

Reassessing John Bell's Response to EPR

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Abstract

I identify the error that John Bell suspected in the defiance of his inequality by quantum probabilities when presuming local realism, in the form he had presented it in his original publication of 1964. While he was aware of one functional restriction among the three spin-products involved in deriving the inequality, there are actually three such binding restrictions, and he neglected all three when assessing it. The situation is resolved by linear programming computations that support the locality of quantum phenomena. The inequality is not defied by quantum probabilities, the quantum expectations are found to be stable, and local realism is preserved.

Keywords

Bell Original Inequality, Locality, Linear Programming

1. Introduction

There are two notable features of John Bell's assessment of his own momentous work [1] [2]. While he actively defended his discoveries against emerging criticism that he considered unwarranted [3] [4], he suspected that something might be wrong with the result he had discovered, and the conclusions it implied. It concerned him that if the property of non-locality of quantum mechanics were different than that of classical mechanics where locality is presumed, there must be some boundary scale at which the structure of physical laws change. He published this concern [5] in 1971, and republished it [6] in 1987 before his death, surmising that the quandary would be resolved in time:

“A possibility is that we will find where the boundary is. More plausible to me is that we will find that there is no boundary. It is hard for me to envisage intelligible discourse about a world with no classical part—no base of given events, be they only mental events in a single consciousness, to be correlated. On the other hand, it is easy to imagine that the classical domain could be

extended to cover the whole. The wave function would prove to be a provisional or incomplete description of the quantum mechanical part, of which an objective account would become possible. It is this possibility, of a homogeneous account of the world, which is for me the chief motivation of the study of the so-called ‘hidden variable’ possibility.”

While he deferred to his mathematical derivation of the famous inequality and its defiance by quantum probabilities, he could also be suspicious of his own capacity for assuring himself. He modestly joked of his credentials in a letter to Dr. Itamar Pitowski that has been reported in a historical commentary by Marian Kupczynski [7]: “I have much difficulty in understanding your proposal—not least because my competence in mathematics does not go beyond kindergarten.”

All modesty and joking aside, Bell has been appropriately lionized for his accomplishments, as have been John Clauser, Alain Aspect, and Anton Zeilinger among the many who have continued his line of enquiry. These have supported his conclusion that “no local hidden-variable theory can reproduce all the experimental predictions of quantum mechanics”. As it turns out, however, it was Bell’s intuition, not his mathematics that have won the day. Despite such eminent support for his legacy, I have made bold to challenge the works of these leading physicists as mistaken, in five published papers [8]-[12] and with a wide-ranging reassessment of the situation in a book [13] which extends them.

In this present article, I reassess Bell’s original published article [1] of 1964, providing complete detail of the errors it contains. I shall presume the reader’s basic familiarity with the context of the relevant experiments in which the primacy of observed magnetic “spin-products” of electrons comes to the fore, and with the computational method of linear programming.

2. The Setup of Bell’s Gedankenexperiment

Bell’s original article assessed the possible results observed in a *gedankenexperiment* conducted on a pair of electrons that are ejected in opposite directions from their common source toward distant observation stations, A and B . Here they engage a pair of Stern-Gerlach magnets simultaneously (and impossibly) at three distinct directional alignment pairings at the two stations, denoted by $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}, \mathbf{c})$, and $(\mathbf{b}, \mathbf{c})$. In Bell’s setup, these would yield the observed spin-products $A(\mathbf{a})B(\mathbf{b})$, $A(\mathbf{a})B(\mathbf{c})$, and $A(\mathbf{b})B(\mathbf{c})$. The numerical values of A and B designating the observed spin as down or up in each case equal -1 or $+1$, as do their products. While we can actually perform this experiment at any one of the directional pairings of the magnets envisaged, we cannot engage all three of them simultaneously. We can only think about it. Moreover, quantum theory proclaims nothing specifically about such *joint* pairing results, because the matrix operators that define the three of them do not commute. Nonetheless, the thought experiment was proposed so to assess the validity of the EPR claim: that the valid probabilistic prognosis of quantum theory about its results can be explained by the

influence of unknown and unobserved supplementary variables pertinent to conditions at the two stations. These are typically denoted by a vector λ , whose possible values are assessed with a rotationally invariant density $\rho(\lambda)$.

For the record, the quantum probabilities pertinent to paired electron spins observed at a relative angle $(\mathbf{d}_A, \mathbf{d}_B)$ between the magnet directions $\mathbf{d}_A$ and $\mathbf{d}_B$ at the two stations are:

Joint spin probabilities:

$$\begin{aligned} P[(A(\mathbf{d}_A) = +1)(B(\mathbf{d}_B) = +1)] \\ = P[(A(\mathbf{d}_A) = -1)(B(\mathbf{d}_B) = -1)] = \frac{1}{2} \sin^2[(\mathbf{d}_A, \mathbf{d}_B)/2], \end{aligned}$$

and

$$\begin{aligned} P[(A(\mathbf{d}_A) = +1)(B(\mathbf{d}_B) = -1)] \\ = P[(A(\mathbf{d}_A) = -1)(B(\mathbf{d}_B) = +1)] = \frac{1}{2} \cos^2[(\mathbf{d}_A, \mathbf{d}_B)/2]. \end{aligned}$$

(1)

Marginal spin probabilities:

$$P[(A(\mathbf{d}_A) = +1)] = P[(B(\mathbf{d}_B) = +1)] = 1/2. \quad (2)$$

and Conditional spin probabilities:

$$P[(A(\mathbf{d}_A) = +1) | (B(\mathbf{d}_B) = +1)] = \sin^2[(\mathbf{d}_A, \mathbf{d}_B)/2] \neq P[(A(\mathbf{d}_A) = +1)] = 1/2,$$

and

(3)

$$P[(A(\mathbf{d}_A) = +1) | (B(\mathbf{d}_B) = -1)] = \cos^2[(\mathbf{d}_A, \mathbf{d}_B)/2], \text{ which is different still.}$$

Further relevant is the

Expectation of spin-product observations:

$$E[A(\mathbf{d}_A)B(\mathbf{d}_B)] = -\cos[(\mathbf{d}_A, \mathbf{d}_B)]. \quad (4)$$

On account of the symmetric structure of this distribution, the specification of any single joint probability for a result at A and B identifies the entire distribution of four probabilities, as would the mere specification of the expectation of the spin product.

The mathematical conclusion of Bell's assessment was his famous inequality, formulated in terms of the quantum theoretic expectations of the spin-products as

$$1 + E[A(\mathbf{b})B(\mathbf{c})] \geq |E[A(\mathbf{a})B(\mathbf{b})] - E[A(\mathbf{a})B(\mathbf{c})]|. \quad (5)$$

Another of many subsequent forms of the inequality specifies it in terms of the probabilities for specific paired spin outcomes, as

$$\begin{aligned} P[(A(\mathbf{a}) = +1)(B(\mathbf{b}) = +1) | \theta(\mathbf{a}, \mathbf{b})] + P[(A(\mathbf{b}) = +1)(B(\mathbf{c}) = +1) | \theta(\mathbf{b}, \mathbf{c})] \\ \geq P[(A(\mathbf{a}) = +1)(B(\mathbf{c}) = +1) | \theta(\mathbf{a}, \mathbf{c})], \end{aligned} \quad (6)$$

where $\theta(\mathbf{d}_A, \mathbf{d}_B)$ is the relative angle between the magnet directions at stations A and B .

The first Equation (5) displays the form expressed in Bell's original published

article of 1964. The second Equation (6) is a form reported in a university level textbook by Holbrow *et al.* [14]. The relation between the two formats comes from an application of the inverse relation $\theta(\mathbf{a}, \mathbf{b}) = \text{Arccos}(-E[A(\mathbf{a})B(\mathbf{b})])$ at the various paired magnet angles, and then

$$P[(A(\mathbf{a}) = +1)(B(\mathbf{b}) = +1) | \theta(\mathbf{a}, \mathbf{b})] = (1/2) \sin^2(\theta/2).$$

It is important to highlight the status of what is being assessed by Bell's inequality as a "thought experiment", because this has come to be denied for various reasons in subsequent literature that has furthered the discussions. Nonetheless, it is evident that the theorem relies on such a threesome of imagined paired-spin-observations on a single pair of electrons. For Bell embeds into its derivation his recognition that the value of $A(\mathbf{a}, \lambda)$ must equal negative the value of $B(\mathbf{a}, \lambda)$, so to respect the quantum specification of $E[A(\mathbf{a})B(\mathbf{a})] = -\cos(0) = -1$. Such a condition would pertain only if it were the same pair of electrons being considered to be observed engaging with the Stern-Gerlach magnets at stations A and B in the three components of the imagined setup. So to allow me to proceed with my assessment of the situation, I shall defer to an **Appendix 1** a commentary on an important and influential article of Hess and Philipp [15] which has challenged this claim. This follows the Acknowledgments at the conclusion of the article.

There is reason to propose consideration of such an experiment that cannot possibly be engaged. We are only *thinking* about such a three-pronged experiment on the electron pair. Its consideration was designed so to assess Einstein's claim that the structure of mechanical laws should be recognized as local at the quantum scale as well as at the scale of classical mechanics. We want to assess the EPR proposition that the state of the pair in a real experiment would be uniquely identified, no matter how the directional angles of the magnets are set. It is proposed that it would be influenced by supplementary variables, honoring the locality of the mechanical processes at each of the two stations.

Standard quantum theory denies this EPR proposition, tendering that the state of the paired electrons is *not* identified uniquely. The pair reside merely in a probabilistic superposition of states. The EPR challenge proposed, rather, that there *is* some state of the electron pair that is determined not only by the angle pairing of the magnets they pass, but also by further supplementary variables that current quantum theory does not completely specify. In this view, the probabilities are not superposing properties of the electron pair, but rather represent considered scientific symmetric uncertainties about the influences of the unspecified variables.

We are ready to clarify the context to which Bell's inequality pertains, and to expose the error involved in its evaluation in terms of quantum probabilities that respect the principle of local realism.

3. The Realm of Possibilities for Bell's Experiment

Let's now examine what could possibly happen if we conducted Bell's gedanken-experiment at any three angle pairings of the Stern-Gerlach magnets at stations A

and B . We are interested specifically in the pairings $(\mathbf{a}, \mathbf{b})$, $(\mathbf{b}, \mathbf{c})$ and $(\mathbf{a}, \mathbf{c})$ apropos to his inequality (5). To assess these, we first display all six possible individual spin results if we were to imagine observing the electron spins at the two stations with their paired magnet directions set in all three directions $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. We will then compute the implied spin-products at the three associated direction pairings relevant to Bell. The listing presumes local realism via its designation that the value of $A(\mathbf{a})$ would be the same when pairing with $B(\mathbf{b})$ or with $B(\mathbf{c})$, presumably determined locally.

Displayed below is a (10×8) matrix that is partitioned horizontally into six and four rows, and vertically into a pair of four columns. We call this the *realm matrix* for the spin observations and their products at the two stations. Its columns denote possible observable situations of the (10×1) vector of quantities named on its left. We can ignore the vertical partitioning of this realm for the moment. The first six rows in the upper partition exhibit spin observation components of conceivable experiments on a pair of electrons imagined to be conducted at stations A and B . The first three rows of the eight columns of the matrix correspond to the Cartesian product $\{+1, -1\}^3$ of possibilities for spin results at three magnet directions at station A . These are followed in each column by their negated values for observations at station B at the same corresponding magnet directions. Such a feature is specified by the quantum probabilities designated in Equations (1) when the directions at A and B are identical, and is also commonly observed experimentally at parallel magnet settings. It was recognized by Bell in his insisted equation $A(\mathbf{a}, \lambda) = -B(\mathbf{a}, \lambda)$.

$$\mathbf{R} \begin{pmatrix} A(\mathbf{a}) \\ A(\mathbf{b}) \\ A(\mathbf{c}) \\ B(\mathbf{a}) \\ B(\mathbf{b}) \\ B(\mathbf{c}) \\ \vdots \\ A(\mathbf{a})B(\mathbf{b}) \\ A(\mathbf{b})B(\mathbf{c}) \\ A(\mathbf{a})B(\mathbf{c}) \\ A(?)B(*) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & \cdots & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & \cdots & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & -1 & 1 & 1 & \cdots & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & \cdots & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & \cdots & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & \cdots & -1 & -1 & -1 & -1 \end{pmatrix} \quad (7)$$

The first three rows in the lower partition then exhibit corresponding *spin-product* observations at the specific magnet direction pairings relevant to Bell's inequality. They are computed as term-by-term products of the rows of possibilities for their multiplicands. The final peculiar row is to be discussed.

We are ready now to notice something unusual about these spin-product possibilities, which has not been recognized by those who subscribe to the supposed defiance of Bell's inequality by quantum probabilities!

Quantum theory specifies the expected value of each of the spin-products that

could be generated in a real experiment, whose possibilities are displayed in the first three rows of the lower partition. Specifically, for any one of them,

$E[A(\mathbf{d}_A)B(\mathbf{d}_B)] = -\cos(\mathbf{d}_A, \mathbf{d}_B)$, where $(\mathbf{d}_A, \mathbf{d}_B)$ denotes the relative angle between the directions of the magnet settings at stations A and B . No one disputes these.

To be specific, the theory identifies the expectation value for each row quantity designated in the lower partition of the matrix for chosen magnet directions $\mathbf{a}$, $\mathbf{b}$, or $\mathbf{c}$, if each angle pairing were chosen on its own for a real experiment with a pair of electrons. However, *quantum theory asserts nothing specific about the component-product of any two of these rows!* That would represent the product of two such spin-products generated on a single pair of electrons. For the quantum probabilities associated with each row derive from a distinct quantum operator, applied to a paired electron state measured in a single paired experiment. And none of the operators identifying these rows of product observations commutes with any other.

This is to say, a simultaneous experiment yielding *the product of any two electron spin-products* cannot be conducted, and is not even being imagined here. We know this as a matter of experimental fact, irrespective of any recourse to quantum theory. Non-commutability of the quantum operators merely verifies this fact. Moreover, quantum theory abstemiously avoids saying anything precise about the results of such an experiment that cannot be conducted. This position results from the uncertainty principle. The theory makes no specific claims about the results of impossible experiments.

Why bring this up now? Examine the first three rows of observation possibilities displayed in the lower partition of the realm matrix (7). Notice that each of the first three rows of quantity possibilities in the lower partition is constrained to equal the negative value of the product of corresponding components in the other two rows of possibilities. Each one of the three spin-products is determined as a function value of the other two rows, their componentwise product.

This product constraint derives from applying the principle of local realism to this procedure for generating the observation possibilities of the spin-products. Our computational method has embedded the fact that the value of $A(\mathbf{a})$ involved in computing $A(\mathbf{a})B(\mathbf{b})$ in each column, for example, is the same as its value incorporated into the computation of $A(\mathbf{a})B(\mathbf{c})$ for that column. For the local context underlying the observation of $A(\mathbf{a})$ is identical in the two scenarios. True, quantum theoretic probabilities *can* be asserted for the quantities designated for any two rows of the lower partition. However, they are *not* asserted for the *product* of any two rows!

This functional restriction means that if we use quantum probabilities to specify agreeable expectations for $E[A(\mathbf{a})B(\mathbf{b})]$ and for $E[A(\mathbf{b})B(\mathbf{c})]$ then we may not use unfettered quantum probabilities to specify the expectation $E[A(\mathbf{a})B(\mathbf{c})]$ as well. For the value of $A(\mathbf{a})B(\mathbf{c})$ is constrained algebraically by the equation

$$A(\mathbf{a})B(\mathbf{c}) = -A(\mathbf{a})B(\mathbf{b})A(\mathbf{b})B(\mathbf{c}) \quad (8a)$$

a product of non-commuting operations that the theory does *not* assess.

In fact, John Bell did use *this* constraining result in the valid derivation of his inequality, on the basis of an application of (4). However, he did not realize its full implications in assessing the defiance of the inequality! For he routinely used the quantum expectations for all three spin-products in evaluating components of the inequality in the gedankenexperiment. He proceeded as if the possibilities for $A(\mathbf{a})B(\mathbf{c})$ companion with those of $A(\mathbf{a})B(\mathbf{b})$ and $A(\mathbf{b})B(\mathbf{c})$ were functionally independent of them, i.e., as if the row of its possibilities were that displayed in the *fourth row* of the lower partition in the realm matrix. These are possibilities for a *different* quantity, the peculiar final component of the quantity vector denoted by $A(?)B(*)$ in Equation (7).

Interestingly, this row of the realm matrix is identical with the first row of the realm matrix, the possibilities for $A(\mathbf{a})$. The quantum expectation for this quantity, of course, equals 0. However, this is *not* $E[A(\mathbf{a})B(\mathbf{c})]$.

Moreover, Bell did not recognize two companion functional restrictions among the three spin-products that are involved in his inequality at all! These are the restrictions

$$A(\mathbf{b})B(\mathbf{c}) = -A(\mathbf{a})B(\mathbf{b})A(\mathbf{a})B(\mathbf{c}) \text{ for } B(\mathbf{b}) = -A(\mathbf{b}) \text{ and } A^2(\mathbf{a}) = 1 \quad (8b)$$

$$\text{and } A(\mathbf{a})B(\mathbf{b}) = -A(\mathbf{b})B(\mathbf{c})A(\mathbf{a})B(\mathbf{c}) \text{ for } A(\mathbf{b}) = -B(\mathbf{b}). \quad (8c)$$

These restrictions are evident algebraically, as well as by examination of the realm matrix in Equation (7).

Let me be clear. Quantum theory *does make* appropriate and accurate prognostications about the results of quantum experiments that can actually be conducted. It provides expectation values for results of specific paired experiments that yield any one of the spin-product observations that can actually be generated: either $A(\mathbf{a})B(\mathbf{b})$, or $A(\mathbf{b})B(\mathbf{c})$, or $A(\mathbf{a})B(\mathbf{c})$. It also would allow the assertion of quantum expectation values for spin-product values of *any two* paired component experiments of Bell's gedankenexperiment. Their paired possibilities are exhibited in two corresponding rows of the lower partition of the realm matrix. It even would allow assertions of quantum expectations for sequences of spin-products on distinct pairs of electrons. However, it *does not* permit itself to assert freely *all three* such expectations for a single pair of electrons simultaneously in an evaluation of Bell's inequality. For each of their possibility vectors must equal negative the component-product of the other two.

4. What Does Quantum Theory Assert about Bell Products?

Nonetheless, the mathematical implications of what the theory does assert can be made precise using computational procedures of linear programming. They derive from a theorem of Bruno de Finetti, which he called "the fundamental theorem of probability." Here is how it works.

Examine once again the lower horizontal partition of the realm matrix displayed as Equation (7). This partition is itself partitioned vertically into two halves

of four columns, for which each half of the first three rows constitutes an identical reflection of the other. (The fourth row for the “peculiar” quantity named $A(?)B(*)$ is irrelevant at the moment.) Now eliminate the redundant right partition from consideration, and examine only the four columns of the top three rows of the left half lower partition. These constitute the vectors of joint possible observations of the spin-products that are pertinent to the Bell inequality, in either form. The mathematical expectation of any one of these products must equal a convex combination of the components in its corresponding row of possibilities in the left partition of the realm matrix. What convex combination? There are many that could suffice, and we can identify them precisely, computationally.

When we do assert quantum expectations for any two of these spin-products, we can design a linear programming computation to identify the extreme bounds on the cohering expectation of a third that are actually motivated by quantum theory. These are associated with the convex coefficient vectors that yield extreme values of an objective function in an LP computation. Following are the details for one specific example showing how this is done, using magnet directions at which it has been thought that Bell’s inequality is defied.

Designate the first three row vectors of the lower left partition of the realm matrix by $\mathbf{r}_{A(a)B(b)}$, $\mathbf{r}_{A(b)B(c)}$, and $\mathbf{r}_{A(a)B(c)}$, respectively. In a context where the relative angles of the Stern-Gerlach magnets are specified as $(\mathbf{a}, \mathbf{b}) = \pi/3$ and $(\mathbf{a}, \mathbf{c}) = 2\pi/3$, for example, the quantum theoretic assertions of $E[A(\mathbf{a})B(\mathbf{b})] = -\cos(\pi/3) = -0.5$ and of $E[A(\mathbf{a})B(\mathbf{c})] = -\cos(2\pi/3) = +0.5$, when assessed on their own. In this context, then the angle $(\mathbf{b}, \mathbf{c})$ also equals $\pi/3$, the difference between the former two angles. But what is the value of $E[A(\mathbf{b})B(\mathbf{c})]$?

It is true that a univocal assertion of the quantum expectation $E[A(\mathbf{b})B(\mathbf{c})]$ on its own would also equal $-\cos(\pi/3) = -0.5$, a value that would conjoin with the quantum expectations $E[A(\mathbf{a})B(\mathbf{b})]$ and $E[A(\mathbf{a})B(\mathbf{c})]$ to defy Bell’s inequality. Inserting these three values into the inequality (1) would make the left-hand-side equal $1 - 0.5 = 0.5$, while the right-hand-side would equal the absolute value $|-0.5 - (+0.5)| = +1$, and it appears that the inequality would be defied. *However, quantum theory does not make this assertion of $E[A(\mathbf{b})B(\mathbf{c})]$ in the context of Bell’s imagined experiment when the former two expectations are asserted!*

The coherent implications of the quantum assertions regarding $A(\mathbf{a})B(\mathbf{b})$ and $A(\mathbf{a})B(\mathbf{c})$ yield only an interval of values for the companion assertion of $E[A(\mathbf{b})B(\mathbf{c})]$. This turns out to be the interval $[0, 1]$ which does not include the unfettered expectation value of -0.5 . The endpoints of this interval derive as bounds from the following linear programs:

$$\text{Find the vectors } \mathbf{q}_4 \text{ that yield } \left(\begin{matrix} \min \\ \max \end{matrix} \right) E[A(\mathbf{b})B(\mathbf{c})] = \mathbf{r}_{A(b)B(c)} \mathbf{q}_4,$$

subject to the linear constraints

$$r_{A(a)B(b)}q_4 = -0.5 \quad \text{and} \quad r_{A(a)B(c)}q_4 = +0.5$$

along with the unit-sum constraint $\mathbf{1}_4^T \mathbf{q}_4 = 1$ on non-negative $\mathbf{q}_4$.

We will name these solution vectors $\mathbf{q}_{\min}(\mathbf{b}, \mathbf{c})$ and $\mathbf{q}_{\max}(\mathbf{b}, \mathbf{c})$, and report them here in their transposed forms:

$$\begin{pmatrix} \mathbf{q}_{\min}^T(\mathbf{b}, \mathbf{c}) \\ \mathbf{q}_{\max}^T(\mathbf{b}, \mathbf{c}) \end{pmatrix} = \begin{pmatrix} 0.25 & 0.50 & 0 & 0.25 \\ 0 & 0.75 & 0.25 & 0 \end{pmatrix}.$$

These yield a bounding interval of $[0, 1]$ for the objective function $r_{A(b)B(c)}q_4$ that coheres with the asserted quantum expectations of $E[A(\mathbf{a})B(\mathbf{b})]$ and $E[A(\mathbf{a})B(\mathbf{c})]$.

Every number in this interval for $E[A(\mathbf{b})B(\mathbf{c})]$ cohering with the specified quantum theoretic expectations for the other two spin-products satisfies Bell's inequality. In Bell's original form of Equation (5), this requires

$$1 + E[A(\mathbf{b})B(\mathbf{c})] \geq |E[A(\mathbf{a})B(\mathbf{b})] - E[A(\mathbf{a})B(\mathbf{c})]| = |-0.5 - 0.5| = 1,$$

an equation that is satisfied everywhere in the cohering quantum interval $[0, 1]$ for $E[A(\mathbf{b})B(\mathbf{c})]$.

As to the other form of Bell's inequality, we can find that by evaluating $\theta(\mathbf{b}, \mathbf{c}) = \text{Arccos}(-E[A(\mathbf{b})B(\mathbf{c})])$ at the end points of its bounding quantum interval, $[0, 1]$. These yield $\text{Arccos}(0) = \pi/2$, and $\text{Arccos}(-1) = \pi$. Thus, a lower extreme value of

$$P[(A(\mathbf{b}) = +1)(B(\mathbf{c}) = +1) | \theta(\mathbf{b}, \mathbf{c})] = \frac{1}{2} \sin^2(\pi/4) = 0.25,$$

and an upper extreme value of $\frac{1}{2} \sin^2(\pi/2) = 0.5$. Both of these endpoints of the cohering quantum interval for $P[(A(\mathbf{b}) = +1)(B(\mathbf{c}) = +1) | \theta(\mathbf{b}, \mathbf{c})]$ are smaller than or equal to the sum

$$\begin{aligned} & P[(A(\mathbf{a}) = +1)(B(\mathbf{b}) = +1) | \theta(\mathbf{a}, \mathbf{b})] + [(A(\mathbf{a}) = +1)(B(\mathbf{c}) = +1) | \theta(\mathbf{b}, \mathbf{c})] \\ &= \frac{1}{2} [\sin^2(\pi/6) + \sin^2(\pi/3)] = 0.5, \end{aligned}$$

as required by Bell's inequality in this form. Quantum probabilities do not defy the inequality.

However, this assessment does not exhaust what quantum theory has to say about the situation. As noted, quantum theory supports the assessment of *any two* of the spin-products specified by the Bell inequalities. The results just reported express the implications of jointly asserting the expectations $E[A(\mathbf{a})B(\mathbf{b})]$ and $E[A(\mathbf{a})B(\mathbf{c})]$ for its cohering assertion of $E[A(\mathbf{b})B(\mathbf{c})]$. *In addition, the theory also provides* asserted expectation values for each of the other two pairs of spin-products, and similar programming problems would yield the companion bounded assertions of the expectation for the remaining one in each case. The

solutions to all three paired min-max linear programming problems yield bounding convex combination vectors $\mathbf{q}_{\min}$ and $\mathbf{q}_{\max}$ that identify the complete solution to Bell's experiment. They are displayed in **Table 1**. Row headings such as $P[A(\mathbf{a})B(\mathbf{b})++]$ denote conditional probabilities such as $P[(A(\mathbf{a})=+1)(B(\mathbf{b})=+1)|\theta(\mathbf{a},\mathbf{b})]$. Comments follow the Table.

Table 1. $\begin{pmatrix} \min \\ \max \end{pmatrix}$ LP solution vectors, and implications for spin-product expectations and for the Bell inequalities at magnet angles $(\mathbf{a},\mathbf{b})=\pi/3$, $(\mathbf{a},\mathbf{c})=2\pi/3$, and $(\mathbf{b},\mathbf{c})=\pi/3$.

	$\mathbf{q}_{\min}(\mathbf{b},\mathbf{c})$	$\mathbf{q}_{\max}(\mathbf{b},\mathbf{c})$	$\mathbf{q}_{\min}(\mathbf{a},\mathbf{b})$	$\mathbf{q}_{\max}(\mathbf{a},\mathbf{b})$	$\mathbf{q}_{\min}(\mathbf{a},\mathbf{c})$	$\mathbf{q}_{\max}(\mathbf{a},\mathbf{c})$
$\mathbf{q}_1$	0.25	0	0.25	0	0.75	0.5
$\mathbf{q}_2$	0.5	0.75	0.25	0	0	0.25
$\mathbf{q}_3$	0	0.25	0	0.25	0.25	0
$\mathbf{q}_4$	0.25	0	0.5	0.75	0	0.25
	*****	*****	*****	*****	*****	*****
$E[A(\mathbf{a})B(\mathbf{b})]$	-0.5	-0.5	0	1	-0.5	-0.5
$E[A(\mathbf{b})B(\mathbf{c})]$	0	1	-0.5	-0.5	-0.5	-0.5
$E[A(\mathbf{a})B(\mathbf{c})]$	0.5	0.5	0.5	0.5	-1	0
	*****	*****	*****	*****	*****	*****
LHSBellIneq	1	2	.5	.5	.5	.5
RHSBellIneq	1	1	.5	.5	.5	.5
	*****	*****	*****	*****	*****	*****
$P[A(\mathbf{a})B(\mathbf{b})++]$	0.125	0.125	0.25	.5	0.125	0.125
$P[A(\mathbf{b})B(\mathbf{c})++]$	0.25	.5	0.125	0.125	0.125	0.125
$P[A(\mathbf{a})B(\mathbf{c})++]$	0.375	0.375	0.375	0.375	0	0.25
	*****	*****	*****	*****	*****	*****
LHSBellProbs	0.375	0.625	0.375	0.625	0.25	0.25
RHSBellProbs	0.375	0.375	0.375	0.375	0	.25

In its first four rows, Table 1 exhibits six columns of the paired solution vectors to the three min-max programming problems we have formulated, with their several computed implications listed below them. Row headings identify the content of each element of that row, and use concise notation to denote the joint probability $P[(A(\mathbf{a})=+1)(B(\mathbf{b})=+1)|\theta(\mathbf{a},\mathbf{b})]$ by $P[A(\mathbf{a})B(\mathbf{b})++]$. Among them are the heading labels LHSBellIneq and RHSBellIneq for the two sides of the Bell Inequality in its original form (5) involving expectations, and LHSBellProbs and RHSBellProbs for the sides of the inequality (6) in its probability form. In every column of the extreme $\mathbf{q}$ vectors, the left-hand side of the

inequality is greater than or equal to the right-hand-side, in both its forms. Bell's inequality is satisfied at every point in the convex hull of probability distributions that quantum theory actually supports.

Next, **Figure 1**, displays the convex hull of the vectors of expected spin-products that are associated with these six solution vectors. Rather than specifying a unique triple of expectations for the three spin-products entailed in Bell's inequality, quantum theory designates an entire convex polytope of triple-expectations that it supports, all of which satisfy the inequality. It is notable that the triple of expectations for three separate experiments on different electron pairs, designated by a star, does not sit within the hull. It would constitute an incoherent assessment of expectations for the components of Bell's gedankenexperiment.

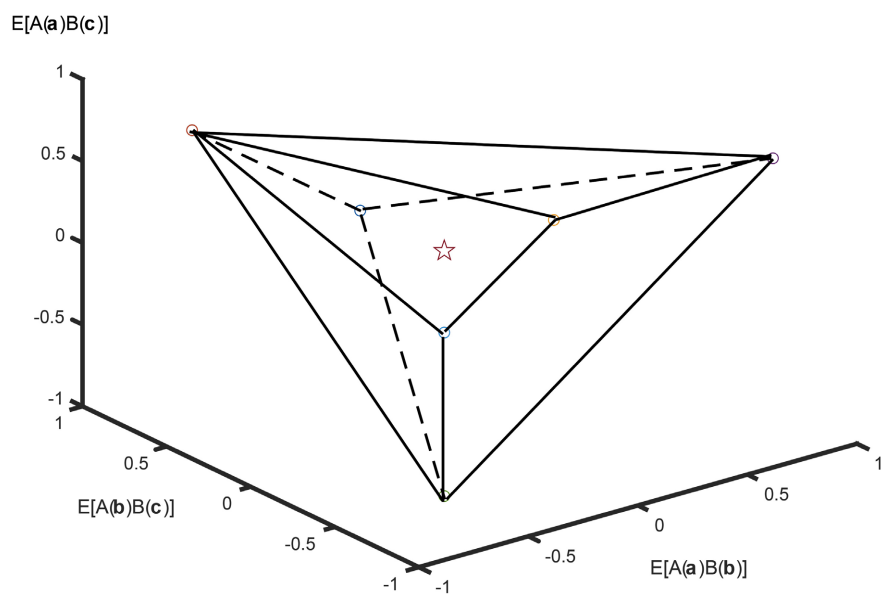


Figure 1. The convex hull of expected spin-product vectors supported by quantum theory for Bell's electron pair, thought to be "observed" at three paired magnet directions. The naive application of three quantum expectations that ignore the functional restrictions embedded in the problem sits outside the hull. The triple $(-0.5, -0.5, +0.5)$ can be seen sitting directly above the lower front edge of the hull, and is incoherent.

5. Bell's Original Concerns with Small Angles

Bell's original article did not display an example such as this, for which his inequality might be defied. (As we have seen, it is not so defied.) Rather, he was avowedly concerned with the stationarity of quantum prognostication at very small differences between two of the magnet directions in three components of the gedanken runs, specified by $\mathbf{b}$ and $\mathbf{c}$. In concluding his quantum theoretic assessment of the inequality he had derived in the form of our Equation (1), Bell remarked rather cryptically that the expectation $E[A(\mathbf{b})B(\mathbf{c})]$ "cannot be stationary at its minimum value (which is -1 when $\mathbf{b} = \mathbf{c}$), and cannot equal the

quantum mechanical value” which is $-\cos(\mathbf{b}, \mathbf{c})$. He merely recognized that if the direction $\mathbf{b}$ is quite close to $\mathbf{c}$, then the right-hand side of his inequality (5) is near to the angular distance $|\mathbf{b} - \mathbf{c}|$. So it appeared that the left-hand-side would be much smaller than the right, *not larger* as the inequality requires. We shall reassess the situation in an example shortly.

The allusion to non-stationarity pertains to the fact that if the magnet directions were actually identical ($\mathbf{b} = \mathbf{c}$) then inequality (5) would say $0 \geq 0$, which *is* true; whereas he thought that if the paired magnet directions specify an angle merely close to 0 (however close) it remains false. *This is not true*. Although Bell was surely on the mark with this concern, his analysis was let down by inattention to the complete set of functional restrictions entailed among the three spin-products in the setup of his gedankenexperiment. We will need to generate an example to investigate the matter.

Suppose $(\mathbf{a}, \mathbf{b}) = 0.8888888$ (in units of radians) and $(\mathbf{a}, \mathbf{c}) = 0.8888887$, which makes the angle $(\mathbf{b}, \mathbf{c}) = -0.0000001$. Then naively inserting the quantum expectations for individual spin-products, it appears that the left-hand-side of the inequality yields

$$\text{LHS} = 1 - \cos(-0.0000001) = 4.9960 \times 10^{-15},$$

while the right-hand-side yields

$$\text{RHS} = |-\cos(0.8888888) + \cos(0.8888887)| = 7.7637 \times 10^{-8},$$

which is a *larger* value to be sure. This RHS value is indeed quite close to $|\mathbf{b} - \mathbf{c}|$ which equals 1×10^{-7} , just as Bell had noticed. If these evaluations of both sides of the inequality were correct, the inequality would be defied. Well, something must be wrong, as Bell himself suspected in the quotation that introduced this article. The structure of a linear programming problem will sort this out for us.

Let us begin analysis by specifying the quantum assertions of $E[A(\mathbf{a})B(\mathbf{b})]$ and $E[A(\mathbf{a})B(\mathbf{c})]$ at these directional pairings. Since $A(\mathbf{b})B(\mathbf{c})$ is constrained to equal negative the product of these two spin-product observations, $-A(\mathbf{a})B(\mathbf{b})A(\mathbf{a})B(\mathbf{c})$, quantum theory says nothing specific about it, for the operators that generate these two product-multiplicands do not commute. However, we can employ a linear programming routine to determine precise bounds on the cohering quantum assertion of $E[A(\mathbf{b})B(\mathbf{c})]$ that the two presumed quantum expectations imply. We shall need to assess additionally two similar computations that begin with specifications of quantum assertions for the other two choices of the angle pairings at two of the three stations as well. Now the orders of magnitude of several component results in this computation are quite small, requiring that we present details with extreme precision, to many decimal places.

Consider again these relative magnet angles on the order at which Bell thought his inequality was defied: $(\mathbf{a}, \mathbf{b}) = 0.8888888$ and $(\mathbf{a}, \mathbf{c}) = 0.8888887$. Then with greater precision, the quantum theoretic evaluation of the right-hand-side of

Bell's inequality (1), an absolute value, equals

$$| -\cos(0.8888888) + \cos(0.8888887) | = 7.763718345987769 \times 10^{-8},$$

when reported through sixteen non-zero decimal places, twenty-three decimal places in all. Concomitantly, only bounds can be computed for the quantum theoretic evaluation of the left-hand-side of the inequality that agree with these assessments. These derive from solutions to the linear programs

$$\begin{pmatrix} \min \\ \max \end{pmatrix} 1 + E[A(\mathbf{b})B(\mathbf{c})] = 1 + \mathbf{r}_{A(\mathbf{b})B(\mathbf{c})} \mathbf{q}_4,$$

subject to the linear constraints

$$\mathbf{r}_{A(\mathbf{a})B(\mathbf{b})} \mathbf{q}_4 = -\cos(0.8888888) \text{ and } \mathbf{r}_{A(\mathbf{a})B(\mathbf{c})} \mathbf{q}_4 = -\cos(0.8888887),$$

along with the unit constraint $\mathbf{1}_4^T \mathbf{q}_4 = 1$ on non-negative $\mathbf{q}_4$.

The bounding interval for the objective function $1 + \mathbf{r}_{A(\mathbf{b})B(\mathbf{c})} \mathbf{q}_4$ that coheres with the constraining quantum probabilities is quite wide, with a lower bound of $7.763718368192229 \times 10^{-8}$ at the minimum problem solution, and the sizeable value of 0.739449682495256 at the maximum problem solution. Every number in this interval motivated by quantum theoretic probabilities exceeds the right-hand-side of Bell's inequality, beginning with the sixteenth decimal place at the lower bound! Let's exhibit the right-hand-side of the inequality just below the minimum value of the left-hand side together, for edification:

$$\text{minimum LHS} = 7.763718368192229 \times 10^{-8}$$

$$\text{while the RHS} = 7.763718345987769 \times 10^{-8},$$

a smaller number beginning at the sixteenth decimal place. The upper bound on the right hand side cohering with the quantum probability specifications is sizeable, viz., 0.739449682495256.

The inequality is surely satisfied by every expectation complex that quantum theory supports, contrary to Bell's assessment which had ignored the constraining equations we have identified.

This does not complete our assessment of the quantum probabilities that are relevant. Next, we presume quantum probabilities for spin products at another chosen pair of these related magnet angles at which Bell thought his inequality was defied, $(\mathbf{a}, \mathbf{b}) = 0.8888888$ (in units of radians) and $(\mathbf{b}, \mathbf{c}) = -0.0000001$. Then the left-hand-side of Bell's inequality equals

$$\text{LHS} = 1 - \cos(-0.0000001) = 4.996003610813204 \times 10^{-15},$$

which is quite a small number, to be sure.

However, the bounds on the right-hand-side of the inequality that is enclosed in the absolute value operator is even smaller, ... zero in fact! This derives from the linear programming setup

$$\begin{pmatrix} \min \\ \max \end{pmatrix} E[A(\mathbf{a})B(\mathbf{b})] - E[A(\mathbf{a})B(\mathbf{c})] = [\mathbf{r}_{A(\mathbf{a})B(\mathbf{b})} - \mathbf{r}_{A(\mathbf{a})B(\mathbf{c})}] \mathbf{q}_4$$

subject to the linear constraints

$$\mathbf{r}_{A(a)B(b)}\mathbf{q}_4 = -\cos(.8888888) \text{ and } \mathbf{r}_{A(b)B(c)}\mathbf{q}_4 = -\cos(-0.0000001),$$

along with the unit constraint $\mathbf{1}_4^T\mathbf{q}_4=1$ on non-negative $\mathbf{q}_4$.

The solutions of both of these computations yield an extreme value of 0, both as the minimum and the maximum value of the objective function. The right-hand-side of the inequality prescribed by quantum theoretic probabilities under these conditions *must equal* 0. This is to say, that when the directions of the magnet settings $\mathbf{b}$ and $\mathbf{c}$ are as near to one another as we have prescribed here, then the expectations $E[A(a)B(b)]$ and $E[A(a)B(c)]$ must be identical in the context of Bell's gedankenexperiment. It is the coherent implications of quantum expectations that assure this. Again, the left-hand-side surely exceeds the right-hand-side of Bell's inequality, which must equal 0. Again, Bell's inequality is satisfied!

Finally, consider the last two magnet pairings among the three at which Bell thought his inequality was defied, $(\mathbf{a},\mathbf{c})=0.8888887$ (in units of radians) and $(\mathbf{b},\mathbf{c})=-0.0000001$. This is essentially the same problem we have just assessed. The specified constraining condition on $E[A(b)B(c)]$ is identical in both. It is only the relative angle pairing of the chosen spin-product expectation that is different and whose quantum expectation is presumed. In this case the left-hand-side of the inequality is the same as previously,

$$1 - \cos(-0.0000001) = 4.996003610813204 \times 10^{-15}.$$

By now not surprisingly, the bounds on the right-hand-side of the inequality, which is enclosed in an absolute value operator, are again both equal to zero. Again, the coherency of quantum theoretic expectations requires equality of the two spin-product expectations in the gedanken context. These results derive from the linear programming computations

$$\left(\begin{array}{c} \min \\ \max \end{array} \right) E[A(a)B(b)] - E[A(a)B(c)] = [\mathbf{r}_{A(a)B(b)} - \mathbf{r}_{A(a)B(c)}]\mathbf{q}_4$$

subject to the linear constraints

$$\mathbf{r}_{ac}\mathbf{q}_4 = -\cos(0.8888887) \text{ and } \mathbf{r}_{bc}\mathbf{q}_4 = -\cos(-0.0000001),$$

along with the unit constraint $\mathbf{1}_4^T\mathbf{q}_4=1$ on non-negative $\mathbf{q}_4$.

Both the lower bound and the upper bound on the objective function are equal to zero. Bell's inequality is satisfied *whenever* the direction of the magnet pairing $(\mathbf{a}, \mathbf{b})$ is very close to that of $(\mathbf{a}, \mathbf{c})$. And the quantum probabilities are stationary, allaying his concern in his response to EPR.

Not only is the inequality satisfied, but the spectre of supposed instability of the quantum solution when the two angle pairings are identical, $(\mathbf{b} = \mathbf{c})$, does not arise. The applicable quantum expectations then yield the truism $0 \geq 0$, at this extreme, and they satisfying the inequality continuously as the magnet direction

$\mathbf{b}$ approaches $\mathbf{c}$.

Table 2 presents a complete display of these computational results. It mimics the structure of the results presented in **Table 1** for the simpler problem which requires fewer decimal places.

Table 2. $\begin{pmatrix} \min \\ \max \end{pmatrix}$ LP solution vectors at magnet angles $(\mathbf{a}, \mathbf{b}) = 0.8888888$, $(\mathbf{a}, \mathbf{c}) = 0.8888887$, and $(\mathbf{b}, \mathbf{c}) = -0.0000001$, and their implications for spin product expectations and Bell's Stationarity Concerns.

	$q_{\min}(\mathbf{b}, \mathbf{c})$	$q_{\max}(\mathbf{b}, \mathbf{c})$	$q_{\min}(\mathbf{a}, \mathbf{b})$	$q_{\max}(\mathbf{a}, \mathbf{b})$	$q_{\min}(\mathbf{a}, \mathbf{c})$	$q_{\max}(\mathbf{a}, \mathbf{c})$
q_1	.815137559966890	.630275158752372	.815137598785479	.815137598785479	.815137559966888	.815137559966888
q_2	0	.184862401214518	0	0	0	0
q_3	.000000038818592	.184862440033110	0	0	0	0
q_4	.184862401214518	0	.184862401214516	.184862401214516	.184862440033107	.184862440033107
	*****	*****	*****	*****	*****	*****
$E[A(\mathbf{a})B(\mathbf{b})]$	-.630275119933780	-.630275119933780	-.630275197570963	-.630275197570963	-.630275119933780	-.630275119933780
$E[A(\mathbf{b})B(\mathbf{c})]$	-.999999922362816	-.260550317504744	-.999999999999995	-.999999999999995	-.999999999999995	-.999999999999995
$E[A(\mathbf{a})B(\mathbf{c})]$	-.630275197570964	-.630275197570964	-.630275197570963	-.630275197570963	-.630275119933781	-.630275119933781
	*****	*****	*****	*****	*****	*****
LHSBellIneq	.000000077637184	.739449682495256	.000000000000005	.000000000000005	.000000000000005	.000000000000005
RHSBellIneq	.000000077637184	.000000077637184	0	0	0	0
	*****	*****	*****	*****	*****	*****
$P[(\mathbf{ab})++ \theta]$	.092431220016555	.092431220016555	.092431200607259	.092431200607259	.092431220016555	.092431220016555
$P[(\mathbf{bc})++ \theta]$	.000000019409296	.184862420623814	.000000000000001	.000000000000001	.000000000000001	.000000000000001
$P[(\mathbf{ac})++ \theta]$	.092431200607259	.092431200607259	.092431200607259	.092431200607259	.092431220016555	.092431220016555
	*****	*****	*****	*****	*****	*****
LHSBellProbs	.092431239425851	.277293640640369	.092431200607261	.092431200607261	.092431220016556	.092431220016556
RHSBellProbs	.092431200607259	.092431200607259	.092431200607259	.092431200607259	.092431220016555	.092431220016555

Precise exhibition of the rounding in the final digit for the two sides of Bell's inequality can be found in Section 5 of the text. Roundings for the Bell Probabilities are accurate as shown.

6. Concluding Comments

The implications of what I have discovered for our understanding of quantum particle behavior are stunning, only because the physics community has been taken in by a mistaken assessment of the functionally constrained quantum expectations appropriate to Bell's inequality. In response, most have contorted their construal of the non-locality of quantum mechanics to account for it. The touted defiance of locality of quantum behavior derives from a mathematical error. There is nothing wrong with Bell's inequality, which applies to quantum probabilities relevant to a thought experiment. There is nothing wrong with the probabilities for the results of real quantum experiments that are promoted by quantum theory. It is their misapplication in assessing the expectations involved in Bell's inequality that has led to the misunderstanding.

The most serious deep error arising from the supposed defiance of the inequality has been the widely publicized rejection of the locality of quantum mechanical processes, which had been promoted by Einstein. His unwavering insistence on locality had led to his relegation within the esteem of the physics community. The mistaken conclusion has by now been long advanced even in introductory level university textbooks, misdirecting young researchers. But there is a myriad of related confusions that have conspired to misconstrue the understanding of mathematical physics and its valid results. These include the characterization of valid quantum probabilities as supporting the so-called physical entanglement of particle behavior. The assessed paired spin behavior recognized in our Equation (1) by the joint probability specification of

$$P\left[\left(A(\mathbf{d}_A) = +1\right)\left(B(\mathbf{d}_B) = -1\right)\right] = P\left[\left(A(\mathbf{d}_A) = -1\right)\left(B(\mathbf{d}_B) = +1\right)\right]$$

is not a prescription that is unusual to quantum phenomena. Rather it is a feature of exchangeable distributions that represent a symmetric structure of uncertainty about observable phenomena at any scale of experience whatever. It is not the distant phenomena that are entangled. It is our symmetric uncertain attitude toward their situations that is entangled. This was one of the features of Einstein's larger viewpoint that have come to be disdained on account of the mistaken defiance of Bell's inequality.

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Conflicts of Interest

The author declares no conflicts of interest regarding this publication.

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Appendix 1. Commenting on an Important Paper

I make the following brief remarks about the important and influential article of Hess and Philipp [15], “The Bell Theorem as a Special Case of a Theorem of Bass”, with due respect. Their article promotes the view that Bell's inequality should pertain to the observation of relevant spin-products observed at any triple of angle directions and for any three distinct pairs of electrons. I contest this. In the following remarks I will presume the reader's familiarity with the details and notation of their article, which is available online using the doi service. Here goes.

The authors (hereafter identified as “H-P”) introduce their analysis with an assertion that “The classical inequalities of Bell [4] are derived by the use of probability theory, in essence by the use of very elementary facts about random variables within the framework pioneered by Kolmogorov. The derivation of the inequalities does not involve physics or quantum mechanics, yet the inequalities have assumed an important role for the foundations of quantum mechanics.” I must protest that this is not true, for the following reason.

The pathbreaking article of Bell developed the inequality firstly by defining the expectation of any spin-product such as $A(\mathbf{a})B(\mathbf{b})$ via the designation $P(\mathbf{a}, \mathbf{b}) = \int d\lambda \rho(\lambda) A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda)$ in an equation he numbered as (2). Bell subsequently identified that this expectation value can reach the value of -1 when the magnet directions $\mathbf{a} = \mathbf{b}$ only if $A(\mathbf{a}, \lambda) = -B(\mathbf{a}, \lambda)$. This identification

must be recognized from its subscribing to the quantum expectation $E[A(\mathbf{a})B(\mathbf{b})] = -\cos(\mathbf{a}, \mathbf{b})$, which we printed as Equation (6) in this present article. Under the condition that $\mathbf{a} = \mathbf{b}$ considered by Bell here, this is correctly recognized as $-\cos(0) = -1$. This is clearly a condition that would pertain to a spin-product $A(\mathbf{a})B(\mathbf{a})$ *only if* it were the same electron pair whose spin is observed with the paired magnet directions set at $(\mathbf{a}, \mathbf{b})$ and at $(\mathbf{b}, \mathbf{c})$. If they were the spins of different electron pairs being observed at two stations, involving $B(\mathbf{b})$ for one pair and involving $A(\mathbf{b})$ for the other, there would be no such restriction. The subsequent analytics in Bell's derivation of his inequality would then not follow. Enough said about this.

So much for the prelude of [15]. Now the subsequent analytic derivations involve a different misunderstanding that derives from the use of insufficient notation. We have recognized the inequality of Bell to involve the expectations of three spin-products, $A(\mathbf{a})B(\mathbf{b})$, $A(\mathbf{a})B(\mathbf{c})$ and $A(\mathbf{b})B(\mathbf{c})$. However, while H-P denote the value of $A(\mathbf{a})$ and of $B(\mathbf{c})$ in each such instance merely as A and C , respectively, they also denote the values of *both* $B(\mathbf{b})$ and of $A(\mathbf{b})$ merely by the identical denotation " B ", ignoring the fact that each of these is necessarily the negative of the other in order to make Bell's inequality pertain.

One technical consequence of this unfortunate designation is that their Equation (9) which they identify as "of course, one of the celebrated Bell inequalities" has a minus sign on its right-hand-side where it should be a plus sign.

This technical error is not so important. What is really important is their claim that the inequality should apply to any measurements of spin-products at any three angle pairings $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}, \mathbf{c})$ and $(\mathbf{b}, \mathbf{c})$ for three *differing* electron pairs. They think that the Aspect type empirical analysis, which presume this, is correct and definitive. It is not. This is made clear in the context of a CHSH format in Chapter 1 of [13] and in [9].

There is still more that should be said. The articles of Bass and Vorob'ev that are cited by H-P are not really relevant to the situation. The quantum distributions for three pairs of spin-products on distinct electron pairs that can actually be observed are *not* marginal to any coherent joint distribution over all three. It has been recognized since the article of Fine [16] that quantum theory specifies no such joint distribution. Specifying one would require assertions regarding the joint results of quantum operators on the Hilbert space of state vectors that do not commute. Nonetheless, we can characterize an entire space of joint distributions that cohere with the assertions quantum theory does provide for any single pair of photons at each of the three designated angle pairings. The three individual joint distributions on any pair of polarization products are not marginal to any of them. The analyses of Bass and of Vorob'ev which characterize joint distributions that agree with any finite specifications of marginal distributions is irrelevant to Bell's inequality.