Journal of Modern Physics, 2026, 17(8), 997-1027
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

The True Schwarzschild Solution and Its Subsequent Geometry

Jean-Pierre Petit, Gilles d'Agostini, Jean-Marc Pechinot

Manaty Research Group, Glanon, France
Email: Jean-pierre.petit@manaty.net

How to cite this paper: Petit, J.-P., d'Agostini, G. and Pechinot, J.-M. (2026) The True Schwarzschild Solution and Its Subsequent Geometry. *Journal of Modern Physics*, **17**, 997-1027.
https://doi.org/10.4236/jmp.2026.178045

Received: May 14, 2026
Accepted: August 25, 2026
Published: August 28, 2026

Abstract

We present the solution to Einstein's stationary, spherically symmetric field equation, expressed using variables initially chosen by Schwarzschild. We show that the geodesics of the Flamm surface are not the geodesics of the hypersurface solution and how the Flamm's 2D cross-section allows us to understand the topology of the 4D object.

Keywords

Schwarzschild Metric, Schwarzschild Geometry, Throat Sphere

1. Introduction

This article will be particularly long. We will begin by revisiting how Karl Schwarzschild constructed the first exact solution to Einstein's equation in 1916—a stationary, spherically symmetric solution [1].

Very few people have followed his calculations in detail, as they require close attention. Indeed, Schwarzschild performs several successive changes of variable, making it quite difficult to follow his work. We shall therefore go through this again in detail, specifying his assumptions and variables at each step. This yields the line element shown in Equation (3), which is what Schwarzschild should have written when expressing his solution metric in terms of his coordinates (t, r, θ, φ).

The reason why he did not do so is then explained. Quite simply, his primary goal was to recover the approximate solution published a few months earlier by Einstein, which accounted for the advance of Mercury's perihelion [2]. To achieve this, he needed to calculate the geodesic trajectories of a test mass (Mercury) around the mass (the Sun). He chose to do this not using the radial coordinate *r*, but rather using what he termed an intermediate quantity R (*Hilfsgröße*), a choice he explained in a letter sent to Einstein as early as December 1915 [3] (see **Figure**

8), even before the article had been published by the journal.

It is absolutely essential to understand that the real formulation of this metric solution, whose construction is presented in details in the present paper, is nothing other than the true form of his original solution—expressed in the coordinates (t, r, θ, φ) he had chosen, and that this calculation involves no extraneous input or additional geometric or physical assumptions. The invariant remains the length s. What is truly extraordinary is that no one has revisited his calculation in the way we are about to do together.

Naturally, this analysis is inextricably linked to the black hole model. The present work stems from an article published in 1989 by the Canadian mathematician L.S. Abrams [4]—a paper that went entirely unnoticed at the time. Abrams pointed out what he regarded as an error in D. Hilbert's work [5]. However, the issue is more complex; it is not, strictly speaking, an error. In fact, Hilbert based his geometry not on length measurement but on a bilinear form—an approach that appeared more general to him—which entails the extension of geodesics along autoparallel curves. This aspect will be analyzed in a future article, as will the reinterpretation of the solution by Kruskal and matters relating to Birkhoff's theorem. The present article is confined to a precise examination of the solution constructed by Karl Schwarzschild in February 1916 [1] and a similarly precise description of Flamm's surface [6].

2. Detailed Re-Examination of Karl Schwarzschild's Calculation

Figure 1 shows how Schwarzschild presents his solution [1]. To convince the reader of the importance of this examination, we deemed it appropriate to also provide the German versions of the passages from the articles.

We felt it important to present an excerpt from the original article, published in German, noting that the English translation of this document was only made available in 1999, 83 years after its publication. The corresponding excerpt, in English translation, follows on **Figure 1**.

where the auxiliary quantity

$$R = (3x_1 + \rho)^{1/3} = (r^3 + \alpha^3)^{1/3}$$

has been introduced.

When one introduces these values of the functions f in the expression (9) of the line element and goes back to the usual polar co-ordinates one gets the line element that forms the exact solution of Einstein's problem:

$$ds^2 = (1 - \alpha/R)dt^2 - \frac{dR^2}{1 - \alpha/R} - R^2(d\vartheta^2 + \sin^2\vartheta d\phi^2), \quad R = (r^3 + \alpha^3)^{1/3}. \qquad (14)$$

The latter contains only the constant α that depends on the value of the mass at the origin.

Figure 1. The way in which Karl Schwarzschild presented his exterior metric solution in January 1916 [1].

It seems essential to draw attention to the fact that this form of the Schwarzschild metric, described in the literature as "standard", is expressed using a variable *R*, auxiliary quantity (*Hilfsgröbe*), and not the variable *r* [1], explicitly designated by Schwarzschild as a radial variable. See **Figure 2**.

§3. If one calls t the time, x, y, z, the rectangular co-ordinates, the most general line element that satisfies the conditions 1-3 is clearly the following:

$$ds^2 = Fdt^2 - G(dx^2 + dy^2 + dz^2) - H(xdx + ydy + zdz)^2$$

where F, G, H are functions of $\boxed{r = \sqrt{x^2 + y^2 + z^2}}$

Figure 2. The true Schwarzschild's radial coordinate [1].

In his article, Schwarzschild presents the above expression as, in his view, the most general form of the solution to Einstein's spherically symmetric and time-translation invariant equation. He then performs a series of changes of variable, for the sake of computational convenience, which means that, in this complex process, we lose the true final expression of his solution, expressed in its initial coordinate system, with its variable *r* and not this intermediate quantity *R*. We will reconstruct it precisely. He goes from its coordinates (t, r, θ, ϕ) to a set (x_4, x_1, x_2, x_3) according to **Figure 3**.

One puts:

$$x_1 = \frac{r^3}{3}, \quad x_2 = -\cos\vartheta, \quad x_3 = \phi. \tag{7}$$

Figure 3. The first Schwarzschild's coordinate change [1].

Which gives, see **Figure 4**:

In the new polar co-ordinates the line element reads:

$$ds^2 = Fdx_4^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right)dx_1^2 - Gr^2\left[\frac{dx_2^2}{1 - x_2^2} + dx_3^2(1 - x_2^2)\right], \tag{8}$$

for which we write:

$$ds^2 = f_4 dx_4^2 - f_1 dx_1^2 - f_2 \frac{dx_2^2}{1 - x_2^2} - f_3 dx_3^2(1 - x_2^2) \tag{9}$$

Then f_1 , $\boxed{f_2 = f_3}$, f_4 are three functions of x_1

Figure 4. Schwarzschild's line element with his new space coordinates [1].

In the following pages, he calculates the Christoffel symbols, introduces all of this into the field equation, solves the equations, and obtains the expressions of his functions f_1, f_2, f_3, f_4. See **Figure 5**.

By exploiting this result, and taking into account de $f_3 = f_2, x_4 = t$, we obtain:

$$\begin{aligned} ds^2 = {} & \left[1 - \alpha(3x_1 + \rho)^{-1/3}\right]dt^2 - \frac{(3x_1 + \rho)^{-4/3}}{1 - \alpha(3x_1 + \rho)^{-1/3}}dx_1^2 \\ & -(3x_1 + \rho)^{2/3}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned} \tag{1}$$

Using the change mentioned on **Figure 6**:

By integrating once more,

$$f_2 = \lambda(3x_1 + \rho)^{2/3}. \quad (\lambda \textit{ integration constant})$$

The condition at infinity requires: $\lambda = 1$. Then

$$f_2 = (3x_1 + \rho)^{2/3}. \tag{10}$$

Hence it results further from (c") and (d)

$$\frac{\partial f_4}{\partial x_1} = \alpha f_1 f_4 = \frac{\alpha}{f_2^2} = \frac{\alpha}{(3x_1+\rho)^{4/3}}.$$

By integrating while taking into account the condition at infinity

$$f_4 = 1 - \alpha(3x_1 + \rho)^{-1/3}. \tag{11}$$

Hence from (d)

$$f_1 = \frac{(3x_1+\rho)^{-4/3}}{1-\alpha(3x_1+\rho)^{-1/3}}. \tag{12}$$

Figure 5. The *f* functions [1].

$$\rho = \alpha^3 \tag{13}$$

Figure 6. From original document [1].

We give for the first time, presented rigorously in a peer-reviewed scientific journal, 110 years after the publication of the original article, using its variables.

$$\left(t, r = \sqrt{x^2+y^2+z^2} \geq 0, \theta, \phi\right) \tag{2}$$

the true solution, given in 1916 by Karl Schwarzschild, of Einstein's homogeneous equation, in spherical symmetry and invariant under time translation.

$$\left(\begin{array}{l} \mathrm{d}s^2 = \dfrac{\left(r^3+\alpha^3\right)^{1/3}-\alpha}{\left(r^3+\alpha^3\right)^{1/3}} c^2\mathrm{d}t^2 - \dfrac{r^4\mathrm{d}r^2}{\left(r^3+\alpha^3\right)\left[\left(r^3+\alpha^3\right)^{1/3}-\alpha\right]} \\ \quad -\left(r^3+\alpha^3\right)^{2/3}\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right) \end{array}\right) \tag{3}$$

Following Schwarzschild, Einstein's homogeneous equation—assuming spherical symmetry and invariance under time translation—is presented here in its author's original coordinates. This is the first time in 110 years that this form—despite its essential nature—has been presented in an article.

The invariant remains the length *s*. It is simply a reformulation of the exact solution provided by Schwarzschild, expressed not in terms of his intermediate quantity *R*, but in terms of his true radial coordinate:

$$r = \sqrt{x^2+y^2+z^2} \geq 0. \tag{4}$$

We have represented and highlighted it in red in the hope that the many specialists will notice this expression, which is being presented to them for the first time. At this point, a question immediately arises: Why didn't Schwarzschild present it in this form, with its variable $r = \sqrt{x^2+y^2+z^2}$, in his article, instead of using this intermediate variable *R*? The answer is found in a letter

Schwarzschild sent to Einstein on December 22, 1915 [3]. He wrote, see **Figure 7**:

As follows, it had beside your α yet a second term, and the problem was physically undetermined. From this I made at once by good luck a search for a full solution. A not too difficult calculation gave the following result: It gave only a line element, which fulfills your conditions 1) to 4), as well as field- and determinant equations, and at the null point and only in the null point is singular.

If:

$$x_1 = r\cos\phi\cos\theta, \quad x_2 = r\sin\phi\cos\theta, \quad x_3 = r\sin\theta$$

$$R = (r^3+\alpha^3)^{1/3} = r\left(1+\frac{1}{3}\frac{\alpha^3}{r^3}+\dots\right)$$

then the line element becomes:

$$ds^2 = \left(1-\frac{\gamma}{R}\right)dt^2 - \frac{dR^2}{\left(1-\frac{\gamma}{R}\right)} - R^2(d\theta^2+\sin^2\theta d\phi^2)$$

R, θ, ϕ are not "allowed" coordinates, with which one must build the field equations, because they do not have the determinant = 1, however the line element expresses itself as the best.

Figure 7. Letter from Schwarzschild to Einstein, December 2015 [3].

Einstein and Schwarzschild were in contact and exchanged letters regularly. In his letter of December 22, 1915, Schwarzschild responded to the article Einstein had just published on November 25, in which he calculated the perihelion advance of Mercury using an approximate solution [2]. Schwarzschild was preparing to send what was then the exact solution. But as he specified in his article, he knew that the radius, for the Sun, was limited to 3 km. He therefore decided to present his result using an intermediate variable R by performing a series expansion, as indicated in the article. We thus find here the origin of what would later be called the "standard form of the Schwarzschild solution".

3. December 1916: David Hilbert's Impact on Geometric Cosmology [5]

During 1915, for seven months, through numerous exchanges—first in person, then through copious correspondence—a very close relationship developed between Einstein and Hilbert, in an atmosphere of great mutual respect. Through Einstein's words, Hilbert discovered the link between advanced mathematics and the development of a new physics, toward which he had always been skeptical. Einstein spoke of the project in which he had become completely invested: a geometric translation, within a framework of covariance of the force of gravity. Intrigued, Hilbert had access to the most advanced mathematical tools of his time. It was a world in which Einstein, first and foremost a fantastic physicist endowed with exceptional intuition, always admitted to feeling uneasy. In this context, Hilbert conceived the project of integrating the two forces known at the time—gravitation and electromagnetism—into a single formalism. Today we would call this a "theory of everything". For him, cosmic geometry is that of a four-dimensional space, designated by the letters (w_1, w_2, w_3, w_4), where the indices 1 to 3 refer to space and where w_4 is the time coordinate. Space is quasi-Euclidean and stationary (invariant under time translation): the phenomenon of cosmic expansion had not yet been discovered by Edwin Hubble. The geometry of the cosmos is determined by a number of functions, the set forming a tensor $g_{\mu\nu}$, from which one can calculate the components of the Ricci tensor, which Hilbert denotes by $K_{\mu\nu}$. The contraction of this tensor gives the Ricci scalar, designated by the letter *K*.

Hilbert then published a long article on November 20, 1915 [7] entitled "Foundations of Physics". While Einstein struggled as best he could with tensors, aided by his friend the mathematician Grassmann, Hilbert took a further step into abstraction by deriving a field equation from a variational calculus, centered on an action constructed around a "world function" *H*:

$$\int H\sqrt{|g|}\,\mathrm{d}\omega \tag{5}$$

$$|g| = |\det g_{\mu\nu}| \tag{6}$$

See **Figure 8**.

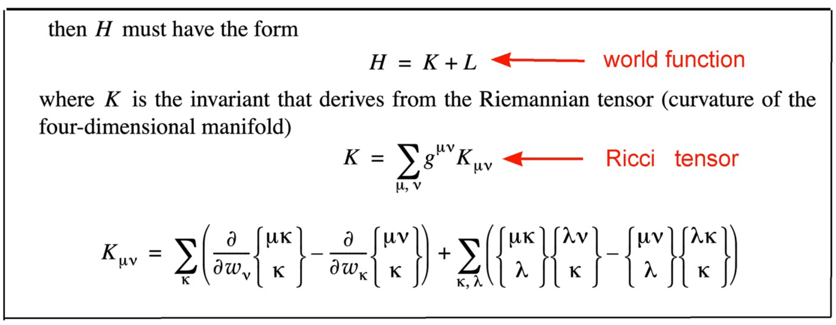


Figure 8. The Hilbert's world function [7].

His stroke of genius was to identify this world function as the sum of two terms, the first being the Ricci scalar and the second the Lagrangian of matter, defining a field where, in principle, gravitation and electromagnetism are ambitiously intertwined. The publication of this article was a real shock for Einstein, who, when he sent his own five days later, on November 25, 1915 [8], presenting the goal of his quest—the famous field equation of this general relativity, which completes his special relativity—discovered that Hilbert had beaten him to it. See **Figure 9**.

If we replace the letter *K* with the letter *R* (the Ricci tensor and scalar *r*), and setting:

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = \chi T_{\mu\nu} \tag{7}$$

The reader will recognize the field equation of general relativity with which they are familiar. Einstein then wrote Hilbert a letter of protest, and the following weeks reflected a certain amount of confusion. However, by mutual agreement, to preserve the feelings of esteem that bound them, the decision was made that this equation would bear only Einstein's name. In the literature, the reader will now find references to the Einstein-Hilbert collaboration.

Using the notation introduced earlier for the variational derivatives with respect to the $g^{\mu\nu}$, the gravitational equations, because of (20), take the form

$$[\sqrt{g}K]_{\mu\nu} + \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0. \qquad (21)$$

The first term on the left hand side becomes

$$[\sqrt{g}K]_{\mu\nu} = \sqrt{g}\left(K_{\mu\nu} - \frac{1}{2}K g_{\mu\nu}\right),$$

Figure 9. The Hilbert's field equation [7].

History makes its choices. It is, indeed essential to note that at no point in Hilbert's article does the word "metric" appear, nor expression beginning with "ds^2 =".

Schwarzschild then published, in quick succession, in January and February 1916, his two solutions to this field equation [1] [9]. Hilbert felt compelled to urgently incorporate this into a second version of his paper, bearing the same title, which he published on December 23, 1916 [5]. Only the solution to the homogeneous equation [1] held his attention. He made no mention of the second.

For Hilbert, a solution to the field equation was not a metric but a bilinear form which, in his view, determined the geometry. This is a crucial point that must be emphasized. Hilbert is, first and foremost, a mathematician. It therefore seems natural to him to consider the most general formulation from a purely mathematical perspective, specifically, a geometry derived from a bilinear form that could have any sign. He operates, in other words, like someone living in the world of mathematics. Einstein, by contrast, inhabits the world of physics. That is where the entire difference lies. See **Figure 10**.

The new $g_{\mu\nu}$ $(\mu, \nu = 1, 2, 3, 4)$ —the gravitational potentials of Einstein—shall then all be real functions of the real variables x_s $(s = 1, 2, 3, 4)$ of such a type that, in the representation of the quadratic form

$$G(X_1, X_2, X_3, X_4) = \sum_{\mu\nu} g_{\mu\nu} X_\mu X_\nu \qquad (28)$$

as a sum of four squares of linear forms of the X_s, three squares always occur with positive sign, and one square with negative sign: thus the quadratic form (28) provides our four dimensional world of the x_s with the metric of a pseudo-geometry. The determinant g of the $g_{\mu\nu}$ turns out to be negative.

Figure 10. Hilbert's Quadratic form [5].

The tedious presentation of these numerous plates is intended to show how Hilbert's highly personal conception of this "geometric cosmology", very different from that of Einstein [8], Schwarzschild [1], Droste [10], Flamm [6], Weyl [11], and others, profoundly influenced the development of this discipline. We will show later how the main proponents of these ideas echoed, without citing their source, the choices made by Hilbert.

Thus, in the plate above, we find the origin of the change in signature. Among the authors cited, this signature was ubiquitous:

$$\text{The « old » signature: } (+---) \tag{8}$$

In the sentences underlined in red in the excerpt from Hilbert's December 1916 article, he explicitly introduces what will become the "modern" signature:

$$(+++-) \text{ or } (-+++) \tag{9}$$

This change will occur gradually and will eventually become widespread in the postwar period, even though, at least in cosmology, there is no article that provides an argument in favor of such a change.

But what does Hilbert mean by pseudo-geometry? The answer is in the following excerpt. See **Figure 11**.

and give the corresponding meaning to the $g_{\mu\nu}$.

In the case of the pseudo-Euclidean geometry we have

$$g_{\mu\nu} = \delta_{\mu\nu},$$

where

$$\delta_{\mu\nu} = 1, \qquad \delta_{\mu\nu} = 0 \quad (\mu \neq \nu).$$

For every metric in the neighborhood of this pseudo-Euclidean geometry the ansatz

$$g_{\mu\nu} = \delta_{\mu\nu} + \varepsilon h_{\mu\nu} + \ldots \tag{37}$$

is valid, where ε is a quantity converging to zero, and $h_{\mu\nu}$ are functions of the w_s. I make the following two assumptions about the metric (37):

I. The $h_{\mu\nu}$ shall be independent of the variable w_4.

II. The $h_{\mu\nu}$ shall show a certain regular behavior at infinity.

Figure 11. Hilbert's pseudo (Euclidean) geometry [5].

In equation (37) of the extract, it is clear that for Hilbert the universe is not only stationary (independent of its time coordinate w_4) but practically Euclidean. All this relativistic physics introduced by Einstein manifests itself only as a tiny perturbation, through the presence of a term $g_{44} = \varepsilon h_{44}$ [5]. The conception of spacetime introduced by Hermann Minkowski [12] in 1909[1], with its metric, is completely neglected. See Minkowski's statement on **Figure 12**.

These elements were very well assimilated by people like Schwarzschild [1]. See **Figure 13**.

In Schwarzschild's work, as with Schwarzschild and all others at that time, the assumptions are unambiguous:

- The coordinates are defined in $\mathbb{R}^4$.
- The element of length ds, associated with the hypersurface, is fundamentally real.
- We are looking for curves that minimize the length.

[1]Minkowski died a few months later from appendicitis that degenerated into a generalized infection. Antibiotics had not yet been invented. A detail of crucial importance.

The axiom means that at every point in the universe the expression

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

is always positive, or, equivalently, that the speed v is always less than c. Thus, c is the upper limit for the speed of any substantial point, and this is precisely where its true meaning lies.

$$d\tau = \frac{1}{c}\sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}$$

The integral $\int d\tau$ of this quantity taken on the world line will be called the proper time (Eingeinzeit) of the substantial point.

Figure 12. Minkowski's axiomatic definition [12] of spacetime (1909).

On the Gravitational Field of a Mass Point according to Einstein's Theory †

by K. Schwarzschild

(Communicated January 13th, 1916 [see above p. 42].)

translation‡ and foreword by

S. Antoci* and A. Loinger**

Foreword. This fundamental memoir contains the ORIGINAL form of the solution of Schwarzschild's problem. It is regular in the whole space-time, with the only exception of the origin of the spatial co-ordinates; consequently, it leaves no room for the science fiction of the black holes. (In the centuries of the decline of the Roman Empire people said: "Graecum est, non legitur"...).

§1. In his work on the motion of the perihelion of Mercury (see Sitzungsberichte of November 18th, 1915) Mr. Einstein has posed the following problem:

Let a point move according to the prescription:

$$\left\{\begin{array}{l} \delta \int ds = 0, \\ \textit{where} \\ ds = \sqrt{\Sigma g_{\mu\nu} dx_\mu dx_\nu} \quad \mu,\nu = 1,2,3,4, \end{array}\right. \tag{1}$$

where the $g_{\mu\nu}$ stand for functions of the variables x, and in the variation the variables x must be kept fixed at the beginning and at the end of the path of integration. In short, the point shall move along a geodesic line in the manifold characterised by the line element ds.

Figure 13. Schwarzschild foreword [1].

In short, it's very clear and simple:

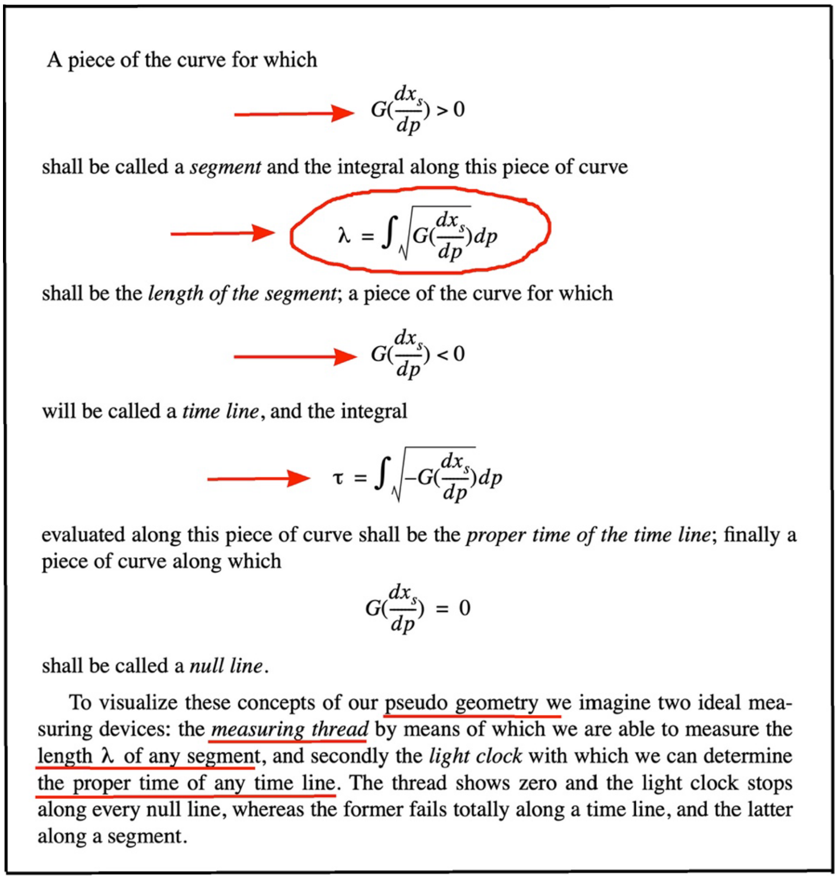
When the element ds ceases to be real, we are outside the hypersurface.

In doing so, he is merely conforming to the prestigious geometrization of physics associated with the lecture Minkowski gave in 1909 [12], just months before his death.

Since Hilbert defines his geometry not in terms of an elementary length d*s*, but rather based on a bilinear form, he arrives at what appears to him to be the most general geometry. It therefore seems logical to him to define two lengths rather than one, a step he takes by introducing an arbitrary sign change. See **Figure 14**.

For Hilbert, it is possible to define two real lengths from different expressions, depending on the sign of the quadratic form *G*. When this form is negative, this length becomes the proper time τ and a real value, measured along timelines, time-like trajectories, and is then obtained by inserting a negative sign in front of it. In the domain of definition, when this form is positive, the scalar obtained is then called the length of a segment, along curves described today as being "space-like". Hilbert is therefore the first to achieve an extension of the solution, now called an "extension to the whole of spacetime". These are thus two different interpretations of the solution. In Schwarzschild's work, the result of the variational calculus is limited to the determination of authentic geodesics where the length is minimized:

$$\delta \int \mathrm{d}s = \delta \int \sqrt{g_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}} = 0 \tag{10}$$



A piece of the curve for which

$$G\left(\frac{dx_s}{dp}\right) > 0$$

shall be called a *segment* and the integral along this piece of curve

$$\lambda = \int \sqrt{G\left(\frac{dx_s}{dp}\right)} dp$$

shall be the *length of the segment*; a piece of the curve for which

$$G\left(\frac{dx_s}{dp}\right) < 0$$

will be called a *time line*, and the integral

$$\tau = \int \sqrt{-G\left(\frac{dx_s}{dp}\right)} dp$$

evaluated along this piece of curve shall be the *proper time of the time line*; finally a piece of curve along which

$$G\left(\frac{dx_s}{dp}\right) = 0$$

shall be called a *null line*.

To visualize these concepts of our pseudo geometry we imagine two ideal measuring devices: the *measuring thread* by means of which we are able to measure the length λ of any segment, and secondly the *light clock* with which we can determine the proper time of any time line. The thread shows zero and the light clock stops along every null line, whereas the former fails totally along a time line, and the latter along a segment.

Figure 14. Hilbert's two different lengths [5].

Hilbert constructs the set of curves directly from the quadratic form, without worrying about whether it is positive or negative.

$$\delta \int \left(g_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu} \right) = 0 \tag{11}$$

Hilbert was also the first to introduce the concept of the light cone, which henceforth has an interior and an exterior, where curves of a spatial nature then unfold.

4. Hilbert's Construction of the Solution to the Field Equation, Stationary and Spherically Symmetric [5]

For Hilbert, a solution to the field equation is a quadratic form, which can then have any sign. Using polar coordinates, he writes, see **Figure 15**.

According to Schwarzschild the most general metric conforming to these assumptions is represented in polar coordinates, where

$$w_1 = r\cos\vartheta$$
$$w_2 = r\sin\vartheta\cos\varphi$$
$$w_3 = r\sin\vartheta\sin\varphi$$
$$w_4 = l,$$

by the expression

$$F(r)dr^2 + G(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + H(r)dl^2 \tag{42}$$

where $F(r)$, $G(r)$, $H(r)$ are still arbitrary functions of r.

Figure 15. Hilbert's coordinate choice [5].

He denotes his time coordinate by the letter l. This coordinate does not intervene at any point in his calculation, since the solution was assumed to be independent of it. For Hilbert, what determines the geometry is a quadratic form. It is therefore in this form that he presents his result (the typographical errors of the original manuscript, reproduced in the English translation, have been corrected):

$$G(\mathrm{d}r,\mathrm{d}\theta,\mathrm{d}\vartheta,\mathrm{d}l) = \frac{r}{r-\alpha}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\vartheta^2 + \frac{r-\alpha}{r}\mathrm{d}l^2 \tag{12}$$

And it is only at this stage that Hilbert adopts the interpretation of relativity suggested in 1902 by the mathematician Henri Poincaré [13]. For Poincaré, as or Hilbert, the time coordinate is purely imaginary ($l = it$). See **Figure 16**.

If we take as integrals of (44) $m = \alpha$, where α is a constant, and $w = 1$, which evidently is no essential restriction, then for $l = it$ (43) results in the desired metric in the form first found by Schwarzschild

$$G(dr, d\vartheta, d\varphi, dt) = \frac{r}{r-\alpha}dr^2 + r^2d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 - \frac{r-\alpha}{r}dt^2 \tag{45}$$

Figure 16. When Hilbert identifies [5] its own result to the Schwarzschild's one.

Like Hilbert, Schwarzschild adopts what appears to him to be the most general form of the solution to Einstein's equation without second member, stationary and spherically symmetric. See **Figure 17**.

When one goes over to polar co-ordinates according to $x = r\sin\vartheta\cos\phi$, $y = r\sin\vartheta\sin\phi$, $z = r\cos\vartheta$, the same line element reads:

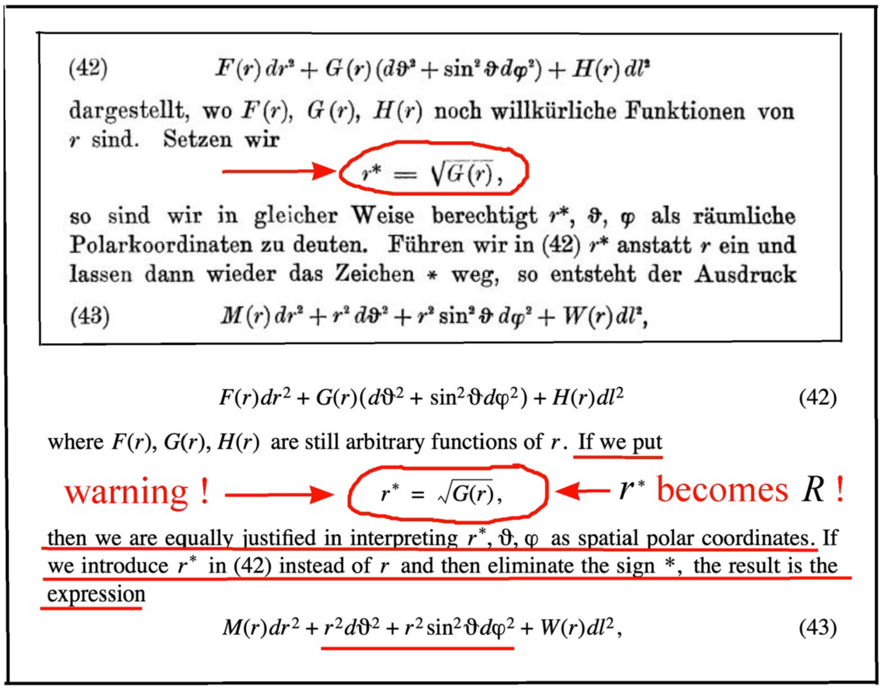
$$ds^2 = Fdt^2 - (G + Hr^2)dr^2 - Gr^2(d\vartheta^2 + sin^2\vartheta d\phi^2). \tag{6}$$

Figure 17. Schwarzschild's line element [1].

As can be seen, in Schwarzschild coordinates $(t, r, \theta, \vartheta)$, this corresponds to

$$Gr^2 = \left(r^3 + \alpha^3\right)^{2/3} \neq r^2 \tag{13}$$

In his calculation, Hilbert introduces a simplification concerning the coefficient of $\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\vartheta^2\right)$ which seemed to him likely to converge more rapidly toward what he identified as Schwarzschild's result in $(t, r, \theta, \vartheta)$. See **Figure 18**.



(42) $F(r)\,dr^2 + G(r)\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) + H(r)\,dl^2$

dargestellt, wo $F(r)$, $G(r)$, $H(r)$ noch willkürliche Funktionen von r sind. Setzen wir

$$r^* = \sqrt{G(r)},$$

so sind wir in gleicher Weise berechtigt r^*, ϑ, φ als räumliche Polarkoordinaten zu deuten. Führen wir in (42) r^* anstatt r ein und lassen dann wieder das Zeichen $*$ weg, so entsteht der Ausdruck

(43) $M(r)\,dr^2 + r^2\,d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 + W(r)\,dl^2,$

$$F(r)dr^2 + G(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + H(r)dl^2 \tag{42}$$

where $F(r)$, $G(r)$, $H(r)$ are still arbitrary functions of r. If we put

warning ! $r^* = \sqrt{G(r)},$ r^* becomes R !

then we are equally justified in interpreting r^*, ϑ, φ as spatial polar coordinates. If we introduce r^* in (42) instead of r and then eliminate the sign $*$, the result is the expression

$$M(r)dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 + W(r)dl^2, \tag{43}$$

Figure 18. Hilbert's approximation [5].

He then identifies his variable r^* with r. At a time when no one could have imagined for a single moment the existence of astronomical objects whose characteristic scale would be anything other than an infinitesimal perturbation, this error had little importance. But in reality, this amounts to identifying $r = \sqrt{x^2 + y^2 + z^2} \geq 0$ with $R = \left(r^3 + \alpha^3\right)^{1/3} \geq \alpha$ (the intermediate Schwarzschild variable). This point was only pointed out in 1989 by L.S. Hence, the expression of Hilbert solution, as presented in its Equation (45) is not strictly identical to the Schwarzschild's one since it is expressed with the intermediary variable *R*, as noticed in 1989 by the Canadian mathematician Abrams [4], 73 years after the publication of Hilbert's paper.

5. Impact on Physicists

The first person to widely publicize these different works on this cosmology was Arthur Eddington [14], as early as 1919. His work testifies to his extraordinary openness to the science of his time. Representing a considerable undertaking, the book presents the tools of differential geometry upon which the progress represented by general relativity is based. However, on page 139, in Chapter III, he reproduces the manner in which Hilbert treats this solution of Einstein's equation. See Figure 19.

CH. III THE FIELD OF AN ISOLATED PARTICLE 139

In flat space-time the interval, referred to spherical polar coordinates and time, is

$$ds^2 = -dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 + dt^2. \qquad (38.11)$$

If we consider what modifications of this can be made without destroying the spherical symmetry in space, the symmetry as regards past and future time, or the static condition, the most general possible form appears to be

$$ds^2 = -U(r)\,dr^2 - V(r)\,(r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2) + W(r)\,dt^2, \qquad (38.12)$$

where U, V, W are arbitrary functions of r. Let

$$r_1^2 = r^2 V(r).$$

Then (38.12) becomes of the form

$$ds^2 = -U_1(r_1)\,dr_1^2 - r_1^2\,d\theta^2 - r_1^2\sin^2\theta\,d\phi^2 + W_1(r_1)\,dt^2, \qquad (38.13)$$

where U_1 and W_1 are arbitrary functions of r_1. There is no reason to regard r in (38.12) as more immediately the counterpart of r in (38.11) than r_1 is. If the functions U, V, W differ only slightly from unity, both r and r_1 will have approximately the properties of the radius-vector in Euclidean geometry; but no length in non-Euclidean space can have exactly the properties of a Euclidean radius-vector, and it is arbitrary whether we choose r or r_1 as its closest representative. We shall here choose r_1, and accordingly drop the suffix, writing (38.13) in the form

$$ds^2 = -e^{\lambda}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 + e^{\nu}\,dt^2, \qquad (38.2)$$

where λ and ν are functions of r only.

Figure 19. From Eddington's book [14], 1919.

One should note the old signature $(+---)$ as well as the use of exponentials e^{λ} and e^{ν} to ensure invariance when coordinates and functions belong to the real domain.

The same formulation appears identically in the book published in 1934 by the mathematician Richard Tolman [15], see on this book, Equations (94.2) and (95.1).

6. Geometric Interpretation of the Schwarzschild Solution

The young German mathematician Ludwig Flamm provided [6], immediately after publication of Schwarzschild's papers, the geometric interpretation of the two solutions [1] [9]. Since these are invariant under time translation, the study reduces to three-dimensional geometric structures undergoing translation in time. The 3D structure results from the joining of two manifolds along their common boundary, which is the spherical surface of the star. The author then considers the section of these two objects at constant θ, which yields 3D objects. See **Figure 20**.

§2. Now we can, in an analogous way, examine the first case which was treated by S c h w a r z s c h i l d, the gravitational field of a point mass. The line-element in its simplest form has the same structure as above and reads

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

Figure 20. L.Flamm's choice for the external metric representation [6].

Let us note in passing that Flamm adopts the signature $(+---)$ and identifies the trajectories of test masses with geodesics minimizing the length s, identified with the proper time τ. See **Figure 21**.

Therefore, if the elementary rulers and clocks are influenced by the gravitational field, then the propagation of light must be influenced in exactly the same way, so that when compared to each other they do not show any differences. The same must hold for all the other physical phenomena. This might be why one can describe the motion of a point particle by the simple variational equation

$$\delta \int_{P_1}^{P_2} ds = 0, \qquad (5)$$

Figure 21. Flamm's variational procedure [6].

He then embeds the object into $\mathbb{R}^3$. See **Figure 22**.

The spatial part of the line-element is again of a non-Euclidean nature. Once more, due to the spherical symmetry it seems to be enough to restrict the consideration to the metric properties in an arbitrary planar section through the origin, *i.e.*, to $\vartheta = \frac{\pi}{2}$. Now one obtains

$$d\sigma_e{}^2 = \frac{dR^2}{1 - \frac{\alpha}{R}} + R^2\, d\varphi^2. \qquad (4)$$

This line-element is also of the form (2'), because one can put

$$\cos^2\chi = 1 - \frac{\alpha}{R},$$

and it is also identical with a line-element on a surface of rotation. Denoting by z the coordinate in the direction of the rotational axis, the equation for the meridional curve follows from

$$\frac{dz}{dR} = \operatorname{tg}\chi = \sqrt{\frac{\alpha}{R - \alpha}}$$

as

$$z^2 = 4\alpha(R - \alpha).$$

This describes a parabola with parameter

$$p = 2\alpha,$$

Figure 22. L. Flamm's embedment operation [6].

He then obtains the meridian (r, z) of this manifold with boundary in the form of a sideways parabola. See **Figure 23**.

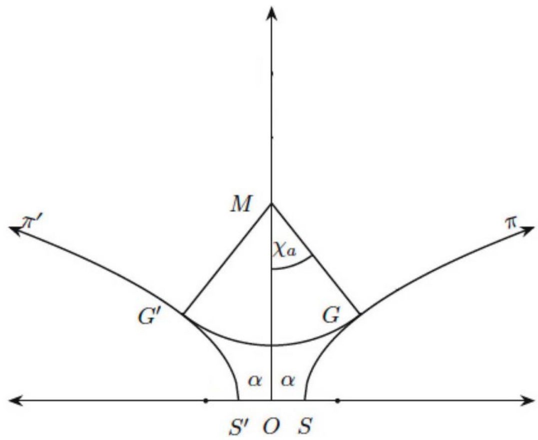


Figure 23. Flamm's medidian [6].

This Schwarzschild geometry is thus perfectly defined. The points G and G' represent the ends of the segment corresponding to the diameter of the sphere along which the two 3D surface elements are joined. The circular arc GG' completes the parabolic portions Gp and G'p'.

The Flamm surface is often described in the literature as a pedagogical artifact intended to illustrate how masses curve the surrounding space. When the spacetime geodesics arising from the 4D hypersurface corresponding to the exterior metric are projected, one obtains curves suggestive of an attractive effect, with the classical perihelion precession, as shown on **Figure 24**.

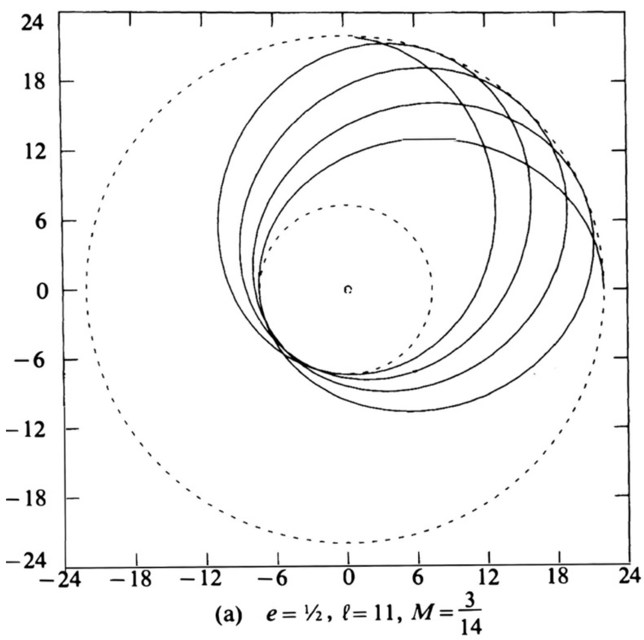


Figure 24. 2D (R, φ) projection of 4D Schwarzschild geodesic [16].

The system of geodesics of the Flamm surface was first studied only very recently [17], 113 years after its first presentation by its author. Later (2019!) this study was extended [18]. In this last paper, the geodesics are grouped into two families, depending on whether or not they cross the throat circle. The authors used a color code to distinguish portions of curves suggesting attraction, and at greater distance, repulsion. Consequently, they cannot constitute a pedagogical image intended to illustrate the way masses modify the geometry of the vacuum surrounding them. See **Figure 25**.

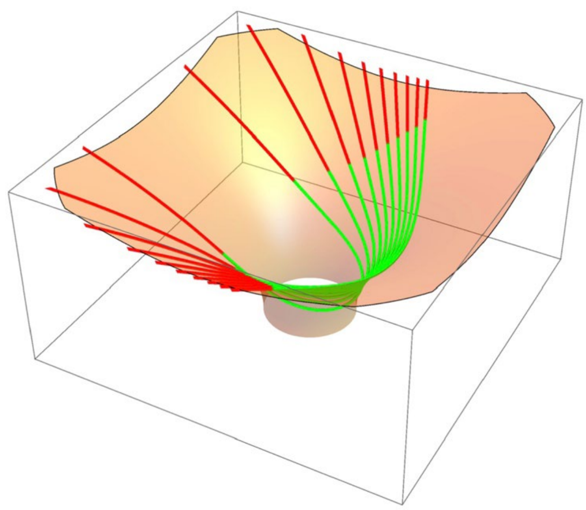


FIG. 15: Arcs of regular geodesics with positive (negative) normal curvatures are shown in green (red) on Flamm's paraboloid.

Figure 25. Flamm's surface geodesics. first family [18].

In **Figure 15** of the cited reference, the authors represented in green the portions of geodesics that do not cross the throat sphere, whose concavity is directed toward the symmetry axis (suggesting attraction). At larger distances, this concavity reverses (red color: repulsion).

This shows that the purely spatial geodesics of the Flamm surface must not be identified with the back-projections onto that surface of the 2D projections, in (R,φ), of spacetime geodesics. Under these conditions, what information does the Flamm surface provide? It is of a topological nature. Since Flamm's analysis shows that the 4D geodesics necessarily project onto this surface, conversely there cannot exist real geodesics crossing the throat sphere that correspond to the region $0 < R < \alpha$ in the representation (t,R,θ,φ).

Schwarzschild's approach is perfectly consistent from the astrophysical point of view. He sought to construct two stationary, spherically symmetric solutions of Einstein's equation. He constructed the interior metric solution [9] in the coordinate system (t,χ,θ,φ).

$$\mathrm{d}s^2 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2 c^2\mathrm{d}t^2 - \hat{R}^2\left(\mathrm{d}\chi^2 + \sin^2\chi\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\mathrm{d}\varphi^2\right) \quad (14)$$

With:

$$R = R_a \sin\chi \quad (15)$$

R_a being the ray of the star. With:

$$\hat{R}^2 = \frac{3c^2}{8\pi G\rho} \quad (16)$$

By going back to the system (t, R, θ, φ) it comes:

$$\begin{aligned}\mathrm{d}s^2 &= \left[\frac{3}{2}\sqrt{1+\frac{8\pi G\rho R_a^2}{3c^2}} - \frac{1}{2}\sqrt{1+\frac{8\pi G\rho R^2}{3c^2}}\right]^2 c^2\mathrm{d}t^2 \\ &+ \frac{\mathrm{d}R^2}{1-\frac{8\pi G\rho R^2}{3c^2}} - R^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\end{aligned} \quad (17)$$

The geodesics of the two solutions join along the sphere representing the boundary of the star, assimilated to a sphere of radius R_a, filled with an incompressible material of constant density ρ. The geometric solution presented by Schwarzschild is illustrated in the following figure, expressing the union of a portion of the Flamm surface with a spherical cap. The global result is a contractible object, as shown on **Figure 26**.

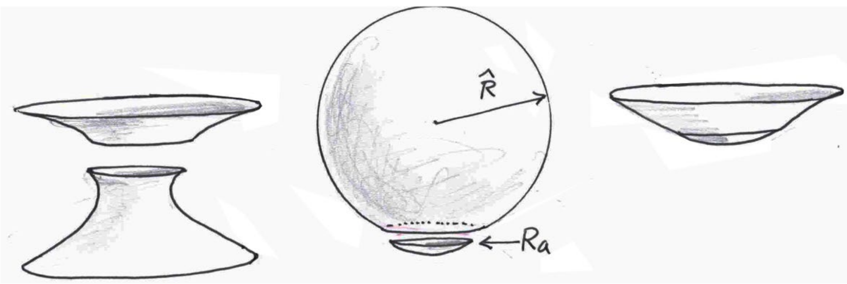


Figure 26. The global geometry of the Schwarzschild's solution.

This constitutes the true geometry of "Schwarzschild spacetime".

7. The Flamm Surface as a Whole

However, some authors considered the isolated exterior metric as a possible geometric description of masses: first H. Weyl in 1917[2] [11], then Einstein and Rosen in 1935 [19]. In the postwar period, the isolated exterior metric solution was considered either as describing a "frozen snapshot" of an object undergoing rapid implosion [20], or as a wormhole [21]. See the Kruskal's wormhole on **Figure 27**.

[2]Using the isotropic coordinate system he introduced, and noting the possible inversion of the time factor on a second sheet, Weyl was the first to suggest a two-sheeted covering structure involving an inversion of the time coordinate x^4. This will be analyzed in a future article; detailing it here would make the exposition overly cumbersome.

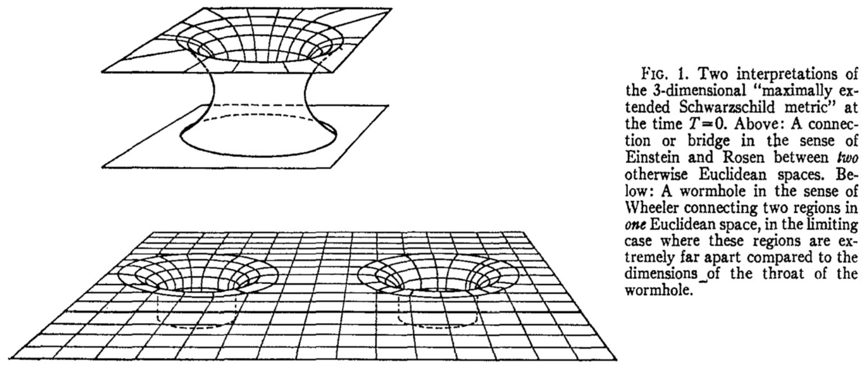


Figure 27. Kruskal wormhole [21].

In both cases, this leads us to consider the Flamm surface in its entirety. Below are its geodesics when they cross the throat circle, as shown on **Figure 28**.

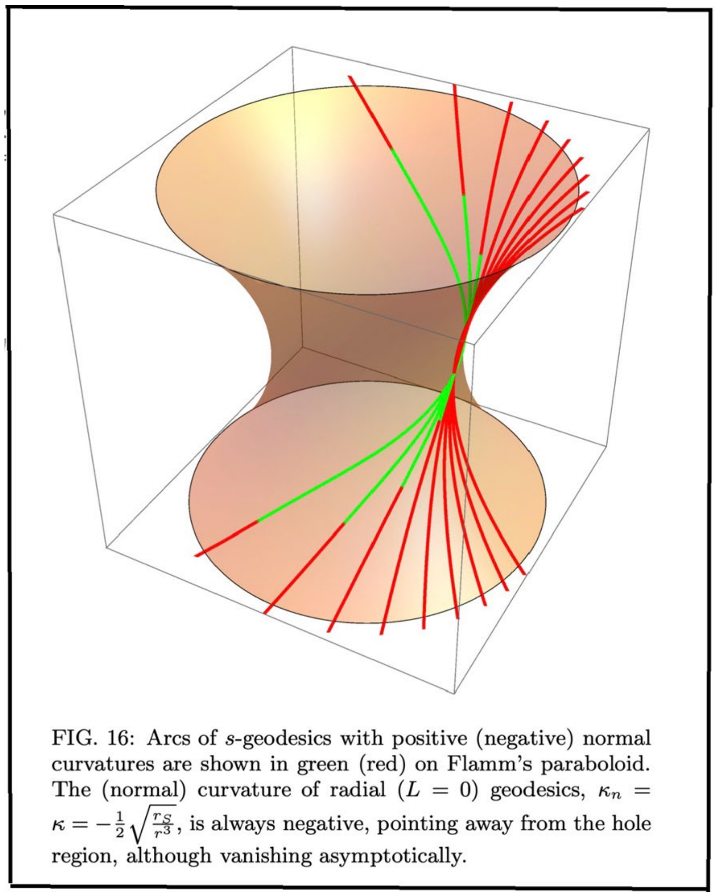


Figure 28. Flamm's surface geodesics. Second family [18].

The same color coding is found again: green when these geodesics direct their concavity toward the axis (attraction), and red when this concavity reverses (repulsion). This is also visible when these Flamm surface geodesics are projected. See **Figure 29**.

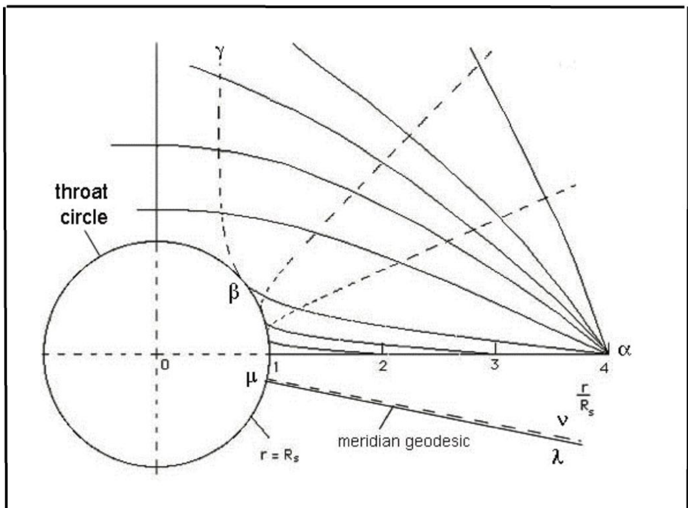


Fig. 4. Projection of geodesics.

Figure 29. Geodesic projection of Flamm surface [17].

In these figures the purely spatial geodesics of the Flamm surface must not be identified with the back-projections onto that surface of the 2D projections, in (R,φ), of spacetime geodesics. This shows that since the 4D geodesics necessarily project onto this surface, there cannot exist real geodesics crossing the throat sphere that correspond to the region $R<\alpha$ in the representation (t,R,θ,φ) with $(0<R<\alpha)$.

When one constructs the curves solving the Lagrange equations, one finds in the projection (R,φ) real curve segments that give the illusion that test masses penetrate into the region $0<R<\alpha$. This illusion disappears when one constructs 3D images of these geodesics in a coordinate system (t,R,φ), as shown on **Figure 30**.

In **Figure 30(A)**, the trajectory of a test mass converging toward the boundary and then emerging from it has been represented. Both operations occur, with this coordinate *t*, over an infinite time interval. In **Figure 30(B)**, one has the impression that the test mass "bounces" off this boundary. In **Figure 30(C)**, this projection **Figure 30(B)** of the geodesic has been back-projected onto the Flamm surface along the green curve, which therefore appears distorted. It should be kept in mind that this is not a geodesic of the Flamm surface.

If one considers separating the two sheets of the hypersurface associated with this isolated exterior metric solution, it then becomes appropriate to introduce a new variable [17]:

$$R=\propto\left(1+\ln ch\rho\right) \tag{18}$$

$$\mathrm{d}s^2=\frac{\ln ch\rho}{1+\ln ch\rho}c^2\mathrm{d}t^2-\alpha^2\left[\frac{1+\ln ch\rho}{\ln ch\rho}th^2\rho\mathrm{d}\rho^2+\left(1+\ln ch\rho\right)^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right)\right] \tag{19}$$

The throat sphere corresponds to the value $\rho=0$. The two sheets are distinct and correspond to $\rho>0$ and to $\rho<0$. At the throat:

$$G_{tt} \propto \frac{\rho^2}{2} \to +0 \qquad g_{\rho\rho} \to 2 \tag{20}$$

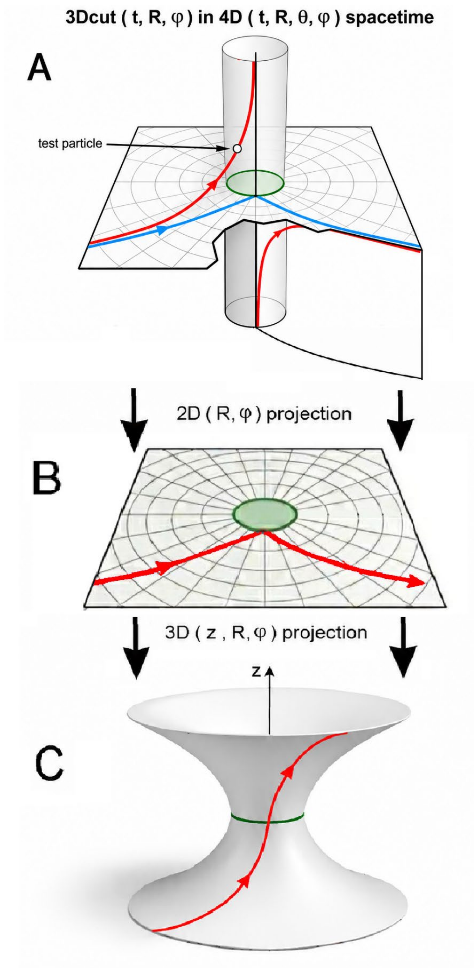


Figure 30. 4D geodesics and projections.

If one restricts oneself to the spatial part described by the metric:

$$\mathrm{d}\sigma^2 = \alpha^2 \left[\frac{1+\ln ch\rho}{\ln ch\rho} th^2 \rho \mathrm{d}\rho^2 + \left(1+\ln ch\rho\right)^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \right] \tag{21}$$

it is regular everywhere. Let us now consider the projections in (ρ,φ) of spacetime geodesics with $\theta = \pi/2$. In a representation (t,R,θ,φ), the projections (R,φ) are solutions of the classical equation:

$$\mathrm{d}\varphi = \pm \frac{\mathrm{d}R}{R^2 \sqrt{\frac{c^2 l^2}{h^2} + \frac{\alpha}{h^2 R} - \frac{1}{R^2} + \frac{\alpha}{R^3}}} \tag{22}$$

where l and h are parameters of the solution curve. Let ϕ denote the angle formed by the tangent to the curve and the radius vector. At the throat:

$$tg\,\phi = \alpha \left|\frac{\mathrm{d}\phi}{\mathrm{d}R}\right|_{R=\alpha} = \frac{h}{\alpha c l} \tag{23}$$

This quantity is finite. This is what gives the impression, when projecting the geodesic lying on the other sheet, of a discontinuity. It seems to bounce on the throat sphere. But this is only an artifact resulting from the embedding into $\mathbb{R}^3$. If one instead adopts a representation (t,ρ,θ,φ) and projects the geodesics with $\theta=\pi/2$ onto the plane (ρ,φ), this angular discontinuity disappears. Let β denote the angle formed by the tangent to the curve and the radius vector in this representation.

$$(tg\,\beta)_{\rho\to 0}=\frac{\rho\mathrm{d}\varphi}{\mathrm{d}\rho}=\frac{R\mathrm{d}\varphi}{\mathrm{d}R}\frac{\rho}{R}\frac{\mathrm{d}R}{\mathrm{d}\rho}=\frac{h}{\alpha^2 cl}\rho^2\to 0 \tag{24}$$

One can therefore draw the corresponding projection of the geodesic. See **Figure 31**.

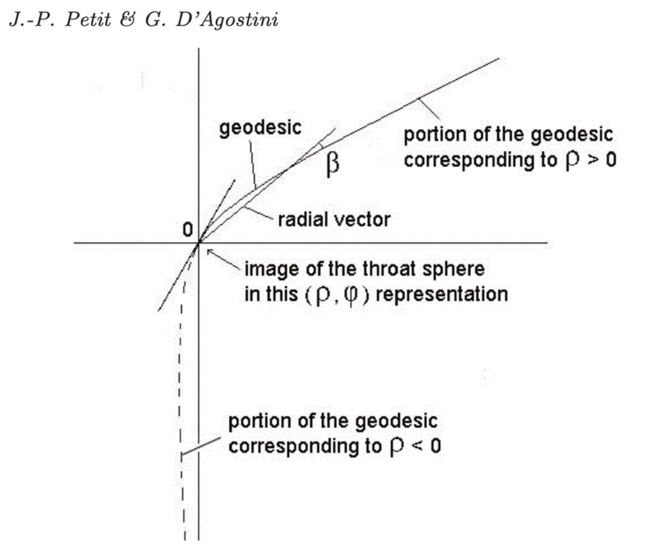


Fig. 9. Geodesical path in a $[\rho,\varphi]$ representation (plane, $\theta=\pi/2$).

Figure 31. Projection of spatiotemporal geodesics in the representation (ρ,φ) [17].

This illustrates the interpretation of the Schwarzschild exterior metric solution as a two-sheeted covering of a manifold. Such a situation could be visualized by drawing the geodesics $(\rho>0)$ on the front side of a sheet of paper and extending them with the curves $(\rho<0)$ on the back side (dashed curve). The throat sphere is then reduced to a point. What opposes our intuition is that this sphere reduced to a point nevertheless possesses a nonzero area $4\pi\alpha^2$, so that the manifold is locally non-contractible even though it visually appears contractible. A similar representation would be obtained by exploiting the true expression of the Schwarzschild exterior metric solution in coordinates (t,r,θ,φ). See **Figure 10**. In a projection (r,φ), using the true variable $r=\sqrt{x^2+y^2+z^2}\geq 0$ the interior of the sphere of area $4\pi\alpha^2$ would likewise reduce to a point.

These elements show that artifacts may arise when one attributes physical reality to a representation obtained through embedding in $\mathbb{R}^3$. That being said, can the Schwarzschild exterior metric solution be considered in isolation? The first response was given in 1939 by Oppenheimer and Snyder [20]. In article [22], a second interpretation is proposed, outlining a model for a neutron star reaching Schwarzschild criticality [9], leading to the establishment of a passage toward a second sheet. Whatever physical meaning one assigns to this model, the following figure illustrates the P-symmetry of the two sheets. In three-dimensional space, four points form a tetrahedron whose volumetric orientation is linked to the orientation of its faces. When this tetrahedron crosses the throat sphere, its orientation becomes inverted, showing that the two sheets are related by an enantiomorphic relationship, as shown on **Figure 32**.

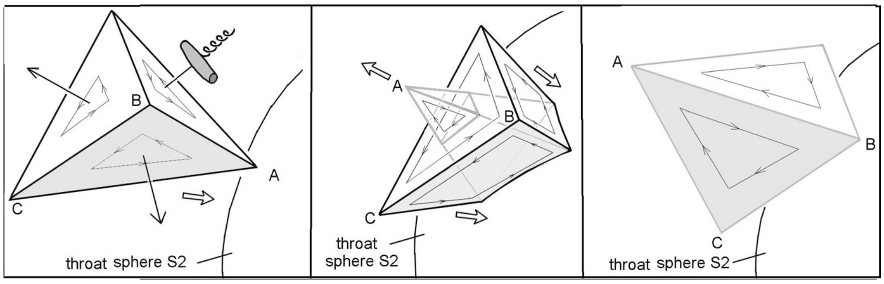


Figure 32. Crossing the throat sphere produces space-inversion [23].

In general, a choice of coordinates leads to a representation of the solution-object that is both geometric and physical. Each such choice corresponds to a particular way of conceiving spacetime, both geometrically and physically. When the stationary, spherically symmetric exterior solution to Einstein's equations is viewed in isolation as a physical object, a black hole, this choice of a geometry that allows for contraction implies a swapping of the roles of space and time coordinates within the object; this issue does not arise in interpretations derived from Flamm's, in which the object is non-contractile and possesses no interior.

8. Conclusion

We must now draw conclusions from this review of the various works in question. This could be summarized by saying that "everything depends on what one calls spacetime". As defined in 1909 by H. Minkowski, it is a hypersurface defined by a real elementary line element: $\mathrm{d}s^2 \geq 0$. This was also the position held by Schwarzschild, Einstein, Droste, Weyl, Flamm, and many others at the time. In December 1916, the mathematician D. Hilbert proposed that the solution hypersurface be defined directly from the bilinear form—representing what is today referred to as the "extension of the solution to the whole of spacetime". Consequently, the signature ceases to act as a rigid constraint and can take either the form $(+---)$ or $(-+++)$. This corresponds to the standard view of spacetime expressed through the geometric interpretation of the Schwarzschild solution known as the black hole model. This article demonstrates that this interpretation differs from the reading of the geometry provided by Flamm. Achieving this "extension to the whole of spacetime" requires that geodesics penetrating "into the interior"—beyond the Schwarzschild sphere—connect differently. That is, instead of extending onto the second sheet of Flamm's surface (organized around a throat sphere and corresponding to a wormhole), they extend into a region of spacetime "where time and space coordinates exchange roles". These correspond to two different topological options. Mathematically, Flamm's physical interpretation corresponds to a non-contractile geometric object, as can be seen directly from the line element in **Figure 10**, whereas the black hole model reflects the preservation of the object's contractibility, at the cost of swapping the roles of the coordinates. This article represents only the beginning of a broader line of reasoning linked to the black hole model. The ultimate goal is to clarify the nature of the geometric extension associated with this model. A future article will address the Kruskal extension and aspects related to Birkhoff's theorem.

Author Contributions

J.-P. Petit is the author of the article. G. D'Agostini verified the calculations, and J.-M. Pechinot contributed to the production of some of the figures.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Schwarzschild, K. (1916) Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin.

[2] Einstein, A. (1915) Erlklärung der Perihelbewegung des Merkur aus allgemeinen Relarivitätstheorie. Königlish Preubische Akademie der Wissenschaften (Berlin) Sitzungberichet, 831-838.

[3] Schwarzschild, K. (1915) Letter to A. Einstein. Collected Papers of Albert Einstein, The Berlin Years, Correspondence 1914-1918, Document n°169. https://einsteinpapers.press.princeton.edu/vol6-doc

[4] Abrams, L.S. (1989) Black Holes: The Legacy of Hilbert's Error. *Canadian Journal of Physics*, **67**, 919-926.

[5] Hilbert, D. (1916) Die Grundlagen der Physik (Esrte Mitteilung). Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse.

[6] Flamm, L. (1916) Beiträge zur Einsteinschen Gravitationstheorie. *Physikalische Zestschrift*, **17**, 448-454.

[7] Hilbert, D. (1915) Die Grundlagen der Physik (Esrte Mitteilung). Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse, 20 November 1915, 395-407.

[8] Einstein, A. (1915) Die Feldgleichungen des Gravitation (The Field Equation of Gravitation). *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, Berlin, 844-847.

[9] Schwarzschild, K. (1916) Über das Gravitionalsfeld einer Kugel Aus incompressibler Flü-

ssigkeit nach der Einsteinschen Theorie. Sitzung der phys. Math. Klasse v.23 märz.

[10] Droste, J. (1917) The Field of a Single Centre in Einstein's Theory of Gravitation and the Motion of a Particle in That Field. *Proceedings of the Koninklijke Nederlandsche Akademie van Wetenschappen*, **19**, 197-215.

[11] Weyl, H. (1917) Zur Gravitationstheorie. *Annalen der Physik*, **359**, 117-145. https://doi.org/10.1002/andp.19173591804

[12] Minkowski, H. (1909) Raum und Zeit. *Physikalische Zeitschrift*, **10**, 75-88.

[13] Poincaré, H. (1902) La mesure du temps. *Revue de métaphysique et de morale*, **10**, 1-13.

[14] Eddington, A. (1919) The Mathematical Theory of Relativity. Cambridge University Press.

[15] Tolman, R.C. (1946) Relativity, Thermodynamics and Cosmology. The Clarendon Press.

[16] Chandrasekhar, S. (1992) The Mathematical Theory of Black Holes. Oxford University Press.

[17] Petit, J.P. and D'Agostini, G. (2015) Cancellation of the Singularity of the Schwarzschild Solution with Natural Mass Inversion Process. *Modern Physics Letters A*, **30**, Article ID: 1550051.

[18] Eufrasio, R.T., Mecholsky, N.A. and Resca, L. (2018) Curved Space, Curved Time, and Curved Space-Time in Schwarzschild Geodetic Geometry. *General Relativity and Gravitation*, **50**, Article No. 159. https://doi.org/10.1007/s10714-018-2481-2

[19] Einstein, A. and Rosen, N. (1935) The Particle Problem in the General Theory of Relativity. *Physical Review*, **48**, 73-77. https://doi.org/10.1103/physrev.48.73

[20] Oppenheimer, J.R. and Snyder, H. (1939) On Continued Gravitational Contraction. *Physical Review*, **56**, 455-459. https://doi.org/10.1103/physrev.56.455

[21] Kruskal, M.D. (1960) Maximal Extension of Schwarzschild Metric. *Physical Review*, **119**, 1743-1745. https://doi.org/10.1103/physrev.119.1743

[22] Petit, J. and D'Agostini, G. (2025) Alternatives to Black Holes: Gravastars and Plugstars. *Journal of Modern Physics*, **16**, 1479-1490. https://doi.org/10.4236/jmp.2025.1610072

[23] Koiran, P., Zejli, H., Levy, J., Margnat, F., Duval, M. and Zejli, H. (2024) PT-Symmetry in One-Way Wormholes. *Annals of Physics*, **470**, Article ID: 169765. https://doi.org/10.1016/j.aop.2024.169765

Supplementary Information

wobei die Hilfsgröße

$$R = (3x_1 + \rho)^{1/3} = (r^3 + \alpha^3)^{1/3}$$

eingeführt ist.

Setzt man diese Werte der Funktionen f im Ausdruck (9) des Linienelements ein und kehrt zugleich zu gewöhnlichen Polarkoordinaten zurück, so ergibt sich das Linienelement, welches die strenge Lösung des Einsteinschen Problems bildet:

$$ds^2 = (1 - \alpha/R)dt^2 - \frac{dR^2}{1 - \alpha/R} - R^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2), \quad R = (r^3 + \alpha^3)^{1/3}. \quad (14)$$

Dasselbe enthält die eine Konstante α, welche von der Größe der im Nullpunkt befindlichen Masse abhängt.

Figure S1. German version of **Figure 1** (the line element expressed with the intermediate quantity R).

§ 3. Nennt man die Zeit t, die rechtwinkligen Koordinaten x, y, z, so ist das allgemeinste Linienelement, welches die Forderungen 1—3 erfüllt, offenbar das folgende:

$$ds^2 = Fdt^2 - G(dx^2 + dy^2 + dz^2) - H(xdx + ydy + zdz)^2$$

wobei F, G, H Funktionen von $r = \sqrt{x^2 + y^2 + z^2}$ sind.

Figure S2. German version of **Figure 2** (the true Schwarzschild's radial coordinate r).

Man setze

$$x_1 = \frac{r^3}{3}, \quad x_2 = -\cos\vartheta, \quad x_3 = \phi. \quad (7)$$

Figure S3. German version of **Figure 3** (Schwarzschild's first coordinate change).

In den neuen Polarkoordinaten lautet das Linienelement:

$$ds^2 = Fdx_4^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right)dx_1^2 - Gr^2\left[\frac{dx_2^2}{1 - x_2^2} + dx_3^2(1 - x_2^2)\right], \quad (8)$$

wofür wir schreiben wollen:

$$ds^2 = f_4 dx_4^2 - f_1 dx_1^2 - f_2\frac{dx_2^2}{1 - x_2^2} - f_3 dx_3^2(1 - x_2^2). \quad (9)$$

Dann sind f_1, $\boxed{f_2 = f_3}$, f_4 drei Funktionen von x_1

Figure S4. German version of **Figure 4** (the three functions f).

Nochmals integriert

$$f_2 = \lambda(3x_1+\rho)^{2/3}. \qquad (\lambda \text{ Integrationskonstante})$$

Die Bedingung im Unendlichen fordert: $\lambda = 1$. Also

$$f_2 = (3x_1+\rho)^{2/3}. \qquad (10)$$

Damit ergibt sich weiter aus (c″) und (d)

$$\frac{\partial f_4}{\partial x_1} = \alpha f_1 f_4 = \frac{\alpha}{f_2^2} = \frac{\alpha}{(3x_1+\rho)^{2/3}}.$$

Integriert in Rücksicht auf die Bedingung im Unendlichen

$$f_4 = 1-\alpha(3x_1+\rho)^{-1/3}. \qquad (11)$$

Nunmehr aus (d)

$$f_1 = \frac{(3x_1+\rho)^{-4/3}}{1-\alpha(3x_1+\rho)^{-1/3}}. \qquad (12)$$

Figure S5. German version of **Figure 4** (expressions of the three functions *f*) [1].

Danach hätte es außer Ihrem α noch eine zweite gegeben und das Problem wäre physikalisch unbestimmt. Daraufhin machte ich einmal auf gut Glück den Versuch einer vollständigen Lösung. Eine nicht zu große Rechnerei ergab folgendes Resultat: Es gibt nur ein Linienelement, das Ihre Bedingungen 1) bis 4) nebst Feld- und Determinantengl. erfüllt und im Nullpunkt und nur im Nullpunkt singulär ist.

Sei:

$$x_1 = r\cos\phi\cos\theta \qquad x_2 = r\sin\phi\cos\theta \qquad x_3 = r\sin\theta$$

$$R = (r^3+\alpha^3)^{1/3} = r\left(1+\frac{1}{3}\frac{\alpha^3}{r^3}+\dots\right)$$

dann lautet das Linienelement:

$$ds^2 = \left(1-\frac{\gamma}{R}\right)dt^2 - \frac{dR^2}{1-\frac{\gamma}{R}} - R^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

R, θ, ϕ sind keine „erlaubten" Koordinaten, mit denen man die Feldgleichungen bilden dürfte, weil sie nicht die Determinante 1 haben, aber das Linienelement schreibt sich in ihnen am schönsten.

Figure S6. German version of **Figure 8** (the letter to Einstein).

so muß H die Gestalt haben

$$H = K + L \longleftarrow \text{world function}$$

wo K die aus dem Riemannschen Tensor entspringende Invariante (Krümmung der vierdimensionalen Mannigfaltigkeit)

$$K = \sum_{\mu,\nu} g^{\mu\nu} K_{\mu\nu}, \longleftarrow \text{Ricci tensor}$$

$$K_{\mu\nu} = \sum_{\varkappa}\left(\frac{\partial}{\partial w_\nu}\left\{\begin{matrix}\mu\varkappa\\ \varkappa\end{matrix}\right\} - \frac{\partial}{\partial w_\varkappa}\left\{\begin{matrix}\mu\nu\\ \varkappa\end{matrix}\right\}\right) + \sum_{\varkappa,\lambda}\left(\left\{\begin{matrix}\mu\varkappa\\ \lambda\end{matrix}\right\}\left\{\begin{matrix}\lambda\nu\\ \varkappa\end{matrix}\right\} - \left\{\begin{matrix}\mu\nu\\ \lambda\end{matrix}\right\}\left\{\begin{matrix}\lambda\varkappa\\ \varkappa\end{matrix}\right\}\right)$$

Figure S7. German version of **Figure 9** (the Hilbert's world function).

Unter Verwendung der vorhin eingeführten Bezeichnungsweise für die Variationsableitungen bezüglich der $g^{\mu\nu}$ erhalten die Gravitationsgleichungen wegen (20) die Gestalt

(21)
$$[\sqrt{g}\,K]_{\mu\nu} + \frac{\partial\sqrt{g}\,L}{\partial g^{\mu\nu}} = 0.$$

Das erste Glied linker Hand wird

$$[\sqrt{g}\,K]_{\mu\nu} = \sqrt{g}\,(K_{\mu\nu} - \tfrac{1}{2}\,K g_{\mu\nu}),$$

Figure S8. German version of Figure 10 (The Hilbert's Field Equation).

Die neuen $g_{\mu\nu}$ (μ, $\nu = 1, 2, 3, 4$) — die Einsteinschen Gravitationspotentiale — sollen dann sämtlich reelle Funktionen der reellen Variabeln x_s ($s = 1, 2, 3, 4$) sein von der Art, daß bei der Darstellung der quadratischen Form

(28)
$$G(X_1, X_2, X_3, X_4) = \sum_{\mu\nu} g_{\mu\nu} X_\mu X_\nu$$

als Summe von vier Quadraten linearer Formen der X_s stets drei Quadrate mit positivem und ein Quadrat mit negativem Vorzeichen auftritt: die quadratische Form (28) liefert somit für unsere vierdimensionale Welt der x_s die Maßbestimmung einer Pseudogeometrie. Die Determinante g der $g_{\mu\nu}$ fällt negativ aus.

Figure S9. German version of Figure 11 (Hilbert's quadratic form).

Im Falle der pseudo-Euklidischen Geometrie haben wir

$$g_{\mu\nu} = \delta_{\mu\nu},$$

worin

$$\delta_{\mu\mu} = 1, \quad \delta_{\mu\nu} = 0 \quad (\mu \neq \nu)$$

bedeutet. Für jede dieser pseudo-Euklidischen Geometrie benachbarte Maßbestimmung gilt der Ansatz

(37)
$$g_{\mu\nu} = \delta_{\mu\nu} + \varepsilon h_{\mu\nu} + \cdots,$$

wo ε eine gegen Null konvergierende Größe und $h_{\mu\nu}$ Funktionen der w_s sind. Über die Maßbestimmung (37) mache ich die folgenden zwei Annahmen:

I. Die $h_{\mu\nu}$ mögen von der Variabeln w_4 unabhängig sein.

II. Die $h_{\mu\nu}$ mögen im Unendlichen ein gewisses reguläres Verhalten zeigen.

Figure S10. German version of Figure 12 (Hilbert's pseudo Euclidean space).

Das Axiom bedeutet, dass an jedem Punkt im Universum der Ausdruck

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

ist immer positiv, oder, gleichbedeutend damit, dass die Geschwindigkeit v immer kleiner als c ist. Somit ist c die obere Grenze für die Geschwindigkeit eines beliebigen wesentlichen Punktes, und genau darin liegt seine wahre Bedeutung.

$$d\tau = \frac{1}{c}\sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}$$

Das Integral $\int d\tau$ dieser Größe entlang der Weltgeraden wird als Eigenzeit des substanziellen Punktes bezeichnet.

Figure S11. German version of **Figure 13** (Minkowski's axiomatic definition of spacetime).

Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie.

Von K. Schwarzschild.

(Vorgelegt am 13. Januar 1916 [s. oben S. 42].)

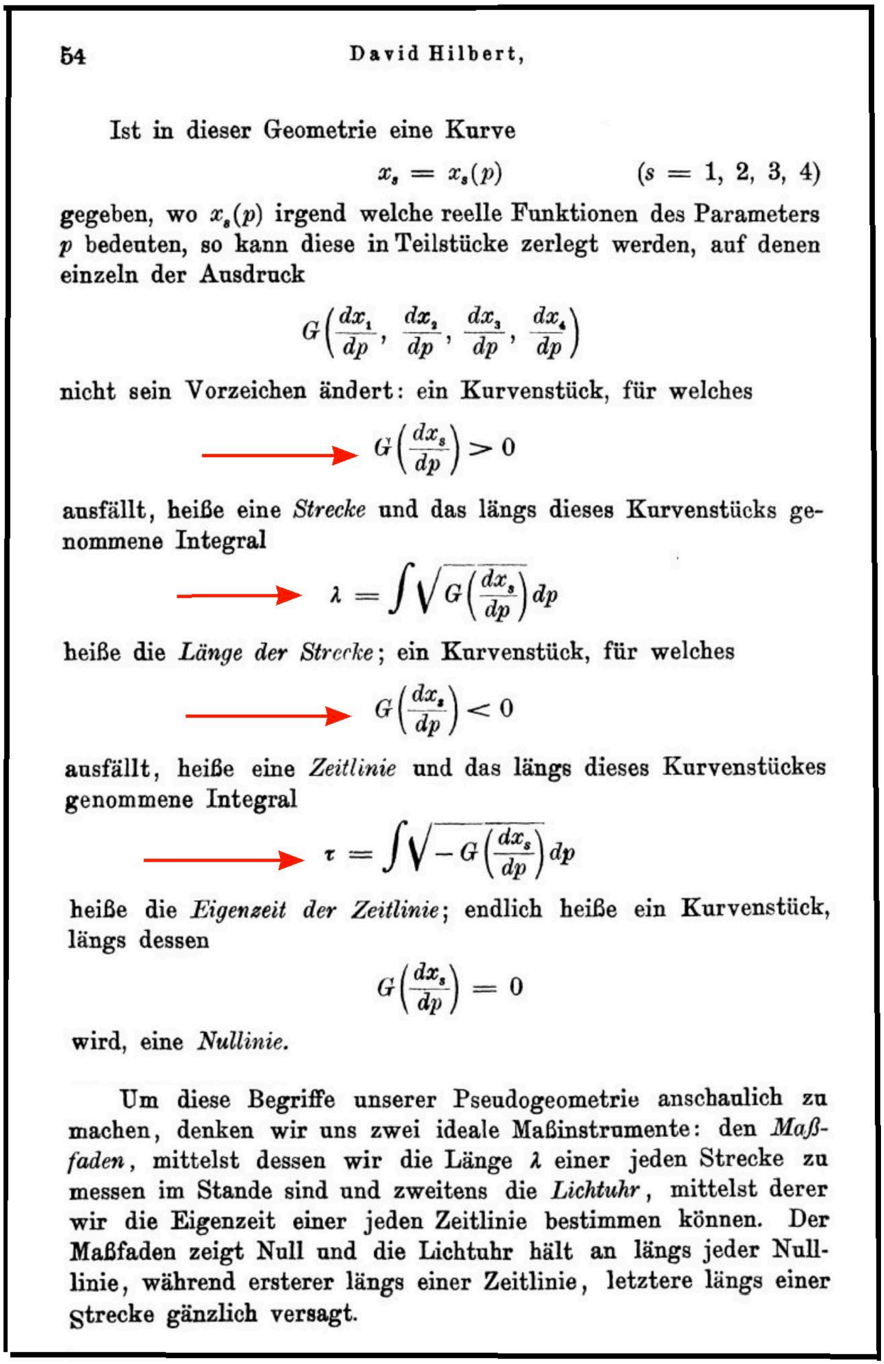
§ 1. Hr. Einstein hat in seiner Arbeit über die Perihelbewegung des Merkur (s. Sitzungsberichte vom 18. November 1915) folgendes Problem gestellt:

Ein Punkt bewege sich gemäß der Forderung

$$\left.\begin{aligned} &\delta \int ds = 0, \\ \text{wobei} \quad &ds = \sqrt{\sum g_{\mu\nu} dx_\mu dx_\nu} \qquad \mu, \nu = 1, 2, 3, 4 \end{aligned}\right\} \quad (1)$$

ist, $g_{\mu\nu}$ Funktionen der Variabeln x bedeuten und bei der Variation am Anfang und Ende des Integrationswegs die Variablen x festzuhalten sind. Der Punkt bewege sich also, kurz gesagt, auf einer geodätischen Linie in der durch das Linienelement ds charakterisierten Mannigfaltigkeit.

Figure S12. German version of **Figure 14** (foreword of the Schwarzschild's paper).



54 David Hilbert,

Ist in dieser Geometrie eine Kurve

$$x_s = x_s(p) \qquad (s = 1, 2, 3, 4)$$

gegeben, wo $x_s(p)$ irgend welche reelle Funktionen des Parameters p bedeuten, so kann diese in Teilstücke zerlegt werden, auf denen einzeln der Ausdruck

$$G\left(\frac{dx_1}{dp}, \frac{dx_2}{dp}, \frac{dx_3}{dp}, \frac{dx_4}{dp}\right)$$

nicht sein Vorzeichen ändert: ein Kurvenstück, für welches

$$G\left(\frac{dx_s}{dp}\right) > 0$$

ausfällt, heiße eine *Strecke* und das längs dieses Kurvenstücks genommene Integral

$$\lambda = \int \sqrt{G\left(\frac{dx_s}{dp}\right)}\, dp$$

heiße die *Länge der Strecke*; ein Kurvenstück, für welches

$$G\left(\frac{dx_s}{dp}\right) < 0$$

ausfällt, heiße eine *Zeitlinie* und das längs dieses Kurvenstückes genommene Integral

$$\tau = \int \sqrt{-G\left(\frac{dx_s}{dp}\right)}\, dp$$

heiße die *Eigenzeit der Zeitlinie*; endlich heiße ein Kurvenstück, längs dessen

$$G\left(\frac{dx_s}{dp}\right) = 0$$

wird, eine *Nullinie.*

Um diese Begriffe unserer Pseudogeometrie anschaulich zu machen, denken wir uns zwei ideale Maßinstrumente: den *Maßfaden*, mittelst dessen wir die Länge λ einer jeden Strecke zu messen im Stande sind und zweitens die *Lichtuhr*, mittelst derer wir die Eigenzeit einer jeden Zeitlinie bestimmen können. Der Maßfaden zeigt Null und die Lichtuhr hält an längs jeder Nulllinie, während ersterer längs einer Zeitlinie, letztere längs einer Strecke gänzlich versagt.

Figure S13. German version of **Figure 15** (Hilbert's definition of lengths).

auf den Koordinatenanfangspunkt.

Nach Schwarzschild ist die allgemeinste diesen Annahmen entsprechende Maßbestimmung in räumlichen Polarkoordinaten, wenn

$$
\begin{aligned}
w_1 &= r\cos\vartheta\\
w_2 &= r\sin\vartheta\cos\varphi\\
w_3 &= r\sin\vartheta\sin\varphi\\
w_4 &= l
\end{aligned}
$$

gesetzt wird, durch den Ausdruck

$$F(r)\,dr^2 + G(r)\,(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2) + H(r)\,dl^2 \tag{42}$$

dargestellt, wo $F(r)$, $G(r)$, $H(r)$ noch willkürliche Funktionen von r sind.

Figure S14. German version of **Figure 16** (Hilbert's coordinates' choice).

If we take as integrals of (44) $m = \alpha$, where α is a constant, and $w = 1$, which evidently is no essential restriction, then for $l = it$ (43) results in the desired metric in the form first found by Schwarzschild

$$G(dr, d\vartheta, d\varphi, dt) = \frac{r}{r-\alpha}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\,d\varphi^2 - \frac{r-\alpha}{r}dt^2 \tag{45}$$

Figure S15. German version of **Figure 17**-German version of **Figure 17** (Hilbert's identification of his solution to Schwarzschild's).

Wenn man zu Polarkoordinaten gemäß $x = r\sin\vartheta\cos\phi$, $y = r\sin\vartheta\sin\phi$, $z = r\cos\vartheta$ übergeht, lautet dasselbe Linienelement:

$$ds^2 = F dt^2 - (G + Hr^2)\,dr^2 - Gr^2\,(d\vartheta^2 + \sin^2\vartheta\,d\phi^2). \tag{6}$$

Figure S16. German version of **Figure 18** (Schwarzschild's line element).

$$F(r)\,dr^2 + G(r)\,(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2) + H(r)\,dl^2 \tag{42}$$

dargestellt, wo $F(r)$, $G(r)$, $H(r)$ noch willkürliche Funktionen von r sind. Setzen wir

$$r^* = \sqrt{G(r)},$$

so sind wir in gleicher Weise berechtigt r^*, ϑ, φ als räumliche Polarkoordinaten zu deuten. Führen wir in (42) r^* anstatt r ein und lassen dann wieder das Zeichen $*$ weg, so entsteht der Ausdruck

$$M(r)\,dr^2 + r^2\,d\vartheta^2 + r^2\sin^2\vartheta\,d\varphi^2 + W(r)\,dl^2, \tag{43}$$

Figure S17. German version of **Figure 19** (Hilbert's approximation).

§2. Nun können wir in analoger Weise den ersten Fall untersuchen, der von Schwarzschild behandelt wurde, nämlich das Gravitationsfeld einer Punktmasse. Das Linienelement hat in seiner einfachsten Form dieselbe Struktur wie oben und lautet:

$$d\,s^2 = \left(1 - \frac{\alpha}{R}\right)\,d\,t^2 - \frac{d\,R^2}{1 - \dfrac{\alpha}{R}} - R^2\left(d\,\vartheta^2 + \sin^2\vartheta\,d\,\varphi^2\right).$$

Figure S18. German version of **Figure 21** (Flamm's choice for the metric representation).

Wenn also elementare Lineale und Uhren vom Gravitationsfeld beeinflusst werden, muss die Ausbreitung des Lichts in exakt derselben Weise beeinflusst werden, sodass sie im Vergleich keine Unterschiede aufweisen. Dasselbe gilt für alle anderen physikalischen Phänomene. Dies könnte erklären, warum sich die Bewegung eines Punktteilchens durch die einfache Variationsgleichung beschreiben lässt.

$$\delta \int_{P_1}^{P_2} d\,s = 0, \tag{5}$$

Figure S19. German version of **Figure 22** (Flamm's variational procedure).

Der räumliche Anteil des Linienelements ist wiederum nichteuklidischer Natur. Aufgrund der sphärischen Symmetrie scheint es wiederum ausreichend zu sein, die Betrachtung auf die metrischen Eigenschaften in einem beliebigen ebenen Schnitt durch den Ursprung zu beschränken, $\vartheta = \pi/2$. un erhält man

$$d\,{\sigma_e}^2 = \frac{d\,R^2}{1 - \dfrac{\alpha}{R}} + R^2\,d\,\varphi^2. \tag{4}$$

Bezeichnet man mit z die Koordinate in Richtung der Rotationsachse, so ergibt sich die Gleichung für die Meridionalkurve aus

$$\frac{d\,z}{d\,R} = \operatorname{tg}\chi = \sqrt{\frac{\alpha}{R - \alpha}}$$

$$z^2 = 4\alpha(R - \alpha).$$

Dies beschreibt eine Parabol mit Parameter

$$p = 2\alpha,$$

Figure S20. German version of **Figure 23** (Flamm's embedment).