

Risk Measure Contextuality by Quantum Weak Value in Quantum Decision Theory

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Abstract

This paper introduces decision-making and risk-measure models based on advanced quantum theory, which address the contextuality of decisions more flexibly than previous approaches. Contextuality affects how risk is perceived, and changes in decision-making are modeled using quantum time evolution and weak values. Numerical simulations reveal that the weak value model captures contextual shifts in risk magnitude in both directions—amplification and reduction—which cannot be expressed by simple projection or time evolution alone. Empirical applications include moral hazard and inverse moral hazard, where contextuality renders the risk either more or less severe, such as governmental monetary aids examples.

Keywords

Quantum Decision Theory, Weak Value, Risk Measure, Time Evolution, Moral Hazard

1. Introduction

This research develops a new quantum decision-making model for addressing contextuality (such as new information and others) using time evolution and quantum weak values. This study explores a practical application concerning the emergence of both moral hazard and “inverse moral hazard” within risk-related decision-making. Conventional insurance or mutual aid systems, such as risk-sharing and pooling, can foster complacency and a false sense of security, thereby inducing moral hazard—a phenomenon observed in Swedish local governments. Conversely, the recent developments within the European Monetary Union exemplify the opposite effect, termed inverse moral hazard (see Section 5). From economic and psychological perspectives, insurance conditions and systemic incentives constitute the contextuality for risk-bearing entities, influencing the ampli-

fication or mitigation of perceived risk. To formalize these dynamics, the proposed quantum model employs unitary and Hermitian transformation parameters to characterize the transition of operators. From the late 1990s through the 2000s, concepts from quantum theory emerged as problem-solving methods in game theory, leading to the development of Quantum Decision Theory (QDT). This approach has been famously applied to the “coin flip problem” (Meyer [1]) and the “prisoner’s dilemma” (Eisert *et al.* [2]). Other researchers, such as Guevara [3], Cheon and Iqbal [4], and Cheon and Takahashi [5] [6], advanced game and decision theories by comparing Pareto optimal and Nash equilibrium solutions. The application of these models was further expanded by Aerts *et al.* [7] in conjunction with research on fallacies and the disjunction effect, and by Eichberger and Pirner [8] to provide a solution for the Ellsberg paradox (Ellsberg [9]).

While some economic and psychological reasoning has been discussed (Ashtiani and Azgomi [10]; Whittle-Walls [11]), the use of quantum theory is often perceived as a mathematical tactic that introduces more parameters. Nevertheless, in addition, such concepts as Born’s rule, quantum superposition, interference, entanglement, and projection have provided practical alternative models for decision-making, resolving various paradoxes and fallacies. This research introduces time evolution and quantum weak values for the first time as tools for more flexible decision modeling. Such a development not only enriches the economic and psychological interpretations of QDT but also broadens the scope of decision models regarding contextuality, particularly in the cases of both moral hazard and inverse moral hazard could happen.

The core of the quantum model proposed in this paper is outlined as follows. This paper represents (measures) risk using the following form:

$$\rho = \langle \varphi | R | \varphi \rangle,$$

where ρ is a risk measure, R is the risk operator, and φ is risk-state. As Schrödinger equation indicates, in case the one period time evolution from initial time t_{ini} to final time t_f , the time evolution and the results are described as:

$$|\varphi(t_f)\rangle \approx U |\varphi(t_{ini})\rangle,$$

where $U \approx e^{-i\frac{H}{\hbar}(t_f - t_{ini})}$, Schrödinger equation: $i\hbar \frac{d}{dt} |\varphi(t)\rangle = H |\varphi(t)\rangle$. H is Hermitian and U is unitary transform. The concept of the weak value, defined by Aharonov *et al.* (1988), allows the weak value risk measure to be calculated as follows: using the concept, weak value risk measure ρ^{wv} is calculated as the below with quantum risk measures $\rho^{ini}(t=t_{ini})$ and $\rho^f(t=t_f)$:

$$\rho^{wv} = \frac{\langle \varphi_f | R | \varphi_{ini} \rangle}{\langle \varphi_f | \varphi_{ini} \rangle}, \quad \rho^{ini} = \langle \varphi_{ini} | R | \varphi_{ini} \rangle, \quad \rho^f = \langle \varphi_f | R | \varphi_f \rangle.$$

where $|\varphi_{ini}\rangle = |\varphi(t_{ini})\rangle$, $|\varphi_f\rangle = |\varphi(t_f)\rangle$. The change from ρ^{ini} through ρ^{wv} to ρ^f can be treated as the risk felt or observed has changed by economic or

psychological contextuality during the period $[t_{ini}, t_f]$. Here the examples of contextuality are information and others. The risk belief update could be translated into the risk operator and at the end the changes of operators express the contextuality (see Section 3). In a typical time evolution calculated by a unitary transformation, the magnitude of the risk measure after evolution is generally no larger than the initial magnitude. However, the weak value demonstrates the possibility of the risk magnitude being amplified. Consequently, this allows for a more flexible risk model under conditions of contextuality.

The rest of the paper is structured as follows. Section 2 presents previous research and preliminaries. Section 3 develops the quantum model using time evolution and quantum weak values. Section 4 provides examples and implication of the model, and Section 5 offers mathematical, economic, and psychological considerations. Section 6 summarizes the paper.

2. Previous Research and Preliminary

2.1. Quantum Theory

For a fundamental overview of quantum mechanics, see, for instance, Gliner [12]. Quantum mechanics posits that physical quantities, such as the state of a wave, are determined only through the act of measurement. The wave function ψ is defined by treating the physical properties of matter as a wave. Using Dirac's bracket notation, the state is represented by a "ket" $|\psi\rangle$ and its dual "bra" $\langle\psi|$. These vectors belong to Hilbert space, representing the states of wave systems.

Next, we introduce Born's rule. When measuring a physical quantity, assuming the system state $|\psi\rangle$ is expanded using the basis states $|\psi_i\rangle (i=1,2,\dots)$, we have $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + \dots$. The probability of obtaining a specific measured value c_j is given by $\langle\psi_j|\psi\rangle^2 = |c_j|^2$, which is the core interpretation of Born's rule the below.

$$\sum_i |c_i|^2 = 1,$$

Expected average of the observation value of an operator A in state is calculated by $\langle\psi|A|\psi\rangle$.

Quantum theory allows for the superposition of multiple bases, leading to interference between them. Consequently, the law of total probability does not always hold, unlike in classical probability. While entanglement is not the primary focus of this paper, entangled states (quantum entanglement)—crucial for resolving the Prisoner's Dilemma—are generally key to addressing various decision paradoxes. Furthermore, the projection principle (wavefunction collapse) refers to the transition of a state from a superposition to a specific eigenstate upon measurement, satisfying Born's rule.

Finally, time evolution is governed by the Schrödinger equation:

$$i\hbar \frac{d}{dt} |\varphi(t)\rangle = H |\varphi(t)\rangle$$

where the Hamiltonian is typically Hermitian.

2.2. Risk in Quantum Decision Theory

In quantum mechanics, the expected value of an observable is obtained through measurement, and the numerical result may vary across different measurement trials. This system resembles human decision-making, which led to the development of Quantum Decision Theory (QDT). In QDT, the expected payoff of action is expressed as $\langle \psi | A | \psi \rangle$ by treating A as an operator, where the decision state is represented by $|\psi\rangle$. This paper employs a risk operator R and defines the risk measure as $\langle \psi | R | \psi \rangle$ (Yamashita [13]).

Early discussions on how quantum theory addresses decision-making paradoxes and fallacies were summarized by Pothos and Busemeyer [14]. Bruza *et al.* [15] further explored the comparison between classical and quantum approaches. More advanced surveys of QDT, including the theoretical justification for using quantum mechanics in social sciences, are provided by Ashtiani and Azgomi [10] and Whittle-Walls [11].

2.3. Related Papers

To date, quantum probability, superposition, interference, entanglement, and projection have been utilized to provide new interpretations of empirical decision-making enigmas (Ashtiani and Azgomi [10]; Eichberger and Pirner [9]; Yamashita [16]). In this context, the application of quantum weak values represents a novel methodology. Key references for weak values include Aharonov *et al.* [17] [18], along with recent developments in physics by Rosales-Zárate *et al.* [19], Chen *et al.* [20], Zhu *et al.* [21], and Dressel *et al.* [22].

While time evolution is based on the Schrödinger equation, this paper also considers cases where the Hamiltonian is non-Hermitian. Pati *et al.* [23] provide useful tools for non-Hermitian systems using weak values. The use of weak values can “amplify” measurement results, a technique widely applied in experimental physics (Ferrie and Combes [24]). Finally, applying advanced quantum theory to decision problems does more than introduce new modeling tactics; it enriches the conceptual discourse of QDT, such as the discussion on contextual variability by Whittle-Walls [11].

3. The Model

3.1. Formulation of Risk (Risk Measure)

Considering risk state $|\varphi\rangle$ in Hilbert space which consists of a set of basic risk states $|\varphi_i\rangle$, in whose superposition the square of the coefficient expresses the probability of each basic risk state to happen. In order to depict the risk measure under the state of $|\varphi\rangle$ by an operator, as an action with payoffs, R is set as the risk measure operator by an action operator with payoff A . The risk measure ρ is the following form:

$$\rho = \langle \psi | R | \psi \rangle,$$

For instance, in case $R = A^2 - \langle \varphi | A | \varphi \rangle^2$, ρ is expressing as $\langle \psi | R | \psi \rangle = \text{Var}(A)$, where supposing $\text{Var}(A)$ as a statistical variance of payoffs by action A to be risk.

The risk measure is characterized by several concepts, such as the characteristics of convex or law invariant (see Arzner *et al.* [25], Foellmer and Knispel [26], Jouini *et al.* [27], Kusuoka [28], Mastrogiamco and Gianin [29], Ravanelli and Svindland [30] and Rockafellar [31]). Particularly, the characteristics below are important.

Monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$,

Cash Invariance: $\rho(X + m) = \rho(X) - m$ (m : Cash),

Normalization: $\rho(0) = 0$, and

Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$ (λ : scalar).

In the case wherein the above top three are true, it is called monetary. In the case wherein all the above are true, it is called convex. In the case wherein the below is also true, it is called coherent:

Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ (λ : scalar).

Among classical risk measures, variance is only convex. VaR is only monetary. CVaR is monetary, convex and coherent. Regarding the quantum risk measure, monotonicity in terms of $\langle \varphi | R | \varphi \rangle$ formulation, if there is an order relationship between two Hermitian operators $X \leq Y$ (meaning the difference $Y - X$ is a positive semi-definite operator), then for any normalized state, $\rho(X) \leq \rho(Y)$ holds. Cash invariance and normalization can be met for normal R . While a simple expectation value like $\langle \varphi | X | \varphi \rangle$ and $\langle \varphi | Y | \varphi \rangle$ are linear with respect to X and Y , this changes when to use a “function” of that operator. In the case

$R = f(A) = A^2 - A$, as the function satisfies

$f(\lambda X + (1 - \lambda)Y) \leq \lambda f(X) + (1 - \lambda)f(Y)$, R is said to be operator convex for any Hermitian operators. Positive homogeneity is secured by the linear characteristic of the risk measure formulation.

3.2. Time Evolution and Weak Value

Schrödinger equation:

$$i\hbar \frac{d}{dt} |\varphi(t)\rangle = H |\varphi(t)\rangle \quad (H : \text{Hamiltonian})$$

indicates time dependance of the state $|\varphi\rangle$. Typically, the observation result is real number so H : Hamiltonian is typically Hermitian operator. One period time evolution from initial time t_{ini} to final time t_f is described as below:

$$|\varphi(t_f)\rangle \approx U |\varphi(t_{ini})\rangle,$$

where $U \approx e^{-i\frac{H}{\hbar}(t_f - t_{ini})}$;

where in case H is Hermitian, U is unitary. The risk measure of time t_{ini} and t_f , are calculated as $\rho^{ini} = \langle \varphi_{ini} | R | \varphi_{ini} \rangle$ and $\rho^f = \langle \varphi_f | R | \varphi_f \rangle$, respectively ($\varphi_{ini} = \varphi(t_{ini})$, $\varphi_f = \varphi(t_f)$). The concept of a weak value is defined by Aharonov

et al. (1988) and using the concept, weak value risk measure ρ^{wv} is calculated as the below:

$$\rho^{wv} = \frac{\langle \varphi_f | R | \varphi_{ini} \rangle}{\langle \varphi_f | \varphi_{ini} \rangle}.$$

The change from ρ^{ini} through ρ^{wv} toward ρ^f can be treated as that the risk felt or observed has changed by economic or psychological contextuality during the time period from $[t_{ini}, t_f]$. (See the next sub-section)

3.3. Model Implication

The time evolution model captures the change in the risk measure from ρ^{ini} to ρ^f , reflecting contextual shifts such as the revelation of new information or the implementation of a new scheme. This framework introduces a novel perspective on risk measurement. A unitary transformation functions similarly to a rotation in Hilbert space; applying a unitary operator changes the “direction” of the state but leaves its norm invariant. Consequently, in such cases, the magnitudes of ρ^{ini} to ρ^f and remain essentially the same. In contrast, ρ^{wv} offers greater flexibility in this regard, as detailed in the following section.

Mathematically,

$$\begin{aligned} \rho^f &= \langle \varphi_f | R | \varphi_f \rangle = \langle \varphi_{ini} U^\dagger | R | U \varphi_{ini} \rangle = \langle \varphi_{ini} | U^\dagger R U | \varphi_{ini} \rangle, \text{ and} \\ \rho^{wv} &= \frac{\langle \varphi_f | R | \varphi_{ini} \rangle}{\langle \varphi_f | \varphi_{ini} \rangle} = \frac{\langle \varphi_{ini} U^\dagger | R | \varphi_{ini} \rangle}{\langle \varphi_{ini} U^\dagger | \varphi_{ini} \rangle} = \langle \varphi_{ini} | \frac{U^\dagger R}{\langle \varphi_{ini} U^\dagger | \varphi_{ini} \rangle} | \varphi_{ini} \rangle \end{aligned}$$

are lead and the time evolution could be translated into the risk operator evolution. To investigate the risk operator change, the cause of change would be found out as information or affection of belief update into the risk operator. The absolute value of the inner product after inserting a unitary matrix will never exceed the maximum possible inner product (which is 1 for normalized states). At most, it can only be equal to it, proved by using the Cauchy-Schwarz inequality. (Also see **Appendix C**)

Another critical implication concerns the real and imaginary parts of the quantum risk measure. Here is a breakdown of what the real and imaginary parts represent. In physics, the real part corresponds to a shift in the measurement entity: pointer’s position. This is often interpreted as the “effective value” of the observable. The imaginary part corresponds to a shift in the pointer’s momentum. Practically, the real part is emphasized because it acts as an extension of the classical average. (In modern quantum metrology, the imaginary part is equally vital.)

Risk measures, particularly ρ^{wv} can possess both real and imaginary components. When H is Hermitian, the risk measure ρ^f is expected to be real numbers. However, in the case of that H is non-Hermitian, the resulting risk measure may be complex. In these instances, the real part is extracted to represent the observed result. The imaginary part denotes a transient state which, regarding ρ^{wv} , can

be interpreted as the “direction” or “rate” of the contextual change.

4. Examples and Implication

Let us consider two-dimensional Hilbert space ($\dim H = 2$). The basic states are set as $|\varphi_0\rangle$ and $|\varphi_1\rangle$. The current state $|\varphi\rangle$ is described by

$|\varphi\rangle = x|\varphi_0\rangle + \sqrt{1-x^2}|\varphi_1\rangle$. This means that the risk-state is x^2 probability for $|\varphi_0\rangle$ and $1-x^2$ probability for $|\varphi_1\rangle$. Depicting its risk measure as $\rho = \langle\varphi|R|\varphi\rangle$, where R is the risk measure operator by an action with payoff matrix A . $|\varphi_0\rangle$

and $|\varphi_1\rangle$ stand for $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively, and supposing

$$\langle\varphi_0|R|\varphi_1\rangle = \langle\varphi_1|R|\varphi_0\rangle = T.$$

$$\begin{aligned} \rho &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} \langle\varphi_0|R|\varphi_0\rangle & \langle\varphi_0|R|\varphi_1\rangle \\ \langle\varphi_1|R|\varphi_0\rangle & \langle\varphi_1|R|\varphi_1\rangle \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \\ &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} \rho_0 & T \\ T & \rho_1 \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \\ &= x^2\rho_0 + (1-x^2)\rho_1 + 2x\sqrt{1-x^2}T \end{aligned}$$

where setting $\langle\varphi_0|R|\varphi_0\rangle = \rho_0$, $\langle\varphi_1|R|\varphi_1\rangle = \rho_1$, $\langle\varphi_0|R|\varphi_1\rangle = \langle\varphi_1|R|\varphi_0\rangle = T$,

respectively. In the time evolution, these states are set as initial (at time t_{ini}). The measurement will finally show the (average) risk $\langle\varphi|R|\varphi\rangle$ and that consists of ρ_0 , T and ρ_1 , with their weight parameter x (supposed to be $0 \leq x \leq 1$). In addition, T is expressing interaction term between the two risk-bases. In this section, the results are compared to the projection example Eichberger and Pirner [9] or Yamashita [16] in **Appendix A**. As a whole, detailed calculation is in **Appendix B**. Details of Hermitian and unitary matrices in this example are in **Appendix C**.

4.1. One Time Period Evolution and Weak Values

The time evolution can be expressed by using a unitary matrix. In this example, the time evolution matrix W and its conjugate transpose $W^\dagger$ are set as below:

$$W = \begin{pmatrix} a & -e^{i\omega}\sqrt{1-a^2} \\ \sqrt{1-a^2} & ae^{i\omega} \end{pmatrix}, \quad W^\dagger = \begin{pmatrix} a & \sqrt{1-a^2} \\ -e^{-i\omega}\sqrt{1-a^2} & ae^{-i\omega} \end{pmatrix}.$$

where parameter a is an evolution parameter. from ρ^{ini} , ρ^{wv} and ρ^f are calculated as the below:

$$\begin{aligned} \rho^{ini} &= \langle\varphi_{ini}|R|\varphi_{ini}\rangle \\ &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} \rho_0 & T \\ T & \rho_1 \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \\ &= x^2\rho_0 + (1-x^2)\rho_1 + 2x\sqrt{1-x^2}T \end{aligned}$$

$$\begin{aligned}\rho^{wv} &= \frac{\langle \varphi_f | R | \varphi_{ini} \rangle}{\langle \varphi_f | \varphi_{ini} \rangle} = \frac{\langle \varphi_{ini} | W^\dagger R | \varphi_{ini} \rangle}{\langle \varphi_{ini} | W^\dagger | \varphi_{ini} \rangle} \\ &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} a & \sqrt{1-a^2} \\ -e^{-i\omega} \sqrt{1-a^2} & ae^{-i\omega} \end{pmatrix} \begin{pmatrix} \rho_0 & T \\ T & \rho_1 \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \\ &\div \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} a & \sqrt{1-a^2} \\ -e^{-i\omega} \sqrt{1-a^2} & ae^{-i\omega} \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix}\end{aligned}$$

Real part of coefficient of ρ_0 :

$$\text{Re} \left(\frac{ax^2 - e^{-i\omega} x \sqrt{1-x^2} \sqrt{1-a^2}}{ax^2 + \sqrt{1-x^2} + e^{-i\omega} \{a(1-x^2) - x \sqrt{1-x^2} \sqrt{1-a^2}\}} \right),$$

Real part of coefficient of ρ_1 :

$$\text{Re} \left(\frac{x \sqrt{1-x^2} \sqrt{1-a^2} + e^{-i\omega} a(1-x^2)}{ax^2 + \sqrt{1-x^2} + e^{-i\omega} \{a(1-x^2) - x \sqrt{1-x^2} \sqrt{1-a^2}\}} \right),$$

Others, see **Appendix B**.

$$\begin{aligned}\rho^f &= \langle \varphi_f | R | \varphi_f \rangle = \langle \varphi_{ini} | W^\dagger R W | \varphi_{ini} \rangle \\ &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} a & \sqrt{1-a^2} \\ -e^{-i\omega} \sqrt{1-a^2} & ae^{-i\omega} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \rho_0 & T \\ T & \rho_1 \end{pmatrix} \begin{pmatrix} a & -e^{i\omega} \sqrt{1-a^2} \\ \sqrt{1-a^2} & ae^{i\omega} \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix}\end{aligned}$$

Coefficient of ρ_0 : $x^2 a^2 - 2xa \sqrt{1-x^2} \sqrt{1-a^2} \cos(\omega) + (1-x^2)(1-a^2)$,

Coefficient of T : See **Appendix B**,

Coefficient of ρ_1 : $x^2(1-a^2) + 2xa \sqrt{1-x^2} \sqrt{1-a^2} \cos(\omega) + (1-x^2)a^2$.

In order to illustrate above, several numerical results (supposing interaction $T = 0$ and $x = a$) are shown in **Table 1** and **Figure 1** show ρ^{wv} .

Table 1. Comparison summary for ρ^{ini} , ρ^{wv} and ρ^f .

Conditions	Risk * regarding ρ_0 (upper line) and ρ_1 (lower line) (interaction $T=0$)		
	Initial, Standard ρ^{ini}	Weak Value ρ^{wv} , Real Part	Final, Time Evolved ρ^f
$x = a$,	a^2	$2a^2 - 1^{**}$	$(2a^2 - 1)^2$
$\omega = 0$	$1 - a^2$	$2(1 - a^2)^{***}$	$4a^2(1 - a^2)$
$x = a$,	a^2	a^2	$2a^4 - 2a^2 + 1$
$\omega = \pi/2$	$1 - a^2$	$1 - a^2$	$2a^2(1 - a^2)$
$x = a$,	a^2	1	1
$\omega = \pi$	$1 - a^2$	0	0

In case ρ^{wv} , the real part is shown. *: Risk is shown by coefficients of ρ_0 and ρ_1 . **, ***: Only these two are beyond the range of from 0 to 1, even though $0 \leq a \leq 1$.

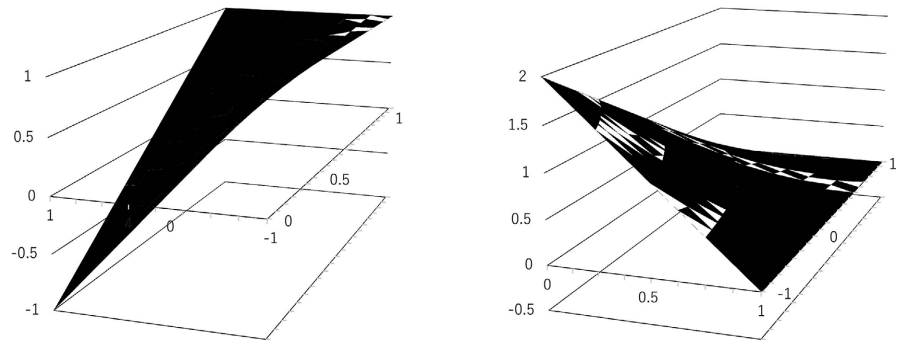


Figure 1. Illustration of real part of ρ_0 coefficient regarding ρ^{wv} . Used $x=a$ so the real part of the coefficient of ρ_0 is $a^2 - (1-a^2)\cos(\omega)$ (left), and that of ρ_1 is $(1-a^2)(1+\cos(\omega))$ (right). In the left figure, horizontal axis of depth direction represents $\cos(\omega)$ and horizontal axis of right direction represents a^2 . In the right figure, horizontal axis of right direction represents $\cos(\omega)$ and horizontal axis of depth direction represents a^2 .

4.2. Weak Value vs. Projection

As Eichberger and Pirner [9] and Yamashita [16] treated, the quantum projection could use for the observation change from objective view to subjective view. According to **Appendix A**, y is set as a parameter of subjective view. Objective view parameter is x , and they are combined if subjective view of the risk is projected by objective view. The projected risk measure ρ^P is the below.

$$\rho^P = \left\{ y^2 \rho_0 + (1-y^2) \rho_1 + 2y\sqrt{1-y^2} T \right\} \\ \times \left\{ x^2 y^2 + (1-y^2)(1-x^2) + 2xy\sqrt{1-x^2}\sqrt{1-y^2} \cos(\theta) \right\}$$

Supposing $x^2 = y^2$ and using the parameter characters of a and ω instead of x and θ , **Table 2** shows the comparison of the projection and the weak value.

Table 2. Comparison Summary for ρ^P and ρ^{wv} . As a reference, ρ^f is also shown.

Conditions	Risk * regarding ρ_0 (upper line) and ρ_1 (lower line) (interaction $T=0$)		
	Projection ρ^P	Weak Value ρ^{wv} , Real Part	Final, Time Evolved ρ^f
$x=a$, $\omega=0$	a^2 $1-a^2$	$2a^2-1^{**}$ $2(1-a^2)^{***}$	$(2a^2-1)^2$ $4a^2(1-a^2)$
$x=a$, $\omega=\pi$	$a^2\{2a^4-2a^2+1\}$ $(1-a^2)\{2a^4-2a^2+1\}$	1 0	$2a^4-2a^2+1$ $2a^2(1-a^2)$
$x=a$, $\omega=\pi/2$	$a^2(2a^2-1)^2$ $(1-a^2)(2a^2-1)^2$	a^2 $1-a^2$	1 0

In case ρ^{wv} , the real part is shown. *: Risk is shown by coefficients of ρ_0 and ρ_1 . **, ***: Only these two are beyond the range of from 0 to 1, even though $0 \leq a \leq 1$.

5. Implication, Empirical Insights and Future Challenges

Regarding unitary or Hermitian transformations in Hilbert space, the magnitude of risk in this model remains either constant or decreases. This is supported by the data presented in [Table 1](#) and [Table 2](#).

As shown in [Table 1](#), the sum of the coefficients for ρ_0 and ρ_1 is consistently 1 for each parameter set (x, ω) . Except for the cases marked with ** and ***, the magnitude of each coefficient for ρ_0 or ρ_1 ranges between 0 and 1 as a varies from 0 to 1. In [Table 2](#), for the ρ^P case, the sum of the coefficients for ρ_0 and ρ_1 is always $2a^4 - 2a^2 + 1$ for each parameter set (x, ω) . Here, each coefficient, as well as their sum, remains within the range of $[0, 1]$ as a changes. This suggests that, if no new information added or no information dissipation, time evolution or projection keep the risk measure (or the inner product) the same size or smaller. This coincides with Hamiltonian is Hermitian.

However, there are notable exceptions. In the cases marked * and ** in [Table 1](#), the magnitudes exceed the $[0, 1]$ range, even though their sum remains 1. During the transition from the initial to the final state, the magnitude of a coefficient can exceed 1. Furthermore, when T is non-zero, interference effects occur, potentially causing more dramatic shifts in the total risk. This related to the denominator, which is the overlap (inner product) between the initial and final states (If the final state that is nearly orthogonal to the initial state, the denominator enlarges the size), and the negative probability and the coefficient of ρ_1 could be negative, to balance the exceeding. Example explanation could be that the same information could be understood differently.

Next discusses the application to practical cases. Typical moral hazard and inverse moral hazard examples are the followings. On the one hand, Pettersson-Lidbom [\[32\]](#) showed that each of Swedish local governments, to which the central government distributed a large number of fiscal transfers (help), so called “soft-budget constraint,” on average, increased its debt by more than 20 percent by going from a previous hard-budget to a soft-budget constraint. On the other hand, Beetsma *et al.* [\[33\]](#) described that in a report in June 2012, the then president of the European Council, Van Rompuy, identified an integrated budgetary framework as one of four building blocks to consolidate the EMU. Shortly after, in December 2012, the “Four Presidents’ Report” (Van Rompuy *et al.* [\[34\]](#)) discussed the gradual creation of a central fiscal capacity aimed at both promoting structural reforms and mitigating asymmetric shocks. The empirical interpretation of these numerical examples is as follows. An initial risk measure may be altered by the introduction of new information or changes in contextuality, which affect the risk operator. For instance, in a risk-sharing pool, the participants’ perception of risk may be amplified by moral hazard or reduced by inverse moral hazard. This suggests that contextuality inherently drives changes in risk perception (the risk measure).

- Projection Case: When risk transitions from ρ^{ini} or ρ^P due to contextuality, the total risk may decrease but never increase. (For instance, no new information nor dissipation because the projection is related to just a

subjective and an objective view.)

- Time Evolution Case: Similarly, when risk transitions from ρ^{ini} or ρ^f , the total magnitude of risk may decrease or remain constant, but it cannot be amplified. (Again for instance, no new information nor dissipation and this is the case of that Hamiltonian is Hermitian.)
- Weak Value Case: In contrast, when risk transitions from ρ^{ini} or ρ^{wv} , certain parameters allow the magnitude of a specific risk to exceed its original value. This amplification cannot be achieved through Hermitian or unitary transformations alone. Specifically, in the examples in **Table 1**, the coefficient of ρ_0 for ρ^{wv} can reach nearly 2, well above the standard limit of 1. (It could happen incidentally the mind treat the probability of ρ_0 larger by the sacrifice of the probability ρ_1 .)
- Empirically, the first case suggests an improvement in the risk environment, such as through inverse moral hazard. The second case represents a mere adjustment of weights between ρ_0 and ρ_1 and without changing the overall risk magnitude. The third case—the weak value model—captures an escalation of risk, representing scenarios where moral hazard worsens the situation.

In this way, time evolution and weak values provide a more flexible framework for QDT, capable of modeling both risk amplification and reduction in the presence of moral hazards. There are many empirical phenomena regarding monetary unions and government aids inverse moral hazard so as the economic matures the third case goes rare.

Regarding future challenges, the treatment of non-Hermitian Hamiltonians (meaning the system is not closed and is open) warrants further investigation (also about negative probability). While extracting the real part to represent risk remains the standard approach, the analytical formulation is often complex. Additionally, further research could explore more profound economic interpretations of QDT enabled by weak values.

6. Summary

This paper introduces decision-making and risk-measure models utilizing advanced quantum theory, specifically quantum weak values. These models capture the contextuality of decisions more naturally by allowing for both the amplification and reduction of perceived risk, thereby offering a novel addition to the QDT framework. While classical utility theory remains a standard tool for modeling decision-making and risk, quantum theory has increasingly been applied to address irrational behaviors and fallacies. The empirical applications discussed herein regarding moral hazard and inverse moral hazard demonstrate that this model provides a more holistic and flexible perspective on risk.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A: Quantum Projection

The projection can be expressed by using Hermitian matrices. If R is the risk measure operator of objective view and the projection matrix is V , based on Eichberger and Pirner [9] and Yamashita [16], the risk measure operator will be VRV for the subjective state of view. For the Hermitian matrix, here

$$V = \begin{pmatrix} y^2 & e^{-i\theta} y \sqrt{1-y^2} \\ e^{i\theta} y \sqrt{1-y^2} & 1-y^2 \end{pmatrix} \text{ is prepared. The parameter } y^2 \text{ is regarding}$$

subjective point of view. In case the risk states are prepared with the weights of x^2 and $1-x^2$, for objective point of view and y^2 and $1-y^2$ for subjective point of view. Another parameter θ is, economically speaking, contextuality of the economic situation, leading how different those two views are.

The risk measure of subjective view is the projection of objective view toward subjective view. The risk measure changes as below.

$$\begin{aligned} \rho^p &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} y^2 & e^{-i\theta} y \sqrt{1-y^2} \\ e^{i\theta} y \sqrt{1-y^2} & 1-y^2 \end{pmatrix} \begin{pmatrix} \rho_0 & T \\ T & \rho_1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} y^2 & e^{-i\theta} y \sqrt{1-y^2} \\ e^{i\theta} y \sqrt{1-y^2} & 1-y^2 \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \end{aligned}$$

Appendix B: Calculation Details

$$\begin{aligned} \rho^{ini} &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} \rho_0 & T \\ T & \rho_1 \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \\ &= x^2 \rho_0 + (1-x^2) \rho_1 + 2x\sqrt{1-x^2} T \end{aligned}$$

$$\begin{aligned} \rho^{wv} &= \frac{\langle \varphi_f | R | \varphi_{ini} \rangle}{\langle \varphi_f | \varphi_{ini} \rangle} = \frac{\langle \varphi_{ini} | W^\dagger R | \varphi_{ini} \rangle}{\langle \varphi_{ini} | W^\dagger | \varphi_{ini} \rangle} \\ &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} a & \sqrt{1-a^2} \\ -e^{-i\theta} \sqrt{1-a^2} & ae^{-i\theta} \end{pmatrix} \begin{pmatrix} \rho_0 & T \\ T & \rho_1 \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \\ &\quad \div \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} a & \sqrt{1-a^2} \\ -e^{-i\theta} \sqrt{1-a^2} & ae^{-i\theta} \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \\ &= \rho_0 \left(\frac{ax^2 - e^{-i\theta} x \sqrt{1-x^2} \sqrt{1-a^2}}{ax^2 + \sqrt{1-x^2} + e^{-i\theta} \{a(1-x^2) - x\sqrt{1-x^2} \sqrt{1-a^2}\}} \right) \\ &\quad + T \left(\frac{x^2 \sqrt{1-a^2} + ax \sqrt{1-x^2} + e^{-i\theta} (ax \sqrt{1-x^2} - \sqrt{1-a^2} (1-x^2))}{ax^2 + \sqrt{1-x^2} + e^{-i\theta} \{a(1-x^2) - x\sqrt{1-x^2} \sqrt{1-a^2}\}} \right) \\ &\quad + \rho_1 \left(\frac{x \sqrt{1-x^2} \sqrt{1-a^2} + e^{-i\theta} a (1-x^2)}{ax^2 + \sqrt{1-x^2} + e^{-i\theta} \{a(1-x^2) - x\sqrt{1-x^2} \sqrt{1-a^2}\}} \right) \end{aligned}$$

For instance,

$$\begin{aligned}
 \operatorname{Re}(\rho_0 \text{ coefficient}) &= \frac{A}{B} \\
 A &= \left(ax^2 - x\sqrt{1-x^2}\sqrt{1-a^2} \cos(\omega) \right) \left(ax^2 + x\sqrt{1-x^2}\sqrt{1-a^2} \right) (1 - \cos(\omega)) \\
 &\quad + \cos(\omega) a - x\sqrt{1-x^2}\sqrt{1-a^2} \left(-x\sqrt{1-x^2}\sqrt{1-a^2} + a(1-x^2) \right) \sin^2(\omega) \\
 B &= \left(\left(ax^2 + x\sqrt{1-x^2}\sqrt{1-a^2} \right) (1 - \cos(\omega)) + \cos(\omega) a \right)^2 \\
 &\quad + \sqrt{1-x^2} + \left(-x\sqrt{1-x^2}\sqrt{1-a^2} + a(1-x^2) \right)^2 \sin^2(\omega) \\
 \rho^f &= \langle \varphi_f | R | \varphi_f \rangle = \langle \varphi_{ini} | W^\dagger R W | \varphi_{ini} \rangle \\
 &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} a & \sqrt{1-a^2} \\ -e^{-i\omega}\sqrt{1-a^2} & ae^{-i\omega} \end{pmatrix} \begin{pmatrix} \rho_0 & T \\ T & \rho_1 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} a & -e^{i\omega}\sqrt{1-a^2} \\ \sqrt{1-a^2} & ae^{i\omega} \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \\
 &= \rho_0 \left\{ x^2 a^2 - 2ax\sqrt{1-x^2}\sqrt{1-a^2} \cos(\omega) + (1-x^2)(1-a^2) \right\} \\
 &\quad + 2T \left\{ xa^2\sqrt{1-a^2} + x\sqrt{1-x^2}(2a^2-1)\cos(\omega) - (1-x^2)a\sqrt{1-a^2} \right\} \\
 &\quad + \rho_1 \left\{ x^2(1-a^2) + 2ax\sqrt{1-x^2}\sqrt{1-a^2} \cos(\omega) + (1-x^2)a^2 \right\} \\
 \rho^p &= \begin{pmatrix} x & \sqrt{1-x^2} \end{pmatrix} \begin{pmatrix} y^2 & e^{-i\theta}y\sqrt{1-y^2} \\ e^{i\theta}y\sqrt{1-y^2} & 1-y^2 \end{pmatrix} \begin{pmatrix} \rho_0 & T \\ T & \rho_1 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} y^2 & e^{-i\theta}y\sqrt{1-y^2} \\ e^{i\theta}y\sqrt{1-y^2} & 1-y^2 \end{pmatrix} \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \\
 &= \left\{ y^2 \rho_0 + (1-y^2) \rho_1 + 2y\sqrt{1-y^2} T \right\} \\
 &\quad \times \left\{ x^2 y^2 + (1-y^2)(1-x^2) + 2xy\sqrt{1-x^2}\sqrt{1-y^2} \cos(\theta) \right\}
 \end{aligned}$$

Appendix C: Characteristics of Hermitian Matrix and Unitary Matrix Used in the Examples

The projection uses a Hermitian matrix, and here

$$V = \begin{pmatrix} y^2 & e^{-i\theta}y\sqrt{1-y^2} \\ e^{i\theta}y\sqrt{1-y^2} & 1-y^2 \end{pmatrix} \text{ is prepared.}$$

For V , $V^\dagger = V$, $V^\dagger V = VV = V$ ($V^\dagger$ is a conjugate and transpose of V).

V has two eigen values and they are 1 and 0.

Eigen value decomposition with $P_0 + P_1 = I$ (The former is P_0 , the latter is P_1):

$$V = 0 \cdot \begin{pmatrix} 1-y^2 & -e^{i\theta}y\sqrt{1-y^2} \\ -e^{-i\theta}y\sqrt{1-y^2} & y^2 \end{pmatrix} + 1 \cdot \begin{pmatrix} y^2 & e^{-i\theta}y\sqrt{1-y^2} \\ e^{i\theta}y\sqrt{1-y^2} & 1-y^2 \end{pmatrix}$$

The time evolution uses a unitary matrix, and here

$$W = \begin{pmatrix} a & -e^{i\omega}\sqrt{1-a^2} \\ \sqrt{1-a^2} & ae^{i\omega} \end{pmatrix} \text{ is set.}$$

For W , $W^\dagger W = I$ ($W^\dagger$ is a conjugate and transpose of W). If a is set as $\cos(\varpi)$ and when ω is set as 0, W becomes $\begin{pmatrix} \cos(\varpi) & -\sin(\varpi) \\ \sin(\varpi) & \cos(\varpi) \end{pmatrix}$, and this means the rotation matrix with the rotation degree ϖ .