

Vibrotransport Solutions of Wave Equations and Their Properties by Subsonic Velocities

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Abstract

The vibrotransport sources of disturbances in various media are the most common. They are associated with moving objects of oscillation whose speed can be subsonic, sonic, supersonic, and in media with several sonic speeds (elastic, for example) and transonic. Here, fundamental and regular vibrotransport solutions of the wave equation are constructed, which describe the dynamics of the medium during the movement of a source, which is concentrated at a point, moves at a constant speed V , and vibrates at a constant frequency ω . The type of equations depends on the Mach number $M = V/c$, where c is the sound speed in the medium. The value of M essentially affects the type of equations to be solved: elliptical for $M < 1$, parabolic for $M = 1$, and hyperbolic for $M > 1$. Green's functions are constructed that describe the dynamics of the medium during the motion of a vibration source concentrated at a point in the subsonic range of speeds in 3D spaces. On this basis, general solutions of the vibrotransport equation are constructed under the action of both spatially distributed moving vibration sources and concentrated on moving surfaces and lines. A mathematical description of the Doppler effect with a graphical illustration is given. The constructed solutions allow one to construct solutions to many equations of continuum mechanics and field theory for studying wave processes excited by various types of moving sources of oscillations in media, and should find wide application in solving various engineering and technical problems.

Keywords

Wave Equation, Vibrotransport Solution, Mach Number, Green's Function, Fourier Transformation, Helmholtz Equations, Subsonic Solutions, Doppler's Effect

1. Introduction

The study of wave propagation processes in continuous media and electromagnetic fields leads to the solution of systems of partial differential equations of various types and the determination of their solutions in the form of vector fields that describe various characteristics of wave processes. These can include, for example, displacements and velocities, as in elastic and multicomponent media, or electromagnetic field strengths, the variation of which in space and time allows for the modeling of such processes and their study using mathematical methods.

As is well known, any vector field

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^3 u_j(\mathbf{x}, t) \mathbf{e}_j$$

can be represented through scalar and vector potentials in the form [1]:

$$\mathbf{u}(\mathbf{x}, t) = \text{grad } \varphi(\mathbf{x}, t) + \text{rot } \boldsymbol{\psi}(\mathbf{x}, t) \quad (1.1)$$

which describe dilatational and vortex waves in the medium. In isotropic media, they typically satisfy the wave equations:

$$\square_{c_\varphi} \varphi = f(\mathbf{x}, t), \quad \square_{c_\psi} \boldsymbol{\psi} = \mathbf{g}(\mathbf{x}, t) \quad (1.2)$$

since the wave propagation velocity in such media is always finite and does not depend on the direction of wave propagation. The speed of these waves can vary, as in elastic media, where shear waves, described by a vector potential, propagate more slowly than dilatational waves. In an electromagnetic medium, described by Maxwell's equations, they are the same. This makes the study of solutions to wave equations necessary and relevant.

Among the active sources of disturbances in various environments, the most common are transport ones, which are associated with moving sources (loads), the shape of which does not change over time, and the speed of movement can be different: subsonic, sonic, supersonic, and in environments with several sonic velocities (elastic, for example), also transonic.

In works [2]-[4], transport solutions of wave equations and equations of elasticity theory are constructed, and a method of boundary integral equations for solving stationary transport subsonic and supersonic boundary value problems in domains with cylindrical boundaries has been developed. It should be noted that the number of works on the study of the impact of transport loads on the environment has been growing in recent decades due to the intensive construction of high-speed roads and underground transport highways, and has a fairly extensive bibliography, which can be found in articles and monographs [5]-[8].

There is another class of disturbance sources (acting forces and loads), which is very important for applications, that not only move at different speeds, but also pulsate (vibrate, oscillate) with a certain frequency. Examples include various electromagnetic emitters, moving elementary particles, moving vibratory transport, etc. Therefore, mathematical modeling of such processes, taking into account the type of source, its speed of movement, and vibration frequency, is relevant. Math-

ematical models of such processes are few and require further development [9]-[11].

In my article [12], written in Russian, I constructed vibrotransport solutions of this equation for spaces of different dimensions. Here, we present different subsonic solutions in three-dimensional space for emitters of any form, which may be described by regular and singular generalized functions.

Here, fundamental and regular *vibrotransport solutions* of the wave equation are constructed in the subsonic case in 3D space. Green's functions are constructed that describe the wave field during the movement of a vibration source concentrated at a point. On this basis, general solutions of the *vibrotransport wave equation* are constructed for the action of both moving vibration sources distributed in space and concentrated on moving surfaces and lines. Obtained solutions allow us to construct solutions to many equations of continuum mechanics for this type of moving disturbance sources in media by using scalar and vector potentials (1.1) and have wide applications in solving various engineering problems.

It should be noted that Blokhintsev [9] constructed particular transport and vibration transport solutions to wave equations in 3D space when solving acoustic equations for moving sources to study sound propagation processes in air. However, he introduced vibration sources differently than in this work. Therefore, the solutions constructed here have different forms.

2. D'Alembert's Wave Equation and Its Properties

We consider D'Alembert wave equation:

$$\square_c u \equiv \Delta u - c^{-2} \frac{\partial^2 u}{\partial t^2} = g(x, t), \quad x \in R^3, \quad t \in R^1. \quad (2.1)$$

Here, $\square_c$ is wave operator, Δ is the Laplace operator, $g(x, t)$ is a locally integrable function. Equation (1) is strictly hyperbolic, and its class of solutions contains functions discontinuous in derivatives. The discontinuity surfaces F in R^4 are the characteristic surfaces of Equation (2.1), which satisfy the characteristic equation in the space $R^{N+1} = \{(x_1, \dots, x_N, \tau \equiv ct)\}$ [13]:

$$v_\tau^2 = \sum_{j=1}^3 v_j^2, \quad (2.2)$$

where $v(x, \tau) = (v_1, v_2, v_3, v_\tau)$ is normal vector to F . It corresponds to a cone of characteristic normal—the *sound cone*, for which $v_\tau = v_4 < 0$. In R^3 such surfaces move with unit velocity over $\tau = ct$:

$$1 = -v_\tau / \|v\|_3, \quad \|v\|_3 = \sqrt{\sum_{j=1}^3 v_j^2}. \quad (2.3)$$

In R^3 , they correspond to *wave fronts* (F_t), moving with speed c over time t . The Hadamard continuity conditions are satisfied on them:

$$[u(x, t)]_{F_t} = 0, \quad [\dot{u}]_{F_t} = -cn_i [u_{,i}]_{F_t}. \quad (2.4)$$

Here, $[f(x, t)]_{F_t}$ is the jump f on F_t :

$$\left[f(x, t) \right]_{F_t} = f^+(x, t) - f^-(x, t) = \lim_{\varepsilon \rightarrow +0} (f(x + \varepsilon n, t) - f(x - \varepsilon n, t)), \quad x \in F_t,$$

$n(x, t)$ is a unit vector normal to F_t , directed in the direction of propagation of the wave front:

$$n_i = \frac{v_i}{\|v\|_N} = \frac{\text{grad } F_t}{\|\text{grad } F_t\|}, \quad i = 1, 2, 3. \quad (2.5)$$

The last equality is valid if the wave front equation can be represented as $F_t(x, t) = 0$ under the condition of existence $\text{grad } F_t$. The class of such solutions of hyperbolic equations is called a *shock wave*. On their fronts, the derivatives of functions and even the functions themselves can undergo jumps.

From the second condition (2.4), follow that on the fronts:

$$\dot{u}^- + cn_i u_{,i}^- = \dot{u}^+ + cn_i u_{,i}^+. \quad (2.6)$$

If in front of the wave front $u \equiv 0$ (medium at rest), this equality gives a useful relation at the wave front:

$$(\text{grad } u, n) = -c^{-1} \dot{u}, \quad x \in F_t.$$

Note that the tangent derivatives to the characteristic surface, due to the continuity of u , are also continuous, i.e.,

$$\gamma_\tau [u_{,\tau}]_F = -\gamma_j [u_j]_F, \quad \forall \gamma: (v, \gamma) = 0. \quad (2.7)$$

In particular, if $\gamma = \gamma^j = (-v_j, v_\tau \delta_1^j, v_\tau \delta_2^j, v_\tau \delta_3^j)$, this leads to conditions of the form:

$$[-u_{,\tau} v_j + u_j v_\tau]_F = 0 \Rightarrow n_j [\dot{u}]_{F_t} = c [u_j]_{F_t}. \quad (2.8)$$

Solutions of the wave Equation (1) that satisfy the conditions on the shock wave fronts are hereinafter referred to as *classical* solutions.

3. Statement of the Vibrotransport Problem

Let us consider vibrotransport solutions of the wave equation (2.1).

Definition 1. Let us call the source function *vibrotransport* if it is represented as

$$g(x, t) = g(x_1, x_2, x_3 - Vt) e^{i\omega t}, \quad (3.1)$$

where V is the velocity of the source movement along the X_3 axis, $\omega > 0$ is the frequency of its oscillations. At $\omega = 0$, the source function is the *transport function*.

If the source function in (3.1) has the form (3.1), then it is natural to look for a solution to the wave Equation (3.1) in a similar form:

$$u(x, t) = u(x_1, x_2, x_3 - Vt) e^{i\omega t}. \quad (3.2)$$

To do this, we introduce the mobile coordinate system $(x_1, x_2, z = x - M\tau)$, $\tau = ct$, $M = V/c$ is the Mach number.

We call the source *subsonic* if the Mach number is $M < 1$, *hypersonic* if $M > 1$,

and *sonic* if $M = 1$.

In the new coordinate system, the solution looks like:

$$u = u(x_1, x_2, z) e^{i w \tau}, \quad w = \omega / c. \quad (3.3)$$

Then, as follows from (3.1), the amplitude of the oscillations is the solution of the *vibrotransport equation* (VTE):

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + (1 - M^2) \frac{\partial^2 u}{\partial z^2} + 2i w M \frac{\partial u}{\partial z} + w^2 u = g(x, z), \quad x \in R^2, z \in R^1. \quad (3.4)$$

We denote $m = \sqrt{|1 - M^2|}$. Then, depending on the velocity of the source, we have three different equations:

Subsonic elliptical for $M < 1$:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + m^2 \frac{\partial^2 u}{\partial z^2} + 2i w M \frac{\partial u}{\partial z} + w^2 u = g(x, z), \quad x \in R^2, z \in R^1; \quad (3.5)$$

Hypersonic hyperbolic at $M > 1$:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - m^2 \frac{\partial^2 u}{\partial z^2} + 2i w M \frac{\partial u}{\partial z} + w^2 u = g(x, z), \quad x \in R^2, z \in R^1; \quad (3.6)$$

Sound parabolic at $M = 1$:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + 2i w M \frac{\partial u}{\partial z} + w^2 u = g(x, z), \quad x \in R^2, z \in R^1; \quad (3.7)$$

It is required to construct a solution of these equations for any right-hand sides from the class of generalized slow-growth functions $S'(R^3)$ [14].

4. Fundamental Solutions of the Vibrotransport Equation—Fourier Transformation

To construct solutions to VTE, we construct the Green's function—the fundamental solution $U(x, t)$ of this equation with the delta function on the right side:

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + (1 - M^2) \frac{\partial^2 U}{\partial z^2} + 2i w M \frac{\partial U}{\partial z} + w^2 U = \delta(x) \delta(z), \quad x \in R^2, z \in R^1, \quad (4.1)$$

which satisfies certain attenuation conditions at infinity, different for each case. Then, using the property of the Green function, we will construct VTE solutions for mobile vibration sources distributed in limited volumes or concentrated on curved lines.

To construct solutions, we use the Fourier transform of generalized functions, which for regular generalized functions from Lebesgue space $L_1(R^3)$ coincides with the classical Fourier transform:

$$\begin{aligned} \bar{f}(\xi, \zeta) &= \int_{R^3} f(x, z) \exp(i(x_1 \xi_1 + x_2 \xi_2 + z \zeta)) dx_1 dx_2 dz \\ f(x, z) &= \frac{1}{(2\pi)^3} \int_{R^3} \bar{f}(\xi, \zeta) \exp(-i(x_1 \xi_1 + x_2 \xi_2 + z \zeta)) d\xi_1 d\xi_2 d\zeta \end{aligned} \quad (4.2)$$

To restore the originals, we use the properties of linear transformations of Fou-

rier variables in the space of the Fourier transformation, namely [14]:

$$\begin{aligned}
 F^{-1}\left[f^*(A(\xi+d))\right] &= \int_{R^3} f^*(A(\xi+d)) e^{i(\xi,x)} dV(\xi) \\
 &= |\det A|^{-1} \int_{R^3} f^*(\zeta) e^{i(A^{-1}\zeta-d,x)} dV(\zeta) \\
 &= \frac{e^{-i(d,x)}}{|\det A|} \int_{R^3} f^*(\zeta) e^{i(\zeta,(A^{-1})^T x)} dV(\zeta) \\
 &= \frac{e^{-i(d,x)}}{|\det A|} f\left((A^{-1})^T x\right).
 \end{aligned} \tag{4.3}$$

The Fourier transform of VTE (5.1) has the following form:

$$-\left(\|\xi\|^2 + (1-M^2)\zeta^2 - 2wv\zeta - w^2\right)\bar{U} = 1, \quad \xi \in R^2, \zeta \in R^1. \tag{4.4}$$

From where it follows:

$$\text{at } M < 1 \quad \bar{U} = -\frac{1}{\|\xi\|^2 + m^2\zeta^2 - 2wM\zeta - w^2}, \tag{4.5}$$

$$\text{at } M > 1 \quad \bar{U} = -\frac{1}{\|\xi\|^2 - m^2\zeta^2 - 2wM\zeta - w^2}, \tag{4.6}$$

$$\text{at } M = 1 \quad \bar{U} = -\frac{1}{\|\xi\|^2 - 2w\zeta - w^2}. \tag{4.7}$$

The appearance of the original depends on the dimension of the space in which this equation is considered.

Here, we consider the subsonic case $V < c$, i.e., $M < 1$.

5. Subsonic Solutions of the Vibrotransport Equation and Their Properties

Let us construct the Green's function $U(x, z)$ —the fundamental solution of VTE (4.1) satisfying the radiation conditions at infinity. To do this, we will find the transformation $U(x, z) = F^{-1}[\bar{U}(\xi, \zeta)]$ using the property of linear transformations of coordinates in the space of Fourier transforms (4.3).

Theorem 1. *Green's function of Equation (4.8) has the form:*

$$\begin{aligned}
 U(x, z) &= -\frac{1}{(2\pi)^3} \int_{R^3} \frac{e^{-i\xi z} e^{-i(\xi,x)}}{\|\xi\|^2 + m^2\zeta^2 - 2wM\zeta - w^2} d\xi_1 d\xi_2 d\zeta \\
 &= -\frac{e^{i(wMz/m^2)}}{(2\pi)^3 m} \int_{R^3} \frac{e^{-i\xi z/m} e^{-i(\xi,x)}}{\|\xi\|^2 + \zeta^2 - (w/m)^2} d\xi_1 d\xi_2 d\zeta.
 \end{aligned} \tag{5.1}$$

Proof: For $M < 1$, we transform (4.5) to a form convenient for constructing the original:

$$\bar{U}(\xi, \zeta) = -\frac{1}{\|\xi\|^2 + m^2\zeta^2 - 2wM\zeta - w^2} = -\frac{1}{\|\xi\|^2 + m^2\left(\zeta - wM/m^2\right)^2 - (w/m)^2};$$

$$\begin{aligned}
U(x, z) &= -\frac{1}{(2\pi)^N} \int_{R^3} \frac{e^{-i\zeta z} e^{-i(\xi, x)}}{\|\xi\|^2 + m^2 (\zeta - wM/m^2)^2 - (w/m)^2} d\xi_1 d\xi_2 d\zeta \\
&= -\frac{e^{i(wMz/m^2)}}{(2\pi)^N} \int_{R^3} \frac{e^{-i\zeta z} e^{-i(\xi, x)}}{\|\xi\|^2 + m^2 \zeta^2 - (w/m)^2} d\xi_1 d\xi_2 d\zeta \\
&= -\frac{e^{i(wMz/m^2)}}{(2\pi)^N m} \int_{R^3} \frac{e^{-i\zeta z/m} e^{-i(\xi, x)}}{\|\xi\|^2 + \zeta^2 - (w/m)^2} d\xi_1 d\xi_2 d\zeta.
\end{aligned} \tag{5.2}$$

Here, we used the substitution of variables $\zeta = \frac{\zeta}{m} - wM/m^2$. Note that here, under the sign of the integral, there is a Fourier transform of the fundamental solution of the three-dimensional Helmholtz equation:

$$\Delta W + k^2 W = \delta(y), \quad k = \frac{w}{m}, \quad y \in R^3.$$

The solution of this equation satisfying the Sommerfeld radiation conditions has the following form, and it is unique [14]:

$$W(y) = \frac{\exp(-ik\|y\|)}{4\pi\|y\|}, \quad y \in R^3. \tag{5.3}$$

Its Fourier transform has the form:

$$\bar{W} = -\frac{1}{\|\xi\|^2 + \zeta^2 - (k + i0)^2}, \quad y \in R^3. \tag{5.4}$$

From formula (5.2), taking into account (5.4) and (5.3), the formula of *Theorem 1* follows:

$$U(x, z) = U(x_1, x_2, z) = -\frac{e^{iwm^{-2}Mz}}{4\pi\sqrt{z^2 + m^2 r^2}} \exp\left(-\frac{iw}{m^2} \sqrt{z^2 + m^2 r^2}\right). \tag{5.5}$$

5.1. Solutions of Homogeneous Subsonic VTE

Now, we build solutions of a Homogeneous VTE:

$$\frac{\partial^2 u^0}{\partial x_1^2} + \frac{\partial^2 u^0}{\partial x_2^2} + 2i wM \frac{\partial u^0}{\partial z} + w^2 u^0 = 0, \quad x \in R^2, z \in R^1; \tag{5.6}$$

In the space of Fourier transforms, it has the form:

$$-\left(\|\xi\|^2 + (1 - M^2)\zeta^2 - 2wv\zeta - w^2\right)\bar{u}^0 = 0, \quad \xi \in R^N, \zeta \in R^1. \tag{5.7}$$

The solution of this equation:

$$\bar{u}^0 = \beta(\xi, \zeta) \delta_S(\xi, \zeta)$$

is a singular generalized function—a *simple layer* on a surface S on which

$$\left(\|\xi\|^2 + (1 - M^2)\zeta^2 - 2wv\zeta - w^2\right) = \|\xi\|^2 + m^2 \left(\zeta - wM/m^2\right)^2 - (w/m)^2 = 0. \tag{5.8}$$

Here, the density of a simple layer $\beta(\xi, \zeta)$ is an arbitrary function integrable on S . Accordingly, its original is a surface integral on S :

$$u^0(x, z) = \int_S \beta(\xi, \zeta) e^{-i(\xi, x)} e^{-i\zeta z} dS(\xi, \zeta), \quad \xi = (\xi_1, \xi_2). \quad (5.9)$$

Note that Equation (5.8) is the equation of an ellipsoid centered at the point $(0, 0, \zeta = wM/m^2)$:

$$\|\xi\|^2 + m^2 \zeta^2 = (w/m)^2, \quad \zeta = \zeta - wM/m^2. \quad (5.10)$$

Solutions of the homogeneous Helmholtz equation can also be used to construct $u^0(x, z)$:

$$\Delta u^0(y) + k^2 u^0(y) = 0. \quad (5.11)$$

Its solutions may be decomposed into series according to spherical harmonics and spherical Bessel functions [15]:

$$\begin{aligned} u(y) &= \sum_{n,m} a_n j_n(k\|y\|) P_n^m(\cos\theta) e^{im\varphi} \\ &= \sum_{n,m} a_n j_n(k\|y\|) P_n^m\left(\frac{y_3}{\|y\|}\right) (\cos\varphi + i\sin\varphi)^m \\ &= \sum_{n,m} a_n \frac{j_n(k\|y\|)}{\|y\|_2^m} P_n^m\left(\frac{y_3}{\|y\|}\right) (y_1 + iy_2)^m, \quad \|y\|_2 = \sqrt{y_1^2 + y_2^2} \end{aligned} \quad (5.12)$$

Here, $P_n^m(\cos\theta)$ are the attached Legendre polynomials, θ, φ are angular spherical coordinates. From the formula (15) follows $y = (x, z/m)$:

$$u^0(x, z) = e^{i(wMz/m^2)} \sum_{n,l} a_n j_n\left(\frac{w}{c} \sqrt{z^2 + m^2 r^2}\right) P_n^l\left(\frac{z}{\sqrt{z^2 + \mu^2 r^2}}\right) \frac{(x_1 + ix_2)^l}{r^l}, \quad (5.13)$$

where $r = \sqrt{x_1^2 + x_2^2}$, the coefficients of a_n are arbitrary complex numbers.

5.2. The General Solution of Subsonic VTE

Let us prove the following theorem.

Theorem 2. *The solution of subsonic VTE has the form:*

$$u(x, z) = U(x, z) * g(x, z) + u^0(x, z). \quad (5.14)$$

If $g(x, z)$ is a regular function and $g(x, z) \in L_1(R^3)$, then

$$U(x, z) * g(x, z) = \int_{R^3} U(x - y, z - h) g(y, h) dy_1 dy_2 dh. \quad (5.15)$$

If $g(x, z)$ is a singular function centered on the surface S :

$g(x, z) = \alpha(x, z) \delta_S(x, z)$, $\alpha(x, z) \in L_1(S)$, then

$$U(x, z) * g(x, z) = \int_S U(x - y, z - h) g(y, h) dS(y, h). \quad (5.16)$$

If $g(x, z)$ is a singular function centered on the curve l :

$g(x, z) = \beta(x, z) \delta_l(x, z)$, $\beta(x, z) \in L_1(l)$, then

$$U(x, z) * g(x, z) = \int_l U(x - y, z - h) g(y, h) dl(y, h). \quad (5.17)$$

Proof. Let us denote the $VT(\partial_1, \partial_2, \partial_z)$ differential operator VTE (4.1). Substituting (5.14) into (4.1) and using the convolution differentiation property [14],

we obtain the required:

$$u_2(x, z) = u(x, z) - u_1(x, z).$$

If $u(x, z)$ is any solution (5.1), then $u_2(x, z) = u(x, z) - u_1(x, z)$ is the solution of a homogeneous VTE. Therefore, $u_1(x, z) = u(x, z) - u_2(x, z)$, i.e., has a similar $u(x, z)$.

5.3. The Doppler Effect

Let's denote $r(x)/z = tg\varphi(x, z)$, $r = \sqrt{x_1^2 + x_2^2}$, where $\varphi(x, z)$ is the angle that forms the radius vector of the point (x, z) with axis Z. Then, the Green's function

$$U(x_1, x_2, z) = -\frac{1}{4\pi\sqrt{z^2 + m^2r^2}} \exp\left(iwm^{-2}z\left(M - \sqrt{1 + m^2tg\varphi^2}\right)\right). \quad (5.18)$$

As we can see, the phase surface $tg\varphi^2 = a^2, \forall a$. This is a cone:

$$x_1^2 + x_2^2 = a^2z^2 = a^2(x_3 - Vt)^2, \quad V > 0. \quad (5.19)$$

If we fix the observation point (x_1, x_2, x_3) and measure the time-arriving signal at this point, then it is described by the function

$$\begin{aligned} U(x_1, x_2, x_3, t) &= U(x, x_3 - Vt)e^{i\omega t} \\ &= -\frac{\exp\left(iwm^{-2}\left(Mx_3 - \sqrt{(x_3 - Vt)^2 + m^2r^2}\right)\right)}{4\pi\sqrt{(x_3 - Vt)^2 + m^2r^2}} e^{i\omega t(1 - (M/m)^2)}. \end{aligned} \quad (5.20)$$

At $V = 0, M = 0$, we obtain a fundamental solution to the Helmholtz equation [14]:

$$U(x_1, x_2, x_3, t) = -\frac{\exp\left(ik\left(-\sqrt{(x_3)^2 + r(x)^2}\right)\right)}{4\pi\sqrt{x_1^2 + x_2^2 + x_3^2}} e^{i\omega t}, \quad k = \frac{\omega}{c}. \quad (5.21)$$

At $\omega = 0, V > 0, 0 < M < 1$,

$$U(x_1, x_2, x_3, t) = U(x, x_3 - Vt) = -\frac{1}{4\pi\sqrt{(x_3 - Vt)^2 + (1 - M)^2r(x)^2}}. \quad (5.22)$$

This is a subsonic transport solution of the wave equation [2].

Figure 1 shows the real $RU(xn, zx, M, \omega) = \text{Re}U(x, z)$ and imaginary $IU(xn, zx, M, \omega) = \text{Im}U(x, z)$ parts of $U(x, z)$ by $x = xn = (1, 1)$ for different $z = zn$ at Mach number $M = 0.1$ and frequencies $\omega = 1$ and $\omega = 10$. In a moving coordinate system, the oscillation frequency increases in front of the source of vibrations and decreases behind it. But in the original fixed X coordinate system, the picture is different.

Figure 2 shows the waveform of the signal at a fixed point on the X_3 axis over time $t = tn$ at Mach number $M = 0.8$ and vibration frequency $\omega = 10$. Here, $RU(x_1, x_3, t, M, \omega) = \text{Re}U(x_1, x_2, x_3, t)$, $x_3 = 0$, $x_1 = (1, 1)$, $t = tn$.

It clearly demonstrates an increase in the frequency and amplitude of vibration when approaching a vibration source and, conversely, their decrease when it is

removed.

As is well known, the pressure in the air satisfies the wave equation [10]. This phenomenon is called the *Doppler effect*—an increase in tone (frequency) and volume (amplitude) when approaching a vibration source and, conversely, a decrease in tone and volume when it is removed.

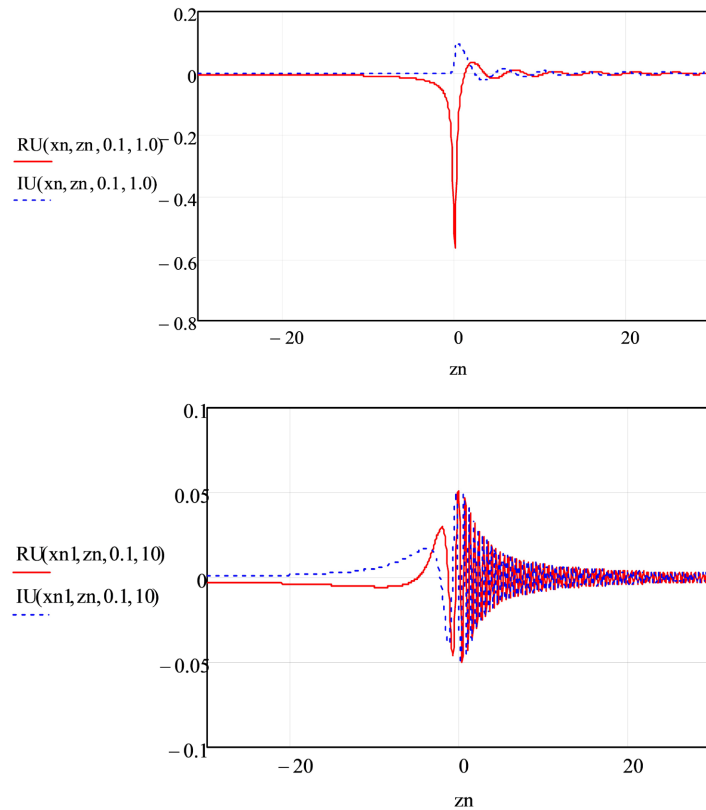


Figure 1. $U(x, z)$ at $M = 0.1$, $\omega = 1; 10$.

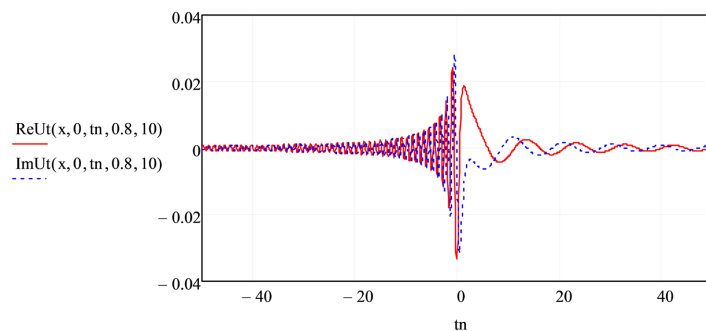


Figure 2. The oscillogram $U(x, 0, t)$ at $M = 0.8$ and $\omega = 10$.

6. Conclusions

Note that wave equations are widely used in the study of wave processes in acoustic, liquid, elastic, and electromagnetic environments as potentials of vector fields that describe their motion. Therefore, the constructed analytical solutions should

find wide application in the study of vibration transport processes in such media.

Transport and vibrotransport loads are one of the most common sources of disturbances in environments, and the bibliography of studies in this area has become increasingly extensive in recent years. The obtained results can be used to study the electromagnetic fields of various light emitters and radio wave emitters located on mobile objects (trains, cars, ships, etc.), as well as to solve transport boundary value problems of electrodynamics in limited areas based on the method of boundary integral equations, which the authors plan to do in the near future using the method of generalized functions, similar to [2]-[4].

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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