

Adaptive Investment Strategies for Transitioning from Fossil-Fuels to Cleaner Energies: An Application of Conjugate Utilities

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How to cite this paper: Moagi, G.S., Doctor, O. and Lungu, E. (2026) Adaptive Investment Strategies for Transitioning from Fossil-Fuels to Cleaner Energies: An Application of Conjugate Utilities. *Journal of Mathematical Finance*, 16, 57-79.
<https://doi.org/10.4236/jmf.2026.162004>

Received: January 29, 2026

Accepted: March 30, 2026

Published: April 2, 2026

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Abstract

We have formulated an investment model that applies a conjugate utilities approach to illustrate that wealth can still grow in transitioning portfolios. We have proved that despite one asset being phased out, a unique solution for the wealth process exists. We have determined an optimal pair (π_1^*, π_2^*) which guarantees a growing total wealth $W(t)$. We have also shown that high stock volatility is detrimental to the return on investment. The results of this study indicate the possibility of portfolio replication while aligning with sustainable development goals, SDG7 and SDG13.

Keywords

Energy Transition, Sustainable Finance, Conjugate Utility Theory, Girsanov Theorem, Stochastic Differential Equations, Portfolio Optimization

1. Introduction

Reducing carbon emission deposition into the atmosphere requires countries to align their emissions with the Paris accord [1], adopting and embracing renewable energy policies. In response to the impending climate crisis, countries are introducing policies to re-evaluate the effectiveness of existing policies where these are available or to introduce green policies if there are no policies in place. Fossil fuel energy generators likewise are realigning their focus from traditional fossil fuel-based assets towards more sustainable, eco-friendly alternatives [2] [3]. This shift partly responds to the increasing regulatory and societal pressures to adopt more responsible, and environmentally-conscious practices and partly to the need to introduce cheaper technologies in energy generation. However, this challenge pre-

sents a unique set of challenges to both public and private energy generators who must now reallocate capital to renewables in order to generate long-term value [4] [5]. Countries are introducing incentives and financial instruments to ensure a smooth adoption and expansion of the renewable energy industry and to encourage individuals to invest in clean power generation in order to take advantage of the new energy order. Although the pace of introducing these incentives varies among countries, the effort within these countries is commendable as individuals are borrowing money to invest in rooftop solar generation. Even the climate skeptics are reconsidering their positions as a result of extreme events such as droughts, hurricanes, fires, extremely low temperatures in places, etc. Credible efforts towards achieving sustainable development goals, seven and thirteen are demonstrated by initiatives such as the interdisciplinary network of independent German research institutes which is focusing on providing information on Germany's energy transformation [6], an initiative designed to measure the pace of transformation. China being one of the main contributors of greenhouse gases, is in fact the world's largest investors in renewable energies [7]. Other notable initiatives include India's rooftop solar project at the individual level, an initiative which is the country's major focus of the renewable energy strategy [8] and Denmark's wind energy project [9]. Involving individuals to invest in National Power plants is a known strategy to reduce risk of loss by the power co operations. However, despite the strategies mentioned above, and others elsewhere including Africa, renewable energy generation accounts for only 30% of the global power mix [10]. Research is being conducted to understand and monitor the evolving terrain as traditional energy sources make way for renewable energy sources [11] [12]. A study by Feng *et al.* [13] has shown how China has made notable advancements in developing green finance, and how it has positioned itself as a front-runner in environmentally sustainable economic instruments like green credits and bonds. Understanding the mechanisms and the potential impacts of green bonds on the overall investment strategy is crucial for power generators looking to align their portfolios with sustainable and responsible practices [14] [15]. Ameur *et al.* [16] used the conditional value-at-risk ratio to study the benefits of including green assets to conventional portfolios and showed that these green assets can reduce portfolio risk in the short-medium term. However, this study did not provide insights why green assets fail to reduce risk in the long term. The advancement of research in applying diversification and portfolio optimization to energy transition from fossil fuels to renewables has gained significance [17] [18]. Daywes *et al.* [19] combined techniques to analyze risk and developed a power generation optimization strategy of a portfolio including hydro, wind and solar as portfolio assets. Their findings suggested that financial factors, particularly debt financing and the structure of cash flows, play a significant role in shaping the correlation between energy resources in a portfolio and diversification across complementary energy sources.

These advancements in research have helped to identify optimal investment

strategies, to balance risks and returns, and to maximize the efficiency and effectiveness of renewable energy portfolios. By diversifying the energy portfolio, investors can reduce their reliance on a single energy source and mitigate risks associated with price volatility and supply disruptions [19]. In this study, we aim to leverage conjugate utility functions to study a problem of replicating portfolios and to demonstrate how to manage the phase out rate, and ensure a growing economy.

2. Formulation of the Problem

Consider a power generating company taking investment positions in two risky assets and a risk-free asset. One stock investment, in fossil fuels, is being phased out gradually as the company migrates to cleaner energies and the other risky asset is an investment in either cleaner energies or other non energy assets. The bond acts as a buffer against stock market volatility. In this study we do not include the buying and selling of assets. Even though the second stock is risky, the investor is however anticipating the potential for capital appreciation from it as the other stock depreciates due to migration to green energy.

Let $L_+ = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space on a finite time horizon, $0 \leq t \leq T$. Based on the description of the assets, the problem described herein is a quasi self-financing portfolio problem because the phase down is externally induced while the gain in value of the second stock is internal. Let $S_0(t)$ represent the bond price, $S_1(t)$ the price of the stock which is being phased out and $S_2(t)$ the price of the stock in other investments excluding fossil fuels at time t . The second asset can be considered as the green asset in the study by Ameer *et al.* [16].

We assume the dynamics for the three assets as follows:

$$\begin{cases} dS_0(t) = S_0(t)r(t)dt; & S_0(0) = 1 \\ dS_k(t) = S_k(t)[\mu_k(t)dt + \sigma_k(t)dB_k(t)]; & S_k(0) > 0 \end{cases}$$

where $r(t) < \mu_k$ for $k = 1, 2$, is the interest rate of the bond, $\mu_1(t)$, $\mu_2(t)$ are the growth rates of the stocks $S_1(t), S_2(t)$, respectively, $\sigma_1(t) > 0$ and $\sigma_2(t) > 0$, are the volatilities of the stocks $S_1(t)$ and $S_2(t)$, respectively. We assume that $\mu_1 \geq r$ and the rates $r(t)$, $c(t)$, $\sigma_1(t)$ and $\sigma_2(t)$ are adapted and bounded processes. The Brownian motions of the stocks are correlated with a correlation coefficient ρ such that $dB_i(t) \cdot dB_j(t) = \rho_{i,j}dt$ for $i, j \in \{1, 2\}$ and $i \neq j$ where $\rho \in [-1, 1]$.

2.1. Derivation of the Total Wealth Process $W(t)$

Let $W(t)$ be the total wealth invested in the three assets, the bond, $S_0(t)$, a depreciating stock, $S_1(t)$, and a stock, $S_2(t)$, representing other investments excluding fossil fuels at time t . Let $\pi_1(t)$ and $\pi_2(t)$ be the amounts of money invested in the stocks $S_1(t)$ and $S_2(t)$ at time t , respectively. The amount at any time t held in the bond is $W - \pi_1 - \pi_2$.

2.1.1. The Phase Out Rate and Its Characteristics

We propose a phase out function

$$c(t) = \gamma \log_2 \left(1 + \frac{t}{T} \right), \text{ where } 0 \leq \gamma < 1, \quad (1)$$

for the following reasons:

- $\log_2 \left(1 + \frac{t}{T} \right), 0 \leq \gamma < 1$ is an increasing function which doubles at the terminal time, T .
- The parameter γ , for $0 \leq \gamma < 1$, defines risk averse (for small values of γ) and risk seeking investments (for values of γ close to 1).

The choice of $c(t)$ is in line with the Sustainable Development Goals (SDGs) 7 and 13 [20].

2.1.2. The Stock Prices

The following are proposed altered stock prices based on the phase out rate:

$$dS_1(t) = S_1(t) [\hat{\mu}_1(t) dt + \hat{\sigma}_1(t) dB_1(t)] \quad (2)$$

$$dS_2(t) = S_2(t) [\hat{\mu}_2(t) dt + \hat{\sigma}_2(t) dB_2(t)] \quad (3)$$

where

$$\hat{\mu}_1(t) = (1 - c(t))\mu_1, \quad \hat{\mu}_2(t) = (1 + c(t))\mu_2$$

$$\hat{\sigma}_1(t) = (1 - c(t))\sigma_1, \quad \hat{\sigma}_2(t) = (1 + c(t))\sigma_2.$$

Let

$$S(t) = \begin{bmatrix} S_1(t) \\ S_2(t) \end{bmatrix} \in \mathbb{R}^2, \quad (4)$$

then we can write the risky assets as:

$$dS(t) = \begin{bmatrix} (1 - c(t))\mu_1 S_1(t) \\ (1 + c(t))\mu_2 S_2(t) \end{bmatrix} dt + \begin{bmatrix} (1 - c(t))\sigma_1 S_1(t) & 0 \\ 0 & (1 + c(t))\sigma_2 S_2(t) \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}. \quad (5)$$

We solve

$$\begin{bmatrix} (1 - c(t))\sigma_1 S_1(t) & 0 \\ 0 & (1 + c(t))\sigma_2 S_2(t) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (1 - c(t))\mu_1 S_1(t) \\ (1 + c(t))\mu_2 S_2(t) \end{bmatrix}.$$

and obtain a unique solution for u_1 and u_2 given by:

$$u_1 = \frac{\mu_1}{\sigma_1}, \quad u_2 = \frac{\mu_2}{\sigma_2}.$$

Let

$$\begin{aligned} M_t &= \exp \left[-\int_0^t u dB_s - \frac{1}{2} \int_0^t u^2 ds \right], \quad t \leq T \\ &= \exp \left[-\frac{\mu_1}{\sigma_1} B_1(T) - \frac{\mu_2}{\sigma_2} B_2(T) - \frac{1}{2} \left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} \right) T \right]. \end{aligned}$$

Hence if we define

$$d\mathbb{Q} = \exp \left[-\frac{\mu_1}{\sigma_1} B_1(T) - \frac{\mu_2}{\sigma_2} B_2(T) - \frac{1}{2} \left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} \right) T \right] dP$$

and

$$d\hat{B}(t) = \begin{bmatrix} \frac{\mu_1}{\sigma_1} \\ \frac{\mu_2}{\sigma_2} \end{bmatrix} dt + dB(t),$$

where $dB(t) = \begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}$, then the Novikov condition holds and we conclude that $\hat{B}(t)$ is a Brownian motion w.r.t the probability measure $\mathbb{Q}$ and the system (5) becomes

$$dS(t) = \begin{bmatrix} (1-c(t))\sigma_1 S_1(t) & 0 \\ 0 & (1+c(t))\sigma_2 S_2(t) \end{bmatrix} d\hat{B}(t).$$

Showing that the risky assets are martingales w.r.t $\mathbb{Q}$.

2.1.3. The Wealth Process

The total wealth process influenced by the price rates $\frac{dS_k}{S_k}$, $k=0,1,2$ is given by:

$$\begin{aligned} dW(t) = & \left[r(t)W(t) + \pi_1(t)((1-c(t))\mu_1(t) - r(t)) \right. \\ & \left. + \pi_2(t)((1+c(t))\mu_2(t) - r(t)) \right] dt + \pi_1(t)(1-c(t))\sigma_1(t)dB_1(t) \quad (6) \\ & + \pi_2(t)(1+c(t))\sigma_2(t)dB_2(t) \quad \text{with } W(0) = w, 0 \leq \gamma < 1. \end{aligned}$$

We note specifically from (6) that the phase out rate affects both the drifts and volatilities of the risky assets. Specifically, $c(t)$ decreases the drift of the fossil fuel asset and dampens the noise part. It increases the drift of the non fossil fuel asset and increases the noise effect. This is in line with the market sentiments as interest wanes in fossil fuel energy assets. Note that the coefficients of Equation (6)

$$\begin{aligned} b(w) = & r(t)W(t) + \pi_1(t)((1-c(t))\mu_1(t) - r(t)) \\ & + \pi_2(t)((1+c(t))\mu_2(t) - r(t)), \\ \hat{\sigma}_1(w) = & \pi_1(t)(1-c(t))\sigma_1(t) \quad \text{and} \quad \hat{\sigma}_2(w) = \pi_2(t)(1+c(t))\sigma_2(t) \end{aligned}$$

satisfy the linear growth conditions and therefore Equation (6) has a global solution, defined for all $t \geq 0$. (See appendix for details).

Equation (6) is similar to the wealth process studied by Duffie D. *et al.* [21]. The departure from Duffie D. *et al.* [21] is that Duffie D. *et al.* [21] deposited proceeds from the stocks into the bond account while our approach aggregates the contributions from each asset while phasing out S_1 by the factor $\gamma c(t)$ at time t .

The problem is to find optimal balances $\pi_1^*(t)$ and $\pi_2^*(t)$ at time t which ensure that the total wealth process is increasing towards the targeted amount at time $t = T$. We want to apply a strategy which allows the power generating company to continuously adjust the strategy in order to shorten the time to maturity towards the targeted amount by varying γ . However, if the science or economic environment suggests that the current phase out rate, $c(t)$, is inadequate then the investor should have an opportunity to adjust $c(t)$ by a factor γ . The weakness of our approach is that adjusting $c(t)$ affects both risky assets. The optimal decision selection of a strategy is therefore a function of the history of the decisions made previously, based on observed returns at the material time.

Proposition 2.1 *Arbitrage free market*

For a fixed time $T > 0$, we define an equivalent probability measure $\mathbb{Q}$ to $\mathbb{P}$ on a filtration $\mathcal{F}_T$. $\mathbb{Q}$ is continuous w.r.t $\mathbb{P}|_{\mathcal{F}_T}$ if $\forall a \in \mathcal{F}_T$, $\mathbb{P}(a) = 0 \Rightarrow \mathbb{Q}(a) = 0$ given that the processes $\mu_1(t)$, $\mu_2(t)$, $\sigma_1(t)$, $\sigma_2(t)$, $r(t)$ and $c(t)$ are deterministic and bounded on $[0, T]$, with $\sigma_1(t) > 0$, $\sigma_2(t) > 0$ and $|c(t)| < 1 \forall t \in [0, T]$. Therefore the discounted price processes are martingales, and the market is arbitrage-free.

We define the market price of risk vector $\bar{\Theta}(t)$ with $\psi(s)$ and $\zeta(s)$ to be adapted processes given below:

$$\bar{\Theta}(t) = \begin{bmatrix} \psi(t) \\ \zeta(t) \end{bmatrix}$$

where

$$\psi(t) = \frac{\mu_1(t)(1-c(t)) - r(t)}{(1-c(t))\sigma_1(t)}, \quad \zeta(t) = \frac{\mu_2(t)(1+c(t)) - r(t)}{(1+c(t))\sigma_2(t)}. \quad (7)$$

The existence of a risk-neutral measure $\mathbb{Q}$ requires Radon-Nikodym derivative Z_T to be a true martingale which is guaranteed if $\psi(t)$ and $\zeta(t)$ satisfy the Novikov condition:

$$\mathbb{E} \left[\exp \left\{ \frac{1}{2} \int_0^T |\psi(t)|^2 dt \right\} \right] < \infty, \quad \mathbb{E} \left[\exp \left\{ \frac{1}{2} \int_0^T |\zeta(t)|^2 dt \right\} \right] < \infty. \quad (8)$$

Proof. Based on the model assumptions $\psi(t)$ and $\zeta(t)$ are deterministic and bounded processes for all $0 \leq t \leq T$. For any deterministic process, the expectation is a finite constant:

- 1) The integral of the squared market price of risk over time is finite

$$\mathbb{E} \left[\int_0^T (\psi(t))^2 dt \right] < \infty.$$

- 2) The exponential of this integrand is also finite:

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T (\psi(t))^2 dt \right) \right] < \infty.$$

Hence the Novikov condition is satisfied, and Z_T is a true martingale. $\square$

The Radon-Nikodym derivative of the risk neutral measure $\mathbb{Q}$ w.r.t $\mathbb{P}$,

$Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$ is given by

$$Z_T := \exp \left\{ -\int_0^T \psi(s) dB_1(s) - \frac{1}{2} \int_0^T \psi^2(s) ds - \int_0^T \zeta(s) dB_2(s) - \frac{1}{2} \int_0^T \zeta^2(s) ds \right\}.$$

defined in vector form as

$$Z_T := \exp \left\{ -\int_0^T \Theta(s) \cdot dB(s) - \frac{1}{2} \int_0^T \|\Theta(s)\|^2 ds \right\}$$

where the correlated Brownian motions vector is given as

$$d\vec{B}(t) = \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}.$$

The Brownian motions are correlated with the correlation coefficient matrix

$$d\vec{\rho}(t) = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

where $\rho \in [-1, 1]$ and $dB_1(t) \cdot dB_2(t) = \rho dt$.

We define the relationship between the measure $\mathbb{P}$ and the risk neutral measure $\mathbb{Q}$ defined by the Girsanov theorem [22] as

$$dB_1^{\mathbb{Q}}(t) = dB_1(t) + \psi(t) dt; \quad t \leq T$$

$$dB_2^{\mathbb{Q}}(t) = dB_2(t) + \zeta(t) dt; \quad t \leq T$$

for all $t \in [0, T]$. Given the process $W(t)$ in (6), substituting the Brownian motions using the Girsanov transformation above to neutralizes the wealth dynamics below to

$$\begin{aligned} dW(t) = & \left[r(t)W(t) + \pi_1((1-c(t))\mu_1 - r(t)) + \pi_2((1+c(t))\mu_2 - r(t)) \right] dt \\ & + \pi_1(1-c(t))\sigma_1(t)dB_1^{\mathbb{Q}}(t) - \pi_1(1-c(t))\sigma_1(t)\psi(t)dt \\ & + \pi_2(1+c(t))\sigma_2(t)dB_2^{\mathbb{Q}}(t) - \pi_2(1+c(t))\sigma_2(t)\zeta(t)dt \end{aligned} \quad (9)$$

substituting the market price of the risks $\psi(t)$ and $\zeta(t)$, and grouping the dt terms the equation neutralizes the drift to the following discounted wealth process

$$dW(t) = r(t)W(t)dt + \pi_1(1-c(t))\sigma_1(t)dB_1^{\mathbb{Q}}(t) + \pi_2(1+c(t))\sigma_2(t)dB_2^{\mathbb{Q}}(t) \quad (10)$$

which is a martingale.

Since the Girsanov's Theorem only shifts the drift and does not change the diffusion the correlation between the two Brownian motions remains exactly ρ .

3. Construction of the Value Wealth Function

Definition 3.1. Let Θ be a set of strategies for the investment process in (10). A strategy is admissible if and only if $W(t) \geq 0$, $\forall t \in [0, T]$ and

$$\mathbb{E} \left[\int_0^T e^{\int_s^T r(x) dx} W(t) dt \mid \mathcal{F}_T \right] = W(T), \text{ for } s, t \in [0, T].$$

Definition 3.2. We denote the value function $u(x)$ by

$$u(x) = \sup_{\pi_1^*, \pi_2^* \in \mathcal{A}} \mathbb{E} \left[\int_0^T e^{\int_s^t r(x) dx} U(W(t)) dt \right]$$

where $\mathcal{A}$ is a set of admissible portfolios and $U(\cdot): \mathbb{R} \mapsto \mathbb{R}$ is the generalized utility function which is strictly increasing, strictly concave, continuously differentiable satisfying the conditions $U'(0) = \lim_{x \rightarrow 0} U'(x) = \infty$ and

$U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0$. Then $u(x)$ is again a utility function provided it satisfies the asymptotic elasticity condition [23].

Definition 3.3. Let $V(y)$ be the Legendre-transform of the function $-U(-x)$ which is continuously differentiable, decreasing, strictly convex function satisfying the conditions $V'(0) = -\infty$, $V'(\infty) = 0$ such that $V(0) = U(\infty)$ and $V(\infty) = U(0)$ then the functions $U(x)$ and $V(y)$ are conjugates and can be written

$$V(y) = \sup_{x > 0} [U(x) - xy], \quad y > 0, \quad (11)$$

and

$$U(x) = \inf_{y > 0} [V(y) + xy], \quad x > 0. \quad (12)$$

Alternatively, the relationship between $U(x)$ and $V(y)$ is given by the relationship (see [24] for more details):

$$V(t, y) = \sup_{x > 0} [U(t, I(y)) - yI(t, y)] \quad (13)$$

where $I(t, y)$ is the inverse of the derivative of $U(t, y)$ given by

$$I(t, y) = \left(\frac{\partial U(t, y)}{\partial y} \right)^{-1}. \text{ The relationship (13) is more amenable to application}$$

of the conjugate utility theory as we shall demonstrate in Section (4.1).

Restated Problem Based on Conjugate Utility Functions

We define the pricing kernel $G(t)$ as in [24] by

$$G(t) = e^{-\int_0^t r(s) ds} \phi(t) \quad (14)$$

where,

$$\begin{aligned} \phi(t) = \exp \left\{ -\frac{1}{2} (\psi^2(s) + \zeta^2(s) + 2\psi(s)\zeta(s)\rho) ds \right. \\ \left. - \int_0^t \psi(s) dB_1(s) - \int_0^t \zeta(s) dB_2(s) \right\}. \end{aligned}$$

The problem is to find a unique optimal pair (π_1^*, π_2^*) from the set $\mathcal{A}$ such that for a pre-determined terminal wealth $\mathbb{E}[G(t)U(W(t))] = W(T)$, $t \in [0, T]$ the investor is able to achieve the stated goal, that is, to realize the targeted terminal wealth $W(T)$.

Specific Problem: Find an optimal pair (π_1^*, π_2^*) such that

$$\sup_{\pi_1^*, \pi_2^* \in \mathcal{A}} \mathbb{E}[G(t)U(W(t))] = W(T)$$

and

$$\mathbb{E}[G(s)W(s)|\mathcal{F}_t] \leq w + \mathbb{E}[G(T)W(T)] \text{ for } t < T.$$

First, we want to prove that the optimal pair (π_1^*, π_2^*) exists is unique and does not depend on the noise. We approach this through the theory of conjugate utilities stated in Definition (3.3).

Lemma 3.1. *Let C_x and D_y be subsets of $L_+(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$ which are convex, solid and closed in the topology of convergence measure. Define $g(x) \in C_x$, $0 \leq g(x) \leq X_T$, $x = x(\pi_1, \pi_2, c(t))$ and $h(y) \in D_y$, $0 \leq h(y) \leq Y_T$, $y = y(\pi_1, \pi_2, c(t))$. Let $U(x)$ and $V(y)$ be the functions defined in (11) and (12) such that*

$$u(x) = \sup_{g \in C_x} [U(g)], \quad (15)$$

$$v(y) = \inf_{h \in D_y} [V(h)]. \quad (16)$$

Then the optimal set (π_1^, π_2^*) for $g(x)$ is the same as that for $h(y)$, that is $\hat{g}(x) = I(\hat{h}(y))$ is the unique optimal solution of (10).*

Proof. Let $\hat{g}(x)$ and $\hat{h}(y)$ be the optimal values such that for $g(x) \in C_x$ and $h(y) \in D_y$, $g(x) \leq \hat{g}(x)$ and $h(y) \leq \hat{h}(y)$. For any $h \in D_y$

$$\mathbb{E}[hI(\hat{h}(y))] \leq xy = \mathbb{E}[\hat{h}(y)I(\hat{h}(y))]. \quad (17)$$

At the maximum point, we have

$$\mathbb{E}[\hat{g}(x)\hat{h}(y)] = xy.$$

For any $g(x) \in C_x$ we have

$$\mathbb{E}[g\hat{h}(y)] \leq xy.$$

This is in line with (17) where $g(x)$ was at its optimal but $h(y)$ was not. From the dual relationship (12) with y at optimal value, we have

$$U(y) \leq V(\hat{h}(y)) + g\hat{h}(y). \quad (18)$$

Taking expectation of (18), we have

$$\begin{aligned} \mathbb{E}[U(y)] &\leq v(y) + xy \\ &= \mathbb{E}[v(\hat{h}(y)) + \hat{h}(y)I(\hat{h}(y))] \\ &= \mathbb{E}[U(I(\hat{h}(y)))] \\ &= \mathbb{E}[U(\hat{g}(x))] \end{aligned}$$

implying that $\hat{g}(x) = I(\hat{h}(y))$. This concludes the proof. $\square$

4. Procedure for Constructing the Dual Function $\tilde{V}(\alpha)$

We want a step-by-step procedure for constructing the dual utility function

$\tilde{V}(\alpha)$. This is demonstrated by Lemma 4.1 below:

Lemma 4.1. Let $U(t, x)$ and $V(t, y)$ be the functions defined in (15) and (16). Define the auxiliary function as:

$$\xi(t, x) = \mathbb{E} \left[e^{\int_0^t \beta ds} \tilde{U}(x_T) \right] \quad (19)$$

for a non-negative β with $x(t)$ defined in terms of the pricing kernel $G(t)$ as:

$$x(t) = \alpha e^{\int_0^t \beta ds} G(t), \quad (20)$$

where $G(t)$ is given by (14). The constructed value dual function is given by

$$V(w) = \inf_{\alpha > 0} \{ \tilde{V}(\alpha) + \alpha w \} = \tilde{V}(\alpha^*) + \alpha^* w.$$

Proof. Substituting (14) in (20) we can express $x(t)$ as

$$\begin{aligned} x(t) = \alpha \exp \left\{ \int_0^t (\beta - r) ds - \frac{1}{2} \int_0^t (\psi^2 + \zeta^2 + 2\psi\zeta\rho) ds \right. \\ \left. - \int_0^t \psi dB_1(s) - \int_0^t \zeta dB_2(s) \right\}, \end{aligned} \quad (21)$$

which is a solution of the stochastic differential equation

$$\begin{aligned} \frac{dx(t)}{x(t)} &= [(\beta - r)dt - \psi(t)dB_1(t) - \zeta dB_2(t)] \\ x(0) &= 1. \end{aligned}$$

Taking the expectation of (21), we obtain

$$\begin{aligned} \mathbb{E}[x(t)] &= \mathbb{E} \left[\alpha \exp \left\{ \int_0^t \left[(\beta - r) - \frac{1}{2} (\psi^2 + \zeta^2 + 2\psi\zeta\rho) \right] ds \right. \right. \\ &\quad \left. \left. - \int_0^t \psi dB_1(s) - \int_0^t \zeta dB_2(s) \right\} \right] \\ &= \alpha \exp \left\{ \int_0^t \left[(\beta - r) - \frac{1}{2} (\psi^2 + \zeta^2 + 2\psi\zeta\rho) \right] ds \right\}, \text{ for } s \in [0, T]. \end{aligned} \quad (22)$$

Comparing Equations (19) and (22), we obtain

$$\tilde{V}(\alpha) = \mathbb{E} \left[e^{-\int_0^T \beta(t) dt} \tilde{U}(x_T) \right].$$

For the utility function $\tilde{U}(t, x)$ given by

$$\tilde{U}(t, x) = U(t, I(x)) - xI(t, x).$$

Following Kwak and Lim [24] the relationship between the dual function $\tilde{V}(\alpha)$ and the value function $V(w)$ can be written as,

$$V(w) = \inf_{\alpha > 0} \{ \tilde{V}(\alpha) + \alpha w \} = \tilde{V}(\alpha^*) + \alpha^* w \quad (23)$$

where α^* is obtained by differentiating $V(w)$ for a given $\tilde{V}(\alpha)$. Once α^* is calculated we can find the optimal wealth (15) for a known utility function. This point is illustrated by the example given in the section below. $\square$

4.1. Application of the Procedure in Subsection 4

We want to illustrate the procedure for calculating $V(w)$ in (23) and the associated optimal triple with a specific example.

Example 4.1. Suppose the utility function is given by the power utility function:

$$U(t, w) = am \frac{w^{1-b}}{1-b} e^{-\lambda t} \quad (24)$$

for some constants $m \neq 0$, $a \neq 0$, $b \in (0, 1)$ and $\lambda > 0$. If $\lambda < 0$ then $U(t, w)$ cannot be of elasticity < 1 . We want to find the dual function $\tilde{V}(\alpha)$ and the optimal amounts π_1^* , π_2^* .

Solution 4.1. We can calculate $I(t, w)$ from the utility in (24) using the formula [23].

$$I(t, w) = \left(\frac{\partial U(t, w)}{\partial w} \right)^{-1}.$$

This gives

$$I(t, w) = \left(\frac{w}{am} \right)^{\frac{-1}{b}} e^{\frac{-\lambda t}{b}}.$$

The dual function $\tilde{V}(\alpha)$ is given by,

$$\begin{aligned} \tilde{V}(\alpha) &= \mathbb{E}[\tilde{U}(\alpha G(t))] \\ &= \mathbb{E} \left[am \frac{\left(\frac{\alpha G(t)}{am} \right)^{\frac{-1}{b}} e^{\frac{-\lambda t}{b}}}{1-b} - \alpha G(t) \left(\frac{\alpha G(t)}{am} \right)^{\frac{-1}{b}} e^{\frac{-\lambda t}{b}} \right] \\ &= \mathbb{E} \left[\frac{am}{1-b} \left(\frac{\alpha G(t)}{am} \right)^{1-\frac{1}{b}} e^{\frac{-\lambda t}{b}} - \left(\frac{\alpha G(t)}{am} \right)^{1-\frac{1}{b}} e^{\frac{-\lambda t}{b}} \right] \\ &= \mathbb{E} \left[\left(\frac{\alpha G(t)}{am} \right)^{1-\frac{1}{b}} e^{\frac{-\lambda t}{b}} \left(\frac{am}{1-b} - 1 \right) \right]. \end{aligned} \quad (25)$$

Using Equation (20), the pricing kernel is given by

$$G(t) = \frac{x(t)}{\alpha} e^{-\beta t}. \quad (26)$$

Solving for $x(t)$ from (25) and (26), we obtain a general formula for the value function as

$$\tilde{V}(\alpha) = \mathbb{E} \left[\left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right) x(t)^{1-\frac{1}{b}} e^{-t \left(\beta \left(1 + \frac{1}{b} \right) + \frac{\lambda}{b} \right)} \right].$$

Defining the auxiliary function $\xi(t, x)$ where $x(t) = x$ as

$$\begin{aligned}\xi(t, x) &= \mathbb{E} \left[\left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right) x^{1-\frac{1}{b}} f(t) \right] \\ &= \left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right) x^{1-\frac{1}{b}} f(t).\end{aligned}\quad (27)$$

It is easy to show that $\xi(t, x)$ satisfies the equation

$$\xi_t - (\beta - r)x\xi_x + \frac{1}{2}(\psi^2 + \zeta^2 + 2\psi\zeta\rho)x^2\xi_{xx} = 0 \quad (28)$$

$$\text{where } \xi_t = \frac{\partial \xi(t, x)}{\partial t}, \quad \xi_x = \frac{\partial \xi(t, x)}{\partial x} \quad \text{and} \quad \xi_{xx} = \frac{\partial^2 \xi(t, x)}{\partial x^2}.$$

Using (27), we can write (28) as:

$$f'(t) = \theta f(t) \quad (29)$$

where

$$\theta = (\beta - r) \left(1 - \frac{1}{b} \right) + \frac{1}{2}(\psi^2 + \zeta^2 + 2\psi\zeta\rho) \frac{1}{b}(1-b)$$

and

$$f(t) = \frac{1}{2} \left(f(0)e^{\theta t} + e^{-t \left(\beta \left(1 + \frac{1}{b} \right) + \frac{\lambda}{b} \right)} \right) \text{ for } t \leq T \text{ and } f(0) \neq 0.$$

Solving (29) and noting that $\xi(0, x) = \tilde{V}(\alpha)$ we obtain

$$\xi(0, x) = \tilde{V}(\alpha) = \left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right) \alpha^{1-\frac{1}{b}} f(0) \quad \text{for } f(0) \neq 0 \quad \text{otherwise the value function does not exist.}$$

From the relation of $V(w)$ and the dual value function in Equation (23) we have,

$$V(w) = \inf_{\alpha > 0} \left\{ \left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right) \alpha^{1-\frac{1}{b}} f(0) + \alpha w \right\}. \quad (30)$$

The optimal α is obtained from (30) and is given by

$$\alpha^* = - \left[wf(0) \left(1 - \frac{1}{b} \right) \left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right) \right]^{-b}.$$

We deduce that for $\xi(0, x) = \tilde{V}(\alpha)$, we have $\xi(t, x) = \tilde{V}(t, \alpha)$ for $t \neq 0$ that is,

$$\begin{aligned}x(t) &= - \left[w(t) f(t) \left(1 - \frac{1}{b} \right) \left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right) \right]^{-b}, \\ x^{\frac{-1}{b}}(t) &= w(t) f(t) \left(\frac{1-b}{b} \right) \left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right)\end{aligned}$$

for $x(t) = \alpha^* e^{\int_0^t \beta ds} G(t)$ and $f(t)$ given above, the optimal terminal wealth is given below by,

$$W^*(T) = \frac{x(T)^{\frac{1}{b}}}{f(T) \left(\frac{1-b}{b} \right) \left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right)} \quad \text{for } t = T. \quad (31)$$

Example 4.2. (Illustration of the procedure 4.1)

Given $a \neq 0$, $m \neq 0$, a utility function $U(t, w) = am \frac{w^{1-b}}{1-b} e^{-\lambda t}$ and the terminal wealth given by $W(T) = \frac{x(T)^{\frac{1}{b}}}{f(T) \left(\frac{1-b}{b} \right) \left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right)}$, find the pair (π_1^*, π_2^*) .

Solution 4.2. Let $W(t) = \frac{x(t)^{\frac{1}{b}}}{f(t)\kappa}$ and $x(t)$ be the auxiliary variable given in (19), where $\kappa = \left(\frac{1-b}{b} \right) \left(\frac{1}{am} \right)^{1-\frac{1}{b}} \left(\frac{am}{1-b} - 1 \right)$.

Then by Ito's formula

$$\begin{aligned} dW(t) = & \frac{x^{\frac{-1}{b}}}{\kappa} \left((f')^{-2} + \frac{\alpha}{f} (\beta - r) + \frac{1}{2} \alpha^2 (\psi^2 + \zeta^2 + 2\psi\zeta\rho) \frac{\left(\frac{-1}{b} - 1 \right)}{b} f^{-1} \right) dt \\ & + \frac{\alpha}{f\kappa} x^{\frac{-1}{b}} \psi(t) dB_1(t) + \frac{\alpha}{f\kappa} x^{\frac{-1}{b}} \zeta(t) dB_2(t). \end{aligned} \quad (32)$$

Comparing (6) and (32) we have,

$$\pi_1^*(t) = \frac{\frac{\alpha}{f} x^{\frac{-1}{b}}}{\sigma_1 f(t) \kappa (1-c(t))} \psi(t), \quad (33)$$

$$\pi_2^*(t) = \frac{\frac{\alpha}{f} x^{\frac{-1}{b}}}{\sigma_2 f(t) \kappa (1+c(t))} \zeta(t). \quad (34)$$

Using (7), we simplify further and write (33) and (34) as

$$\pi_1^*(t) = \frac{aW(t) (\mu_1 (1-c(t)) - r)}{b\sigma_1^2 (1-c(t))^2}. \quad (35)$$

Similarly

$$\pi_2^*(t) = \frac{aW(t) (\mu_2 (1+c(t)) - r)}{b\sigma_2^2 (1+c(t))^2}. \quad (36)$$

Remark 4.2. We note that $\pi_1^*(t)$ and $\pi_2^*(t)$ depend on the wealth process $W(t)$ at each time $t \in [0, T]$ and since $W(t)$ is driven a stochastic noise, $\pi_1^*(t)$ and $\pi_2^*(t)$ are also dependent on the noise.

Remark 4.3. When the optimal investment strategies $\pi_1^*(t)$ and $\pi_2^*(t)$ are divided by the wealth process $W(t)$ we have the resulting expressions representing the proportions of the wealth invested in the respective risky stocks at time t . These are

$$\frac{\pi_1^*(t)}{W(t)} = \frac{a(\mu_1(1-c(t))-r)}{b\sigma_1^2(1-c(t))^2}, \quad \frac{\pi_2^*(t)}{W(t)} = \frac{a(\mu_2(1+c(t))-r)}{b\sigma_2^2(1+c(t))^2},$$

and the proportion on the bond is given by

$$\frac{b\sigma_2^2\sigma_1^2(1+c(t))^2(1-c(t))^2 - a\sigma_2^2(1+c(t))^2(\mu_1(1-c(t))-r)}{b\sigma_1^2\sigma_2^2(1-c(t))^2(1+c(t))^2} - \frac{a\sigma_1^2(1-c(t))^2(\mu_2(1+c(t))-r)}{b\sigma_1^2\sigma_2^2(1-c(t))^2(1+c(t))^2}.$$

From (35) and (36) we obtain an optimal ratio

$$\frac{\pi_1^*(t)}{\pi_2^*(t)} = \frac{((1-c(t))\mu_1 - r)(1+c(t))^2\sigma_2^2}{((1+c(t))\mu_2 - r)(1-c(t))^2\sigma_1^2},$$

where

$$c(t) = \gamma \log_2 \left(1 + \frac{t}{T} \right), \quad t \geq 0.$$

4.2. Results and Simulations

4.2.1. Ratio of π_1/π_2 against Time t

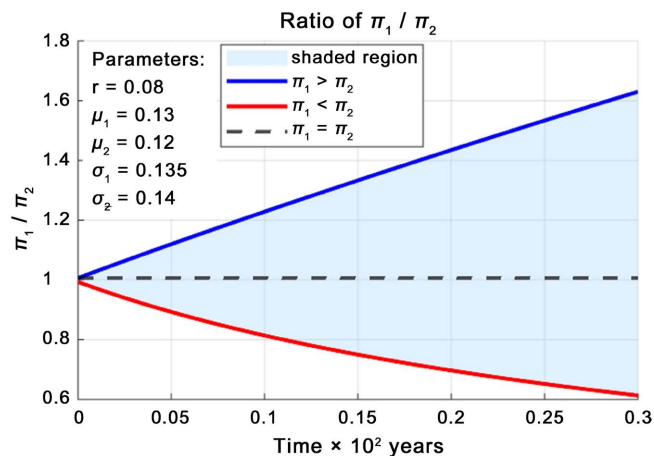


Figure 1. Evolution of the investment ratio π_1/π_2 .

Figure 1 traces the time dynamics of the ratio π_1/π_2 under fixed parameters stated on the graph. We have the following possible scenarios:

- If initially $\pi_1 > \pi_2$ then the ratio π_1/π_2 continues to grow implying that the effort to migrate from fossil fuels is not effective.
- If initially $\pi_1 = \pi_2$ then the effort to migrate is not sufficient as the two risky assets maintain the same ratio of significance.
- If initially $\pi_1 < \pi_2$ then the migration to cleaner energies is more effective.

Based on this, the strategy is to choose an aggressive investment strategy in the asset π_2 .

4.2.2. Simulation of the Wealth Process $W(t)$

The following examples illustrates how this sustained effort of divesting impacts the wealth growth processes.

Example 1: When the initial investments in stock $S_1 > S_2$ i.e. $\pi_1 = 5500$, $\pi_2 = 3500$.

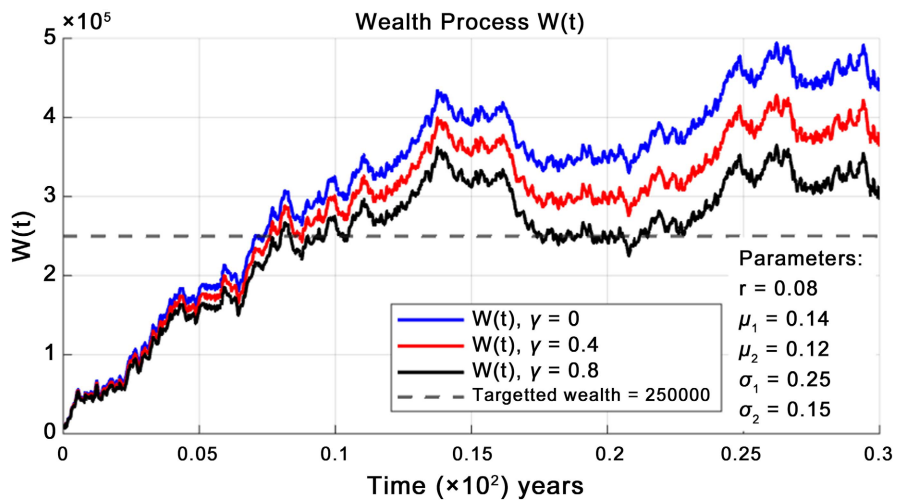


Figure 2. $W(t)$ for $\pi_1 > \pi_2$ With $\mu_1 > \mu_2$ for different phase-out rates γ .

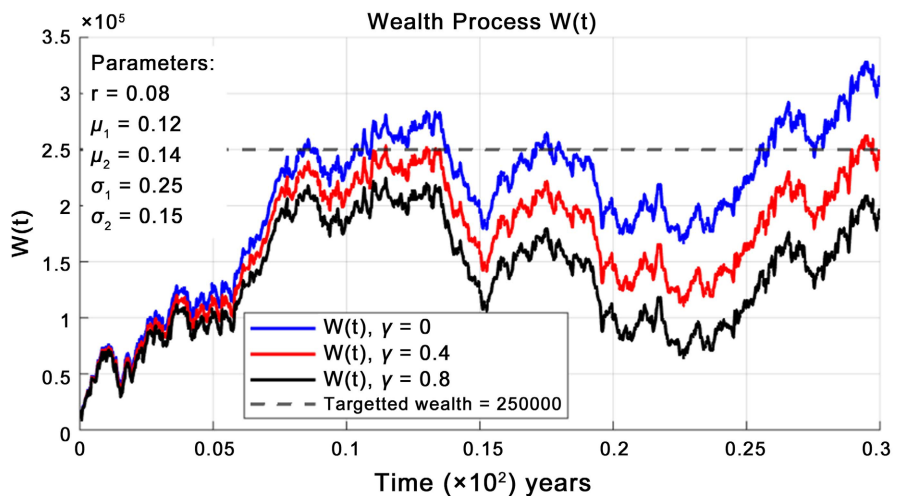


Figure 3. $W(t)$ for $\pi_1 > \pi_2$ With $\mu_1 < \mu_2$ for different phase-out rates γ .

Figure 2 presents a scenario when initially $\pi_1 > \pi_2$ and the drift $\mu_1 > \mu_2$ for varying values of γ . We observe that when $\gamma = 0$ the total wealth grows much faster than for the cases $\gamma > 0$. For $\gamma = 0$ the total wealth process exceeds the target wealth faster than for $\gamma > 0$ although the target wealth is exceeded for all cases under consideration. We observe further that even though the total wealth grows and exceeds the target wealth, the growth eventually mimics a mean reverting process with the wealth process for $\gamma = 0.8$ temporarily below the target wealth.

Figure 3 presents a scenario when initially $\pi_1 > \pi_2$ but $\mu_1 < \mu_2$ for varying values of γ . We can see that initially the total wealth process grows towards the target wealth. However, these wealth processes lose momentum and mimic a mean reverting process only the case $\gamma = 0$ displays mean reverting behavior about the target wealth. The other wealth processes display this behavior below the target wealth.

Example 2: Wealth Dynamics on Increased γ for Equal Initial Investments $S_1 = S_2$ and $\pi_1 = \pi_2 = 4500$

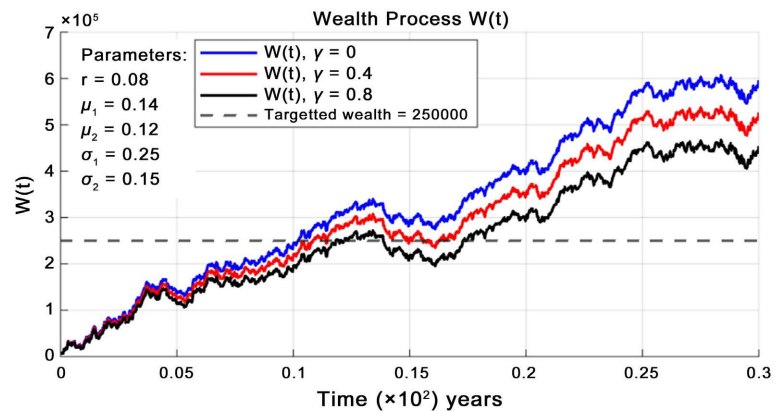


Figure 4. $W(t)$ for equal initial investments with $\mu_1 > \mu_2$.

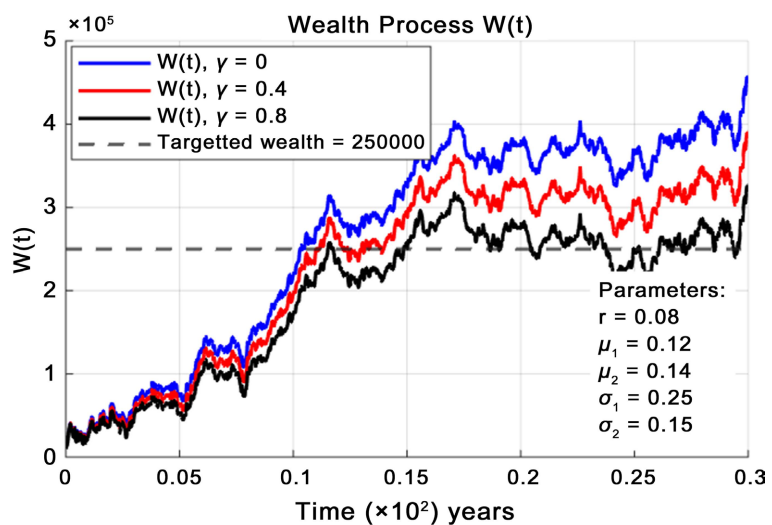


Figure 5. $W(t)$ for equal initial investments with $\mu_1 < \mu_2$.

Figure 4 and **Figure 5** represent scenarios when initially $\pi_1 = \pi_2$. We have computed cases for $\mu_1 > \mu_2$ (**Figure 4**) and $\mu_1 < \mu_2$ (**Figure 5**) for varying values of γ . Clearly for $\gamma = 0$ the target wealth is reached much faster than the other cases, for both $\mu_1 > \mu_2$ and $\mu_1 < \mu_2$. However, **Figure 4** shows that the wealth processes keep growing and they all exceed the target wealth at different times. **Figure 4** represents a risk-averse investor who is inclined towards more predictable returns. The commitment towards migration to cleaner energies is determined by a preference for lower returns with known risks over higher returns with unknown risks.

Figure 5 shows that the wealth processes grow less rapidly than those in **Figure 4**. For small values of γ the wealth processes exceed the target wealth but for large values of γ the wealth processes mimic mean reverting process about the target wealth. **Figure 5** represents risk-tolerant investors who are willing to take a risk for the good of the environment.

Example 3: when the initial investments in stock $S_2 > S_1$ i.e. $\pi_1 = 3500$, $\pi_2 = 5500$.

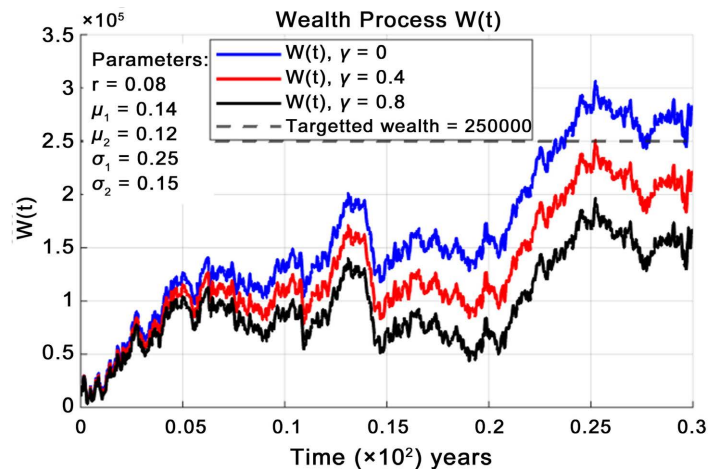


Figure 6. $W(t)$ for $\pi_1 < \pi_2$ with $\mu_1 < \mu_2$.

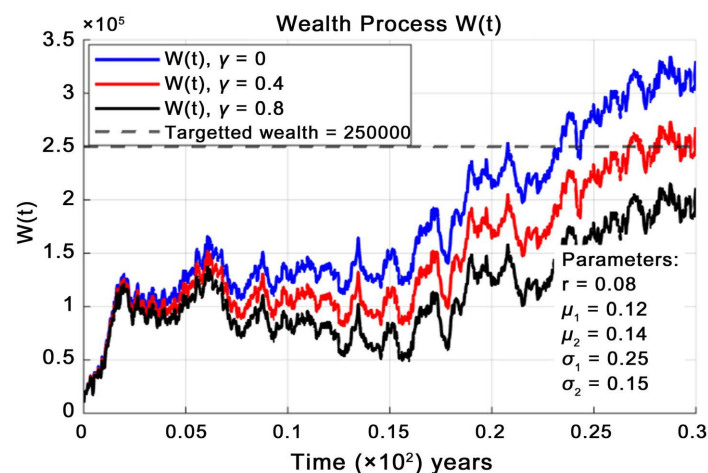


Figure 7. $W(t)$ for $\pi_1 < \pi_2$ with $\mu_1 > \mu_2$.

Figure 6 shows a scenario when initially $\pi_1 < \pi_2$ and the drift $\mu_1 < \mu_2$ for varying values of γ . We observe that when $\gamma = 0$, the total wealth grows and reaches the target much faster than for the cases $\gamma > 0$. When $\gamma = 0.4$ and $\gamma = 0.8$ the total wealth process never exceeds the target at all. **Figure 7** presents a scenario when initially $\pi_1 < \pi_2$ and the drift $\mu_1 > \mu_2$ for varying values of γ . We can see that initially the total wealth processes grow slowly. For $\gamma = 0$ the wealth process eventually exceeds the target wealth. For other values of $\gamma \neq 0$ the wealth processes do not exceed the target wealth although for small values of γ the wealth processes go close to the target wealth.

4.2.3. Strategy 1: Varying Interest Rate

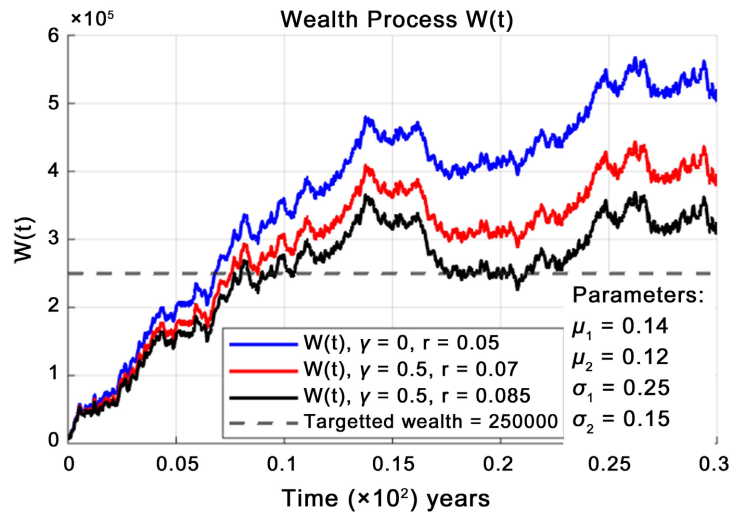


Figure 8. $W(t)$ with varying bond interest rate $r(t)$ for $\gamma = 0.5$.

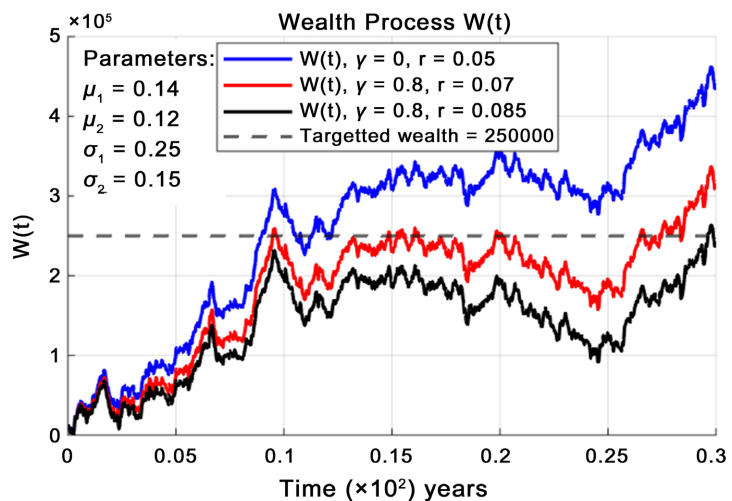


Figure 9. $W(t)$ with varying bond interest rate $r(t)$ for $\gamma = 0.8$.

Figure 8 and **Figure 9** illustrate a scenario for fixed values of, $\gamma = 0.5$ and $\gamma = 0.8$, when the bond interest rate is increased from $r(t) = 0.07$ to

$r(t) = 0.085$ with fixed risky asset interest rates $\mu_1 = 0.14$, $\mu_2 = 0.12$, and fixed volatilities σ_1 and σ_2 . **Figure 8** shows that the wealth increased for the baseline which is considered as a no phase out process. Even though there is growth for the higher interest rate considered, compared to the wealth with interest rate $r(t) = 0.07$ there is still delayed growth in the wealth process on both cases compared to when $\gamma = 0$. This is the case for **Figure 9** where the value of γ is increased to $\gamma = 0.8$. Comparing the two figures, it takes time for the strong phase outs to grow compared to the weak phase out where growth in wealth takes a shorter time to show growth in wealth. This indicates that phasing out has a possibility of bringing gains, but in the long run if done on a large scale and has a possibility of delaying these gains.

4.2.4. Strategy 2 Increased Volatility on the Wealth Process

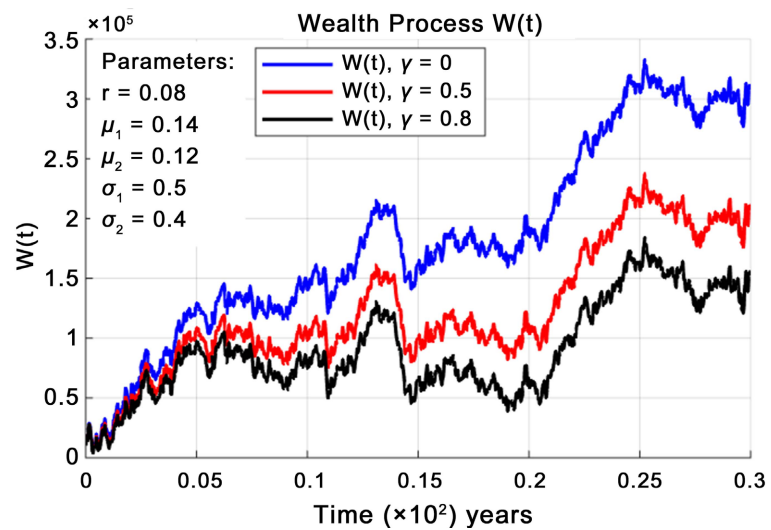


Figure 10. $W(t)$ with volatilities $\sigma_1 = 0.5$ and $\sigma_2 = 0.4$.

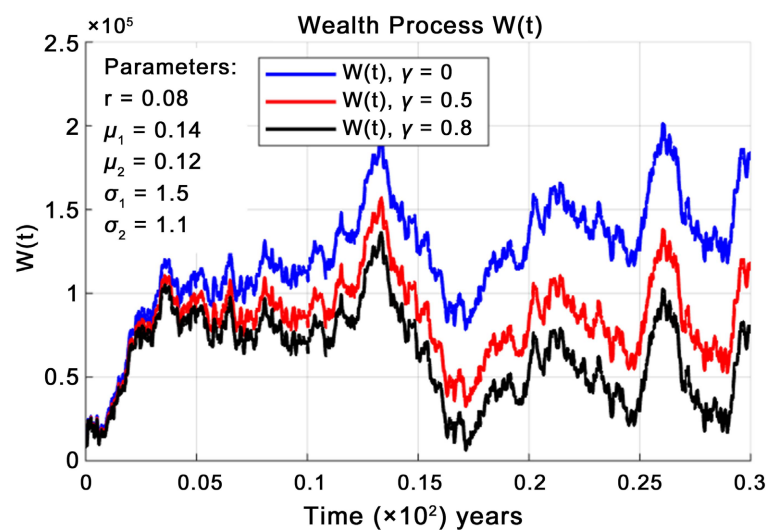


Figure 11. $W(t)$ with volatilities $\sigma_1 = 1.5$ and $\sigma_2 = 1.1$.

Figure 10 and **Figure 11** illustrate the same concept but with increased volatility rates. **Figure 10** with $\sigma_1 = 0.5$, $\sigma_2 = 0.4$ shows fluctuations on the wealth with minimal growth rates. This is the case for **Figure 11** where $\sigma_1 = 1.5$, $\sigma_2 = 1.1$ increased from **Figure 10**. In this figure, fluctuations are characterized by reduced growth which tends to fluctuates around the same point and decreases as the process progresses.

5. Conclusions

This study underscores the critical role that investment strategies can play toward sustainable energy solutions as we transition away from fossil fuel dependency. Through the application of the conjugate utility theory, we have demonstrated how an investor can manage assets and contribute towards reducing dependence and possibly fossil fuels eventual elimination of this energy source with managed risk of loss of wealth. The approaches introduced here not only aid in minimizing potential financial losses but also promote an environmentally responsible investment strategy. We have demonstrated that sometimes the only option to maintain a growing economy that prevents job losses is to reduce the phase down rate. Of course, this is environmentally harmful but it is a question of balancing people's livelihoods and the environment. Our findings further highlight the feasibility of replicating investment portfolios that are aligned with sustainable development goals. By validating these strategies regularly, we have demonstrated their robustness, which is essential for practical application in the real world.

We have obtained unique solutions for the wealth process. We have also shown the negative impact of high stock volatilities which have the effect of negating the gains resulting from bond and non-fossil fuels asset growth. Our major achievement is the construction of the procedure for calculating the dual utility function which plays a vital role in proving the uniqueness of the wealth process. This procedure is demonstrated using the power utility function. However, this procedure is applicable to any utility function satisfying the elasticity condition. This work demonstrates that countries can transition to cleaner energies without the risky of economic decline.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

- [1] United Nations Framework Convention on Climate Change. Key Aspects of the Paris Agreement. <https://unfccc.int/most-requested/key-aspects-of-the-paris-agreement>
- [2] Glemarec, Y. (2010) Financing the Transition to a Low-Carbon Society. *Journal of Renewable and Sustainable Energy*, **2**, Article ID: 031013. <https://doi.org/10.1063/1.3459135>
- [3] Schoenmaker, D. and Schramade, W. (2019) Principles of Sustainable Finance. Oxford University Press.

- [4] Agârbiceanu, S.M. and Păun, T. (2021) The Need for a Paradigm Shift in Finance: Sustainable Corporate Finance. *Management of Sustainable Development*, **13**, 33-38. <https://doi.org/10.54989/msd-2021-0006>
- [5] Balp, G. and Strampelli, G. (2022) Institutional Investor ESG Engagement: The European Experience. *European Business Organization Law Review*, **23**, 869-904. <https://doi.org/10.1007/s40804-022-00266-y>
- [6] Quitzow, L., Canzler, W., Grundmann, P., Leibenath, M., Moss, T. and Rave, T. (2016) The German Energiewende—What's Happening? Introducing the Special Issue. *Utilities Policy*, **41**, 163-171. <https://doi.org/10.1016/j.jup.2016.03.002>
- [7] International Energy Agency (2024) World Energy Investment 2024. IEA. <https://www.iea.org/reports/world-energy-investment-2024>
- [8] Goel, M. (2016) Solar Rooftop in India: Policies, Challenges and Outlook. *Green Energy & Environment*, **1**, 129-137. <https://doi.org/10.1016/j.gee.2016.08.003>
- [9] Government of Denmark. Pioneers in Clean Energy. <https://denmark.dk/innovation-and-design/clean-energy>
- [10] Enerdata. Global Energy Trends (2021) <https://www.enerdata.net/publications/reports-presentations/2021-energy-climate-trends.html>
- [11] Chaudhary, Y.C., Bangi, V.K.T., Guduru, R.K., Aung, K. and Reddy, G.N. (2017) Preliminary Investigation on Generation of Electricity Using Micro Wind Turbines Placed on a Car. *International Journal of Renewable Energy Development*, **6**, 75-81. <https://doi.org/10.14710/ijred.6.1.75-81>
- [12] Chen, J.M. (2021) The Capital Asset Pricing Model. *Encyclopedia*, **1**, 915-933. <https://doi.org/10.3390/encyclopedia1030070>
- [13] Feng, W., Bilivogui, P., Wu, J. and Mu, X. (2023) Green Finance: Current Status, Development, and Future Course of Actions in China. *Environmental Research Communications*, **5**, Article ID: 035005. <https://doi.org/10.1088/2515-7620/acc1c7>
- [14] Alamgir, M. and Cheng, M.-C. (2023) Do Green Bonds Play a Role in Achieving Sustainability? *Sustainability*, **15**, Article No. 10177. <https://doi.org/10.3390/su151310177>
- [15] Tan, X., Dong, H., Liu, Y., Su, X. and Li, Z. (2022) Green Bonds and Corporate Performance: A Potential Way to Achieve Green Recovery. *Renewable Energy*, **200**, 59-68. <https://doi.org/10.1016/j.renene.2022.09.109>
- [16] Ben Ameer, H., Ftiti, Z., Louhichi, W. and Yousfi, M. (2024) Do Green Investments Improve Portfolio Diversification? Evidence from Mean Conditional Value-at-Risk Optimization. *International Review of Financial Analysis*, **94**, Article ID: 103255. <https://doi.org/10.1016/j.irfa.2024.103255>
- [17] Gao, H.O. and Stasko, T.H. (2009) Diversification in the Driveway: Mean-Variance Optimization for Greenhouse Gas Emissions Reduction from the Next Generation of Vehicles. *Energy Policy*, **37**, 5019-5027. <https://doi.org/10.1016/j.enpol.2009.06.067>
- [18] Zhao, W.-H., Shi, Q.-S. and Dai, Q. (2009) Optimization Models for Renewable Generating Technologies Portfolio. 2009 *International Conference on Management Science and Engineering*, Moscow, 14-16 September 2009, 1836-1842. <https://doi.org/10.1109/icmse.2009.5317711>
- [19] Pinheiro Neto, D., Domingues, E.G., Coimbra, A.P., de Almeida, A.T., Alves, A.J. and Calixto, W.P. (2017) Portfolio Optimization of Renewable Energy Assets: Hydro, Wind, and Photovoltaic Energy in the Regulated Market in Brazil. *Energy Economics*, **64**, 238-250. <https://doi.org/10.1016/j.eneco.2017.03.020>

- [20] United Nations (2015) The 17 Sustainable Development Goals.
<https://sdgs.un.org/goals>
- [21] Duffie, D., Fleming, W., Soner, H.M. and Zariphopoulou, T. (1997) Hedging in Incomplete Markets with HARA Utility. *Journal of Economic Dynamics and Control*, **21**, 753-782. [https://doi.org/10.1016/s0165-1889\(97\)00002-x](https://doi.org/10.1016/s0165-1889(97)00002-x)
- [22] Øksendal, B. (2003) The Itô Formula and the Martingale Representation Theorem. In: Øksendal, B., Ed., *Stochastic Differential Equations: An Introduction with Applications*, Springer, 43-63. https://doi.org/10.1007/978-3-642-14394-6_4
- [23] Kramkov, D. and Schachermayer, W. (1999) The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets. *The Annals of Applied Probability*, **9**, 904-950. <https://doi.org/10.1214/aoap/1029962818>
- [24] Kwak, M. and Lim, B.H. (2014) Optimal Portfolio Selection with Life Insurance under Inflation Risk. *Journal of Banking & Finance*, **46**, 59-71.
<https://doi.org/10.1016/j.jbankfin.2014.04.019>

Appendix

A1. Imposed Conditions

- The interest rates $r(t)$, $\mu_1(t)$, $\mu_2(t)$ are measurable w.r.t time and bounded away from zero, therefore there exists $\epsilon > 0$ such that for

$$D := (r(t), \mu_1(t), \mu_2(t)) \text{ we have } |D| \geq \epsilon.$$

- The coefficients $\pi_1(t)$, $\pi_2(t)$, $\sigma_1(t)$ are time measurable and square integrable, that is

$$\int_0^T (\pi_1(t) \sigma_1(t))^2 dt < \infty \text{ and } \int_0^T (\pi_2(t) \sigma_2(t))^2 dt < \infty.$$

- The initial wealth w is independent of the Brownian motions $B_1(t)$ and $B_2(t)$, $t \in [0, T]$.

- The Lipschitz Condition:** The drift term; $b(t, W) = r(t)W(t) + f(t)$, where $f(t) = \pi_1(t)((1-c(t))\mu_1(t) - r(t)) + \pi_2(t)((1+c(t))\mu_2(t) - r(t))$,

and the diffusion coefficient

$$\sigma(t, W) = \begin{bmatrix} (1-c(t))\sigma_1\pi_1(t) \\ (1+c(t))\sigma_2\pi_2(t) \end{bmatrix}.$$

Therefore the Lipschitz condition from Theorem 5.2.1 [2]

$$\begin{aligned} |b(t, W_1) - b(t, W_2)| + |\sigma(t, W_1) - \sigma(t, W_2)| &= |r(t)(W_1 - W_2)| + |\sigma(t) - \sigma(t)| \\ &= |r(t)| \cdot |W_1 - W_2| \\ &\leq D |W_1 - W_2| \end{aligned}$$

since $r(t)$ is bounded for all $t \in [0, T]$ the drift is Lipschitz continuous with constant $|r(t)| \leq D$.

A2. Simulation Methodology

In bridging the analytical derivations with the numerical results, the following were taken into consideration when performing the simulations on the cases considered,

- The wealth process was simulated using the Euler-Maruyama discretized with a constant time step $\Delta t = 0.0003$ over a time horizon $T = [30]$. At each time step t_i , the wealth W_{t_i} is updated using the chosen parameters of the assets.
- The portfolio at every discrete time step Δt rebalanced to ensure that the numerical path tracks the continuous-time theoretical optimum.
- Constraints and non-negativity; We imposed admissibility restrictions by projecting the controls onto the feasible set $w_i^{sim}(t) = \max\{0, w_i(t)\}$ to enforce the no short selling constraint, since the considered parameters all scales above zero and this results in a non-negative wealth ensuring that short selling is not considered in the transition model.
- The initial investments weights $\pi_1(0)$ and $\pi_2(0)$ used were not chosen randomly but calculated at $t = 0$ from the analytical solutions for the simulations considered using Equations (35) and (36).