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A New Approach to Gravitation and Relativity Based upon Enhanced Newtonian Gravity and Special Relativity (II)

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Abstract

The gravitational point theory presented in Part I is supplemented by a description of gravitational forces when observed from non-inertially moving reference frames. The Special Theory of Relativity is extended to cope with linearly and circularly accelerated reference frames. It follows that inertial and heavy mass are only equal on exact circular paths, otherwise they require a relativistic correction, and the stationary observer determines the shape of the rotating circle as a circle with a slightly changed radius, without having to abandon Euclidean geometry. A second supplement concerns the determination of the Lagrange and Hamilton functions of particles or bodies of variable mass as well as the Schrödinger wave equation of the gravitational field which opens up a path to extend classical quantum physics to include gravity.

Keywords

General Relativity, Special Relativity, Schrödinger Equation for Gravity, Quantum Theory of Gravitation, Physics of Neutron Stars

1. Introduction

The theory of gravitation (dubbed ENG) developed in Part I [1], which manages with a constant speed of light and simple Euclidean geometry, yet can calculate redshift, light deflection, perihelion precession and signal propagation times just as precisely as the General Theory of Relativity (GR), is characterised by the fact that the moving mass of light or massive particles depends on the position in the gravitational field. In addition, ENG does not borrow anything from GR, in particular, it does not require any sort of equivalence principle[1] and does not restrict Special Relativity (SR) a priori to local validity. Nothing therefore stands in the way of trying to extend SR, which only applies locally in GR, to non-inertial reference systems. But it is also desirable, because in conjunction with the dynamic equations of ENG, the actual shape of a rotating body could be determined (see Introduction Part I). Since the complete elaboration of an alternative theory of relativity would go beyond the scope of this work, I will limit myself to linear and circularly accelerated reference systems and work out the structurally new basic features (see Sections 2.2-2.4). These include the differential form of the Lorentz transformation and the appearance of the 4-vector of mass and momentum and its time derivative as well as the relativistic energy of a particle or body of variable mass. Further results include:

- for the observer at rest, the radius and circumference of the rotating disc change to the same extent, *i.e.* Euclidean geometry applies,
- the equality of inertial and heavy mass, *i.e.* the equivalence principle applies strictly only to circular motion and must be generalised relativistically.

Note that the treatment of arbitrarily accelerated reference frames, including translation and reflections, will finally lead to differential Poincare transformations and, therefore, cannot alter these conclusions.

Compared to GR, ENG is also characterised by its relatively simple structure inherited from classical point mechanics, which is only supplemented by a differential equation of mass. Provided that Lagrange and Hamiltonian formalism can be explained with variable mass, the formal development of the ENG into a quantum field theory of gravitation appears straightforward. However, if we consider that QED has only found its final form in a long series of papers and over many years, we can only expect this work to illuminate the structural innovations and extensions arising from variable mass for the development of ENG into a quantum field theory of gravitation. This means in the first step to reformulate the point mechanics given in Part I by Lagrangian and Hamiltonian functions for bodies with variable mass (see Sections 3.1-3.3), which are a prerequisite for the construction of a classical field theory with variable mass. Then, in the second step, to derive Schrödinger's wave equation for variable mass (see Subsections 3.4-3.5). From here, but not in this work, the special features variable mass brings along for a quantum (field) theory can be studied and preferably applied to the physics of neutron stars.

We adopt the designations of Part I. Sections, equations, examples, figures, remarks and references of Part I are referenced by a preset I.

[1]See [2], p. 318 or [3], p. 386 for modern, more mathematical formulations. Older and more physical interpretations to be found in [4], p. 37-41 or [5], p. 175.

2. Extending SR to Selected Reference Frames Falling in Gravity

One consequence of the principle of equivalence is that the rate of clock is not constant and, a fortiori, the speed of light is not constant (see reference [I.9] or the discussion in Remark I.2). In GR, therefore, Lorentz transformations can only exist in gravity-free space. Since ENG does not require a principle of equivalence and speed of light is always constant, the way is clear to try to extend SR to non-inertial reference frames.

2.1. Co-Operating Observers

Let us consider any stationary point $\boldsymbol{P}_0$ in the gravitational field of a sun with mass M_S and radius r_S. We choose the connecting line from $\boldsymbol{P}_0$ to the centre of the sun as the x-axis, place the origin at $\boldsymbol{P}_0$, the y-axis as usual vertically anti-clockwise, and the resulting stationary coordinate system is designated O. Next, we consider a (massive) coordinate system Ω, which is incident to the Sun from $\boldsymbol{P}_0$ without initial velocity and whose coordinate axes are aligned with those of the stationary coordinate system, and a (massive) reference frame O', which is falling to the Sun from $\boldsymbol{P}_0$ with an initial velocity different from zero and which again has parallel and aligned axes to O. All coordinate systems are equipped with a laser clock of the same type at their origin, *i.e.* they run at the same frequency ν_0. At the start time, O' and Ω are at the location $\boldsymbol{P}_0$, all clocks show the same time. We note the elapsing clock time and runtime in O with t and τ, resp., that in O' with t' and τ' and that in Ω with t'' and τ''. In the following, the notation $\tau = \tau(t)$ is used and the differential $\mathrm{d}\tau$ stands for $\mathrm{d}\tau/\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}\tau/\mathrm{d}t \cdot \mathrm{d}t/\mathrm{d}r \cdot \mathrm{d}r$; similarly for τ' and τ'' (see I. 62-I.66).

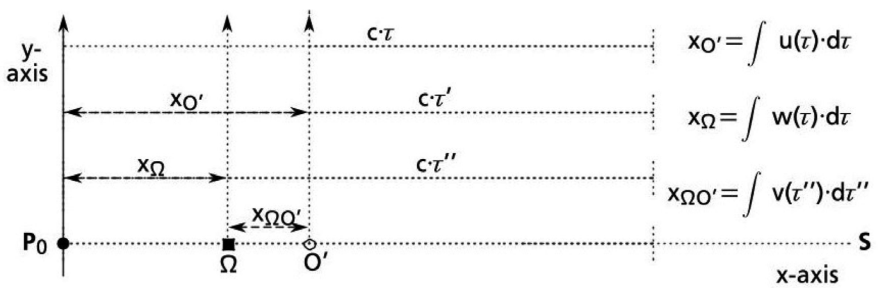

Figure 1. The linearly accelerated case.

At start time, a ray of light is sent from O' to the sun with $\boldsymbol{p}(\tau) = c \cdot m(\tau) \cdot \boldsymbol{t}(\tau) = c \cdot (m(\tau), 0)$, *i.e.* in a radial direction[2]. An observer in O can measure it directly or indirectly by determining the distance to O' and otherwise relying on the measurement of the observer in O'. With the designations from Figure 1, the following therefore applies: $c \cdot \tau = x_{O'}(\tau) + c\tau'$. The observer

[2]Since O emerges from the gravitational reference frame through displacement by a constant vector, the momentum vector $\boldsymbol{p}$ can be easily derived from the momentum vector in the gravitational reference frame. Similar for position vectors.

in Ω proceeds in the same way and writes down: $c\tau'' = x_{\Omega O'}(\tau'') + c\tau'$. Finally, we have the relationship $c\tau = x_{\Omega}(\tau) + c\tau''$, in which all 3 observers work together. These relationships read in differential form

$$\begin{aligned} c\cdot \mathrm{d}\tau &= u(\tau)\mathrm{d}\tau + c\mathrm{d}\tau', \\ c\cdot \mathrm{d}\tau'' &= v(\tau'')\mathrm{d}\tau'' + c\mathrm{d}\tau', \\ c\cdot \mathrm{d}\tau &= w(\tau)\mathrm{d}\tau + c\mathrm{d}\tau''. \end{aligned}$$

Now, let us multiply the first relationship on both sides of the equation by the still unknown function $f_u(\tau)$

$$c\cdot f_u(\tau)\mathrm{d}\tau = u(\tau)\cdot f_u(\tau)\mathrm{d}\tau + c\cdot f_u(\tau)\mathrm{d}\tau'$$

and define the new differential $\mathrm{d}\tau_u := f_u(\tau)\cdot \mathrm{d}\tau$. With the inverse function $\tau = \tau(\tau_u)$ of $\tau_u = \tau_u(\tau)$ we have: $c\cdot \mathrm{d}\tau_u = u(\tau_u)\mathrm{d}\tau_u + c\cdot f_u(\tau_u)\mathrm{d}\tau'$. If we repeat this for the second relationship with $f_v(\tau'')$ or for the third relationship with $f_w(\tau)$, we obtain with $\mathrm{d}\tau''_v := f_v(\tau'')\cdot \mathrm{d}\tau''$ or $\mathrm{d}\tau_w := f_w(\tau)\cdot \mathrm{d}\tau$ and the corresponding inverse functions

$$\begin{aligned} c\cdot \mathrm{d}\tau''_v &= v(\tau''_v)\mathrm{d}\tau''_v + c\cdot f_v(\tau''_v)\mathrm{d}\tau', \\ c\cdot \mathrm{d}\tau_w &= w(\tau_w)\mathrm{d}\tau_w + c\cdot f_w(\tau_w)\mathrm{d}\tau''. \end{aligned}$$

If we now suppress the indices in the new differentials and consider $\int c\cdot \mathrm{d}\tau_w = \int c\cdot \mathrm{d}\tau$, we get

$$\begin{aligned} c\mathrm{d}\tau &= u(\tau)\cdot \mathrm{d}\tau + c\cdot f_u(\tau)\cdot \mathrm{d}\tau', \\ c\mathrm{d}\tau'' &= v(\tau'')\cdot \mathrm{d}\tau'' + c\cdot f_v(\tau'')\cdot \mathrm{d}\tau', \\ c\mathrm{d}\tau &= w(\tau)\mathrm{d}\tau + c\cdot f_w(\tau)\cdot \mathrm{d}\tau'. \end{aligned}$$

Solving the first two of the three new relationships for $\mathrm{d}\tau'$ and the third for $\mathrm{d}\tau''$

$$\begin{aligned} \mathrm{d}\tau' &= \frac{\mathrm{d}\tau - u(\tau)\cdot \mathrm{d}\tau/c}{f_u(\tau)} = \frac{\mathrm{d}\tau'' - v(\tau'')\mathrm{d}\tau''/c}{f_v(\tau'')}, \\ \mathrm{d}\tau'' &= \frac{\mathrm{d}\tau - w(\tau)\cdot \mathrm{d}\tau/c}{f_w(\tau)} \end{aligned} \tag{1}$$

yields with $\mathrm{d}\tau'/\mathrm{d}\tau = (\mathrm{d}\tau''/\mathrm{d}\tau)\cdot(\mathrm{d}\tau'/\mathrm{d}\tau'')$

$$\frac{1-u(\tau)/c}{f_u(\tau)} = \frac{1-w(\tau)/c}{f_w(\tau)}\cdot\frac{1-v(\tau'')/c}{f_v(\tau'')}.$$

If we experimentally set $f_z(a) := \left(1 - z^2(a)/c^2\right)^{1/2}$, we obtain

$$\begin{aligned} &\frac{1-u(\tau)/c}{1+u(\tau)/c} = \frac{1-w(\tau)/c}{1+w(\tau)/c}\cdot\frac{1-v(\tau'')/c}{1+v(\tau'')/c} \\ &\Rightarrow u(\tau) = \frac{v(\tau'') + w(\tau)}{1 + v(\tau'')\cdot w(\tau)/c^2}. \end{aligned} \tag{2}$$

In the limiting case of vanishing or constant gravitation, Ω and O' are inertial frames, so that the law of addition of velocities of special relativity is rediscovered[3]. Consequently, Equation (2) represents the addition law of velocities in gravita-

[3]Using the method presented, it can even be shown that the Lorentz contraction is the only function that satisfies Equation (2).

tional space. The derivation of Equation (1) is not restricted to gravitational fall, but applies to any linear acceleration. As in SR, Equation (1) implies

$$\begin{pmatrix} \frac{\mathrm{d}\tau'}{\mathrm{d}\tau} \\ \frac{\mathrm{d}x'}{\mathrm{d}\tau} \end{pmatrix} = \frac{1}{f(\tau)} \begin{pmatrix} 1 & \frac{-u(\tau)}{c^2} \\ -u(\tau) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} \end{pmatrix}$$
$$\Rightarrow\ c^2\mathrm{d}\tau'^2 - \mathrm{d}x'^2 - \mathrm{d}y'^2 = c^2\mathrm{d}\tau^2 - \mathrm{d}x^2 - \mathrm{d}y^2,$$

a relation which is not only valid for $c = \mathrm{d}x/\mathrm{d}\tau$, but for any particle with $\mathrm{d}x/\mathrm{d}\tau < c$. In the special case of $\mathrm{d}x/\mathrm{d}\tau = u(\tau)$, $\tau'(t) = \int f(\tau(t)) \cdot \mathrm{d}\tau/\mathrm{d}t\,\mathrm{d}t$ *defines the proper time* $\tau'(t)$ *in gravity.*

Since the *clock times* t *and* $t'(t) := \tau'(t)$ *are now fixed* and the contraction acts in the x-direction, which is completely independent of the existence of a differently directed ray of light, the observers in O or O' no longer have any freedom in describing a ray simultaneously emanating from q_0 in any direction other than radial (see Figure 2).

So the observer in O will accept the coordinates given to him by the observer in O' in the y'-direction, but from now on will contract in the x'-direction. And vice versa, if the observer in O' takes over the coordinates from the observer in O.

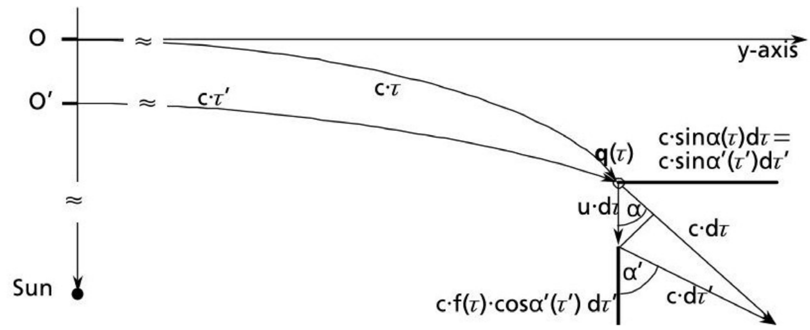

Figure 2. Light curves in O and O' (not to scale).

Let such a light beam be considered; the momentum of the photon is given by $\boldsymbol{p}(\tau) = m(\tau) \cdot c\boldsymbol{t}(\tau) = m(\tau) \cdot c \cdot (\cos\alpha(\tau), \sin\alpha(\tau))$. In Figure 2, $\theta_0 := 90^\circ$, $\alpha_0 := \arcsin(v_0/c)$ is used as an example.

With $c \cdot \boldsymbol{t}'(\tau') = c \cdot (\cos\alpha'(\tau'), \sin\alpha'(\tau'))$, $\boldsymbol{u}(\tau) := u(\tau) \cdot (1,0)$, the following equation applies to the infinitesimal increases at the point $\boldsymbol{q}(\tau)$:

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = c \cdot \begin{pmatrix} \cos\alpha(\tau) \\ \sin\alpha(\tau) \end{pmatrix} \mathrm{d}\tau = u(\tau) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\tau + c \cdot \begin{pmatrix} \cos\alpha'(\tau') \cdot f(\tau) \\ \sin\alpha'(\tau') \end{pmatrix} \mathrm{d}\tau'$$
$$= u(\tau) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\tau + \begin{pmatrix} \frac{\mathrm{d}x'}{\mathrm{d}\tau'} \cdot f(\tau) \\ \frac{\mathrm{d}y'}{\mathrm{d}\tau'} \end{pmatrix} \cdot \frac{\mathrm{d}\tau'}{\mathrm{d}\tau} \mathrm{d}\tau \ \Rightarrow\ \frac{1}{f(\tau)} \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}\tau} - u(\tau) \\ f(\tau) \cdot \frac{\mathrm{d}y}{\mathrm{d}\tau} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}x'}{\mathrm{d}\tau} \\ \frac{\mathrm{d}y'}{\mathrm{d}\tau} \end{pmatrix}. \tag{3}$$

It follows:

$$\left(u(\tau) \cdot \frac{\mathrm{d}\tau}{\mathrm{d}\tau'} + f(\tau) \cdot c \cdot \cos\alpha'(\tau') \right)^2 + c^2 \cdot \sin^2\alpha'(\tau') = c^2 \cdot \left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau'} \right)^2,$$

giving

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}-\frac{u(\tau)/c\cdot f(\tau)\cdot\cos\alpha'(\tau')}{1-u^2(\tau)/c^2}\right)^2$$
$$=\frac{1}{1-u^2(\tau)/c^2}\cdot\left(\sin^2\alpha'(\tau')+\frac{f^2(\tau)\cdot\cos^2\alpha'(\tau')}{1-u^2(\tau)/c^2}\right)$$
$$\Rightarrow\frac{\sqrt{\left(1-\frac{u^2(\tau)}{c^2}-f^2(\tau)\right)\cdot\sin^2\alpha'(\tau')+f^2(\tau)}+\frac{u(\tau)}{c}\cdot f(\tau)\cdot\cos\alpha'(\tau')}{1-u^2(\tau)/c^2}=\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}.$$

Since $f(\tau)$ is already fixed, we get with $Z(\tau(\tau'))=:Z(\tau')$

$$\frac{1+\cos\alpha'(\tau')\cdot u(\tau')/c}{\left(1-u^2(\tau')/c^2\right)^{1/2}}=\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}. \tag{4}$$

Moreover, the transformation law of angles results from Equation (3):

$$\cos\alpha(\tau(\tau'))=:\cos\alpha(\tau')=\frac{u(\tau')/c+\cos\alpha'(\tau')}{1+u(\tau')\cdot\cos\alpha'(\tau')/c},\quad \sin\alpha(\tau')=\frac{\sin\alpha'(\tau')}{\mathrm{d}\tau/\mathrm{d}\tau'}. \tag{5}$$

If we insert the expression $\cos\alpha'(\tau')=\left(\cos\alpha(\tau)-u(\tau)/c\right)/\left(1-u(\tau)\cdot\cos\alpha(\tau)/c\right)$ obtained from Equation (5) into Equation (4), we get the inverse expression to Equation (4)

$$\frac{\mathrm{d}\tau'}{\mathrm{d}\tau}=\frac{1-\cos\alpha(\tau)\cdot u(\tau)/c}{\left(1-u^2(\tau)/c^2\right)^{1/2}}. \tag{6}$$

The inverse expression can also be obtained if an observer in O' reconstructs the light path in O' by observing the relative movement of O with respect to O' and otherwise assumes the light path of O as he perceives it when the light ray departs from O'.

Then the vector equation reads

$$c\cdot\begin{pmatrix}\cos\alpha'(\tau')\\ \sin\alpha'(\tau')\end{pmatrix}\mathrm{d}\tau'=u(\tau')\begin{pmatrix}-1\\0\end{pmatrix}\mathrm{d}\tau'+c\cdot\begin{pmatrix}\cos\alpha(\tau)\cdot f(\tau')\\ \sin\alpha(\tau)\end{pmatrix}\mathrm{d}\tau,$$

from which Equation (6) follows analogue to the previous one.

In addition, Equation (4) contains the formulae for the times which the Minkowski experiment enforces for the horizontal ($\alpha'=0$) and vertical ($\alpha'=90^\circ$) rays of light. We can compactly summarise Equation (3) and Equation (6) to give:

$$\begin{pmatrix}\mathrm{d}\tau'/\mathrm{d}\tau\\ \mathrm{d}x'/\mathrm{d}\tau\\ \mathrm{d}y'/\mathrm{d}\tau\end{pmatrix}=\frac{1}{f(\tau)}\begin{pmatrix}1 & -u(\tau)/c^2 & 0\\ -u(\tau) & 1 & 0\\ 0 & 0 & f(\tau)\end{pmatrix}\begin{pmatrix}1\\ \mathrm{d}x/\mathrm{d}\tau\\ \mathrm{d}y/\mathrm{d}\tau\end{pmatrix} \tag{7}$$
$$\Rightarrow c^2\mathrm{d}\tau'^2-\mathrm{d}x'^2-\mathrm{d}y'^2=c^2\mathrm{d}\tau^2-\mathrm{d}x^2-\mathrm{d}y^2,$$

a relation which is not only valid for $c=\mathrm{d}x/\mathrm{d}\tau$, but for any particle with $\mathrm{d}x/\mathrm{d}\tau<c$.

2.2. Differential Special Lorentz Transformation

Now, O, Ω and O' have only 1 spatial dimension. From Equation (1) follows:

$$\frac{\mathrm{d}\tau'}{\mathrm{d}\tau}=\frac{1-u(\tau)/c}{\left(1-u^2(\tau)/c^2\right)^{1/2}},$$

$$\frac{c\cdot\mathrm{d}\tau'}{\mathrm{d}\tau}=\frac{\mathrm{d}x'}{\mathrm{d}\tau'}\cdot\frac{\mathrm{d}\tau'}{\mathrm{d}\tau}=\frac{\mathrm{d}x'}{\mathrm{d}\tau}=\frac{c-u(\tau)}{\left(1-u^2(\tau)/c^2\right)^{1/2}}=\frac{\mathrm{d}x/\mathrm{d}\tau-u(\tau)}{\left(1-u^2(\tau)/c^2\right)^{1/2}}$$

or in matrix notation with $f(\tau):=\left(1-u^2(\tau)/c^2\right)^{1/2}$:

$$\begin{pmatrix}\mathrm{d}\tau'/\mathrm{d}\tau\\ \mathrm{d}x'/\mathrm{d}\tau\end{pmatrix}=\frac{1}{f(\tau)}\begin{pmatrix}1 & -u(\tau)/c^2\\ -u(\tau) & 1\end{pmatrix}\begin{pmatrix}1\\ \mathrm{d}x/\mathrm{d}\tau\end{pmatrix}, \tag{8}$$

In particular:

$$\left\langle\begin{pmatrix}\mathrm{d}\tau\\ \mathrm{d}x\end{pmatrix},\begin{pmatrix}\mathrm{d}\tau\\ \mathrm{d}x\end{pmatrix}\right\rangle:=c^2\mathrm{d}\tau^2-\mathrm{d}x^2=c^2\mathrm{d}\tau'^2-\mathrm{d}x'^2=\left\langle\begin{pmatrix}\mathrm{d}\tau'\\ \mathrm{d}x'\end{pmatrix},\begin{pmatrix}\mathrm{d}\tau'\\ \mathrm{d}x'\end{pmatrix}\right\rangle,$$

a relation which is not only valid for $c=\mathrm{d}x/\mathrm{d}\tau$, but for any $\mathrm{d}x/\mathrm{d}\tau<c$. If we multiply Equation (8) by $\mathrm{d}\tau/\mathrm{d}t$ and integrate, we obtain the space-time coordinates of O' as a function of the (clock) time t of O:

$$\begin{pmatrix}\tau'(t)-\tau'_0\\ x'(t)-x'_0\end{pmatrix}=\int_{t_0}^{t}\frac{1}{f(t)}\begin{pmatrix}1 & -u(t)/c^2\\ -u(t) & 1\end{pmatrix}\begin{pmatrix}\mathrm{d}\tau/\mathrm{d}t\\ \mathrm{d}x/\mathrm{d}t\end{pmatrix}\mathrm{d}t,$$

here $\mathrm{d}\tau/\mathrm{d}t$ and $\mathrm{d}x/\mathrm{d}t=\mathrm{d}x/\mathrm{d}\tau\cdot\mathrm{d}\tau/\mathrm{d}t$ are given by (I.66). If $(\mathrm{d}x/\mathrm{d}\tau)(\tau(t))=u(\tau(t))$, τ' means the clock time t' in O':

$$t'(t):=\tau'(t)=\int_0^t\left(1-u^2(t)/c^2\right)^{1/2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\mathrm{d}t.$$

Example 1 Clock and scale comparison during acceleration. We consider two reference frames which move away from each other *in gravity-free space* in a straight line with the speed $u(t)=u_0+bt$. We take the direction of movement as the x-axis and before the start the coordinate systems are united at the origin and the x'-axis is parallel to the x-axis. We set $\mathrm{d}m/\mathrm{d}t=0$, thus $\mathrm{d}\tau/\mathrm{d}t=0$. Then, Equation (8) reduces to

$$\begin{pmatrix}\mathrm{d}t'/\mathrm{d}t\\ \dfrac{\mathrm{d}x'}{\mathrm{d}t'}\dfrac{\mathrm{d}t'}{\mathrm{d}t}\end{pmatrix}=\frac{1}{\sqrt{1-\dfrac{u^2(t)}{c^2}}}\begin{pmatrix}1 & -u(t)/c^2\\ -u(t) & 1\end{pmatrix}\begin{pmatrix}1\\ \dfrac{\mathrm{d}x}{\mathrm{d}t}\end{pmatrix}.$$

We set $\mathrm{d}x'/\mathrm{d}t'=0$ and calculate the times t and t' since the start time. We obtain $\mathrm{d}x/\mathrm{d}t=u(t)$ and $\mathrm{d}t'=\mathrm{d}t\cdot\left(1-u^2(t)/c^2\right)^{1/2}$, and thus after integration:

$$t'=\frac{c}{2b}\left[\frac{u(t)}{c}\sqrt{1-\frac{u^2(t)}{c^2}}+\arcsin\frac{u(t)}{c}\right]-\frac{c}{2b}\left[\frac{u_0}{c}\sqrt{1-\frac{u_0^2}{c^2}}+\arcsin\frac{u_0}{c}\right],$$

where the last expression represents the integral constant for the given boundary conditions. Let us develop this expression around u_0 up to the third power of t yielding

$$t' = t\cdot\sqrt{1-\frac{u_0^2}{c^2}} - \frac{b}{2c}\frac{u_0/c\cdot t^2}{\sqrt{1-\frac{u_0^2}{c^2}}} - \frac{b^2}{6c^2}\frac{t^3}{\left(1-\frac{u_0^2}{c^2}\right)^{3/2}}$$

$$= t\cdot\sqrt{1-\frac{u_0^2}{c^2}}\left\{1 - \frac{b}{2c}\frac{u_0/c\cdot t}{1-\frac{u_0^2}{c^2}} - \frac{b^2}{6c^2}\frac{t^2}{\left(1-\frac{u_0^2}{c^2}\right)^{3/2}}\right\},$$

we see that SR represents the linear approximation. At constant acceleration, there is a time t' for which $u(t)=c$, namely

$$t' = \frac{c}{2b}\cdot\left(\pi/2 - \frac{u_0}{c}\sqrt{1-\frac{u_0^2}{c^2}} - \arcsin\frac{u_0}{c}\right).$$

The initial velocity and the acceleration thus determine the end of the visibility of O' for O.

In O, we set $u_E = 1\,\mathrm{m/s}$ as the unit velocity and choose $\mathrm{d}x/\mathrm{d}t = u(t) + u_E$; then $\mathrm{d}x'/\mathrm{d}t$ represents the velocity in the moving reference frame O', as calculated from O, and equals

$$\mathrm{d}x'/\mathrm{d}t = \frac{\mathrm{d}x/\mathrm{d}t - u(t)}{\left(1-u^2(t)/c^2\right)^{1/2}} =: \frac{u_E}{\left(1-u^2(t)/c^2\right)^{1/2}}.$$

With $t_0 = 0 = t_0'$ and $t = 1\,\mathrm{s}$ the integration results in

$$u_E\int_0^1\left(1-u^2(t)/c^2\right)^{-1/2}\mathrm{d}t = x'(t).$$

The metre scale in O thus appears extended in O', or the metre scale of O' shortened in O.

Example 2 Doppler effect. We continue to operate in gravity-free space and neglect the $\mathrm{d}m/\mathrm{d}t$-term. From Equation (8), we obtain the formula for a light wave, $x' = ct'$, which was transmitted by a body in O' accelerating away from O and is travelling towards O:

$$\mathrm{d}x = \frac{1-u(t)/c}{\left(1-u(t)/c\right)^{1/2}}\mathrm{d}t = \frac{\left(1-u(t)/c\right)^{1/2}}{\left(1+u(t)/c\right)^{1/2}}.$$

Integration gives for $u(t) = u_0 + bt$:

$$t' = \frac{c}{b}\left(\sqrt{1-\frac{u^2(t)}{c^2}} + \arcsin\frac{u(t)}{c}\right) - \frac{c}{b}\left(\sqrt{1-\frac{u_0^2}{c^2}} + \arcsin\frac{u_0}{c}\right).$$

If the acceleration and speed of a distant light source are known, which is observed on Earth with the frequency f or period $T = 1/f$, the above formula offers the possibility of experimental verification. A Taylor expansion makes the correction to the classical relativistic Doppler effect formula clear:

$$T' = T \cdot \frac{\sqrt{1-\frac{u_0}{c}}}{\sqrt{1+\frac{u_0}{c}}} \cdot \left\{ 1 - \frac{b}{2c} \frac{T}{1-\frac{u_0^2}{c^2}} - \frac{b^2}{6c^2} \frac{(1-2u_0/c) \cdot T^2}{\left(1-\frac{u_0^2}{c^2}\right)^{3/2}} \right\}.$$

The Doppler effect is somewhat reduced; a similar conclusion can be drawn for the transverse effect.

Remark 1. Of course, the calculations in the above examples can only be accurate to a first approximation. From the perspective of GR, it follows from the principle of equivalence that a gravitational field equivalent to the acceleration b exists, and since GR and ENG agree in the case of weak gravitational fields, the GR theorist concludes that the calculations must be carried out with a $\mathrm{d}m/\mathrm{d}t$ term non-zero in ENG and that, consequently, a different result must also be obtained for the GR calculation. From the perspective of ENG, which does not need a principle of equivalence, it makes sense to couple the variable mass directly to the force field that causes the acceleration, rather than to the equivalent, virtual gravitational field, and to set up a corresponding differential equation. Since GR and ENG do indeed differ in strong gravitational fields and the electromagnetic forces are already approximately 10^{40} times stronger[4] than the gravitational force, direct coupling to the force field appears to be preferable. Thus, the above examples are exact in zeroth order only.

In order to arrive at *relativistic dynamics*, we must consider the energy and momentum of a particle and embed the force equation relativistically. Let us multiply Equation (8) by $m(\tau)$, which is given by (I.64),

$$\begin{pmatrix} m(\tau) \cdot \mathrm{d}\tau'/\mathrm{d}\tau \\ m(\tau) \cdot \mathrm{d}x'/\mathrm{d}\tau \end{pmatrix} = \frac{1}{f(\tau)} \begin{pmatrix} 1 & -u(\tau)/c^2 \\ -u(\tau) & 1 \end{pmatrix} \begin{pmatrix} m(\tau) \\ m(\tau) \cdot \mathrm{d}x/\mathrm{d}\tau \end{pmatrix} =: L_u \begin{pmatrix} m(\tau) \\ p(\tau) \end{pmatrix} \tag{9}$$

and ask for the meaning of the vector on the left. Choosing the transformation into the rest frame of the particle with $\mathrm{d}x/\mathrm{d}\tau = u(\tau)$, it follows that

$$\begin{pmatrix} m(\tau) \cdot \mathrm{d}\tau'/\mathrm{d}\tau \\ m(\tau) \cdot \mathrm{d}x'/\mathrm{d}\tau \end{pmatrix} = f(\tau) \begin{pmatrix} m(\tau) \\ 0 \end{pmatrix} \Rightarrow \frac{\mathrm{d}\tau'}{\mathrm{d}\tau} = f(\tau),\ \frac{\mathrm{d}x'}{\mathrm{d}\tau} = 0.$$

This makes $f(\tau) \cdot m(\tau) =: m_0'(\tau)$ the mass of the particle in the rest frame, *i.e.* the rest mass, and we arrive at

$$m(\tau) = m_0'(\tau)/f(\tau). \tag{10}$$

Consequently, in gravity-free space Equation (10) becomes the SR relations $m(u) = m_0' / (1-u^2/c^2)^{1/2}$, $t' = f(u) \cdot t$, and for light the rest mass $m'(\tau) = 0$ results. Because $\mathrm{d}\tau'/\mathrm{d}\tau = f(\tau)$ holds only for the transformation into the rest frame, it makes sense to consider $m'(\tau) := m(\tau) \cdot \mathrm{d}\tau'/\mathrm{d}\tau$ to represent the mass in O' generally. Let $p'(\tau') = m'(\tau') \cdot \mathrm{d}x'/\mathrm{d}\tau'$ represent the momentum of the particle in O'. With $\tau' = \tau'(\tau)$ we get:

[4] See Equation (57) below.

$$p'\left(\tau'(\tau)\right)=m'\left(\tau'(\tau)\right)\cdot\left(\mathrm{d}x'/\mathrm{d}\tau\right)\left(\tau'(\tau)\right)=m(\tau)\cdot\frac{\mathrm{d}\tau'}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}x'}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}\tau}{\mathrm{d}\tau'}=m(\tau)\cdot\frac{\mathrm{d}x'}{\mathrm{d}\tau},$$

thus we can re-write Equation (9) in the form:

$$\begin{pmatrix} m'(\tau) \\ p'(\tau) \end{pmatrix}=L_u\begin{pmatrix} m(\tau) \\ p(\tau) \end{pmatrix}. \tag{11}$$

Forming the scalar product with the metric $g_{11}=c^2$, $g_{22}=-1$ and $g_{ij}=0$ otherwise, we obtain

$$m'^2(\tau)\cdot c^2-p'^2(\tau)=m^2(\tau)\cdot c^2-p^2(\tau)=m_0'^2(\tau)\cdot c^2. \tag{12}$$

In the presence of gravity, the rest mass is therefore no longer an invariant constant, but an invariant function.

Next, let us calculate the relativistic energy of a particle. With the index E meaning any point in the gravitational field and $v(\tau):=\mathrm{d}x/\mathrm{d}\tau$, (I.62) reads in the radial case

$$\frac{\mathrm{d}p}{\mathrm{d}\tau}=-\frac{\mathrm{d}V}{\mathrm{d}r}\left(r(\tau)\right)\cdot\frac{m_\infty(\tau)\cdot m_E(\tau)}{m_\infty\cdot m_E}\cdot\boldsymbol{e}_r(\tau),$$

$$\frac{\mathrm{d}m}{\mathrm{d}\tau}=-k\cdot\frac{v(\tau)}{c^2}\cdot\frac{\mathrm{d}V}{\mathrm{d}r}\left(r(\tau)\right)\cdot\frac{m_\infty(\tau)\cdot m_E(\tau)}{m_\infty\cdot m_E}\cdot\left\langle\boldsymbol{e}_r(\tau),\boldsymbol{t}(\tau)\right\rangle.$$

For the left-hand side we obtain with $m(\tau)=m_0'(\tau)/f(\tau)$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}\cdot\boldsymbol{v}(\tau)=m_0'(\tau)\cdot v(\tau)\cdot\tau\frac{v(\tau)}{\sqrt{1-\frac{v^2(\tau)}{c^2}}}+\frac{v^2(\tau)\cdot\mathrm{d}m_0'/\mathrm{d}\tau}{\sqrt{1-\frac{v^2(\tau)}{c^2}}},$$

and from this for the work W:

$$\begin{aligned}W(\tau)-W(\tau_E)&=\int_{\tau_E}^{\tau}\left(\frac{m_0'(\tau)\cdot\frac{\mathrm{d}v}{\mathrm{d}\tau}v(\tau)}{\left(1-\frac{v^2(\tau)}{c^2}\right)^{3/2}}+\frac{\mathrm{d}m_0'}{\mathrm{d}\tau}\cdot\frac{v^2(\tau)}{\sqrt{1-\frac{v^2(\tau)}{c^2}}}\right)\mathrm{d}\tau\\&=\left.\frac{m_0'(\tau)\cdot c^2}{\sqrt{1-\frac{v^2(\tau)}{c^2}}}\right|_{\tau_E}^{\tau}-c^2\int_{\tau_E}^{\tau}\frac{\mathrm{d}m_0'}{\mathrm{d}\tau}\sqrt{1-\frac{v^2(\tau)}{c^2}}\mathrm{d}\tau.\end{aligned} \tag{13}$$

The expression for the particle energy reads therefore:

$$E_r(\tau):=\frac{m_0'(\tau)\cdot c^2}{\left(1-v^2(\tau)/c^2\right)^{1/2}}-c^2\int_{\tau_E}^{\tau}\frac{\mathrm{d}m_0'}{\mathrm{d}t}\left(1-v^2(\tau)/c^2\right)^{1/2}\mathrm{d}\tau. \tag{14}$$

$E_r(\tau)=m_0'\cdot c^2/f(\tau)=m(\tau)\cdot c^2$ only results for $\mathrm{d}m_0'/\mathrm{d}t=0$, which is *why we do not use* $\left(E_r(\tau)/c^2,p(\tau)\right)$ *but* $\left(m(\tau),p(\tau)\right)$ *as the four-vector in the gravitational field.* With Equation (10) and the equation

$$\frac{\mathrm{d}m_0'}{\mathrm{d}\tau}=\frac{\mathrm{d}m}{\mathrm{d}\tau}\cdot f(\tau)-m(\tau)\cdot\frac{v(\tau)\cdot\mathrm{d}v/\mathrm{d}\tau}{c^2f(\tau)} \tag{15}$$

we can evaluate the integral and obtain with (I.23):

$$\begin{aligned}
&-c^2\int_{\tau_E}^{\tau}\frac{\mathrm{d}m}{\mathrm{d}t}\cdot\left(1-\frac{v^2(\tau)}{c^2}\right)\mathrm{d}t+\int_{\tau_E}^{\tau}m(\tau)\cdot v(\tau)\cdot\frac{\mathrm{d}v}{\mathrm{d}\tau}\mathrm{d}\tau\\
&=-m(\tau)\cdot c^2+m(\tau_E)\cdot c^2+\int_{\tau_E}^{\tau}\frac{\mathrm{d}m}{\mathrm{d}t}\cdot v^2(\tau)\mathrm{d}t+\left.\frac{m(\tau)\cdot v^2(\tau)}{2}\right|_{\tau_E}^{\tau}-\frac{1}{2}\int_{\tau_E}^{\tau}v^2(\tau)\frac{\mathrm{d}m}{\mathrm{d}t}\mathrm{d}t\\
&=-m(\tau)\cdot c^2+m(\tau_E)\cdot c^2+\frac{1}{2}\left(m(\tau)\cdot v^2(\tau)\Big|_{\tau_E}^{\tau}+\int_{\tau_E}^{\tau}v^2(\tau)\frac{\mathrm{d}m}{\mathrm{d}t}\mathrm{d}t\right)\\
&=:-m(\tau)\cdot c^2+m(\tau_E)\cdot c^2+E_{nr}(\tau)-E_{nr}(\tau_E).
\end{aligned}$$

Thus, $E_r(\tau)=m(\tau_E)\cdot c^2+E_{nr}(\tau)-E_{nr}(\tau_E)$ so that *the relativistic energy function* Equation (14) *of the particle differs from the non-relativistic one* (I.23) *by only a constant*. This is not astonishing, for integrating directly $\mathrm{d}p/\mathrm{d}\tau\cdot v(\tau)\cdot \boldsymbol{t}(\tau)=(\mathrm{d}m/\mathrm{d}\tau\cdot v(\tau)+m(\tau)\cdot\mathrm{d}v/\mathrm{d}\tau)$ yields

$$W(\tau)-W(\tau_E)=\frac{1}{2}\left(m(\tau)\cdot v^2(\tau)+\int_{\tau_E}^{\tau}v^2(\tau)\frac{\mathrm{d}m}{\mathrm{d}t}\mathrm{d}t\right)-\frac{m_E\cdot v_E^2}{2}=E_{nr}(\tau)-E_{nr}(\tau_E).$$

Thus, *whether the work difference is calculated with the relativistic energy function or with the non-relativistic one makes no difference for* $m(\tau)=m_0'(\tau)/f(\tau)$. With (I.23b) it follows:

$$E_r(\tau)=m(\tau_E)\cdot c^2+(m(\tau)-m(\tau_E))\cdot c^2/k, \tag{16}$$

which nicely shows how the famous SR-relation $E=mc^2$ *generalises in the case of variable mass.*

If we repeat the work calculation for the right-hand side of (I.62), carry out the transition to the integration variable r and resolve the mass equation for the effective potential, we obtain the expression

$$-\int_{r_E}^{r(\tau)}\frac{\mathrm{d}V_{ef}}{\mathrm{d}r}\mathrm{d}r:=-\int_{r_E}^{r(\tau)}\frac{\mathrm{d}V}{\mathrm{d}r}\cdot\frac{m_\infty(r)\cdot m_E(r)}{m_\infty\cdot m_E}\mathrm{d}r=\frac{c^2}{k}\int_{r_E}^{r(\tau)}\frac{\mathrm{d}m}{\mathrm{d}r}\mathrm{d}r=\frac{c^2}{k}\cdot m(r)\Big|_{r_E}^{r(\tau)} \tag{17}$$

so that the law of conservation of energy applies again:

$$\begin{aligned}
&E_r(\tau)+V_{ef}(r(\tau))=E_r(\tau_E)+V_{ef}(r(\tau_E))\\
&\Leftrightarrow E_{nr}(\tau)+V_{ef}(r(\tau))=E_{nr}(\tau_E)+V_{ef}(r(\tau_E)).
\end{aligned}$$

Directly from Equation (17), we get an alternative representation of the effective potential:

$$V_{ef}(r(\tau))=-\frac{c^2}{k}\cdot m(r(\tau))+\text{const.}, \tag{18}$$

with the mass given by (I.64), compare (I.58).

Next, if we now form the scalar product of $(\mathrm{d}m/\mathrm{d}\tau,\mathrm{d}p/\mathrm{d}\tau)$ with $(1,\mathrm{d}x/\mathrm{d}\tau)$ and integrate, we obtain:

$$\int_{\tau_0}^{\tau}\left\langle\begin{pmatrix}\mathrm{d}m/\mathrm{d}\tau\\ \mathrm{d}p/\mathrm{d}\tau\end{pmatrix},\begin{pmatrix}1\\ \mathrm{d}x/\mathrm{d}\tau\end{pmatrix}\right\rangle\mathrm{d}\tau = m(\tau)\cdot c^2\Big|_{\tau_0}^{\tau} - E_r(\tau)+E_r(\tau_0)$$
$$= c^2\int_{\tau_0}^{\tau}\frac{\mathrm{d}m_0'}{\mathrm{d}\tau'}\cdot\left(1-v^2(\tau')/c^2\right)^{1/2}\mathrm{d}\tau'. \tag{19}$$

According to Equation (19)*, the invariant function is the relativistic expression of the lost or won energy. Only for* $\mathrm{d}m_0'/\mathrm{d}t=0$ *does the usual result, namely zero, occur for the scalar product of four-force and four-velocity.*

Other than the expression of work Equation (13), clearly Equation (19) is an invariant function; for with

$$L_u^{-1}=\frac{1}{f(\tau)}\begin{pmatrix}1 & u(\tau)/c^2\\ u(\tau) & 1\end{pmatrix},\ L_u^{-1}\circ L_u=\mathrm{Id}$$

we have

$$\left\langle\begin{pmatrix}\mathrm{d}m/\mathrm{d}\tau\\ \mathrm{d}p/\mathrm{d}\tau\end{pmatrix},\begin{pmatrix}1\\ \mathrm{d}x/\mathrm{d}\tau\end{pmatrix}\right\rangle=\left\langle\begin{pmatrix}\mathrm{d}m/\mathrm{d}\tau\\ \mathrm{d}p/\mathrm{d}\tau\end{pmatrix},\left(L_u^{-1}\circ L_u\right)\begin{pmatrix}1\\ \mathrm{d}x/\mathrm{d}\tau\end{pmatrix}\right\rangle$$
$$=\left\langle L_u\begin{pmatrix}\mathrm{d}m/\mathrm{d}\tau\\ \mathrm{d}p/\mathrm{d}\tau\end{pmatrix},L_u\begin{pmatrix}1\\ \mathrm{d}x/\mathrm{d}\tau\end{pmatrix}\right\rangle$$
$$=:\left\langle\begin{pmatrix}\mathrm{d}m_u'/\mathrm{d}\tau\\ \mathrm{d}p_u'/\mathrm{d}\tau\end{pmatrix},\begin{pmatrix}\mathrm{d}\tau_u'/\mathrm{d}\tau\\ \mathrm{d}x_u'/\mathrm{d}\tau\end{pmatrix}\right\rangle.$$

In particular, if $u(\tau)=\mathrm{d}x/\mathrm{d}\tau=:v(\tau)$ and therefore $m_u'=m_0'$, $\tau_u'=\tau'$, $x_u'=x'$, then we have:

$$L_v\begin{pmatrix}1\\ \mathrm{d}x/\mathrm{d}\tau\end{pmatrix}=\begin{pmatrix}\dfrac{\mathrm{d}\tau'}{\mathrm{d}\tau}=f(\tau)\\ \dfrac{\mathrm{d}x'}{\mathrm{d}\tau}=0\end{pmatrix},\ L_v\begin{pmatrix}\mathrm{d}m/\mathrm{d}\tau\\ \mathrm{d}p/\mathrm{d}\tau\end{pmatrix}=\frac{1}{f(\tau)}\begin{pmatrix}\dfrac{\mathrm{d}m}{\mathrm{d}\tau}-\dfrac{v(\tau)}{c^2}\cdot\dfrac{\mathrm{d}p}{\mathrm{d}\tau}\\ -v(\tau)\cdot\dfrac{\mathrm{d}m}{\mathrm{d}\tau}+\dfrac{\mathrm{d}p}{\mathrm{d}\tau}\end{pmatrix}, \tag{20}$$

With Equation (15) follows

$$\frac{\mathrm{d}m}{\mathrm{d}\tau}-\frac{v(\tau)}{c^2}\cdot\frac{\mathrm{d}p}{\mathrm{d}\tau}=\frac{\mathrm{d}m}{\mathrm{d}\tau}\cdot\left(1-\frac{v^2(\tau)}{c^2}\right)-\frac{m(\tau)}{2c^2}\cdot\frac{\mathrm{d}v^2}{\mathrm{d}\tau}=f(\tau)\cdot\frac{\mathrm{d}m_0'}{\mathrm{d}\tau}.$$

This inserted into Equation (20) gives

$$L_v\begin{pmatrix}\mathrm{d}m/\mathrm{d}\tau\\ \mathrm{d}p/\mathrm{d}\tau\end{pmatrix}=\begin{pmatrix}\dfrac{\mathrm{d}m_0'}{\mathrm{d}\tau}\\ \dfrac{m(\tau)}{f(\tau)}\dfrac{\mathrm{d}v}{\mathrm{d}\tau}\end{pmatrix}=\begin{pmatrix}\dfrac{\mathrm{d}m_0'}{\mathrm{d}\tau}\\ \dfrac{\mathrm{d}p'}{\mathrm{d}\tau}\end{pmatrix}, \tag{21}$$

and finally Equation (19) follows again. Thus, during the transformation into the rest frame of a particle, the time derivative of the mass changes into that of the rest mass and the acceleration force exerted on the particle changes into a Minkowski force (see [2] section 4.6), which, since $\mathrm{d}x'/\mathrm{d}\tau=0$, represents a weight force G' acting in O' and pointing towards the centre of gravity according to Figure 1. *Thus, if* $G(\tau)$ *is the weight in O equivalent to the force* $m\cdot\mathrm{d}v/\mathrm{d}\tau$*, it turns into* $G'(\tau)=G(\tau)/f(\tau)$*. In other words, the equivalence principle of GR does not hold in ENG in its strict form*. Below (see Equation (26)) it is proven that in ENG the equality of inertial and heavy mass, found experimentally with a torsion balance on the rotating Earth (see [6]), is valid for circular motion, however.

Remark 2. Equation (8) and Equation (11) represent Lorentz transformations in differential form, which convert the trajectories of light or massive particles falling in the gravitational field (with or without non-zero initial velocity) into those of the reference frame O' moving in a straight line accelerated relative to the reference frame O at rest. *Thus, in addition to the time-dependency of the Lorentz transformation, the four-vectors* $\left(m(\tau), p(\tau)\right)$ *and* $\left(\mathrm{d}m/\mathrm{d}\tau, \mathrm{d}p/\mathrm{d}\tau\right)$ *present the basic extension of SR to gravity.* Again, the restriction to the gravitational acceleration of the reference frame is not essential.

Example 3 Clock comparison with gravity. A clock falls with $v_0 = 0$ at time $t_0 = 0$ from the Earth (whose gravity is neglected) towards the Sun. We ask for the time which the falling clock indicates for an observer on Earth. If we place ourselves in the rest frame of the falling particle, *i.e.* set $x'(\tau) - x'_E = 0$ in Equation (8), then we get $\mathrm{d}x/\mathrm{d}\tau = v(\tau)$, $t_0 = \tau_0 = \tau'_0$:[5]

$$\frac{\mathrm{d}\tau'}{\mathrm{d}\tau} = \frac{1 - v^2(\tau)/c^2}{\left(1 - v^2(\tau)/c^2\right)^{1/2}} = \left(1 - v^2(\tau)/c^2\right)^{1/2} \Rightarrow \tau' = \int_0^\tau \left(1 - v^2(\tau)/c^2\right)^{1/2} \mathrm{d}\tau < \tau.$$

With (I.52), the integration results in the proper time $\tau'(t) = t'(t)$:

$$t'(t) = -\int_{r_E}^{r(t)} \left(1 - v^2(r)/c^2\right)^{1/2} \cdot \frac{m_\infty}{m_\infty(r)} \cdot \frac{m_E}{m(r)} \cdot \frac{\mathrm{d}r}{v(r)}.$$

The difference $\tau'(t) - \tau(t)$ is calculated with (I.28) and $v_0 = 0$:

$$t'(t) - \tau(t) = -\frac{k^{1/2}}{c} \int_{r_E}^{r(t)} \left(\sqrt{1 - \frac{1}{k}\left(1 - \frac{m_E^2}{m^2(r)}\right)} - 1 \right) \cdot \frac{m_\infty}{m_\infty(r)} \cdot \frac{m_E}{m(r)} \cdot \frac{\mathrm{d}r}{\sqrt{1 - \frac{m_E^2}{m^2(r)}}}. \quad (22)$$

$\tau(t)$ is the runtime that the observer on Earth attributes to the falling clock without taking relativity into account. From Equation (22) follows that the relativistic effect is much more pronounced in weak gravitational fields than the mass effect (the relative change of mass $\left(m(t) - m_E\right)/m_E$) or the runtime effect (the relative change of runtime $\left(t - \tau(t)\right)/t$). Clearly, in strong gravitational fields, this is expected to change.

2.3. Extension to Other Reference Frames

In order that the orbital curve for O' can be calculated with the formulae (I.64-66), O' must be equipped with a mass gathered at the origin. Let such an orbital curve starting from the origin of O be characterised by the orbital velocity $u(\tau)$ and the orbital angle $\delta(\tau)$ and corresponding initial values u_0, δ_0. Let the origin of the coordinate system O', whose axes are chosen parallel to those of O, be attached to the trajectory point $\boldsymbol{q}(\tau)$ (see **Figure 3**). In the first step, we rotate O by the angle $\delta(\tau)$ and move it by the distance $-\int u(\tau) \cdot \sin\left(\theta(\tau) - \delta(\tau)\right) \mathrm{d}\tau$

[5]Compare the simplicity and clarity of this derivation with the derivation from GR, e.g. pp. 182-184 in [5].

along the Y-axis:

$$\begin{pmatrix} \mathrm{d}\tau \\ \mathrm{d}X \\ \mathrm{d}Y \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta(\tau) & \sin\delta(\tau) \\ 0 & -\sin\delta(\tau) & \cos\delta(\tau) \end{pmatrix} \begin{pmatrix} \mathrm{d}\tau \\ \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ u(\tau)\cdot\sin\left(\theta(\tau)-\delta(\tau)\right) \end{pmatrix} \mathrm{d}\tau.$$

If (X', Y') designate the coordinates of O' relative to (X, Y), Equation (7) applies

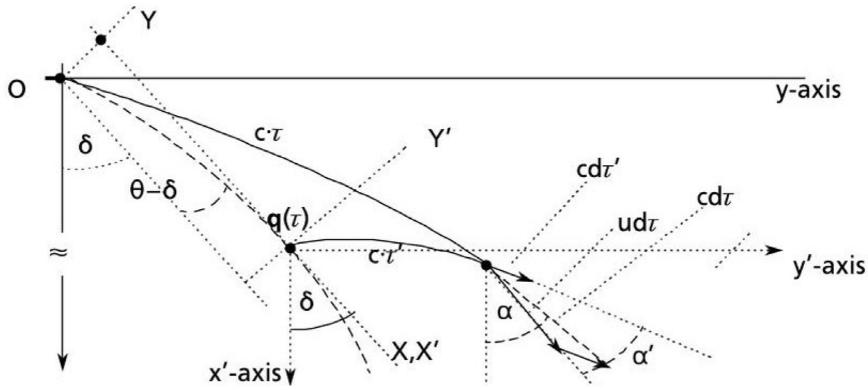

Figure 3. Arbitrarily falling light curves in O and O'.

$$\begin{pmatrix} \mathrm{d}\tau' \\ \mathrm{d}X' \\ \mathrm{d}Y' \end{pmatrix} = \frac{1}{f(\tau)} \begin{pmatrix} 1 & -u(\tau)/c^2 & 0 \\ -u(\tau) & 1 & 0 \\ 0 & 0 & f(\tau) \end{pmatrix} \begin{pmatrix} \mathrm{d}\tau \\ \mathrm{d}X \\ \mathrm{d}Y \end{pmatrix}.$$

In the second step, the coordinates (X, Y) and (X', Y') are converted into the coordinates (x, y) and (x', y') by an inverse translation and rotation:

$$\begin{pmatrix} \mathrm{d}\tau' \\ \mathrm{d}x' \\ \mathrm{d}y' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta(\tau) & -\sin\delta(\tau) \\ 0 & \sin\delta(\tau) & \cos\delta(\tau) \end{pmatrix} \left[\begin{pmatrix} \mathrm{d}\tau' \\ \mathrm{d}X' \\ \mathrm{d}Y' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u(\tau)\cdot\sin\left(\theta(\tau)-\delta(\tau)\right) \end{pmatrix} \mathrm{d}\tau \right].$$

The translations cancel out, the rotations remain. Thus, we have (omitting the variable τ):

$$\begin{pmatrix} \mathrm{d}\tau' \\ \mathrm{d}x' \\ \mathrm{d}y' \end{pmatrix} = \frac{1}{f(\tau)} \begin{pmatrix} 1 & -\frac{u}{c^2}\cdot\cos\delta & -\frac{u}{c^2}\cdot\sin\delta \\ -u\cdot\cos\delta & \begin{matrix}\cos^2\delta \\ +f\cdot\sin^2\delta\end{matrix} & \begin{matrix}\cos\delta\cdot\sin\delta \\ -f\cdot\sin\delta\cdot\cos\delta\end{matrix} \\ -u\cdot\sin\delta & \begin{matrix}\sin\delta\cdot\cos\delta \\ -f\cdot\cos\delta\cdot\sin\delta\end{matrix} & \begin{matrix}\sin^2\delta \\ +f\cdot\cos^2\delta\end{matrix} \end{pmatrix} \begin{pmatrix} \mathrm{d}\tau \\ \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} =: L_{u,\delta} \begin{pmatrix} \mathrm{d}\tau \\ \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}. \quad (23)$$

Again: $c^2\mathrm{d}\tau^2 - \mathrm{d}x^2 - \mathrm{d}y^2 = c^2\mathrm{d}\tau'^2 - \mathrm{d}x'^2 - \mathrm{d}y'^2$, an equation which is valid for light and massive particles. Thus, the Cartesian reference frame O' can be sent from any chosen location $\boldsymbol{q}$ onto any orbit, which is given by the orbital velocity $u(\tau)$ and the orbital angle $\delta(\tau)$ in O. At the same time, a particle may start from $\boldsymbol{q}_0$ whose trajectory is given by its orbital velocity $v(\tau)$ and the angle of inclination $\alpha(\tau)$ and their initial values. With respect to the stationary reference frame O, the four-vectors $(\tau, \boldsymbol{q})$ and $(m(\tau), \boldsymbol{p}(\tau))$ measured in O are transformed into the same four-vectors of O', and vice versa. The only restriction of this work is that no forces other than gravity act on the particle, otherwise additional force potentials coupling to gravity must be introduced. The extended Lorentz transformations are therefore prepared for additional types of force, which is not included in this work.

If we understand O' as the rest frame of the particle, the transformation into the rest frame reads with $\alpha = \delta$ and $u(\tau)\cdot\cos\alpha(\tau)\cdot \mathrm{d}\tau = \mathrm{d}x$, $u(\tau)\cdot\sin\alpha(\tau)\cdot\mathrm{d}\tau = \mathrm{d}y$

$$\begin{pmatrix} \mathrm{d}\tau' \\ \mathrm{d}x' \\ \mathrm{d}y' \end{pmatrix} = \frac{1}{f(\tau)} \begin{pmatrix} 1 & -\frac{u}{c^2}\cdot\cos\alpha & -\frac{u}{c^2}\cdot\sin\alpha \\ -u\cdot\cos\alpha & \begin{matrix}\cos^2\alpha \\ +f\cdot\sin^2\alpha\end{matrix} & \begin{matrix}\cos\alpha\cdot\sin\alpha \\ -f\cdot\sin\alpha\cdot\cos\alpha\end{matrix} \\ -u\cdot\sin\alpha & \begin{matrix}\sin\alpha\cdot\cos\alpha \\ -f\cdot\cos\alpha\cdot\sin\alpha\end{matrix} & \begin{matrix}\sin^2\alpha \\ +f\cdot\cos^2\alpha\end{matrix} \end{pmatrix} \begin{pmatrix} 1 \\ u\cdot\cos\alpha \\ u\cdot\sin\alpha \end{pmatrix} \mathrm{d}\tau = \begin{pmatrix} f(\tau)\cdot\mathrm{d}\tau \\ 0 \\ 0 \end{pmatrix}.$$

Also consider the transformation of $(\mathrm{d}m/\mathrm{d}\tau, \mathrm{d}p/\mathrm{d}\tau)$ into the rest frame. Setting

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} &= \left(\frac{\mathrm{d}m}{\mathrm{d}\tau}u(\tau) + m(\tau)\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)\cdot\boldsymbol{t}(\tau) + m(\tau)\cdot u(\tau)\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\cdot\boldsymbol{n}(\tau) \\ &=: A\cdot\boldsymbol{t}(\tau) + B\cdot\boldsymbol{n}(\tau) = \left(A\cdot\cos\alpha(\tau) - B\cdot\sin\alpha(\tau), A\cdot\sin\alpha(\tau) + B\cdot\cos\alpha\right) \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{1}{f(\tau)} \begin{pmatrix} 1 & -\frac{u}{c^2}\cdot\cos\alpha & -\frac{u}{c^2}\cdot\sin\alpha \\ -u\cdot\cos\alpha & \begin{matrix}\cos^2\alpha \\ +f\cdot\sin^2\alpha\end{matrix} & \begin{matrix}\cos\alpha\cdot\sin\alpha \\ -f\cdot\sin\alpha\cdot\cos\alpha\end{matrix} \\ -u\cdot\sin\alpha & \begin{matrix}\sin\alpha\cdot\cos\alpha \\ -f\cdot\cos\alpha\cdot\sin\alpha\end{matrix} & \begin{matrix}\sin^2\alpha \\ +f\cdot\cos^2\alpha\end{matrix} \end{pmatrix} \begin{pmatrix} \mathrm{d}m/\mathrm{d}\tau \\ A\cdot\cos\alpha - B\cdot\sin\alpha \\ A\cdot\sin\alpha(\tau) + B\cdot\cos\alpha \end{pmatrix} \\ &= \frac{1}{f} \begin{pmatrix} \mathrm{d}m/\mathrm{d}\tau - A\cdot\frac{u}{c^2} \\ -u\cdot\cos\alpha\cdot\mathrm{d}m/\mathrm{d}\tau + A\cdot\cos\alpha - B\cdot f\cdot\sin\alpha \\ -u\cdot\sin\alpha\cdot\mathrm{d}m/\mathrm{d}\tau + A\cdot\sin\alpha + B\cdot f\cdot\cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{d}m_0'/\mathrm{d}\tau \\ \frac{m}{f}\cdot\frac{\mathrm{d}u}{\mathrm{d}\tau}\cdot\cos\alpha - m\cdot u\cdot\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\cdot\sin\alpha \\ \frac{m}{f}\cdot\frac{\mathrm{d}u}{\mathrm{d}\tau}\cdot\sin\alpha + m\cdot u\cdot\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\cdot\cos\alpha \end{pmatrix}. \end{aligned} \tag{24}$$

Example 4 Clock and scale comparison, the general case. At location $\boldsymbol{q}_0$ the particle enters the gravitational field at the angle α_0 with the initial velocity v_0 and the mass $m_0 = m'_0 / f_0$, with these initial values it falls in the gravitational field. The same applies to the reference frame O', which falls from $\boldsymbol{q}_0$ with the relative velocity $u(\tau)$ and the orbital angle $\delta(\tau)$ in the gravitational field. We now consider the particle trajectory given by $\mathrm{d}x/\mathrm{d}\tau = v(\tau)\cdot\cos\alpha(\tau)$, $\mathrm{d}y/\mathrm{d}\tau = v(\tau)\cdot\sin\alpha(\tau)$. With Equation (23) follows:

$$\frac{\mathrm{d}\tau'}{\mathrm{d}\tau} = \frac{1}{f(\tau)}\cdot\left(1 - \frac{u(\tau)v(\tau)}{c^2}\cdot\cos\big(\delta(\tau)-\alpha(\tau)\big)\right),$$

$$\frac{\mathrm{d}x'}{\mathrm{d}\tau} = \frac{1}{f(\tau)}\cdot\Big(-u\cdot\cos\delta + v(\tau)\cdot\big(\big(\cos^2\delta + f\cdot\sin^2\delta\big)\cdot\cos\alpha + \cos\delta\cdot\sin\delta\cdot(1-f)\cdot\sin\alpha\big)\Big),$$

$$\frac{\mathrm{d}y'}{\mathrm{d}\tau} = \frac{1}{f(\tau)}\cdot\Big(-u\cdot\sin\delta + v(\tau)\cdot\big(\big(f\cdot\cos^2\delta + \sin^2\delta\big)\cdot\sin\alpha + \cos\delta\cdot\sin\delta\cdot(1-f)\cdot\cos\alpha\big)\Big).$$

Or in integral form:

$$\tau'(\tau) = \int_0^\tau \frac{1 - u(\tau)\cdot v(\tau)\cdot\cos\big(\delta(\tau)-\alpha(\tau)\big)/c^2}{f(t)}\mathrm{d}\tau,$$

$$x'(\tau) = \int_0^\tau \frac{-u\cdot\cos\delta + v(\tau)\cdot\big(\big(\cos^2\delta + f\cdot\sin^2\delta\big)\cdot\cos\alpha + \cos\delta\cdot\sin\delta\cdot(1-f)\cdot\sin\alpha\big)}{f(t)}\mathrm{d}\tau,$$

$$y'(\tau) = \int_0^\tau \frac{-u\cdot\sin\delta + v(\tau)\cdot\big(\big(f\cdot\cos^2\delta + \sin^2\delta\big)\cdot\sin\alpha + \cos\delta\cdot\sin\delta\cdot(1-f)\cdot\cos\alpha\big)}{f(t)}\mathrm{d}\tau.$$

Let $\tau = \tau_{1m}$ denote the time, so that $x'(\tau_{1m}) = 1\,\mathrm{m}$, then a comparison of the sizes of the scales can already be carried out, because without $f(\tau)$, there is simply the Galilean transformation. The metre unit of O' thus appears shortened in O in the usual way. And if we transform into the rest frame, the following applies:

$$\tau'(t) = -\int_0^t \left(1 - v^2(t)/c^2\right)^{1/2}\frac{\mathrm{d}\tau}{\mathrm{d}t}\mathrm{d}t.$$

Since the speed increases in the case of falling and $\mathrm{d}\tau/\mathrm{d}t$ becomes increasingly smaller than 1, the gravitational runtime effect amplifies the relativistic effect. Only with strong gravitational fields could both effects become equally large.

2.4. Rotating Reference Frame

In gravity-free space, the reference frame O' performs a clockwise orbit around the point $(r,0)$. The radius is r and the circular velocity is ω, so the orbital velocity is $u = r\cdot\omega$. O' starts its orbit at the time $t_0 = 0$ at the point $(0,0)$, so that the orbital angle of the X'-axis relative to the x-axis is $\delta_0 = \pi/2$ (see **Figure 4**). Then $\delta(t) = \delta_0 - \omega\cdot t$. We are interested in *the trajectory in O' of the origin of O*. With $(\mathrm{d}x, \mathrm{d}y) = (0,0)$ and Equation (23), we have

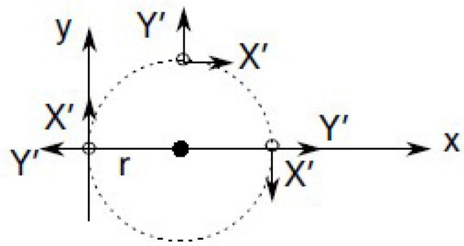

Figure 4. Rotating reference system O' in gravity-free space.

$$\begin{pmatrix} \mathrm{d}t' \\ \mathrm{d}x' \\ \mathrm{d}y' \end{pmatrix} = \frac{1}{f(u)} L_{u,\delta} \begin{pmatrix} \mathrm{d}t \\ 0 \\ 0 \end{pmatrix} = \frac{\mathrm{d}t}{f(u)} \begin{pmatrix} 1 \\ -u \cdot \sin \omega \cdot t \\ -u \cdot \cos \omega \cdot t \end{pmatrix},$$

and integration gives:

$$\begin{pmatrix} t' \\ x'(t) \\ y'(t) \end{pmatrix} = \frac{1}{f(u)} \begin{pmatrix} t \\ r \cdot \cos \omega \cdot t \\ -r \cdot \sin \omega \cdot t \end{pmatrix} + \begin{pmatrix} 0 \\ C_1 \\ C_2 \end{pmatrix}. \tag{25}$$

Now, $x'(0) = 0 = r/f + C_1$ gives $C_1 = -r/f$, and $y'(0) = 0$ gives $C_2 = 0$. Let us return to the coordinates (X', Y') of **Figure 3** by

$$\begin{pmatrix} t' \\ X'(t) \\ Y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \delta(t) & \sin \delta(t) \\ 0 & -\sin \delta(t) & \cos \delta(t) \end{pmatrix} \begin{pmatrix} t' \\ x'(t) \\ y'(t) \end{pmatrix}$$

yielding:

$(\omega \cdot t, \delta(t))$	$(0, \pi/2)$	$(\pi/2, 0)$	$(\pi, -\pi/2)$	$(3\pi/2, -\pi)$
$X'(t)$	0	$-r/f$	0	r/f
$Y'(t)$	0	$-r/f$	$-2r/f$	$-r/f$

From the point of view of O, O' therefore completes a perfect circle with radius r/f. *And for a single revolution in O of* T *seconds, O' needs the runtime* $T' = T/f$. *Because* $2\pi r/u = T$, $2\pi(r/f)/u = T'$, *results that do not call Euclidean geometry into question*. Analogously for a planet that follows an exact orbit around its centre of gravity. Then $\mathrm{d}m/\mathrm{d}\tau = 0$ (see I, Example 8) and with

$$g(r) = \frac{1}{1 + g \cdot M_0/(c^2 \cdot r)}, \; u = \sqrt{\frac{g \cdot M_0}{r \cdot g(r)}}, \; \omega = \frac{u}{r}, \; \tau = \frac{1}{g(r)} \cdot t, \; f = (1 - u^2/c^2)^{1/2}$$

and Equation (25) we obtain for the distance $d'(\tau)$ of the origin of O circulating around the origin of O' and the corresponding runtime:

$$d'(\tau) = \frac{(2 - 2\cos \omega \cdot \tau)^{1/2}}{f} \cdot r, \; \tau' = \frac{\tau}{(1 - u^2/c^2)^{1/2}}.$$

Again a perfect circle results.

It is also interesting to see what form the four-vector $(\mathrm{d}m/\mathrm{d}\tau, \mathrm{d}\boldsymbol{p}/\mathrm{d}\tau)$ takes with the transformation into the orbiting rest frame O'. With $\mathrm{d}u/\mathrm{d}\tau = 0$,

$\alpha(\tau)=\delta(\tau)=\delta_0-\omega\tau$ and Equation (24) follows

$$\begin{pmatrix}\dfrac{\mathrm{d}m}{\mathrm{d}\tau}\\ \dfrac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}\end{pmatrix}\to\begin{pmatrix}\mathrm{d}m_0'/\mathrm{d}\tau\\ -m\cdot u\cdot\dfrac{\mathrm{d}\alpha}{\mathrm{d}\tau}\cdot\sin\alpha\\ m\cdot u\cdot\dfrac{\mathrm{d}\alpha}{\mathrm{d}\tau}\cdot\cos\alpha\end{pmatrix}=-m\cdot r\cdot\omega^2\begin{pmatrix}0\\ -\sin\alpha\\ \cos\alpha\end{pmatrix}=-m\cdot r\cdot\omega^2\begin{pmatrix}0\\ \boldsymbol{n}\end{pmatrix}. \tag{26}$$

Because $g\cdot M\cdot m/r^2=m\cdot r\omega^2$, *Eötvös' theorem of the equality of inertial and heavy mass, found experimentally with a torsion balance on the rotating Earth, is proven to apply in the relativistic setting in the case of circularly accelerated systems.*

Remark 3. Because in the radial case with $\mathrm{d}\alpha/\mathrm{d}\tau=0$ the Minkowski force appears in Equation (24) (see comment following Equation (21)), this also means that in ENG *the equivalence principle of GR can only apply strictly in a circle, or must be generalised to cover* Equation (24).

2.5. Summary

Just as the special theory of relativity is based on Newtonian mechanics, an extension of SR can be based on ENG. The extension concerns reference frames falling in the gravitational field or, in the general case, accelerated reference frames and forces the transition to coordinate differentials and their mapping to those of an accelerated reference frame. It is *the constant speed of light fulfilled by ENG* that allows a Lorentz transformation to be derived in differential form for linearly and circularly accelerated reference frames. Instead of $(E/c^2,p)$, in the case of variable mass $(m(\tau),\boldsymbol{p}(\tau))$ proves to be the 4-vector, which leads to the 4-vector $(\mathrm{d}m/\mathrm{d}\tau,\mathrm{d}\boldsymbol{p}/\mathrm{d}\tau)$ uniting the force and mass equation of ENG. In addition to the differential form, the extension of SR to accelerated reference frames consists in the fact that

- the rest mass of a body is an invariant function,
- the energy of a body amounts to $(m(\tau)-m_0)\cdot c^2/k+m_0c^2$,
- the scalar product of the 4-vectors of force and velocity does not vanish, but is an invariant function representing the lost or won energy.

Of particular interest in this work are circularly accelerated reference frames, be they those in gravity-free space or those moving around a gravitational centre, because with the extended SR the question must be decidable as to what form a rotating disc takes from the point of view of an observer at rest. The result is that radius and periphery increase by the same factor. Finally, the transformation into the rest frame of the rotating disc or the rotating body shows that the equality of inertial and heavy mass is maintained on the circle, but not on deviating orbits, which is why the equivalence principle of GR cannot be generally valid in ENG, but must be corrected relativistically.

In weak gravitational fields such as that of the solar system, the relativistic effect dominates the mass effect, *i.e.* the results obtained with ENG and extended SR can hardly be distinguished from those obtained with SR and classical mechanics that only recognise particles of constant rest mass. Only in strong gravitational fields can this become different, when relative mass changes significantly determine the energy of a body.

As mentioned, the complete elaboration of SR into a theory of relativity generalizing SR to arbitrarily accelerated reference systems was not the aim of this work[6]. The intention, however, is to draw attention to the fact that the experimental evidence of ENG invites us to rethink a general theory of relativity featuring:

- the kinematics can be developed for arbitrary reference frames independently of any dynamics,
- the theory can be applied to gravitation as well as to other forces.

3. The Schrödinger Equation with Variable Mass in the Gravitational Field

If matter is to be treated field-theoretically, the gravitational physics developed in Part I must be reformulated by means of Lagrange formalism. This requires recourse to d'Alembert's principle and an investigation of the consequences of variable mass for the establishment of the Lagrange function and Hamiltonian of a light or massive particle. The embedding of the Hamilton-Jacobi and continuity equations in a partial differential equation of the second order must also be worked out anew in order to obtain a wave equation for massive particles in the gravitational field (see [7] Section 3). We will show that to the classical form of the Schrödinger equation terms are added which reflect the loss or gain of mass in the gravitational field. This does not yet represent a quantisation of gravity, but is a test of the idea that heavy matter can be represented as a field. For other approaches to a gravitative Schrödinger equation (see [8]).

In this section, the Lagrange formalism of mechanics is derived for bodies with variable mass, a transfer to fields is then obvious. As in the foregoing section, the aim is not systematic completeness, but to take the direct route to the gravitational Schrödinger equation. For this reason, the derivation of Lagrange functions for photon and particle is performed in non-relativistic fashion.

We use V to note V_{ef}. We write $\partial/\partial \boldsymbol{q}$ or $\partial/\partial \boldsymbol{v}$ as a gradient after $\boldsymbol{q}$ or $\boldsymbol{v}$, abbreviate $\boldsymbol{q}(\tau) = r(\tau)\cdot \boldsymbol{e}_r(\tau)$, $\boldsymbol{v}(\tau) = \mathrm{d}\boldsymbol{q}/\mathrm{d}\tau = v(\tau)\cdot \boldsymbol{t}(\tau)$, $\mathrm{d}\boldsymbol{v}/\mathrm{d}\tau = \mathrm{d}v/\mathrm{d}\tau \cdot \boldsymbol{t}(\tau) + v(\tau)\cdot \mathrm{d}\alpha/\mathrm{d}\tau \cdot \boldsymbol{n}(\tau)$ and understand the mass as a function of the location $\boldsymbol{q}$ and the time τ with the restriction $\partial m/\partial \tau = 0$; thus: $m(\tau,\boldsymbol{q}) = m(\boldsymbol{q}(\tau))$, $\partial m/\partial \boldsymbol{q}\cdot \boldsymbol{v}(\tau) = \mathrm{d}m/\mathrm{d}r\cdot \boldsymbol{e}_r(\tau)\cdot \boldsymbol{v}(\tau)$. $m(\boldsymbol{q}(\tau)) =: m(\tau)$ means the mass of a photon or a particle. In the latter case, we had set up the additional equation (see I.24 or I.62 and I.64):

$$\frac{\mathrm{d}m}{\mathrm{d}\tau} = \frac{\partial m}{\partial \boldsymbol{q}}\cdot \boldsymbol{v}(\tau) = -\frac{k}{c^2}\frac{\partial V}{\partial \boldsymbol{q}}\cdot \boldsymbol{v}(\tau) \Rightarrow \frac{\partial m}{\partial \boldsymbol{q}} = -\frac{k}{c^2}\frac{\partial V}{\partial \boldsymbol{q}}. \tag{A}$$

Note that we can resolve the mass function according to the velocity independently of the particular choice of central potential (see I.28 and text following Equation I.64)

[6]Such a compendium would have to cover foundations and applications similar to the first 7 sections of [7]; not to mention a corresponding group theoretical extension.

$$v\left(\boldsymbol{q}(\tau)\right)=\sqrt{\frac{c^2}{k}-\left(\frac{c^2}{k}-v_0^2\right)\cdot\frac{m_0^2}{m^2\left(\boldsymbol{q}(\tau)\right)}}\Leftrightarrow m\left(\boldsymbol{q}(\tau)\right):=m_0\cdot\sqrt{\frac{1-k\cdot v_0^2/c^2}{1-k\cdot v^2\left(\boldsymbol{q}(\tau)\right)/c^2}},\quad (\mathrm{B})$$

so that it can be understood as a function of velocity. If we insert this function into (I.62), we obtain the Equations (I.26) for orbital velocity v and angle α without the mass Equation (A) or I.24. Thus, if $\mathrm{d}m/\mathrm{d}\tau$, $\partial m/\partial\boldsymbol{q}$ or $m(\tau)$ will appear in a Lagrangian, Hamiltonian or Schrödinger equation, it can be computed by means of A or B.

3.1. D'Alembert's Principle Revisited

We use $\boldsymbol{X}$ to denote the force acting on a particle and $\delta\boldsymbol{q}$ to denote a virtual displacement; then D'Alembert's principle for a particle with constant mass m reads

$$\left(\boldsymbol{X}\left(\boldsymbol{q}(\tau)\right)-m\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\tau}\right)\cdot\delta\boldsymbol{q}(\tau)=0.$$

For particles of variable mass we must apply

$$\left(\boldsymbol{X}\left(\boldsymbol{q}(\tau)\right)-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(m(\tau)\cdot\boldsymbol{v}(\tau)\right)\right)\cdot\delta\boldsymbol{q}(\tau)=0. \tag{27}$$

For the displacement $\delta\boldsymbol{q}(\tau):=\boldsymbol{q}'(\tau)-\boldsymbol{q}(\tau)$, it is assumed that it disappears at the start or end point of the particle trajectory at time τ_0 or τ_1 respectively. Applied to Equation (27) and with

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\delta\boldsymbol{q}=\frac{\mathrm{d}\boldsymbol{q}'}{\mathrm{d}\tau}-\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\tau}=\delta\left(\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\tau}\right)=\delta(\boldsymbol{v}),$$

we obtain the generalised Lagrange central equation

$$\begin{aligned}&\boldsymbol{X}(\boldsymbol{q})\cdot\delta\boldsymbol{q}-m(\tau)\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\tau}\cdot\delta\boldsymbol{q}-\boldsymbol{v}\cdot\frac{\mathrm{d}m}{\mathrm{d}\tau}\cdot\delta\boldsymbol{q}\\&=\boldsymbol{X}(\boldsymbol{q})\cdot\delta\boldsymbol{q}-m(\tau)\cdot\left(\frac{\mathrm{d}}{\mathrm{d}\tau}(\boldsymbol{v}\cdot\delta\boldsymbol{q})-\delta\left(\frac{v^2}{2}\right)\right)-\boldsymbol{v}\cdot\frac{\mathrm{d}m}{\mathrm{d}\tau}\cdot\delta\boldsymbol{q}=0\\&\Rightarrow\boldsymbol{X}(\boldsymbol{q})\cdot\delta\boldsymbol{q}+m(\tau)\cdot\delta\left(\frac{v^2}{2}\right)-\boldsymbol{v}(\tau)\cdot\frac{\mathrm{d}m}{\mathrm{d}\tau}\cdot\delta\boldsymbol{q}=m(\tau)\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}(\boldsymbol{v}\cdot\delta\boldsymbol{q}).\end{aligned}$$

If we integrate over time and take into account the disappearance of the variation at the boundaries, we obtain for the right-hand side of the last line

$$\begin{aligned}\int_{\tau_0}^{\tau_1}m(\tau)\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\boldsymbol{v}(\tau)\cdot\delta\boldsymbol{q}(\tau)\right)\mathrm{d}\tau&=\left(m\cdot\boldsymbol{v}\cdot\delta\boldsymbol{q}\right)\Big|_{\tau_0}^{\tau_1}-\int_{\tau_0}^{\tau_1}\boldsymbol{v}(\tau)\cdot\delta\boldsymbol{q}(\tau)\frac{\mathrm{d}m}{\mathrm{d}\tau}\mathrm{d}\tau\\&=-\int_{\tau_0}^{\tau_1}\boldsymbol{v}(\tau)\cdot\delta\boldsymbol{q}(\tau)\frac{\mathrm{d}m}{\mathrm{d}\tau}\mathrm{d}\tau\\&\Rightarrow\int_{\tau_0}^{\tau_1}\boldsymbol{X}(\boldsymbol{q})\cdot\delta\boldsymbol{q}+m(\tau)\cdot\delta\left(\frac{v^2}{2}\right)=0,\end{aligned}$$

so that with Newton's gravitational force $\left(\boldsymbol{X}\cdot\delta\boldsymbol{q}=-\partial V/\partial\boldsymbol{q}\cdot\delta\boldsymbol{q}=-\delta V\right)$ the result is

$$\int_{\tau_0}^{\tau_1}\left(-\delta V+\frac{1}{2}\cdot m(\tau)\cdot\delta\left(v^2\right)\right)\mathrm{d}\tau=0. \tag{28}$$

In the case of the photon $\left(\boldsymbol{v}=c\cdot\boldsymbol{t}(\tau),m(\tau)=m\left(\boldsymbol{q}(\tau)\right)\right)$ we get

$$\frac{1}{2}\cdot m(\tau)\cdot\delta\left(v^2\right)=\delta\left(\frac{1}{2}\cdot m(\tau)\cdot v^2\right)-\frac{c^2}{2}\cdot\frac{\mathrm{d}m}{\mathrm{d}\boldsymbol{q}}\cdot\delta\boldsymbol{q}=\delta\left(\frac{1}{2}\cdot m(\tau)\cdot v^2\right)+\frac{1}{2}\cdot\delta V.$$

Thus Equation (28) takes on the form

$$\delta\int_{\tau_0}^{\tau_1}\left(-\frac{V(\boldsymbol{q}(\tau))}{2}+\frac{m(\tau)\cdot v^2(\tau)}{2}\right)\mathrm{d}\tau=0. \tag{29}$$

For massive particles applies with Equation (B):

$$m(\tau)\cdot\delta\left(\frac{v^2}{2}\right)=-\left(1-k\cdot v_0^2/c^2\right)\cdot\frac{c^2\cdot m_0}{k}\cdot\delta\sqrt{\frac{1-k\cdot v^2(\tau)/c^2}{1-k\cdot v_0^2/c^2}}$$

so that Equation (28) can be written in the form

$$\delta\int_{\tau_0}^{\tau_1}\left(-V-\zeta_0\cdot\left(1-k\cdot v^2/c^2\right)^{1/2}\right)\mathrm{d}\tau=0,\ \text{mit}\ \zeta_0:=\left(1-k\cdot v_0^2/c^2\right)^{1/2}\cdot c^2\cdot m_0/k. \tag{30}$$

Note that the classical expression $\delta\int\left(-V(q(\tau))+1/2\cdot m_0\cdot v^2(\tau)\right)\mathrm{d}\tau$ is obtained for small v/c.

From Equation (29) and Equation (30), the Lagrange functions for the realisation of the variation can be read directly.

3.2. Lagrange and Hamilton Function of the Photon

We first consider radially emitted light ($\alpha=0^\circ$ or 180°). With Equation (29), the Lagrange function of the radial photon reads:

$$L_0\left(\boldsymbol{q}(\tau),c\cdot\boldsymbol{t}(\tau)\right)=\frac{1}{2}\cdot\left(c^2\cdot\boldsymbol{t}^2(\tau)\cdot m(\tau)-V(\boldsymbol{q}(\tau))\right),$$

and we obtain the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L_0}{\partial\boldsymbol{v}}=\frac{\mathrm{d}}{\mathrm{d}\tau}(c\cdot\boldsymbol{t}\cdot m)(\tau)=m(\tau)\cdot c\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}\tau}+c\boldsymbol{t}(\tau)\cdot\frac{\mathrm{d}m}{\mathrm{d}\tau}=\frac{\partial L_0}{\partial\boldsymbol{q}}=\frac{1}{2}\cdot\left(c^2\frac{\partial m}{\partial\boldsymbol{q}}-\frac{\partial V}{\partial\boldsymbol{q}}\right). \tag{31a}$$

If we multiply by $\boldsymbol{t}$, we get

$$c\frac{\mathrm{d}m}{\mathrm{d}\tau}-\frac{c^2}{2}\frac{\partial m}{\partial\boldsymbol{q}}\cdot\boldsymbol{t}=\frac{c}{2}\frac{\mathrm{d}m}{\mathrm{d}\tau}=-\frac{1}{2}\frac{\partial V}{\partial\boldsymbol{q}}\cdot\boldsymbol{t}(\tau)\Rightarrow\frac{\partial m}{\partial\boldsymbol{q}}=-\frac{1}{c^2}\frac{\partial V}{\partial\boldsymbol{q}}\Rightarrow\frac{\partial L_0}{\partial\boldsymbol{q}}=-\frac{\partial V}{\partial\boldsymbol{q}},$$

and thus Equation (31a) is identified as the basic Equation (I.20) of the photon. Since L_0 does not explicitly depend on time[7], $\partial L_0/\partial\tau-\mathrm{d}/\mathrm{d}\tau\left(L_0-\boldsymbol{v}\cdot\partial L_0/\partial\boldsymbol{v}\right)=0$. In fact, the missing Lagrange equation

$$\begin{aligned}\frac{\partial L_0}{\partial\tau}-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(L_0-\frac{\partial L_0}{\partial\boldsymbol{v}}\cdot\boldsymbol{v}\right)&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(c^2m(\tau)+V(\boldsymbol{q}(\tau))\right)\\&=-\frac{1}{2}\left(c^2\frac{\mathrm{d}m}{\mathrm{d}\tau}+\frac{\partial V}{\partial\boldsymbol{q}}\cdot\boldsymbol{v}(\tau)\right)\\&=-\frac{1}{2}\left(-\frac{\partial V}{\partial\boldsymbol{q}}\cdot\boldsymbol{v}(\tau)+\frac{\partial V}{\partial\boldsymbol{q}}\cdot\boldsymbol{v}(\tau)\right)=0\\&\Rightarrow\frac{1}{2}\cdot\left(c^2\cdot m(\tau)+V(\boldsymbol{q}(\tau))\right)=\text{const.}\end{aligned} \tag{31b}$$

[7]We need not discuss lost energy here, because of $V=V_{ef}$. See I section 4.

yields the energy theorem (see I.9)[8].

With $\boldsymbol{p}_0(\tau)=\boldsymbol{p}_0(\boldsymbol{q}(\tau),c\boldsymbol{t}(\tau)):=\partial L_0/\partial \boldsymbol{v}=m(\boldsymbol{q}(\tau))\cdot c\boldsymbol{t}$ as canonical momentum, the Hamiltonian $H_0(\tau,\boldsymbol{q},\boldsymbol{p}_0):=\partial L_0/\partial \boldsymbol{v}\cdot c\boldsymbol{t}(\tau)-L_0(\tau)$ and the respective differential equations read:

$$
\begin{aligned}
&H_0(\tau,\boldsymbol{q},\boldsymbol{p}_0)=\frac{1}{2}\cdot\left(\frac{\boldsymbol{p}_0^2(\tau)}{m(\boldsymbol{q}(\tau))}+V(\boldsymbol{q}(\tau))\right),\\
&\frac{\partial H_0}{\partial \boldsymbol{p}_0}=\frac{\boldsymbol{p}_0(\tau)}{m(\boldsymbol{q}(\tau))}=c\boldsymbol{t}(\tau)=\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\tau},\\
&-\frac{\partial H_0}{\partial \boldsymbol{q}}=\frac{1}{2}\cdot\left(\frac{\boldsymbol{p}_0(\tau)}{m(\boldsymbol{q}(\tau))}\right)^2\frac{\partial m}{\partial \boldsymbol{q}}-\frac{1}{2}\cdot\frac{\partial V}{\partial \boldsymbol{q}}=\frac{1}{2}\cdot\left(c^2\frac{\partial m}{\partial \boldsymbol{q}}-\frac{\partial V}{\partial \boldsymbol{q}}\right)=\frac{\partial L_0}{\partial \boldsymbol{q}}=\frac{\mathrm{d}\boldsymbol{p}_0}{\mathrm{d}\tau},\\
&\frac{\mathrm{d}H_0}{\mathrm{d}\tau}=\frac{\partial H_0}{\partial \boldsymbol{p}_0}\cdot\left(-\frac{\partial H_0}{\partial \boldsymbol{q}}\right)+\frac{\partial H_0}{\partial \boldsymbol{q}}\cdot\frac{\partial H_0}{\partial \boldsymbol{p}_0}+\frac{\partial H_0}{\partial \tau}=\frac{\partial H_0}{\partial \tau}=0=-\frac{\partial L_0}{\partial \tau}.
\end{aligned}
\tag{32}
$$

With the Lagrange function L_0 or the Hamiltonian H_0, the correct value of the redshift is obtained, but only half the value of the light deflection for non-radially directed light paths. To correct this, we make the addition (see equation I.18)

$$
L(\tau,\boldsymbol{q}(\tau),c\cdot\boldsymbol{t}(\tau))=L_0(\boldsymbol{q}(\tau),c\cdot\boldsymbol{t}(\tau))-c\boldsymbol{t}(\tau)\cdot\int_{\tau_0}^{\tau}\psi(\tau)\frac{\mathrm{d}E}{\mathrm{d}\tau}\boldsymbol{n}(\tau)\mathrm{d}\tau,\ \frac{\mathrm{d}E}{\mathrm{d}\tau}=c^2\frac{\mathrm{d}m}{\mathrm{d}\tau}. \tag{33}
$$

If we differentiate L according to the velocity $\boldsymbol{v}=c\cdot\boldsymbol{t}$, we obtain the canonical momentum of the photon

$$
\boldsymbol{p}(\tau):=\frac{\partial L}{\partial \boldsymbol{v}}=\boldsymbol{p}_0(\tau)-\int\psi(\tau)\frac{\mathrm{d}E}{\mathrm{d}\tau}\boldsymbol{n}(\tau)\mathrm{d}\tau, \tag{34}
$$

and the Lagrange equations turn out to be identical to (I.21):

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \boldsymbol{v}}=\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}=\frac{\mathrm{d}\boldsymbol{p}_0}{\mathrm{d}\tau}-\psi(\tau)\frac{\mathrm{d}E}{\mathrm{d}\tau}\boldsymbol{n}(\tau)=\frac{\partial L}{\partial \boldsymbol{q}}=\frac{\partial L_0}{\partial \boldsymbol{q}}. \tag{35}
$$

Let us check the explicit time dependence of L:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \boldsymbol{v}}\cdot\boldsymbol{v}(\tau)-L(\tau)\right)&=-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L_0}{\partial \boldsymbol{v}}\cdot\boldsymbol{v}(\tau)-\boldsymbol{v}\cdot\int\psi\frac{\mathrm{d}E}{\mathrm{d}\tau}\cdot\boldsymbol{n}-L_0+\boldsymbol{v}\cdot\int\psi\frac{\mathrm{d}E}{\mathrm{d}\tau}\cdot\boldsymbol{n}\right)\\
&=\frac{\partial L_0}{\partial \tau}=0.
\end{aligned}
$$

So, the energy theorem applies again. The canonical momentum $\boldsymbol{p}(\tau)$ is thus unambiguously determined, and a component is added to the mechanical momentum of the photon which has no influence on the redshift, but is only evident in the deflection of light. With the definition

$H:=\partial L/\partial \boldsymbol{v}\cdot\boldsymbol{v}(\tau)-L(\tau)=\boldsymbol{p}\cdot\boldsymbol{v}(\tau)-L(\tau)=\boldsymbol{p}_0\boldsymbol{v}(\tau)-L_0(\tau)$ follows:

$$
H(\boldsymbol{q},\boldsymbol{p},\tau)=\frac{v^2(\tau)\cdot m(\boldsymbol{q}(\tau))+V(\boldsymbol{q}(\tau))}{2}
$$

[8]Classically, $mv^2/2$ refers to the potential V, but for the photon in the gravitational field according to Equation (29), $c^2m(\tau)/2$ refers to $V/2$.

$$=\frac{1}{2}\left(\frac{\left(\boldsymbol{p}(\tau)+\int\psi(\tau)\frac{\mathrm{d}E}{\mathrm{d}\tau}\boldsymbol{n}(\tau)\mathrm{d}\tau\right)^2}{m(\boldsymbol{q}(\tau))}+V(\boldsymbol{q}(\tau))\right), \tag{36}$$

so that with Equation (32) and $\partial L/\partial \boldsymbol{q}=\partial L_0/\partial \boldsymbol{q}$ the canonical equations read:

$$\begin{aligned}
&\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}=-\frac{\partial H}{\partial \boldsymbol{q}}=\frac{\partial L}{\partial \boldsymbol{q}},\\
&\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\tau}=\frac{\partial H}{\partial \boldsymbol{p}}=\frac{\boldsymbol{p}(\tau)+\int\psi(\tau)\frac{\mathrm{d}E}{\mathrm{d}\tau}\boldsymbol{n}(\tau)\mathrm{d}\tau}{m(\tau)}=\boldsymbol{v}(\tau),\\
&\frac{\partial H}{\partial \tau}=-\frac{\partial L}{\partial \tau}=0.
\end{aligned} \tag{37}$$

3.3. Lagrange Function and Hamiltonian of the Massive Particle

Analogous to the procedure for the photon in Equation (33), we supplement the term $\int\psi(\tau)\cdot\mathrm{d}\epsilon/\mathrm{d}\tau\cdot\boldsymbol{n}(\tau)$, where ϵ is given by (see I.23)

$$\epsilon(\tau)=\int_{\tau_0}^{\tau}v^2(\tau')\cdot\frac{\mathrm{d}m}{\mathrm{d}\tau'}\mathrm{d}\tau'.$$

As a new Lagrange function, we thus get with Equation (30)

$$L(\boldsymbol{q}(\tau),\boldsymbol{v}(\tau))=-\zeta_0\cdot\sqrt{1-k\cdot\frac{v^2(\tau)}{c^2}}-\boldsymbol{v}(\tau)\cdot\int_{\tau_0}^{\tau}\psi(\tau)\frac{\mathrm{d}\epsilon}{\mathrm{d}t}\cdot\boldsymbol{n}(\tau)\mathrm{d}\tau-V(\boldsymbol{q}(\tau)), \tag{38}$$

which with $\partial L/\partial \boldsymbol{v}=m(\boldsymbol{q})\boldsymbol{v}-\int\psi(\tau)\cdot\mathrm{d}\epsilon/\mathrm{d}t\cdot\boldsymbol{n}(\tau)\mathrm{d}\tau$ gives the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \boldsymbol{v}}=\frac{\mathrm{d}m}{\mathrm{d}\tau}\cdot\boldsymbol{v}(\tau)+m(\tau)\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\tau}-\psi(\tau)\cdot\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}\cdot\boldsymbol{n}(\tau)=\frac{\partial L}{\partial \boldsymbol{q}}=-\frac{\partial V}{\partial \boldsymbol{q}}, \tag{39}$$

corresponding directly to (I.21). Now, substituting $\mathrm{d}m/\mathrm{d}\tau$ according to Equation (B) yields

$$m(\tau)\left[\frac{k}{c^2}\frac{v(\tau)\cdot\mathrm{d}v/\mathrm{d}\tau}{1-k\cdot v^2(\tau)/c^2}\cdot\boldsymbol{v}(\tau)+\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\tau}\right]=-\frac{\partial V}{\partial \boldsymbol{q}}+\psi(\tau)\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}\boldsymbol{n}(\tau),$$

and after multiplying by $\boldsymbol{t}$ we obtain the differential equation for the orbital velocity (see first line of equation I.26)

$$m(\tau)\frac{\mathrm{d}v}{\mathrm{d}\tau}\left[1+\frac{k\cdot v^2(\tau)/c^2}{1-k\cdot v^2(\tau)/c^2}\right]=-\frac{\partial V}{\partial \boldsymbol{q}}\cdot\boldsymbol{t}(\tau)\rightarrow\frac{\mathrm{d}v}{\mathrm{d}\tau}=-\frac{1-k\cdot v^2(\tau)/c^2}{m(\tau)}\cdot\frac{\partial V}{\partial \boldsymbol{q}}\cdot\boldsymbol{t}(\tau),$$

and after multiplication by $\boldsymbol{n}$ we obtain the differential equation for α:

$$m(\tau)\cdot v(\tau)\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}=-\frac{\partial V}{\partial \boldsymbol{q}}\cdot\boldsymbol{n}(\tau)+\psi(\tau)\frac{\mathrm{d}\epsilon}{\mathrm{d}t}=-\frac{\partial V}{\partial \boldsymbol{q}}\cdot\boldsymbol{n}(\tau)+\psi(\tau)\cdot v^2(\tau)\frac{\mathrm{d}m}{\mathrm{d}\tau}.$$

Finally, substituting $\mathrm{d}m/\mathrm{d}\tau$ and $\mathrm{d}v/\mathrm{d}\tau$ again gives the second line of (I.26)

$$m(\tau)\cdot v(\tau)\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}=-\frac{\partial V}{\partial \boldsymbol{q}}\cdot\boldsymbol{n}(\tau)-\psi(\tau)\cdot k\cdot\frac{v^3(\tau)}{c^2}\cdot\frac{\partial V}{\partial \boldsymbol{q}}\cdot\boldsymbol{t}(\tau).$$

Recall $m(\tau)=m(\boldsymbol{q}(\tau))$; without τ we use $m(\boldsymbol{q})$. With Equation (38) and the canonical momentum

$$\begin{aligned} \boldsymbol{p} &:= \frac{\partial L}{\partial \boldsymbol{v}} = m(\boldsymbol{q})\boldsymbol{v} - \int \psi(\tau)\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}\boldsymbol{n}(\tau)\mathrm{d}\tau \\ &= m_0 \cdot \boldsymbol{v}\sqrt{\frac{1-k\cdot v_0^2/c^2}{1-k\cdot v^2(\tau)/c^2}} - \int \psi(\tau)\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}\boldsymbol{n}(\tau)\mathrm{d}\tau \end{aligned} \tag{40}$$

and the abbreviation $\boldsymbol{A}(\tau) := \int \psi(\tau)\cdot \mathrm{d}\epsilon/\mathrm{d}\tau \cdot \boldsymbol{n}(\tau)\mathrm{d}\tau$, the following expression results for $H := \partial L/\partial \boldsymbol{v}\cdot \boldsymbol{v} - L$:

$$\begin{aligned} H &= m(\boldsymbol{q})\cdot v^2 - \boldsymbol{v}\cdot\boldsymbol{A} + \zeta_0\cdot\left(1-kv^2/c^2\right)^{1/2} + \boldsymbol{v}\cdot\boldsymbol{A} + V(\boldsymbol{q}) \\ &= m(\boldsymbol{q})\cdot v^2 + \zeta_0\cdot\left(1-kv^2/c^2\right)^{1/2} + V(\boldsymbol{q}). \end{aligned} \tag{41}$$

If we express $m(\boldsymbol{q})\cdot\boldsymbol{v}$ by $\boldsymbol{p}+\boldsymbol{A}$, we arrive at the expression of H with the variables $\boldsymbol{p}$ and $\boldsymbol{q}$:

$$H = \frac{(\boldsymbol{p}+\boldsymbol{A})^2}{m(\boldsymbol{q})} + V(\boldsymbol{q}) + \zeta_0\cdot\sqrt{1-\frac{k}{c^2}\frac{(\boldsymbol{p}+\boldsymbol{A})^2}{m^2(\boldsymbol{q})}}, \tag{42}$$

the last term being new. Recall that $m(\boldsymbol{q})$ is the solution of (A). The canonical equations then take the form

$$\begin{aligned} -\frac{\partial H}{\partial \boldsymbol{q}} &= -\frac{\partial V}{\partial \boldsymbol{q}} + \frac{(\boldsymbol{p}+\boldsymbol{A})^2}{m^2(\boldsymbol{q})}\frac{\partial m}{\partial \boldsymbol{q}} - \frac{c^2\cdot m(\boldsymbol{q})}{2k}\cdot\frac{2k}{c^2}\frac{(\boldsymbol{p}+\boldsymbol{A})^2}{m^3(\boldsymbol{q})}\frac{\partial m}{\partial \boldsymbol{q}} = -\frac{\partial V}{\partial \boldsymbol{q}} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}, \\ \frac{\partial H}{\partial \boldsymbol{p}} &= 2\frac{\boldsymbol{p}+\boldsymbol{A}}{m(\boldsymbol{q})} - \frac{c^2\cdot m(\boldsymbol{q})}{2k}\cdot\frac{2k}{c^2}\frac{\boldsymbol{p}+\boldsymbol{A}}{m^2(\boldsymbol{q})} = \frac{\boldsymbol{p}+\boldsymbol{A}}{m(\boldsymbol{q})} = \boldsymbol{v}(\tau) = \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\tau},\ \frac{\mathrm{d}H}{\mathrm{d}\tau} = \frac{\partial H}{\partial \tau} = 0. \end{aligned} \tag{43}$$

Remark 4. Due to $\partial H/\partial\tau = 0$, we have $H = \text{const}$. Since H represents the total energy, $H - V$ is the particle energy. It follows Equation (41) and Equation (B)

$$H - V = m(\boldsymbol{q})\cdot v^2 + \zeta_0\cdot\left(1-kv^2/c^2\right)^{1/2} = \frac{c^2}{k}\cdot m(\boldsymbol{q}) \Rightarrow$$

$$H = \text{const.} = \frac{c^2}{k}\cdot m(\boldsymbol{q}) + V(\boldsymbol{q}) \Rightarrow \frac{\partial m}{\partial \boldsymbol{q}} = -\frac{k}{c^2}\frac{\partial V}{\partial \boldsymbol{q}},$$

so that we have recovered the mass equation, Equation (A), and thus also Equation (B).

The Lagrange equations are second-order differential equations, and $\boldsymbol{q}$ and $\boldsymbol{v}$ can therefore be understood as a function of the start and end point of the variation: $\boldsymbol{q} = \boldsymbol{q}(\tau, \boldsymbol{q}_0, \boldsymbol{q}_1)$, $\boldsymbol{v} = \boldsymbol{v}(\tau, \boldsymbol{q}_0, \boldsymbol{q}_1)$. If you use this to calculate the action integral

$$S := \int_{\tau_0}^{\tau_1} L\left(\tau, \boldsymbol{q}(\tau, \boldsymbol{q}_0, \boldsymbol{q}_1), \boldsymbol{v}(\tau, \boldsymbol{q}_0, \boldsymbol{q}_1)\right)\mathrm{d}\tau,$$

the action function S becomes a function of the start and end points in time and space: $S = S(\tau_0, \boldsymbol{q}_0, \tau_1, \boldsymbol{q}_1)$. If we fix $\tau_0, \boldsymbol{q}_0$ and consider S only as a function of $\tau_1 =: \tau$, $\boldsymbol{q}_1 =: \boldsymbol{q}$, the calculation results in $\partial S/\partial \boldsymbol{q} = \partial L/\partial \boldsymbol{v} = \boldsymbol{p}$ and $\partial S/\partial\tau = -(\partial L/\partial \boldsymbol{v}\cdot\boldsymbol{v} - L) = -H(\tau, \boldsymbol{q}, \boldsymbol{p})$. Or summarised with Equation (42) as Hamilton-Jacobian equation:

$$0=\frac{\partial S}{\partial \tau}+H=\frac{\partial S}{\partial \tau}+\frac{\left(\partial S/\partial \boldsymbol{q}+\boldsymbol{A}\right)^2}{m(\boldsymbol{q})}+V(\boldsymbol{q})+\zeta_0\cdot\sqrt{1-\frac{k}{c^2}\frac{\left(\partial S/\partial \boldsymbol{q}+\boldsymbol{A}\right)^2}{m^2(\boldsymbol{q})}}, \tag{44a}$$

the last term being new. If we develop the root up to order v^4/c^4, we get

$$0=\frac{\partial S}{\partial \tau}+\frac{\left(\partial S/\partial \boldsymbol{q}+\boldsymbol{A}\right)^2}{m(\boldsymbol{q})}+V(\boldsymbol{q})+\zeta_0\cdot\left(1-\frac{k}{2c^2}\frac{\left(\partial S/\partial \boldsymbol{q}+\boldsymbol{A}\right)^2}{m^2(\boldsymbol{q})}-\frac{k^2}{8c^4}\frac{\left(\partial S/\partial \boldsymbol{q}+\boldsymbol{A}\right)^4}{m^4(\boldsymbol{q})}\right). \tag{44b}$$

In the case of a light particle, the Hamilton-Jacobian equation is as follows

$$0=\frac{\partial S}{\partial \tau}+H=\frac{\partial S}{\partial \tau}+\frac{\left(\partial S/\partial \boldsymbol{q}+\boldsymbol{A}\right)^2}{2m(\boldsymbol{q})}+\frac{V(\boldsymbol{q})}{2}. \tag{44c}$$

Since the energy theorem

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \boldsymbol{v}}\boldsymbol{v}(\tau)-L(\tau)\right)=\frac{\partial L}{\partial \tau}=0$$

applies to both light and massive particles, $H=\text{const}$ and

$$S(\tau,\boldsymbol{q})=\int_{q_0}^{q(\tau)}\frac{\partial L}{\partial \boldsymbol{v}}\cdot\mathrm{d}\boldsymbol{q}-H\cdot(\tau-\tau_0).$$

3.4. Continuity Equation for $\mathrm{d}m/\mathrm{d}\tau\neq 0$

To represent a particle, a body or even a star as a matter field, we need a continuity equation in addition to the Hamilton-Jacobi equation. In the particle image, we generate the density ρ by sending one and the same particle with the same initial mass $m(\tau_0)$ but different initial velocity on its journey from an initial point $\boldsymbol{q}_0$. The trajectories created with the velocity field $\boldsymbol{v}(\tau,\boldsymbol{q})$ generated in this way emerge unambiguously from the solutions of the Lagrange equations and the additional mass function. According to the density and volume of a particle of variable mass m, we can imagine the orbits as tubes that change their diameter along the orbit. We will therefore define the density ρ at time τ at location $\boldsymbol{q}$ as the number $N(\tau,\boldsymbol{q})$ of all orbital tubes passing $\boldsymbol{q}$ and divided by the number $N(\Omega(\tau))$ of all orbital tubes in the volume $\Omega(\tau)$ moved along by all particle orbits. This means that the density defined in this way is a density which can fluctuate from time to time, from volume to volume, so that the change over time is generally not equal to zero. The total number or mass of all particles in the volume $\Omega(\tau)$ is then calculated with the mass density $\mu(\tau,\boldsymbol{q}):=m(\tau,\boldsymbol{q})\rho(\tau,\boldsymbol{q})$ as follows

$$\int_{\Omega(\tau)}\rho(\tau,\boldsymbol{q})\mathrm{d}^3q=N(\Omega(\tau)),\ \int_{\Omega(\tau)}\mu(\tau,\boldsymbol{q})\mathrm{d}^3q=M(\Omega(\tau)), \tag{45}$$

In the case of constant particle mass, $m(\tau,\boldsymbol{q})=m_0$, the conservation of the total mass then follows: $M(\Omega(\tau))=N\cdot m_0$, *i.e.* the total time derivative of Equation (45) is zero, from which the continuity equation of the particle image follows:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega(\tau)}\rho(\tau,\boldsymbol{q})\mathrm{d}^3q=0\ \Leftrightarrow\ \frac{\partial\rho}{\partial\tau}+\nabla\cdot\left(\rho(\tau,\boldsymbol{q})\cdot\boldsymbol{v}(\tau,\boldsymbol{q})\right)=0.$$

However, when particles move in the gravitational field, their mass generally changes, so that:

$$0 \neq \frac{\mathrm{d}M}{\mathrm{d}t}\left(\Omega(\tau)\right) = \int_{\Omega(\tau)}\left(\frac{\partial\mu}{\partial\tau}(\tau,\boldsymbol{q}) + \nabla\cdot\left(\mu(\tau,\boldsymbol{q})\cdot\boldsymbol{v}(\tau,\boldsymbol{q})\right)\right)\mathrm{d}^3q. \tag{46a}$$

Let us now recall the expression for the energy that a body loses or gains instantaneously due to its mass in a gravitational field (see I.23a) and take the spatial average of it,

$$\frac{\mathrm{d}E}{\mathrm{d}\tau}\left(\Omega(\tau)\right) := \int_{\Omega(\tau)}\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}\rho(\tau,\boldsymbol{q})\mathrm{d}^3q = \int_{\Omega(\tau)}\frac{\partial m}{\partial\boldsymbol{q}}\cdot\boldsymbol{v}(\tau,\boldsymbol{q})\cdot v^2(\tau,\boldsymbol{q})\cdot\rho(\tau,\boldsymbol{q})\mathrm{d}^3q, \tag{46b}$$

then, according to (I.23b), the following must hold: $c^2/k\cdot\mathrm{d}M/\mathrm{d}\tau = \mathrm{d}E/\mathrm{d}\tau$. We summarise Equation (46a), Equation (46b) in differential form:

$$0 = \frac{\partial\mu}{\partial\tau}(\tau,\boldsymbol{q}) + \nabla\cdot\left(\mu(\tau,\boldsymbol{q})\cdot\boldsymbol{v}(\tau,\boldsymbol{q})\right) - \frac{k}{c^2}\cdot v^2(\tau,\boldsymbol{q})\cdot\frac{\partial m}{\partial\boldsymbol{q}}\cdot\boldsymbol{v}(\tau,\boldsymbol{q})\cdot\rho(\tau,\boldsymbol{q}).$$

which we refer to as Equation (46) in the subsequent text.

Since the mass does not depend partially on time, we obtain (suppressing the dependence on the variables $\tau,\boldsymbol{q}$)

$$0 = m\cdot\frac{\partial\rho}{\partial\tau} + m\cdot\nabla\cdot(\rho\cdot\boldsymbol{v}) + \rho\cdot\frac{\partial m}{\partial\boldsymbol{q}}\cdot\boldsymbol{v}\left(1 - \frac{k\cdot v^2}{c^2}\right).$$

With $\boldsymbol{v} = (\boldsymbol{p}+\boldsymbol{A})/m = (\partial S/\partial\boldsymbol{q}+\boldsymbol{A})/m$ and division of

$$\begin{aligned}0 &= m\cdot\frac{\partial\rho}{\partial\tau} + m\cdot\frac{\partial\rho}{\partial\boldsymbol{q}}\cdot\frac{\boldsymbol{p}+\boldsymbol{A}}{m} + \rho\cdot\nabla\cdot(\boldsymbol{p}+\boldsymbol{A}) - \rho\cdot\frac{\boldsymbol{p}+\boldsymbol{A}}{m}\cdot\frac{\partial m}{\partial\boldsymbol{q}} \\ &\quad + \rho\cdot\frac{\partial m}{\partial\boldsymbol{q}}\cdot\frac{\boldsymbol{p}+\boldsymbol{A}}{m}\cdot\left(1 - \frac{k}{c^2}\frac{(\boldsymbol{p}+\boldsymbol{A})^2}{m^2}\right) \\ &= m\cdot\frac{\partial\rho}{\partial\tau} + \frac{\partial\rho}{\partial\boldsymbol{q}}\cdot(\boldsymbol{p}+\boldsymbol{A}) + \rho\cdot\nabla\cdot(\boldsymbol{p}+\boldsymbol{A}) - \frac{k}{c^2}\cdot\rho\cdot\frac{\partial m}{\partial\boldsymbol{q}}\cdot\frac{(\boldsymbol{p}+\boldsymbol{A})^3}{m^3}\end{aligned}$$

by $m\cdot\rho$ finally results the extended continuity equation

$$\begin{aligned}0 &= \frac{1}{\rho}\frac{\partial\rho}{\partial\tau} + \frac{1}{\rho}\frac{\partial\rho}{\partial\boldsymbol{q}}\cdot\frac{\partial S/\partial\boldsymbol{q}+\boldsymbol{A}}{m} + \frac{\partial^2S/\partial\boldsymbol{q}^2+\nabla\cdot\boldsymbol{A}}{m} \\ &\quad - \frac{k\left(\partial S/\partial\boldsymbol{q}+\boldsymbol{A}\right)^2}{c^2m^2}\cdot\frac{1}{m}\frac{\partial m}{\partial\boldsymbol{q}}\cdot\frac{\partial S/\partial\boldsymbol{q}+\boldsymbol{A}}{m}.\end{aligned} \tag{47a}$$

the last term being new. Thus, with $\mathrm{d}m/\mathrm{d}\tau = 0$ (gravity-free space or circular motion), the classical continuity equation is obtained.

Remark 5. The fourth term in the continuity equation is of the order v^3/c^3 due to Equation (A), which is why it can be neglected compared to the leading terms, at least in weak gravitational fields. If we restrict ourselves to v^2/c^2 terms, we obtain the classical form of the continuity equation, albeit with m as a function:

$$0 = \frac{1}{\rho}\frac{\partial\rho}{\partial\tau} + \frac{1}{\rho}\frac{\partial\rho}{\partial\boldsymbol{q}}\frac{\partial S/\partial\boldsymbol{q}+\boldsymbol{A}}{m(\boldsymbol{q})} + \frac{1}{m(\boldsymbol{q})}\left(\partial^2S/\partial\boldsymbol{q}^2+\nabla\cdot\boldsymbol{A}\right). \tag{47b}$$

In the Hamilton-Jacobi equation, Equation (44a), terms of arbitrarily high order occur; if we also restrict ourselves to v^2/c^2 terms, then from Equation (44b) remains

$$0=\frac{\partial S}{\partial \tau}+\frac{\left(\partial S/\partial \boldsymbol{q}+\boldsymbol{A}\right)^2}{m(\boldsymbol{q})}\left(1-\frac{k\cdot\zeta_0}{2c^2\cdot m(\boldsymbol{q})}\right)+V(\boldsymbol{q})+\zeta_0. \tag{44d}$$

Using Equation (B), we obtain

$$\frac{k\cdot\zeta_0}{c^2\cdot m(\boldsymbol{q})}=\sqrt{1-\frac{k}{c^2}\frac{\left(\partial S/\partial \boldsymbol{q}+\boldsymbol{A}\right)^2}{m^2(\boldsymbol{q})}}.$$

For non-relativistic velocities, the classical form thus follows:

$$0=\frac{\partial S}{\partial \tau}+\frac{\left(\partial S/\partial \boldsymbol{q}+\boldsymbol{A}\right)^2}{2m(\boldsymbol{q})}+V(\boldsymbol{q})+\zeta_0.$$

3.5. The Schrödinger Wave Equation in Gravity

We follow the classical concept for deriving a Schrödinger equation (see for example [7]). The matter wave $\zeta(\tau,\boldsymbol{q})=A(\tau,\boldsymbol{q})\cdot \mathrm{e}^{iG(\tau,\boldsymbol{q})}$ representing the particle is given by the real functions of amplitude $A(\tau,\boldsymbol{q})$ and phase $G(\tau,\boldsymbol{q})$. The dimensionless phase, multiplied by $\hbar$, should fulfill the Hamilton-Jacobi equation, *i.e.* $S(\tau,\boldsymbol{q})=\hbar\cdot G(\tau,\boldsymbol{q})$ is used as the action function. For the amplitude square $A^2(\tau,\boldsymbol{q})$, proportionality to the density of the matter distribution ρ is required. This partial differential equation, which describes the dynamics of the matter wave $\zeta(\tau,\boldsymbol{q})$, will be 1. homogeneous, 2. of second order due to the derivatives occurring in the Hamilton-Jacobi and continuity equations, and thirdly linear due to superponability. If we only consider the Hamilton-Jacobi equation, Equation (44d) and the continuity Equation (47b), *i.e. omit orders higher than* v^2/c^2, this differential equation will have the following form:

$$\begin{aligned}&\left\{a(\tau,\boldsymbol{q})+b_0(\tau,\boldsymbol{q})\cdot\frac{\partial}{\partial\tau}+\sum_k^3 b_k(\tau,\boldsymbol{q})\frac{\partial}{\partial q_k}+c_{00}(\tau,\boldsymbol{q})\frac{\partial^2}{\partial\tau^2}\right.\\&\left.+\sum_k^3 c_{k0}(\tau,\boldsymbol{q})\frac{\partial}{\partial q_k}\frac{\partial}{\partial\tau}+\sum_{k,l}c_{kl}(\tau,\boldsymbol{q})\frac{\partial}{\partial q_k}\frac{\partial}{\partial q_l}\right\}\zeta(\tau,\boldsymbol{q})=0.\end{aligned} \tag{48}$$

The complex coefficient functions are to be determined in such a way that Hamilton-Jacobi and continuity equations are embedded in it, *i.e.* one is found in the real part and the other in the imaginary part of Equation (48). As already mentioned, we require $A^2(\tau,\boldsymbol{q})=\zeta^*(\tau,\boldsymbol{q})\zeta(\tau,\boldsymbol{q})=\rho(\tau,\boldsymbol{q})$. In the first step, we insert $\zeta(\tau,\boldsymbol{q})$ and decompose Equation (48) according to magnitude and phase (we suppress the dependence on the variables $\tau,\boldsymbol{q}$):

$$\begin{aligned}0&=a\zeta+b_0\left(\frac{\zeta}{A}\frac{\partial A}{\partial\tau}+i\zeta\frac{\partial G}{\partial\tau}\right)+\sum_k b_k\left(\frac{\zeta}{A}\frac{\partial A}{\partial q_k}+i\zeta\frac{\partial G}{\partial q_k}\right)\\&+\sum_k c_{k0}\left(\frac{\zeta}{A}\frac{\partial^2 A}{\partial q_k\partial\tau}+i\zeta\frac{\partial^2 G}{\partial q_k\partial\tau}+\frac{\partial\zeta}{\partial q_k}\left(\frac{\zeta}{A}\frac{\partial A}{\partial\tau}+i\zeta\frac{\partial G}{\partial\tau}\right)\right)\\&+\sum_{k,l}c_{kl}\left(\frac{\zeta}{A}\frac{\partial^2 A}{\partial q_k\partial q_l}-\zeta\frac{\partial G}{\partial q_k}\frac{\partial G}{\partial q_l}+i\left(\zeta\frac{\partial^2 G}{\partial q_k\partial q_l}+2\frac{\zeta}{A}\frac{\partial A}{\partial q_k}\frac{\partial G}{\partial q_l}\right)\right).\end{aligned}$$

Since there are no derivatives of the type $\partial^2/\partial q_k\partial\tau$ or $\partial/\partial q_k\cdot\partial/\partial t$ in Equation (47a) and Equation (43), we have $c_{k0}=0$. If we divide by ζ and in the second step decompose all coefficients into real and imaginary parts ($a=a^1+ia^2,\cdots$), we obtain the real part of Equation (48):

$$\begin{aligned}0&=a^1+b_0^1\frac{1}{A}\frac{\partial A}{\partial\tau}-b_0^2\frac{\partial G}{\partial\tau}+\sum_k\left(b_k^1\frac{1}{A}\frac{\partial A}{\partial q_k}-b_k^2\frac{\partial G}{\partial q_k}\right)\\&+\sum_{k,l}\left(c_{kl}^1\left(\frac{1}{A}\frac{\partial^2 A}{\partial q_k\partial q_l}-\frac{\partial G}{\partial q_k}\frac{\partial G}{\partial q_l}\right)-c_{kl}^2\left(\frac{\partial^2 G}{\partial q_k\partial q_l}+\frac{2}{A}\frac{\partial A}{\partial q_k}\frac{\partial G}{\partial q_l}\right)\right)\end{aligned}\tag{49}$$

and as the imaginary part:

$$\begin{aligned}0&=a^2+b_0^2\frac{1}{A}\frac{\partial A}{\partial\tau}+b_0^1\frac{\partial G}{\partial\tau}+\sum_k\left(b_k^2\frac{1}{A}\frac{\partial A}{\partial q_k}+b_k^1\frac{\partial G}{\partial q_k}\right)\\&+\sum_{k,l}\left(c_{kl}^2\left(\frac{1}{A}\frac{\partial^2 A}{\partial q_k\partial q_l}-\frac{\partial G}{\partial q_k}\cdot\frac{\partial G}{\partial q_l}\right)+c_{kl}^1\left(\frac{\partial^2 G}{\partial q_k\partial q_l}+\frac{2}{A}\frac{\partial A}{\partial q_k}\cdot\frac{\partial G}{\partial q_l}\right)\right).\end{aligned}\tag{50}$$

Now, Equation (44d) reads with $m:=m(\boldsymbol{q})$, $V:=V(\boldsymbol{q})$:

$$\begin{aligned}0&=\hbar\frac{\partial G}{\partial\tau}+\frac{\hbar^2(\partial G/\partial\boldsymbol{q})^2+2\hbar\cdot\partial G/\partial\boldsymbol{q}\cdot\boldsymbol{A}}{m}\cdot\left(1-\frac{\zeta_0\cdot k}{2c^2\cdot m}\right)\\&+V+\zeta_0+\frac{\boldsymbol{A}^2}{m}\cdot\left(1-\frac{\zeta_0\cdot k}{2c^2\cdot m}\right)\end{aligned}\tag{51}$$

Comparison of Equation (49) with Equation (51) gives

$$a^1=V+\zeta_0+\frac{\boldsymbol{A}^2}{m}\left(1-\frac{\zeta_0\cdot k}{2c^2\cdot m}\right),\ b_0^2=-\hbar,\ c_{kl}^1=-\frac{\hbar^2}{m}\left(1-\frac{\zeta_0\cdot k}{2c^2\cdot m(\boldsymbol{q})}\right)\cdot\delta_{kl},$$

$$b^2=-\frac{2\hbar}{m}\left(1-\frac{\zeta_0\cdot k}{2c^2\cdot m}\right)\cdot\boldsymbol{A},$$

where $b_0^1=0$, $b_k^1=0$, $c_{kl}^2=0$ can be set if the continuity equation allows it. We see that at least one new term will be added to the Hamilton-Jacobi equation.

With $A^2=\rho$ the continuity equation, Equation (47b), containing no higher order terms than v^2/c^2, reads:

$$0=\frac{1}{A}\frac{\partial A}{\partial\tau}+\frac{\hbar}{2m}\frac{\partial^2 G}{\partial\boldsymbol{q}^2}+\frac{\hbar}{m}\frac{1}{A}\frac{\partial A}{\partial\boldsymbol{q}}\frac{\partial G}{\partial\boldsymbol{q}}+\frac{\boldsymbol{A}}{m}\frac{1}{A}\frac{\partial A}{\partial\boldsymbol{q}}+\frac{\nabla\cdot\boldsymbol{A}}{2m}$$

Because of $b_0^1\neq 1$ we have to multiply the equation by $-\hbar$ and get

$$0=-\frac{\hbar}{A}\frac{\partial A}{\partial\tau}-\frac{\hbar^2}{2m}\frac{\partial^2 G}{\partial\boldsymbol{q}^2}-\frac{\hbar^2}{m}\frac{1}{A}\frac{\partial A}{\partial\boldsymbol{q}}\frac{\partial G}{\partial\boldsymbol{q}}-\frac{\hbar\boldsymbol{A}}{m}\frac{1}{A}\frac{\partial A}{\partial\boldsymbol{q}}-\frac{\hbar\cdot\nabla\cdot\boldsymbol{A}}{2m}\tag{52}$$

Compared with Equation (50) we get

$$a^2=-\frac{\hbar\cdot\nabla\cdot\boldsymbol{A}}{2m},\ b_0^2=-\hbar,\ c_{kl}^1=-\frac{\hbar^2}{2m}\delta_{kl},\ b_k^2=-\frac{\hbar A_k}{m},\ b_k^1=0,$$

where $b_0^1=0$, $c_{kl}^2=0$. The continuity equation is thus completely embedded in Equation (52), no new terms are added.

If the gravitational field is not so strong that $\left(1-k\cdot v_0^2/c^2\right)^{1/2}\cdot m_0/m(\boldsymbol{q})$ becomes smaller than, say, $1-0.01$, corresponding to a velocity of less than 17,000 km/s, we can set

$$1-\frac{\zeta_0\cdot k}{2c^2\cdot m(\boldsymbol{q})}=1-\frac{\left(1-k\cdot v^2/c^2\right)^{1/2}}{2}\simeq\frac{1}{2},\tag{53}$$

and the coefficients b_k^2 and c_{kl}^1 of the Hamilton-Jacobi and the continuity equations coincide. If we now insert these coefficients into Equation (49), we obtain:

$$0=a^1+\hbar\cdot\frac{\partial G}{\partial\tau}+\frac{\hbar^2}{2m}\cdot\left(\frac{\partial G}{\partial\boldsymbol{q}}\right)^2+\frac{\hbar\cdot\boldsymbol{A}}{m}\cdot\frac{\partial G}{\partial\boldsymbol{q}}-\frac{\hbar^2}{2m}\cdot\frac{1}{A}\frac{\partial^2 A}{\partial\boldsymbol{q}^2}.$$

The last term, which does not appear in the Hamilton-Jacobi equation, Equation (51), can be neglected if the following applies:

$$\left|\frac{\partial^2 A/\partial\boldsymbol{q}^2}{A}\right|\ll\left|\frac{\partial G}{\partial\boldsymbol{q}}\right|^2.$$

However, for nearly plane waves $G(\tau,\boldsymbol{q})\simeq\boldsymbol{k}\cdot\boldsymbol{q}-\omega\cdot t$ this is given. This is because, analogous to geometrical optics, $1/A\cdot\partial^2 A/\partial\boldsymbol{q}^2\ll|\boldsymbol{k}|^2$ is required. In the limit of small waves, which is fulfilled for a massive particle, the Hamilton-Jacobian equation arises again and the embedding in a partial differential equation of 2nd order is thus possible.

If we insert the coefficients found into the field equation, Equation (48), as a final step, we obtain an equation which we refer to as the Schrödinger wave equation in gravity:

$$\begin{aligned}&\left\{\frac{\hbar}{i}\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m(\boldsymbol{q})}\cdot\frac{\partial^2}{\partial\boldsymbol{q}^2}+\frac{\hbar}{i}\cdot\frac{\boldsymbol{A}(\tau)}{m(\boldsymbol{q})}\cdot\frac{\partial}{\partial\boldsymbol{q}}+\left(V+\zeta_0+\frac{\boldsymbol{A}^2(\tau)}{2m(\boldsymbol{q})}+\frac{\hbar}{i}\frac{\nabla\cdot\boldsymbol{A}}{2m(\boldsymbol{q})}\right)(\tau,\boldsymbol{q})\right\}\zeta(\tau,\boldsymbol{q})=0\\&=\left\{\frac{\hbar}{i}\frac{\partial}{\partial\tau}+\frac{1}{2m(\boldsymbol{q})}\left(\frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{q}}+\boldsymbol{A}\right)^2+V+\zeta_0\right\}\zeta(\tau,\boldsymbol{q}).\end{aligned}\tag{54}$$

It is valid for non-relativistic velocities and differs structurally from the classical Schrödinger equation with electromagnetic potential only in the variable mass function, which satisfies Equation (A), and the gravitational potential. Note that $\nabla\cdot\boldsymbol{A}=0$ is fulfilled by definition of $\boldsymbol{A}$.

With the product approach $\zeta(\tau,\boldsymbol{q})=T(\tau)\cdot u(\boldsymbol{q})$ and $\partial A/\partial\tau=0$, $T(\tau)=\mathrm{e}^{-iE\tau/\hbar}$ and the stationary differential equation

$$E\cdot u(\boldsymbol{q})=-\left\{\frac{1}{2m(\boldsymbol{q})}\left(\frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{q}}+\boldsymbol{A}\right)^2+V+\zeta_0\right\}u(\boldsymbol{q})\tag{55}$$

result from Equation (54), with E the total energy of the particle.

Example 5. $\mathrm{d}m/\mathrm{d}\tau=0$ implies $\partial m/\partial\boldsymbol{q}=0$, $\boldsymbol{A}=0$ and $\zeta_0=0$ which is true for gravity-free space ($V=0$) or circular motion. For the latter the original gestalt of the Schrödinger wave equation remains:

$$E\cdot u(\boldsymbol{q})=-\frac{\hbar^2}{2m_0}\frac{\partial^2 u}{\partial\boldsymbol{q}^2}+V(\boldsymbol{q})\cdot u(\boldsymbol{q}).\tag{56}$$

If we simplify $V = V_{ef}$ to $V_N = -gMm_0/|\boldsymbol{q}|$, the mathematical solutions are in principle available (see [9] sections 38, 39). However, if we consider, for example, two protons and calculate the magnitude of the ratio of the Coulomb force to the gravitational force:

$$\frac{K_C}{K_N} = \frac{\dfrac{e_p^2/|\boldsymbol{q}|^2}{4\pi\epsilon_0}}{\dfrac{Gm_p^2}{|\boldsymbol{q}|^2}} = \frac{9\times10^9\times\left(1.6\times10^{-19}\right)^2}{6.7\times10^{-11}\times\left(1.67\times10^{-27}\right)^2} = 1.2\times10^{38}, \tag{57}$$

the gravitational potential proves to be approximately 10^{38} times weaker than the Coulomb potential. Similarly, the proton-electron combination yields 2.3×10^{39} so that the discrete gravitative energy spectrum of 2 neutrons must therefore be about 10^{34} times smaller than that of the hydrogen atom. The same applies to non-circular motion according to Equation (54), for in the microcosm τ is identical to t and the local variation of mass neglectable. Consequently, applying the Schrödinger Equation (54) to the terrestrial microcosm makes little sense. Furthermore, since the nuclear forces and the forces acting within protons or neutrons exceed the Coulomb force by several orders of magnitude, the unification of the various forces with a gravitative quantum theory is an endeavour that can bear fruit only in worlds where the gravitational force increases by so many orders of magnitude that it can interact with the electromagnetic, nuclear or strong forces.

I am unable to judge whether it is possible in the laboratory to strip an atomic nucleus of its orbiting electrons and instead have a neutron orbit it, thereby capturing the gravitational energy spectrum. The application of Equation (54) to neutron stars [10] seems more promising, as their density is of the same order of magnitude as that of atomic nuclei or even higher. There, clusters of neutrons would gravitationally interact at close distance (then, the variation of mass is no longer be neglectable in Equation (54)) and such clusters would coalesce in different states of aggregation to form the neutron star. Accordingly, quantum-mechanical gravitational processes must occur within them, whose energy spectrum would have to be of about equal order of magnitude or higher as for nuclear force. Consequently, gravitons would possess much higher energies than in the terrestrial microcosm. And if these, strongly red-shifted, were to leave the gravitational field of the neutron star and propagate freely as gravitational waves, they would have to interact with macroscopic bodies. It is precisely this, the existence of gravitational waves, that has now been proven, albeit for other processes, such as the merger of two black holes or a rotating binary pulsar.

Due to the limited volume and high density of the material of a neutron star, the occurrence of relativistic velocities of neutrons or clusters of neutrons is highly unlikely, meaning that the Schrödinger Equation (54) is the right candidate to describe gravitational interactions within stellar matter. Also note that the physics of the interior of neutron stars is studied, among other things, with the help of nuclear physics [10], meaning that all quantum-theoretical prerequisites (uncertainty principle, commutator relations, statistics, Hilbert space, operator representation of physical variables, Fock space etc.) are physically given in neutron stars. The gravitational Schrödinger equation can thus be formulated as an operator equation, and experts in the physics of neutron stars then have a tool at their disposal with which, firstly, to extend quantum physics to include gravity and, secondly, to study the interaction between gravitational and nuclear forces. Similar applies to black holes, which are analysed using methods from particle physics [11]. Before, however, mathematical studies of the solution behaviour of the Schrödinger Equation (54) with variable mass and gravitational potential V_{ef} [9] are necessary for this or for the laboratory experiments mentioned.

3.6. Summary

Allowing for variable mass in classical mechanics leads to new terms in both the Lagrangian and Hamiltonian functions, as well as in the continuity equation. If one restricts oneself to non-relativistic velocities, one obtains, on the basis of the Hamilton-Jacobi equation and the continuity equation, a Schrödinger equation which differs from the classical one only in terms of the mass varying with position and the gravitational potential. The solutions to the gravitative Schrödinger equation in the terrestrial atomic region exhibit a discrete energy spectrum which is a factor of 10^{36} - 10^{40} smaller than the energy spectrum of the hydrogen atom. In contrast, the gravitational potential in neutron stars reaches the strength of the nuclear force and beyond, whilst the velocity of neutrons and clusters of neutrons remains non-relativistic, so that the gravitational Schrödinger equation can prove its worth there. Unlike core and surrounding electrons in the terrestrial atom, neutrons and neutron clusters move in very close proximity, so that the solutions to the gravitational Schrödinger equation cannot be compared with those of the classical Schrödinger equation. Subject to the discovery of the new solutions of Equation (54), a gravitational quantum theory for the microscopically small world of neutron clusters can be developed on this basis, which correspondingly transitions into the macroscopic world of the inner and outer core of a neutron star, occurring in various states of aggregation. Whilst gravitational and nuclear forces operate in the microcosm, which, thanks to the Schrödinger Equation (54), can now be treated within a combined quantum theory, hydrodynamics, plasma physics and ENG come into play in the macrocosm of a neutron star. Note that GR can only specify initial parameters and define boundary conditions such as mass, density and radius of a neutron star, but cannot shed light on the microscopic quantum dynamics of a neutron star, unlike the gravitational Schrödinger equation derived on the basis of ENG.

4. Conclusions

In this part of the paper, it has been shown that the Special Theory of Relativity

[9] See equations I.58-I.59 for photons and equation I.62b-I.64 for massive particles.

can be generalised to non-inertial reference frames without having to depart from Euclidean geometry, abandon the constancy of the speed of light, or invoke any principle of relativity or equivalence. In doing so, it has been proven that, firstly, except in rotating reference frames, the principle of equivalence is not exactly valid, but must undergo a relativistic correction. And that, secondly, the radius and circumference of a rotating disc stretch by the *same* factor. Together with what was proven in the first part, namely that

- Einstein's justification for non-Euclidean geometry is invalid,
- the new theory of gravitation (dubbed ENG) correctly calculates all four key experiments without sacrificing Euclidean geometry and the constancy of the speed of light, or requiring the rate of an atomic clock to depend on its location in a gravitational field,

the *foundations* of GR appear to be seriously called into question. On the other hand, as discussed, thanks to the inherent great flexibility of the g-matrix with which non-Euclidean geometry is equipped to fit the experimental world, GR has provided valuable services for modelling cosmic events and the universe. Nevertheless, *from the point of view of ENG*, GR appears to be a phenomenological theory that puts causal research on the back burner. For everything that Einstein achieved with the leap into non-Euclidean geometry and the introduction of variable speed of light can also be achieved with Euclidean geometry and constant speed of light and a simple differential equation for mass, which is the cause of such diverse phenomena as redshift and deflection of light or the perihelion precision of the planets. The advantage of a causal theory like ENG becomes particularly clear with the Shapiro experiment, in which the runtime is not explained by a curvature of the space-time universe, but by the period of a radar wave changing in the gravitational field, whereby the distance travelled by the light wave is simply the sum of all periods multiplied by the constant speed of light.

ENG provides an alternative theory with the following additional merits:

- Classical mechanics is supplemented only by a differential equation for mass (concept of variable mass), Lagrangian and Hamiltonian formalisms can be applied without difficulty, and there is no need for new principles,
- the "local only" validity of SR and the equivalence principle of GR are avoided and replaced by a new theory of relativity (generalising SR to non-inertial reference frames as described above for the special cases of linearly or circularly accelerated reference frames), which can be combined with ENG to form a relativistic dynamics, just as the kinematics of SR can be combined with Newtonian mechanics,
- the concept of variable mass is readily incorporated into the Schrödinger wave equation which, in strong gravitational fields of neutron stars, can be further developed into a quantum theory in conjunction with nuclear physics.

Unlike GR, ENG thus distinguishes between gravitation and relativity. Similarly, the quantisation of gravity does not seem to require any truly new approach.

Does ENG have any other advantages? The possibility of explaining a fraction of dark energy or matter has already been mentioned. However, the real innovation of ENG seems to be that the addition of a differential equation for mass is a minimal, though not insignificant, extension to mechanics and any other classical theory. It fills a gap that has existed in SR, for example, since the beginning: there is no differential equation for the mass defect from which the mass loss can be calculated. Instead, the mass loss must be inferred indirectly using the energy and momentum theorem for incoming and outgoing particles. Or taking quantum field theories with their phenomenological procedures for renormalising the mass of a particle, a differential equation of mass could provide a remedy here.

This means that Equation (I.24) has to be adapted to other forces than gravity. What has proven so successful with mass, the gravitative charge should also play a significant role for other types of charges.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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