and  $\theta$  (Flamm's paraboloid).

## 10. A Look Back at the Use of a Stationary Solution to Describe a Highly Unsteady Phenomenon

This question was raised by P. Koiran in 2021 [25]. The use of the Schwarzschild exterior metric solution—a specific form of the solution to Einstein’s vacuum field equations—was proposed in 1939 by Oppenheimer and Snyder [20]. In this context, the variable  $t$  appearing in the line element is intended to correspond to the proper time of a distant observer. Consequently, for such an observer, the time taken for a test mass to fall freely or to escape appears infinite. Based on this, the authors suggest that the solution describes the implosion of a mass—such as a massive star—as a process akin to “freezing the film”; this extremely brief event (lasting a few days) would appear to a distant observer as a virtually static image. However, is this feature an intrinsic property of the solution? The answer is no. Introducing a cross-term involving  $drdt$  alters the scenario, as demonstrated in [25].

- If the sign of  $drdt$  term is negative, the free-fall time becomes finite and is measured in days, whereas the escape time remains infinite.
- If the sign of  $drdt$  term is positive, opposite conclusion

As mentioned in the article, the presence of this term is by no means ruled out by Birkhoff’s theorem. The theorem merely requires the form in question to be isometric to the diagonal form, which is indeed the case.

If the validity of this solution is called into question, what alternative scenario might describe the outcome of a massive star’s implosion? One proposed model is that of “plugstars” [26]. It predicts that all future images of supermassive objects at galactic centers will exhibit the same 3:1 ratio between maximum and minimum dimensions—a prediction that already appears consistent with data from the first two such objects, M87\* and SgrA\*. This model relies on the assumption of constant, or quasi-constant, mass density. Regarding subcritical neutron stars—which could also be described by this model, with a mass limit of around 2.5 solar masses (compared to 3 for a black hole)—this description is not unreasonable; beyond a certain threshold of matter compression, treating the matter as a gas is no longer appropriate. Furthermore, nucleons become so closely packed that quantum effects—which are not accounted for in the gas model—must come into play.

In a future article, we will introduce a time-dependent perturbation term into both the interior and exterior metrics—including cross-terms in both—along with a topological discontinuity at the center. If it can then be shown that the core conversion of the excess positive mass into negative mass, and its subsequent expulsion from the celestial body, tend to cancel out the perturbation term, then self-stability will be demonstrated.

## 11. Conclusions

The aim of this article was to re-examine the geometric significance of the maximal extension of the Schwarzschild solution, starting from a fundamental require-

ment: the geometric nature of an object should not depend on the coordinate system chosen to represent it.

A preliminary study of simple two-dimensional examples has shown that the equations defining autoparallel curves can possess solutions extending beyond the domain where the corresponding metric admits a real Euclidean embedding. In the case of the sphere, the torus, and Flamm's surface, these additional branches can be described using semi-complex extensions of their embedding spaces. Although some coordinates become purely imaginary, their projections onto the real-coordinate subspaces remain real. We have termed these projections "ghost images."

Flamm's surface provides a particularly significant example in this regard. Its real domain  $r > \alpha$  and its semi-complex extension ( $0 < r < \alpha$ ) join continuously at the throat  $r = \alpha$ . These two domains thus appear as two parts of a single, extended geometric construction.

We then revisited the Kruskal construction, detailing separately—line by line—the standard calculation in regions I and II. In its usual physical formulation, the tortoise coordinate is written as follows:

$$r^* = r + \alpha \ln \left| \frac{r}{\alpha} - 1 \right|$$

This compact notation actually encompasses two distinct real expressions on either side of  $r = \alpha$ . Added to this are different sign choices in the definition of the exponential coordinates  $U$  and  $V$ , which allow for the construction of the four sectors of the Kruskal diagram and lead to the swapping of hyperbolic functions between the exterior and interior regions.

We therefore pursued a different approach. Instead of requiring the coordinates to remain real on both sides of  $r = \alpha$ , we retained a single coordinate transformation—derived in the exterior region—and analytically continued it into the domain  $0 < r < \alpha$ . In this continuation, the coordinates  $u$  and  $v$  naturally become purely imaginary. The standard real Kruskal representation can then be recovered via a complex translation of the time variable, an operation that results in the swapping of the hyperbolic sine and cosine functions.

From this perspective, regions II and IV can be interpreted as belonging to a semi-complex extension of spacetime geometry, represented in the standard Kruskal construction by coordinates endowed with a pseudo-real character. The three-dimensional representation:

$$r = H(u, v)$$

...foliated by the curves  $r = \text{const.}$ , makes this structure geometrically visible and shows the continuous connection of the four sectors along  $r = \alpha$ .

Two descriptions of the continuation of the Schwarzschild geometry thus emerge. The conventional description maintains real coordinates throughout the Kruskal manifold and leads, in the domain  $0 < r < \alpha$ , to an inversion of the causal nature of the directions associated with the Schwarzschild coordinates:  $r$

becomes timelike while  $t$  becomes spacelike. By contrast, the description proposed here employs a single analytic change of variables and leads to the interpretation of the interior sectors as belonging to a semi-complex extension.

The question therefore goes beyond the mere elimination of a coordinate singularity at  $r = \alpha$ . It concerns the geometric nature and, ultimately, the physical status attributed to the domain obtained by extending the solution.

A second difficulty then arises. The Schwarzschild solution is stationary, whereas the gravitational collapse supposed to lead to the formation of a black hole is, by nature, a time-dependent phenomenon. Moreover, Eddington-Finkelstein-type representations and the calculations of free-fall time discussed in this article demonstrate that the divergence of Schwarzschild time at the horizon is not an invariant property of the geometry. The interpretation of an essentially dynamic process based on a stationary solution therefore warrants re-examination.

This situation leaves open the possibility of other descriptions of the final state of gravitational collapse. The previously proposed “plugstar” model constitutes one such alternative. In particular, it leads to observational predictions that could, in principle, allow for a distinction between the two scenarios. A dynamic extension of this model, incorporating time-dependent perturbation terms as well as cross-terms in the interior and exterior metrics, will be the subject of future work.

When it comes to choosing between these two interpretations, the principle of Occam’s razor would suggest opting for the simpler one. Furthermore, can Nature accept a physics in which space and time coordinates swap roles? If the black hole model were ever to be abandoned, could the “plugstar” model take its place? If the ratio between the maximum and minimum wavelengths turned out to be significantly close to 3, the question would have to be raised.

*Only Nature holds the answer to these questions.*

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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