

# Complex Field Theory of Dark Matter and Dark Energy: Novel Atomic and Nuclear Models and a Unified Framework for the Four Fundamental Forces

Hossin Abdeldayem 

NASA-Goddard Space Flight Center, Greenbelt, Maryland, USA

Email: [hossin.abdeldayem@nasa.gov](mailto:hossin.abdeldayem@nasa.gov), [hossin@abdeldayem.net](mailto:hossin@abdeldayem.net)

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## Abstract

Complex Field Theory (CFT) describes Dark Matter (DM) and Dark Energy (DE) as complex fields composed of positively and negatively charged complex mass, respectively. These complex fields permeate the entire universe and play fundamental roles in governing the physical characteristics of everything in it. Several aspects of the theory have been discussed in earlier publications. In this paper, we investigate whether the forces associated with DM and DE at the atomic and nuclear scales share the same fundamental origin as the gravitational force at cosmic scales. Within the CFT framework, we derived a positive Yukawa force that balances the attractive Coulomb force acting on ground-state atomic electrons. We also derived an atomic radiation sink that prevents atomic electrons from radiating electromagnetic energy externally, thereby providing a mechanism for atomic stability. The paper concludes that the forces underlying both Bohr's atomic model and the quantum mechanical model are rooted in the same complex nature as the complex charges of DM- and DE, responsible for gravity. Consequently, these forces are interpreted as different manifestations of a single unified origin rather than as distinct physical sources. Bohr's model (1913) was the first successful theory to quantize atomic energy levels and provided a theoretical foundation for Rydberg's empirical formula. However, it incorrectly attributed the stability of ground-state electrons to a balance between the attractive Coulomb force and the centrifugal force of the orbital electrons. Furthermore, it did not provide a physical explanation for the atomic radiationless nature. The quantum mechanical (QM) model, developed by de Broglie, Schrödinger, and Max Born, describes ground-state electrons by a three-dimensional stationary wave function governed by Schrödinger's equation. Within this framework, atomic stability is attributed to the combined effects of the attractive

Coulomb force and the uncertainty principle, which prevents the electron from collapsing into the nucleus. In addition, the stationary nature of the bound-state wave function implies the absence of a time-varying electric dipole moment in the electron probability distribution, thereby preventing external electromagnetic radiation. In this paper, the concepts of CFT, together with the path-integral formulation of Quantum Field Theory, are used to develop a framework for deriving both the ground-state stability and the radiationless nature of atomic electrons. Furthermore, a repulsive Yukawa force is derived to balance the attractive Coulomb force in the ground state. CFT also concluded an alternative model to Yukawa strong nuclear force model and validated it with experimental data. The analysis identifies a common factor of complex nature that unifies the Yukawa force with the energy and momentum associated with the uncertainty principle.

### Keywords

Complex Field Theory, Bohr's Atomic Model, Quantum Mechanics Atomic Model, Quantum Field Theory, Path-Integral, Dark Matter, Dark Energy, Yukawa Potential Energy, Radiation Theory

## 1. Introduction

In 1913, Bohr formulated his atomic model [1] [2], which was the first to explain the hydrogen spectrum successfully and provided a theoretical foundation for Rydberg's empirical formula. The model, incorrectly, assumed that the electrons in the ground state are stable under the balance of their orbital circulation with the attractive Coulomb's force. The model left unresolved the fundamental question of why the accelerated atomic electrons do not radiate energy. Bohr simply postulated, without theoretical justification, that atomic electrons do not emit radiation, contradicting the classical theory of electrodynamics.

In 1923, de Broglie [3] [4] proposed that the electron in a hydrogen atom behaves not as a particle in quantized orbits, as Bohr envisioned, but as a circular stationary wave. Building on this idea, Schrödinger [5] [6] formulated a wave equation in 1926 that described the electron as a three-dimensional stationary wavefunction, " $\psi$ ". His model successfully reproduced Bohr's energy levels and Rydberg's formula. That same year, Max Born [7] interpreted the square of the wavefunction,  $|\psi^2|$ , as representing the probability of finding the electron in a particular location in space.

This publication is the fourth in a series on Complex Field Theory (CFT), which proposes that Dark Matter (DM) and Dark Energy (DE) are complex fields of complex-charged masses ( $i\mu$ ) carrying positive and negative complex charges, respectively. These publications demonstrated that DM and DE play major roles in the physical behaviors of the universe's contents at cosmic and subatomic scales. Their complex masses ( $i\mu$ ) are seeds for real mass formation at a threshold energy [8]. The theory proposes that positive complex charges ( $+iq$ ) are concomitant with

every neutral and positively charged object in the universe, in proportion to its mass, whereas the negative complex charges ( $-iq$ ) are associated with every negatively charged particle. Similar complex charges attract each other, while opposite complex charges repel.

The theory resolved successfully several long-standing physical mysteries in earlier publications at cosmic scales such as the origin of gravity [7] and its field, the expansion of the universe, formulated an alternative model to Higgs mechanism [8] of the mass formation, set a constrained condition on Einstein's mass-energy equivalence ( $E = mc^2$ ) and electron-positron pair production [8], challenged the conventional interpretations of the Z and  $W^\pm$  bosons mass formation as due to vacuum fluctuations and instead attributed their masses to a dense presence of dark matter [8]. Moreover, the theory [9] provided novel explanations for several vaguely understood quantum phenomena such as entanglement, wave-particle duality, tachyons, the "spooky action effect at a distance", and revived the old concept of the ether medium. It also unified Coulomb's and Newton's laws under a single theoretical framework.

This paper offers an alternative theoretical model to Bohr's and the QM models to explain atomic stability, using the CFT's complex charges concept to produce Yukawa repulsive force, which balances Coulomb's attractive force. At the same time, the model introduced an atomic radiation sink that prevents the electromagnetic radiation from being released externally.

**Note:** The natural units ( $\hbar = c = k = \mu_0 = 1$ , where  $k$ =Coulomb constant,  $\mu_0$  is the magnetic permeability) are used throughout the paper.

## 2. Theory

### 2.1. Stability of the Ground State Atomic Electron

The electron in the ground state of the hydrogen atom in Bohr's model is orbiting the nucleus, and Coulomb's attractive force is balanced by the centrifugal force of the orbital motion. Later, de Broglie, Schrodinger, and Max Born developed the quantum mechanical model, where the ground state electrons were given zero orbital momentum and form a stationary electronic cloud around the nucleus. Despite being subject to the single attractive Coulomb force, the electron is not falling into the nucleus because of the uncertainty principle  $\Delta x \Delta p \geq \hbar/2$ , which tells that when the electron gets too close to the nucleus ( $\Delta x \approx 0$ ), the uncertainty in its momentum  $\Delta p$  highly increases, which implies an increase in the electron's kinetic energy that protects the electron and prevents it from falling into the nucleus. The balance between the attractive Coulomb potential and the potential kinetic energy takes place at Bohr's radius.

#### 2.1.1. CFT's Induced Yukawa's Potential

The CFT makes use of the complex charges to introduce a repulsive Yukawa force that balances the attractive Coulomb's force. The scalar complex field is best described by the Klein-Gordon equation [10]-[12] and Yukawa potential can be de-

rived from Klein-Gordon equation [13] [14] as follows:

$$-\frac{\partial^2 \psi}{\partial^2 t^2} + \nabla^2 \psi = m^2 \psi \quad (1)$$

In deriving interaction energies from the path integral for bound states, several approximations are introduced that must be physically justified:

- a) The static limit,
- b) Neglecting the self-interacting terms,
- c) Giving a mass to the photon.

**(a) The static limit** assumes the sources do not change appreciably during the interaction:  $J(x, t) \approx J(x)$  and  $\psi(r, t) \rightarrow V(r)$ . The physical justification of the static limit is attributed to the fact that in a bound state:

- The particles move relatively slowly; the proton in the hydrogen atom is essentially stationary, and the electron velocity  $v \ll c$  thus the interaction can be approximated as instantaneous.
- The binding energy is usually much smaller than the rest energy,
- The radiation effects are suppressed,
- Retardation effects are small.

The time derivative term in Equation (1) becomes zero, then:

$$\nabla^2 V = m^2 V \quad (2)$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) - m^2 V = 0 \quad (3)$$

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r} \frac{\partial V}{\partial r} - m^2 V = 0 \quad (4)$$

Let  $U = rV$

$$\begin{aligned} \frac{dU}{dr} &= r \frac{dV}{dr} + V \\ \frac{d^2 U}{dr^2} &= r \frac{d^2 V}{dr^2} + 2 \frac{dV}{dr} \end{aligned} \quad (5)$$

Substituting (5) into (4)

$$\frac{d^2 U}{dr^2} - m^2 U = 0 \quad (6)$$

where:

$$U = \pm g^2 \exp(\pm mr) \quad (7)$$

$$V = \frac{\pm g^2}{r} \exp(\pm mr) = \frac{\pm g^2}{r} \exp(\pm r/a) \quad (8)$$

Yukawa chose the negative sign to describe the attractive force holding the nucleons inside the nucleus. “ $a$ ” is the shielding distance or the range of the force. And  $g^2 = (-iq_e)(iq_p)$ , where  $(-iq_e)$  and  $(iq_p)$  are the complex charges of the electron and the proton, respectively.

### 2.1.2. Path-Integral Formulation of the Repulsive Complex Charges

The CFT introduces a repulsive Yukawa-type force via the complex charges  $(+iq_p)$

and  $(-iq_e)$  to balance Coulomb's attractive force and protect the atom from collapsing. The path-integral formulations are in **Appendix 1**, which result in the repulsive energy:

$$E = +\frac{q_e q_p}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{\vec{k}^2 + m^2} \quad (9)$$

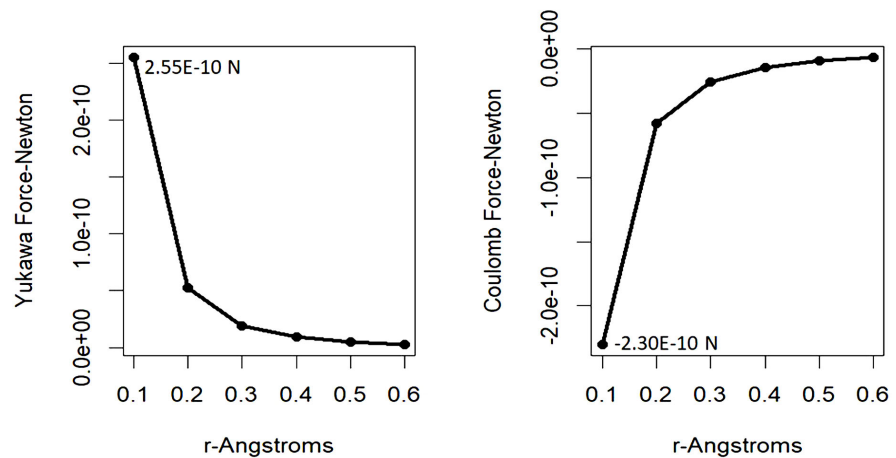
This positive energy is a representation of the repulsive Yukawa force.

### 2.1.3. Path-Integral Formulation for the Attractive Coulomb's Force

The path-integral formulations are shown in **Appendix 2** and calculate the attractive Coulomb interaction between the electron and the proton, which are fermions and Klein-Gordan equation cannot be used in this case. The path integration method is handled in this case by the quantum electrodynamic (QED) approach, where the interaction between the two is a photon of spin 1. The source is represented by the term  $J$ . The ground state amplitude becomes:

$$\begin{aligned} \langle 0 | e^{-iHT} | 0 \rangle &= e^{iW(J)} = e^{-iET} \\ E &= -e^2 \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{\vec{k}^2 + m^2} \end{aligned} \quad (10)$$

where  $W(J)$  is a representation of an integration defined in **Appendix 1**. The negative energy represents the attractive quantum electrodynamic Coulomb force, balanced by Equation (9). **Figure 1** shows the behavior of Coulomb attractive force and the repulsive Yukawa force at different distances from the nucleus. Both are equal at Bohr's radius and Yukawa force is slightly larger at 0.1 Å.



**Figure 1.** Illustrates the behavior of the repulsive Yukawa and attractive Coulomb forces within the hydrogen atom at different distances from the nucleus. At the atomic radius of 0.529 Å, the two forces are equal in magnitude but opposite in direction. At 0.1 Å, the repulsive Yukawa force exceeds that of Coulomb's attractive force.

### 2.1.4. Quantization of the Complex Charges

Bohr, in his atomic model [1] [2], derived the radius of the ground-state hydrogen atom by equating the centrifugal force to the attractive Coulomb force.

$$m_e v^2 = \frac{1}{4\pi\epsilon} \left( \frac{e^2}{r} \right) \quad (11)$$

Since

$$m_e v r = \hbar \quad (12)$$

Bohr's radius at  $r = a$  is:

$$a = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} \quad (13)$$

where  $m_e$  is the electron's mass, and " $a$ " is Bohr's radius = 0.529 Å.

The radial distribution function is:

$$P_{10}(r) = r^2 |R_{10}|^2 \quad (14)$$

$$P_{10}(r) = r^2 |R_{10}|^2 = \frac{4r^2}{a^3} e^{-2r/a} \quad (15)$$

At the most probable value:

$$\begin{aligned} \frac{dP_{10}(r)}{dr} &= \frac{d}{dr} \left( \frac{4r^2}{a^3} e^{-2r/a} \right) = 0 \\ &= \frac{8r}{a^3} e^{-2r/a} \left( 1 - \frac{r}{a} \right) = 0 \end{aligned} \quad (16)$$

Which means that  $r = a$  is the most probable radius (Bohr's radius).

### 2.1.5. Bohr's Radius

The total energy = Kinetic energy + Potential energy

$$E = T + U = \left( \frac{P^2}{2m} \right) - \frac{e^2}{r} \quad (17)$$

The ground state energy of the hydrogen atom is the net value of the repulsive Yukawa potential and the attractive Coulomb potential:

$$-13.6 \text{ eV} = \frac{(-iq_e)(iq_p)}{r} \exp(-r/a) - \frac{e^2}{r} \quad (18)$$

Comparing (17) and (18)

The kinetic energy term is equivalent to Yukawa potential energy:

$$T = \left( \frac{P^2}{2m} \right) = \frac{(-iq_e)(iq_p)}{r} \exp(-r/a) \quad (19)$$

This implies that the kinetic energy within the system arises from the complex charges and their complex fields, which gives rise to the Yukawa positive potential energy  $\left( \frac{|-iq_e||iq_p|}{r} \right) e^{-r/a}$ . Equation (19) is an indication that the energy of a

system resides in the complex field, which is consistent with Schrödinger's expressions for the complex energy and momentum operators, which act on the complex component of the wave function. These expressions suggest that the dynamical

properties of the system reside in the complex part of its wave function. This interpretation was previously proposed in Ref. [9].

According to the Virial theorem, the expectation potential energy is minus twice the expectation kinetic energy (*i.e.*,  $V = -2T$ ), then:

$$T = -\frac{1}{2}V = -\frac{1}{2}\left(\frac{-e^2}{r}\right) = \frac{e^2}{2r} \quad (20)$$

Using Equation (19)

$$\left(\frac{|-iq_e||iq_p|}{r}\right)e^{-r/a} = \frac{e^2}{2r} \quad (21)$$

$$|-iq_e||iq_p| = \frac{e^2}{2}e^{r/a} \quad (22)$$

At the same time, for the ground-state electron to be in a stable state, the repulsive Yukawa force should balance the negative Coulomb force.

$$\begin{aligned} \frac{e^2}{r} &= \frac{|-iq_e||iq_p|}{r^2} \left(\frac{a+r}{a}\right)e^{-r/a} \\ |-iq_e||iq_p| &= \left(\frac{ae^2}{a+r}\right)e^{r/a} \end{aligned} \quad (23)$$

Equating the right sides of equations (22) and (23), then ( $r = a$ )

Substituting  $r = a$  in Equation (23)

$$\begin{aligned} |-iq_e||iq_p| &= \left(\frac{e^2}{2}\right)\exp(1) \\ |-iq_e||iq_p| &= 1.36e^2 \end{aligned} \quad (24)$$

Since

$$\frac{|iq_p|}{|-iq_e|} = \frac{\delta m_p}{\delta m_e} \approx 1836.15 \quad (25)$$

$$\begin{aligned} |iq_e| &= 0.027e \\ \text{and} \\ |iq_p| &= 49.57e \end{aligned} \quad (26)$$

And since

$$\frac{|iq_n|}{|iq_p|} = \frac{\delta m_n}{\delta m_p} \approx 1.001378$$

Then, the estimated complex neutron charge is:

$$|iq_n| = 49.638e$$

## 2.2. CFT's Model of the Strong Nuclear Force

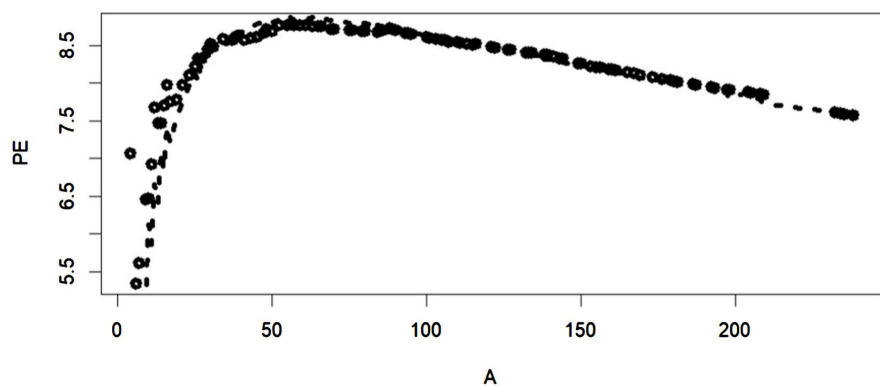
CFT offers an alternative model to Yukawa's nuclear force model [13]. According to CFT [7] each nucleon within the nucleus carries a positive complex charge,

denoted by  $(iq_p)$  for the proton and  $(iq_n)$  for the neutron. These complex charges are nearly equal in magnitude because the proton and neutron masses are nearly equal ( $m_n \approx m_p$ ). The interaction between nucleons arising from these complex charges is predominantly attractive and is approximately  $(49.57)^2 = 2,457$  times stronger than the electrostatic repulsion between protons. This strong, attractive interaction explains why the nuclear force greatly exceeds the proton-proton electrostatic repulsion, thereby ensuring the stability of the atomic nucleus.

The binding energy per nucleon (BE/N) curve is one of the most important experimental observations in nuclear physics. In this work, the experimental BE/N data are compared with the predictions of the CFT model for the nuclear force. The empirical fitting formula (Equation (27)) consists of the attractive neutron contribution in the first term and the attractive proton contribution in the second term, subtracted by the repulsive electrostatic interaction represented by the third term. The resulting fit is shown as the dotted curve, while the experimental data are represented by circular points in **Figure 2**.

$$\text{BE/N} = \left\{ \left( \alpha \exp^{-(A-Z)^{0.194}} \right) \left[ \frac{k(A-Z)((A-Z)-1)}{rA} \right] \frac{(q_n)^2}{2} \right\} + \left\{ \left( \beta \exp^{-Z^{0.267}} \right) \left[ \frac{kZ(Z-1)}{rA} \right] \frac{(q_p)^2}{2} \right\} - \left\{ \left[ \frac{kZ(Z-1)}{rA} \right] \frac{e^2}{2} \right\} \quad (27)$$

where “ $\alpha$ ” =  $10^{-2.44}$  and  $\beta = 10^{-1.735}$  are fitting factors, “ $r$ ” is the radius of the nucleon in each nuclei [  $r = 1.2 \times 10^{-15} A^{1/3}$  ],  $q_n = |iq_n| = 49.638e$  and  $q_p = |iq_p| = 49.57$ , where “ $e$ ” is the proton electric charge,  $A$  is the mass number,  $k$  is Coulomb constant in MeV.m/C<sup>2</sup>, and  $Z$  is the number of protons.



**Figure 2.** Equation (25) is represented by the dotted fitting curve to the circular rounded experimental data points of the nuclear binding energy per nucleon.

### 2.3. The Radiationless Nature of Atomic Electrons

As mentioned earlier, Bohr’s model did not address the issue of atomic radiation, whereas the quantum mechanical model attributes the absence of atomic radiation to the lack of a time-varying electric dipole moment in the stationary electronic wave function. In the CFT framework, the classical theory of electromag-

netic radiation is applied to the complex charges, leading to the prediction of an atomic radiation sink that absorbs the emitted electromagnetic radiation and prevents radiation from escaping into space, thereby creating a closed energy system.

### 2.3.1. Orbital Motion

The classical radiation field theory [15] derives both the electric and magnetic fields from the scalar potential ( $\varphi$ ) and vector potential ( $A$ ), where:

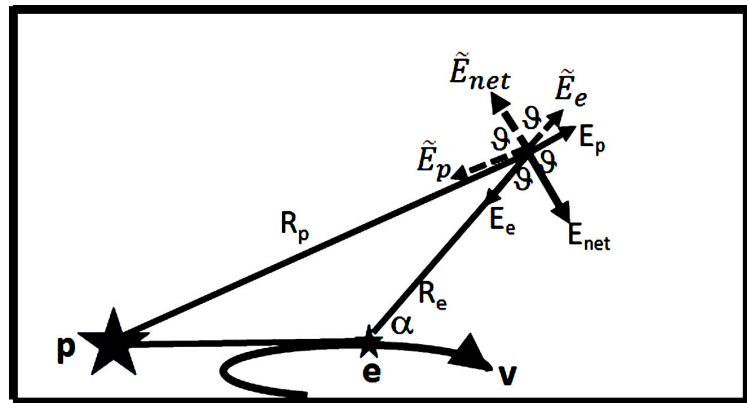
$$E = -\nabla\varphi - \frac{\partial A}{\partial t} \text{ and } B = \nabla \times A \quad (28)$$

For nonrelativistic charged particles [15]-[17],

$$E_{rad}(r, t) = \frac{q}{4\pi c^2 R \epsilon_0} (\hat{n} \times \hat{n} \times \vec{a}) \quad (29)$$

And

$$B = \frac{1}{c} (\hat{n} \times E) \quad (30)$$



**Figure 3.** Electric Fields directions of the electron and the proton at a far distance point.  $E_e, E_p, E_{net}$  and those of the complex charges  $\tilde{E}_e, \tilde{E}_p, \tilde{E}_{net}$  are as shown.

The fields of the electron and the proton at the far-distant point in **Figure 3**, where  $R_e = R_p = R$  are equal ( $E_e = E_p$ ) and their net value “ $E_{net}$ ” makes an equal angle with  $E_e$  and  $E_p$ , respectively. Similar notation  $\tilde{E}_e, \tilde{E}_p$  and  $\tilde{E}_{net}$  for the complex charges are such that  $\tilde{E}_p \gg \tilde{E}_e$  for the complex charge on the proton is much larger than that on the electron and the resultant value  $\tilde{E}_{net}$  makes a small angle with  $\tilde{E}_p$ . To simplify the derivation, take the component of  $\tilde{E}_{net}$  of the resultant complex field opposite to  $E_{net}$ .

$$\begin{aligned} E_{net} &= E_p \cos \theta + E_e \cos \theta \\ &= \frac{e \cos \theta}{4\pi R c^2 \epsilon_0} [\hat{n} \times (\hat{n} \times \vec{a}_p - \hat{n} \times \vec{a}_e)] \end{aligned} \quad (31)$$

Due to the mass difference between the proton and the electron,  $\vec{a}_p \ll \vec{a}_e$ , then:

$$E_{net} \approx \frac{-e \cos \theta}{4\pi R c^2 \epsilon_0} [\hat{n} \times (\hat{n} \times \vec{a}_e)] \quad (32)$$

Similarly, for the complex charges:

$$\begin{aligned}\tilde{E}_{net} &= \tilde{E}_p \cos \theta + \tilde{E}_e \cos \theta \\ &= \frac{iq_p \cos \theta}{4\pi R \epsilon_o} [\hat{n} \times \hat{n} \times \vec{a}_p] - \frac{iq_e \cos \theta}{4\pi \epsilon_o} [\hat{n} \times \hat{n} \times \vec{a}_e] \\ &= \frac{ie \cos \theta}{4\pi R \epsilon_o} [(49.57)(\hat{n} \times \hat{n} \times \vec{a}_p) - (0.027)(\hat{n} \times \hat{n} \times \vec{a}_e)]\end{aligned}\quad (33)$$

Although  $\vec{a}_p \ll \vec{a}_e$  but the constant factor in front of the first term is 1,836 times larger than that of the second term, so, the proton's contribution cannot be ignored.

Consequently,

$$\begin{aligned}B_{net} &= (\hat{n} \times E_{net}) \\ \text{and} \\ \tilde{B}_{net} &= (\hat{n} \times \tilde{E}_{net})\end{aligned}\quad (34)$$

Note that  $E_{net}$  and  $\tilde{E}_{net}$  lie in the plane of polarization containing  $\hat{n}$  (along  $R$ ) and  $\vec{a}$ , while both  $B_{net}$  and  $\tilde{B}_{net}$  are perpendicular to this plane. If we let the angle " $\alpha$ " between  $\hat{n}$  and  $\vec{a}$ . Then

$$B_{net} = \frac{-e \cos \theta}{4\pi R \epsilon_o} [\hat{n} \times (\hat{n} \times \vec{a}_e)] \sin \alpha \quad (35)$$

And

$$\tilde{B}_{net} = \frac{i \cos \theta}{4\pi R \epsilon_o} [(49.57e) \hat{n} \times (\hat{n} \times \vec{a}_p) - (0.027e) \hat{n} \times (\hat{n} \times \vec{a}_e)] \sin \alpha \quad (36)$$

The Poynting vector is defined as:

$$\begin{aligned}S &= \frac{1}{4\pi} (E_{net} \times B_{net}) = \frac{1}{4\pi} [E_{net} \times (\hat{n} \times E_{net})] \\ &= \frac{1}{4\pi} [\hat{n} (E \cdot E) - E (\hat{n} \cdot E)]\end{aligned}\quad (37)$$

If we take the limit of large  $R$  along  $(\hat{n})$  and keep just the radiation terms, noting that  $\hat{n}$ ,  $E_{net}$ , and  $B_{net}$  are mutually perpendicular, or  $\hat{n} \cdot E_{net} = \hat{n} \cdot B_{net} = 0$ , then,

$$S = \frac{1}{4\pi} [\hat{n} (E \cdot E)] = \frac{e^2 \cos^2 \theta \sin^2 \alpha}{(4\pi)^2 R^2 \epsilon_o^2} [\hat{n} \times (\hat{n} \times \vec{a}_e)]^2 \quad (38)$$

And

$$\begin{aligned}\tilde{S} &= \frac{1}{4\pi} (\tilde{E}_{net} \times \tilde{B}_{net}) = \frac{1}{4\pi} [\tilde{E}_{net} \times (\hat{n} \times \tilde{E}_{net})] \\ &= \frac{1}{4\pi} [\hat{n} (\tilde{E}_{net} \cdot \tilde{E}_{net}) - \tilde{E}_{net} (\hat{n} \cdot \tilde{E}_{net})] \\ &= \frac{1}{4\pi} [\hat{n} (\tilde{E}_{net} \cdot \tilde{E}_{net})] \\ &= \frac{-e^2 \cos^2 \theta \sin^2 \alpha}{(4\pi R \epsilon_o)^2} [(49.57) \hat{n} \times \hat{n} \times \vec{a}_p - (0.027) \hat{n} \times \hat{n} \times \vec{a}_e]^2\end{aligned}\quad (39)$$

Note that  $\tilde{S}$  always has a negative sign, which acts as a radiation sink, absorbing the positive radiation " $S$ " of the electron to form a closed energy-conserving system, preserving the atom from losing energy. In conclusion, the dark matter

and dark energy are preserving the stability of the material universe.

### 2.3.2. Linear Oscillations [18]-[21]

The ground-state electron in the quantum mechanical model is not orbiting the nucleus but is connected to the nucleus by a complex field line. This complex field line has its inherent oscillating frequency [9], causing both particles to oscillate with the same frequency along the radius of the atom. Although the electron and the proton carry opposite charges that are equal in magnitude, they still do not form a dipole moment because the masses are different.

This linear oscillation along the radius explains the origin of de Broglie standing wave oscillation of each electron in its specific atomic orbit as the Earth's Schumann resonance of 7.83 Hz [21] with respect to the Sun. Schumann frequency was incorrectly attributed to a resonance in the space between the Earth's surface and the ionosphere. This inherent oscillation of the complex field causes an inherent oscillation of every material particle or object in the universe. The complex fields of DM and DE are the origin of the Ether medium, which links everything in the universe together.

Because the electron and proton are oscillating with the same frequency along the radius, they will have different accelerations with an average acceleration "a" for the linear oscillation given by:

$$\vec{a} = -\omega^2 \vec{x} \quad (40)$$

where  $\omega$  is the angular frequency and "x" is the displacement from equilibrium. The mass of a proton ( $m_p$ ) is  $\approx 1836.15 \times m_e$ , where  $m_e$  is the electron mass. In a two-body system, the center of mass (CM) is maintained stationary. The electron must have a significantly larger amplitude of motion than the proton. Let the electron's position be  $x_e(t) = A_e \cos(\omega t)$  and the proton's position be  $x_p(t) = -A_p \cos(\omega t)$ ; they oscillate in opposite directions to preserve CM stationary. Their respective accelerations are:

$$\vec{a}_e = -\omega^2 \vec{A}_e \cos(\omega t) \text{ and } \vec{a}_p = -\omega^2 \vec{A}_p \cos(\omega t). \quad (41)$$

So, the ratio of accelerations is:

$$\frac{\vec{a}_p}{\vec{a}_e} = \frac{\vec{A}_p}{\vec{A}_e} \quad (42)$$

Since  $A_p \ll A_e$ , it follows that:

$$\vec{a}_p \ll \vec{a}_e \quad (43)$$

Although both oscillate with the same frequency, the lighter electron undergoes greater acceleration than the heavier proton to preserve the correct dynamics of the two-body system. At the same time, the complex charges on each particle are unequal. Consequently, the radiation emitted by each can be modeled as that from a point charge oscillating according to the Larmor formula [18]-[21].

$$P = \frac{q^2 \vec{a}^2}{6\pi} \quad (44)$$

where  $q$  is the charge of the particle and  $a$  is its acceleration.

The positive conventional power radiated by the electron and the proton is:

$$P = \frac{1}{6\pi} \left[ (-e\vec{a}_e)^2 + (+e\vec{a}_p)^2 \right] \approx \frac{1}{6\pi} (-e\vec{a}_e)^2 \quad (45)$$

The negative radiation power by the complex charges is:

$$\begin{aligned} \tilde{P} &= \frac{1}{6\pi} \left[ (-iq_e\vec{a}_e)^2 + (+iq_p\vec{a}_p)^2 \right] \\ &= \frac{-1}{6\pi} \left[ (0.027e\vec{a}_e)^2 + (49.57e\vec{a}_p)^2 \right] \end{aligned} \quad (46)$$

Equations (45) and (46) demonstrate again that the complex charges induce a radiation sink for the linear radiation of the electron and the proton to form a closed, energy-conserving system.

### 3. Conclusions

Complex Field Theory (CFT) employs the path integral formalism of quantum field theory to construct an atomic model that identifies the mechanisms responsible for the stability of the atomic ground state and the radiationless nature of atoms, consistent with both Bohr's model and quantum mechanics. CFT also predicts a positive Yukawa-type force induced by the complex charges, which balances the attractive Coulomb force. Furthermore, the theory demonstrates the existence of a radiation sink that absorbs electromagnetic radiation, thereby preventing its escape and rendering atoms radiationless. In addition, the paper quantizes the complex charges of the electron, proton, and neutron and proposes an alternative model for Yukawa's strong nuclear force. The model provides a good fit to the experimental binding energy per nucleon data.

It is important to note that the forces derived within the Complex Field Theory (CFT) from the complex charges are not fundamentally different from the effective forces invoked in Bohr's model and the quantum mechanical model. This conclusion follows if one recognizes that the momentum and energy arising from the uncertainty principle originate from the complex component of the wave function and from Schrödinger's complex operators for momentum and energy. In other words, both the repulsive Yukawa force and the energy associated with the uncertainty principle have the same underlying complex origin. Consequently, the Yukawa force introduced by CFT should not be regarded as a new fundamental force, but rather as an alternative physical manifestation of the energy and momentum represented by the uncertainty principle. They are, therefore, two different representations of the same underlying phenomenon, both originating from a common complex source.

By reaching this conclusion, we infer that the Dark Matter (DM) and Dark Energy (DE) fields responsible for generating the gravitational force at cosmic scales are also the underlying source of atomic and nuclear stability. Consequently, gravitational and nuclear forces are interpreted as arising from the same fundamental origin.

Furthermore, the mass formation model presented in Reference [9], which describes the transformation of charged complex mass into real mass, also implies simultaneous transformation of complex charges into real electric charges. We then infer that the electromagnetic force likewise originates from the same complex source and is therefore unified with both the gravitational and nuclear forces.

In addition, CFT challenged earlier in Reference [8] the conventional interpretation, which attributes the masses of Z and  $W^\pm$  bosons to vacuum fluctuations, and proposing instead that these masses arise from a dense presence of Dark Matter. In other words, the weak nuclear force originating these heavy bosons is attributed to dense DM and consequently of complex origin. We identify then that weak nuclear force is of the same fundamental source and can also be unified with the above three forces.

Within this framework, the CFT of DM and DE provides a physical base line for the unification of all four fundamental forces of nature. Further theoretical development and experimental investigation are needed to evaluate this framework and in addition explore its implications for other physical phenomena of the Casimir force, the Lamb shift, and the anomalous magnetic moment.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Appendix 1

Starting with Schrodinger's Equation:

$$\begin{aligned}
 i \frac{\partial \psi}{\partial t} &= H\psi \\
 \int_i^f \frac{\partial \psi}{\psi} &= -iH \int dt \\
 \ln \left( \frac{\psi_f}{\psi_i} \right) &= -iHT \\
 \psi_f &= e^{-iHT} \psi_i \\
 \langle \psi_f | \psi_f \rangle &= \langle \psi_f | e^{-iHT} | \psi_i \rangle
 \end{aligned} \tag{A1}$$

The path integral representation of the ground state amplitude is [22]:

$$Z \equiv \langle 0 | e^{-iHT} | 0 \rangle = \int D\varphi e^{i \int d^4x \mathcal{L}(\varphi)} \tag{A2}$$

where is the Lagrangian density:

$$\mathcal{L}(\varphi) = 1/2 \left[ (\partial\varphi)^2 - m^2\varphi^2 \right] \tag{A3}$$

Substitute Equation (3) into (2) in the following special case with the source  $J$

$$Z \equiv \langle 0 | e^{-iHT} | 0 \rangle = \int D\varphi e^{i \int d^4x \left[ \frac{1}{2} \left[ (\partial\varphi)^2 - m^2\varphi^2 \right] + J(\varphi) \right]} \tag{A4}$$

Integrating by parts, Equation (4) becomes:

$$Z \equiv \langle 0 | e^{-iHT} | 0 \rangle = \int D\varphi e^{i \int d^4x \left[ -\frac{1}{2} \varphi (\partial^2 + m^2) \varphi + J\varphi \right]} \tag{A5}$$

where  $Z$  is the ground state amplitude,  $\varphi$  is the field, and  $H$  is the Hamiltonian for a scalar system (spin = 0).

Starting with Gaussian integration:

$$\int_{-\infty}^{\infty} dx e^{-x^2/2} = \sqrt{2\pi} \tag{A6}$$

Which can be extrapolated to:

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2/2 + Jx} = \left( \frac{2\pi}{\alpha} \right)^{1/2} e^{J^2/2\alpha} \tag{A7}$$

Let “ $\alpha$ ” represents a real  $N \times N$  symmetry matrix  $A_{ji}$  and  $x(i, j = 1, \dots, N)$ ,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_N e^{\frac{1}{2} x \cdot A \cdot x + J \cdot x} = \left( \frac{(2\pi)^N}{\det[A]} \right)^{1/2} e^{\frac{1}{2} J \cdot A^{-1} J} \tag{A8}$$

Replacing  $A$  by  $-iA$  and  $J$  by  $iJ$ , then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_N e^{\frac{i}{2} x \cdot A \cdot x + iJ \cdot x} = \left( \frac{(2\pi i)^N}{\det[A]} \right)^{1/2} e^{\frac{-i}{2} J \cdot A^{-1} J} \tag{A9}$$

Comparing (9) and (5), “ $A$ ” plays the role of  $-(\partial^2 + m^2)$ ,  $A^{-1} = D(x-y)$  and  $A * A^{-1} = I$  or

$$A_{ij}A_{jk}^{-1} = \delta_{ik} :$$

$$-(\partial^2 + m^2)D(x-y) = \delta^4(x-y) \quad (\text{A10})$$

$$Z(J) = \mathfrak{Z} e^{-(i/2) \iint d^4x d^4y J(x) D(x-y) J(y)} = \mathfrak{Z} e^{iW(J)} \quad (\text{A11})$$

where,  $\mathfrak{Z}$  is the overall factor with the determinant in (8) and independent of  $J$ .

$$W(J) = -\frac{1}{2} \iint d^4x d^4y J(x) D(x-y) J(y) \quad (\text{A12})$$

and

$$\delta^4(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \quad (\text{A13})$$

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

From Equation (9):

$$W(J) = -\frac{1}{2} \iint d^0x d^0y \iint d^3x d^3y J(x) D(x-y) J(y) \quad (\text{A14})$$

let

$$J(x) = J_1(x) + J_2(x) \quad (\text{A15})$$

$J(x)$  is a time-independent localized delta function  $= iq\delta(x)$  and  $J(y) = iq\delta(y)$  both are in 3-dimensions.

$$\begin{aligned} J_1(x) &= -iq_e \delta(\vec{x}) \\ J_2(y) &= +iq_p \delta(\vec{y}) \end{aligned} \quad (\text{A16})$$

where  $J_1(x)$  is the localized complex charge  $(-iq_e \delta(x))$  of the electron and  $J_2(y)$  is  $(+iq_p \delta(y))$  of the proton.  $W(J)$  will consist of four terms:  $J_1^* J_1$ ,  $J_2^* J_2$ ,  $J_1^* J_2$ , and  $J_2^* J_1$ .

**(b) Neglecting the first two self-interacting terms, which correspond to**

- The energy required to assemble the source,
- Renormalization of particle masses,
- Local energy surrounding each particle.

For a point charge, these terms are formally divergent and it is omitted in bound-state calculation because, one is interested in the relative energy interaction between constituents  $[V(r) = E(r) - E(\infty)]$ . Self-energies:

- Do not depend on separation  $r$ ,
- Merely shift the overall energy baseline,
- Are absorbed into renormalized particle masses.

Thus, they do not affect:

- Forces,
- Levels splitting,
- Binding energies.

And only the cross terms contribute to the observable potential.

$$\begin{aligned}
W(J) &= \frac{q_e q_p}{2} \iint d^0 x d^0 y \int d^3 x \delta(\vec{x}) \int d^3 y \delta(\vec{y}) \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon} \\
&= \frac{q_e q_p}{2} \iint d^0 x d^0 y \int \frac{dk^0}{2\pi} e^{ik^0(x-y)} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{k^2 - m^2 + i\epsilon} \\
&= \frac{q_e q_p}{2} \int d^0 x \int d^0 y \delta(x-y) \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{k^2 - m^2 + i\epsilon} \\
&= \frac{q_e q_p}{2} \int d^0 x \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{k^2 - m^2 + i\epsilon} \\
&\quad k^2 = k_0^2 - \vec{k}^2
\end{aligned} \tag{A17}$$

where  $k_0 = E/c$  is the time component and equal zero for steady state, then  $k^2 = -\vec{k}^2$ :

$$W(J) = -\frac{q_e q_p}{2} \int d^0 x \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{\vec{k}^2 + m^2} \tag{A18}$$

The first integral represents the time, and the infinitesimal ( $i\epsilon$ ) term is ignored since  $\vec{k}^2 + m^2$  is always positive.

$$Z = \mathfrak{Z} e^{iW(J)} = \mathfrak{Z} e^{-iET} \tag{A19}$$

Then

$$E = +\frac{q_e q_p}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{\vec{k}^2 + m^2} \tag{A20}$$

## Appendix 2

**(c) To simplify the derivations, we assume that the photon has a small mass “ $m$ ” and set  $m = 0$  at the end. If the result does not diverge, we will presume that our assumption is correct [22]. The mass term for the photon:**

- Makes integral mathematically well defined,
- Isolates physically meaningful finite contributions, while not interpreted as a real photon mass and at the end we set  $m = 0$ .

For a massless mediator like the photon, the propagator contains  $(1/k^2)$ , which can produce infrared divergences at  $k = 0$ . To control this, one temporally replaces  $1/k^2$  by  $1/(k^2 - m^2)$  in Minkowski spacetime or in Euclidean form  $1/(k^2 + m^2)$ , where “ $m$ ” is an infinitesimal fictitious photon mass. The transformation from Minkowski space (real time) to Euclidean space (imaginary time), Wick rotation mathematical transformation is used *i.e.* ( $t \rightarrow i\tau$ ) or ( $k_0 \rightarrow ik_4$ ), where

- $T$  is the physical time coordinate,
- $\tau$  is the Euclidean (imaginary time),

In Minkowski spacetime with metric  $(+, -, -, -)$  and

$$s^2 = t^2 - x^2 - y^2 - z^2.$$

after substituting  $t = -i\tau$ ,

$s^2 = -(\tau^2 + x^2 + y^2 + z^2)$ . Ignoring the overall minus sign, one obtains the Euclid-

ean distance  $r_E^2 = -(\tau^2 + x^2 + y^2 + z^2)$ , and the Lorentzian geometry becomes an ordinary 4-dimensional Euclidean geometry. A propagator in Minkowski space has the form

$$1/(k^2 - m^2 + i\varepsilon) = 1/(k_o^2 - \mathbf{k}^2 - m^2 + i\varepsilon)$$

Applying  $k_o = ik_4$

Gives  $k_o^2 - \mathbf{k}^2 - m^2 = -(k_4^2 + \mathbf{k}^2 - m^2)$

Defining  $k_E^2 = k_4^2 + \mathbf{k}^2$ , then the propagator becomes:

$1/(k_E^2 + m^2)$ , where

- $k_o$  is the energy component of the four-momentum,
- $k_4$  is the Euclidean time-momentum component.
- Suppresses infinitely long wavelength modes,

The Lagrangian density of the electromagnetic field of spin = 1 is:

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu \quad (\text{A21})$$

where the 4D vector potential  $A^\mu = [\phi(x, t), A(x, t)]^T$  and the field

$$F^{\mu\nu} = \partial^\nu A^\mu(x) - \partial^\mu A^\nu(x)$$

The field theory of the vector meson is:

$$Z = \int DA e^{iS(A)} \equiv e^{iW(J)}$$

and

(A22)

$$S(A) = \int d^4x \mathcal{L} = \int d^4x \left\{ \frac{1}{2} A_\mu \left[ (\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] A_\nu + A_\mu J^\mu \right\}$$

The last integral follows the integration by parts. The square bracket is:

$$\left[ (\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] D_{\nu\lambda}(x-y) = \delta_\lambda^\mu \delta^{(4)}(x-y) \quad (\text{A23})$$

$$\left[ -(k^2 - m^2) g^{\mu\nu} + k^\mu k^\nu \right] D_{\nu\lambda}(k) = \delta_\lambda^\mu \quad (\text{A24})$$

$$W(J) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J^\mu(k) \frac{-g_{\nu\lambda} + k_\mu k_\lambda / m^2}{k^2 - m^2 + i\varepsilon} J^\nu(k) \quad (\text{A25})$$

The current conservation

$\partial_\mu J^\mu(x) = 0$ , which in momentum space is  $k_\mu J^\mu(k) = 0$ , then

The inverse of the differential operator:

$$D_{\nu\lambda}(k) = \frac{-g_{\nu\lambda} + k_\mu k_\lambda / m^2}{k^2 - m^2} \quad (\text{A26})$$

$$D_{\nu\lambda}(x-y) = \int \frac{d^4k}{(2\pi)^4} D_{\nu\lambda}(k) e^{ik(x-y)} \quad (\text{A27})$$

Thus

$$W(J) = -\frac{1}{2} \int d^4x d^4y J(x) D(x-y) J(y) \quad (\text{A28})$$

Following the same argument above:

$$J(x) = J_1(x) + J_2(x) \quad (\text{A29})$$

And ignoring the self-interaction terms among the Js and consider only those interactions between the electron and the proton and considering them being independent of time and localized by a delta function at the locations  $x$  and  $y$ , where  $J_1(x) = -e\delta(x)$  and  $J_2(y) = +e\delta(y)$ , then:

$$\begin{aligned} W(J) &= -\int dx^o dy^o \int d^3x d^3y J_1(x) D(x-y) J_2(y) \\ &= e^2 \int dx^o dy^o \int d^3x \delta(x) \int d^3y \delta(y) \int \frac{d^4k}{(2\pi)^4} D_{\nu\lambda}(k) e^{ik(x-y)} \end{aligned} \quad (\text{A30})$$

$$\begin{aligned} W(J) &= \int dx^o dy^o \int d^3x d^3y J_1(x) D(x-y) J_2(y) \\ &= -e^2 \int dx^o dy^o \int d^3x \delta(x) \int d^3y \delta(y) \int \frac{d^4k}{(2\pi)^4} D_{\nu\lambda}(k) e^{ik(x-y)} \\ &= -e^2 \int dx^o dy^o \int \frac{dk^0}{2\pi} e^{ik^0(x-y)^0} \int \frac{d^3k}{(2\pi)^3} D_{\nu\lambda}(k) e^{i\vec{k} \cdot (\vec{x}-\vec{y})} \\ &= -e^2 \int dx^o \int dy^o \delta(x-y)^0 \int \frac{d^3k}{(2\pi)^3} D_{\nu\lambda}(k) e^{i\vec{k} \cdot (\vec{x}-\vec{y})} \\ &= -e^2 \int dx^o \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{x}-\vec{y})}}{k^2 - m^2 + i\varepsilon} \end{aligned} \quad (\text{A31})$$

$$k^2 = k_0^2 - \vec{k}^2$$

where  $k_0 = E/c$  is the time component and equal zero for steady state, then  $k^2 = -\vec{k}^2$ :

$$W(J) = e^2 \int dx^o \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{x}-\vec{y})}}{\vec{k}^2 + m^2} \quad (\text{A32})$$

The  $i\varepsilon$  can again be dropped since  $\vec{k}^2 + m^2$  is always positive.

$$\begin{aligned} \langle 0 | e^{-iHT} | 0 \rangle &= e^{iW(J)} = e^{-iET} \\ E &= -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{x}-\vec{y})}}{\vec{k}^2 + m^2} \end{aligned} \quad (\text{A33})$$