

Lower and Upper Bounds Estimators for a Real Yield Curve Based on Another Real Yield Curve and Its Break-Even Inflation Rate

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Abstract

We present a method to estimate some lower and upper bounds of a real yield curve. Between these bounds lies a region where it is mathematically impossible to contain real yields that satisfy the stated assumptions. This method is based on another real yield curve and a simple economic model of its break-even inflation rate that is split in inflation expectation, convexity term, and liquidity and inflation risk premiums. This formula is specially useful when one of these curves is difficult to obtain but a respective inflation expectation series is available. We illustrate our method with yield curves from Brazil, since it issues sovereign bonds linked to two different inflation indices each.

Keywords

Term Structure of Real Rates, Inflation-Indexed Bonds, Expected Inflation, Inflation Risk Premium, Liquidity Risk Premium

1. Introduction

Pension funds, insurance and reinsurance companies have been using term structures of interest rates for some years, at least since 2009, the year of Solvency II directives. Term structures can be formed by nominal or real (inflation adjusted) interest rates and are also known by the name of nominal and real yield curves. In a country or market, distinct price indices measure inflation differently and may give rise to more than one real yield curve.

In this article, we present a method to obtain some lower and upper bounds of a real yield curve, that is based on another real yield curve. This method estimates the bounds using a simple economic model of the respective break-even inflation rate, which is split in inflation expectation, convexity term, liquidity risk premium

and inflation risk premium.

The proposed estimator is specially useful when one of the two curves is difficult to obtain, such as in an illiquid market, but a respective inflation expectation series is available. We illustrate our method with yield curves from Brazil, since it issues sovereign bonds linked to two different inflation indices each.

Besides this introduction, this article has four other sections beginning in Section 2, where we present the required theory to support our method. In the next Section 3, we propose our lower and upper bounds estimators which are applied in Section 4. Section 5 summarizes the main results.

2. A Decomposition of the Break-Even Inflation Rate

The results of this section are not new and we closely follows Joyce *et al.* [1] and Vicente & Graminho [2] [3].

Let $P_t^N(\tau)$, for $t, \tau \geq 0$, denote the present value at time t of \$1 receivable τ periods ahead. We denote its continuously compounded zero-coupon nominal yield to maturity by $y_t^N(\tau)$, such that

$$P_t^N(\tau) = \exp(-\tau y_t^N(\tau)).$$

Also, let I_t^A and I_t^M denote two distinct price indices at time t , and let $P_t^A(\tau)$ and $P_t^M(\tau)$ denote the respective present value at time t of $\$ I_{t+\tau}^A / I_t^A$ and $\$ I_{t+\tau}^M / I_t^M$, both receivable τ periods ahead. Similarly to the nominal case, we denote their continuously compounded zero-coupon real yields to maturity by $y_t^A(\tau)$ and $y_t^M(\tau)$, respectively, such that

$$P_t^A(\tau) = \exp(-\tau y_t^A(\tau)); \quad P_t^M(\tau) = \exp(-\tau y_t^M(\tau)).$$

Given the nominal and real yields to maturity, we have the following break-even inflation rates:

$$\eta_t^A(\tau) = y_t^N(\tau) - y_t^A(\tau); \quad (1)$$

$$\eta_t^M(\tau) = y_t^N(\tau) - y_t^M(\tau). \quad (2)$$

Finally, let the inflation rates between the times t and $t + \tau$ be given by

$$\pi_t^A(\tau) = \frac{\ln I_{t+\tau}^A - \ln I_t^A}{\tau} \quad (3)$$

and

$$\pi_t^M(\tau) = \frac{\ln I_{t+\tau}^M - \ln I_t^M}{\tau}. \quad (4)$$

Assumption A1. There are no arbitrage opportunities available in the market where exist the financial assets with prices $P_t^N(\cdot)$, $P_t^A(\cdot)$ and $P_t^M(\cdot)$, and the price indices I_t^A and I_t^M , for $t \geq 0$.

By the Fundamental Theorem of Asset Pricing, assumption **A1** is equivalent to existence of a positive linear pricing rule which, by the Pricing Rule Representation Theorem, is equivalent to the existence of stochastic discount factors, also

called pricing kernels or state-price density, and of a martingale measure, also called risk-neutral pricing. For details see, for instance, Dybvig and Ross [4].

Denote the real price deflator processes by $\{\Psi_t^A, t \geq 0\}$ and $\{\Psi_t^M, t \geq 0\}$ and, similarly, denote the respective nominal price deflators by $\Psi_t^{NA} = \Psi_t^A / I_t^A$ and $\Psi_t^{NM} = \Psi_t^M / I_t^M$.

Let $M_t^A(\tau) = \Psi_{t+\tau}^A / \Psi_t^A$ and $M_t^M(\tau) = \Psi_{t+\tau}^M / \Psi_t^M$ denote the real stochastic discount factors between t and $t + \tau$. Similarly, let $M_t^{NA}(\tau) = \Psi_{t+\tau}^{NA} / \Psi_t^{NA}$ and $M_t^{NM}(\tau) = \Psi_{t+\tau}^{NM} / \Psi_t^{NM}$ denote the nominal stochastic discount factors between t and $t + \tau$.

The nominal prices of the assets, discounted by these deflators are thus martingales such that [2] [3]

$$P_t^A(\tau) = E_t(M_t^A(\tau)) = E_t\left(\frac{\Psi_{t+\tau}^A}{\Psi_t^A}\right) = E_t\left(\frac{\Psi_{t+\tau}^{NA}}{\Psi_t^{NA}} \frac{I_{t+\tau}^A}{I_t^A}\right) = E_t\left(M_t^{NA}(\tau) \frac{I_{t+\tau}^A}{I_t^A}\right)$$

and

$$P_t^N(\tau) = E_t(M_t^{NA}(\tau)) = E_t\left(\frac{\Psi_{t+\tau}^{NA}}{\Psi_t^{NA}}\right) = E_t\left(\frac{\Psi_{t+\tau}^A}{\Psi_t^A} \frac{I_t^A}{I_{t+\tau}^A}\right) = E_t\left(M_t^A(\tau) \frac{I_t^A}{I_{t+\tau}^A}\right),$$

where $E_t(X_s)$ denotes the expectation of X_s given all the information up to time $t \leq s$.

Obviously, we also have that

$$P_t^M(\tau) = E_t(M_t^M(\tau)) = E_t\left(\frac{\Psi_{t+\tau}^M}{\Psi_t^M}\right) = E_t\left(\frac{\Psi_{t+\tau}^{NM}}{\Psi_t^{NM}} \frac{I_{t+\tau}^M}{I_t^M}\right) = E_t\left(M_t^{NM}(\tau) \frac{I_{t+\tau}^M}{I_t^M}\right)$$

and $P_t^N(\tau) = E_t(M_t^{NM}(\tau))$.

As will be clear in the sequence, the key insight of this paper is to notice that the present value at time t of \$1 receivable τ periods ahead is unique, such that

$$P_t^N(\tau) = E_t(M_t^{NA}(\tau)) = E_t(M_t^{NM}(\tau)). \quad (5)$$

As a side note, we observe that these no-arbitrage conditions correspond to the Euler condition in a representative agent macroeconomic model where the real stochastic discount factors are related to the marginal utility of the representative investor. For more details on this interpretation, see Joyce *et al.* [1].

Assumption A2. The random variables $M_t^A(\cdot)$, $M_t^M(\cdot)$, I_t^A and I_t^M , for $t \geq 0$, are conditionally (multivariate) lognormal.

Under assumptions **A1** and **A2**, Vicente and Graminho [2] [3] show that

$$\eta_t^A(\tau) = E_t(\pi_t^A(\tau)) - \frac{\tau}{2} \text{Var}_t(\pi_t^A(\tau)) + \text{Cov}_t(\pi_t^A(\tau), m_t^A(\tau))$$

and

$$\eta_t^M(\tau) = E_t(\pi_t^M(\tau)) - \frac{\tau}{2} \text{Var}_t(\pi_t^M(\tau)) + \text{Cov}_t(\pi_t^M(\tau), m_t^M(\tau)),$$

where $\text{Var}_t(X_s)$ denotes the variance of X_s and $\text{Cov}_t(X_s, Y_s)$ denotes the

covariance of X_s and Y_s , both given all the information up to time $t \leq s$, $m_t^A(\tau) = \ln M_t^A(\tau)$, and $m_t^M(\tau) = \ln M_t^M(\tau)$.

Joyce *et al.* [1] and Vicente and Graminho [2] [3] argue that these results are approximately correct without the lognormal assumption.

Thus, the break-even inflation is composed of three terms. The first one is the expected inflation. The Fisher equation [5] considers only this term, but Joyce *et al.* [1] and Vicente and Graminho [2] [3] give a mathematical justification showing that the break-even inflation is not made only by the expected inflation and other terms should be considered. The second term is known in finance as the convexity bias. The third term is the inflation risk premium (IRP), that represents the additional amount required by economic agents when investing in nominal assets instead of real ones.

These decompositions are complemented by $\ell_t^A(\tau), \ell_t^M(\tau) \geq 0$, the liquidity premium on each respective real asset, which are usually no more liquid than the nominal asset, such that finally we have

$$\eta_t^A(\tau) = E_t(\pi_t^A(\tau)) - \frac{\tau}{2} \text{Var}_t(\pi_t^A(\tau)) + \text{Cov}_t(\pi_t^A(\tau), m_t^A(\tau)) - \ell_t^A(\tau) \quad (6)$$

and

$$\eta_t^M(\tau) = E_t(\pi_t^M(\tau)) - \frac{\tau}{2} \text{Var}_t(\pi_t^M(\tau)) + \text{Cov}_t(\pi_t^M(\tau), m_t^M(\tau)) - \ell_t^M(\tau). \quad (7)$$

These decompositions will be used in the next section in order to search for an estimator for $y_t^M(\tau)$, that is theoretically motivated empirically relevant, when $\eta_t^M(\tau)$ is not observed.

3. A proposed Estimator and Its Respective Lower and Upper Bounds

Calculating $\eta_t^A(\tau) - \eta_t^M(\tau)$ from (1) and (2) and rearranging the terms, we arrive at

$$y_t^M(\tau) = y_t^A(\tau) - [\eta_t^M(\tau) - \eta_t^A(\tau)].$$

That is, one real yield to maturity is equal to the other minus the gap between the respective break-even inflation rates, for any fixed time t and maturity τ .

In the cases where $y_t^A(\tau)$ and $\eta_t^A(\tau)$ are observed but $\eta_t^M(\tau)$ is not, we propose to estimate $y_t^M(\tau)$ by $\hat{y}_t^M(\tau)$, such that

$$\hat{y}_t^M(\tau) = y_t^A(\tau) - [\hat{\eta}_t^M(\tau) - \eta_t^A(\tau)], \quad (8)$$

where the estimator $\hat{\eta}_t^M(\tau)$ of $\eta_t^M(\tau)$ is given by

$$\hat{\eta}_t^M(\tau) = \hat{E}_t(\pi_t^M(\tau)) - \frac{\tau}{2} \widehat{\text{Var}}_t(\pi_t^M(\tau)) + \widehat{\text{Cov}}_t(\pi_t^M(\tau), m_t^M(\tau)) - \hat{\ell}_t^M(\tau), \quad (9)$$

and the right-hand terms are the estimators of the expected inflation, the convexity, the IRP and the liquidity premium, respectively.

Vicente and Graminho [2] [3] estimate the two first terms directly inflation data and from survey data on inflation expectations, as usually made by professional forecasters.

Since we are estimating $y_t^M(\tau)$ in a situation when $\eta_t^M(\tau)$ is not observed, we are usually facing a situation where it is also difficult to find an estimator $\hat{\ell}_t^M(\tau)$. However, in some practical situations it may be sufficient to have upper or lower bounds $\hat{y}_{L,t}^M(\tau) \leq \hat{y}_t^M(\tau) \leq \hat{y}_{U,t}^M(\tau)$. In order to have an idea of these bounds, we propose to evaluate our estimator at two basic references: its minimum value, $\ell_t^M(\tau) = 0$, and the liquidity premium of the more liquid asset, $\ell_t^M(\tau) = \ell_t^A(\tau)$. Obviously, if a reliable estimator of $\ell_t^M(\tau)$ can be obtained, it must be used instead, without prejudice to using other references to improve the analysis.

It remains to estimate the IRP. For this, we begin by taking the natural logarithm of (5) and use the relations just above it to obtain

$$\ln E_t \left(M_t^A(\tau) \frac{I_t^A}{I_{t+\tau}^A} \right) = \ln E_t \left(M_t^M(\tau) \frac{I_t^M}{I_{t+\tau}^M} \right). \quad (10)$$

The terms enclosed by parentheses are lognormal variables because they are products and ratios of lognormal variables. Also, if X is lognormal, then $\ln E(X) = E(\ln X) + \text{Var}(\ln X)/2$ (see [6], for instance). Then we have that

$$\ln E_t \left(M_t^A(\tau) \frac{I_t^A}{I_{t+\tau}^A} \right) = E_t \left(m_t^A(\tau) - \tau \pi_t^A(\tau) \right) + \frac{1}{2} \text{Var}_t \left(m_t^A(\tau) - \tau \pi_t^A(\tau) \right)$$

and

$$\ln E_t \left(M_t^M(\tau) \frac{I_t^M}{I_{t+\tau}^M} \right) = E_t \left(m_t^M(\tau) - \tau \pi_t^M(\tau) \right) + \frac{1}{2} \text{Var}_t \left(m_t^M(\tau) - \tau \pi_t^M(\tau) \right).$$

Replacing the two previous equations in (10), expanding the variances, and solving for the desired IRP, we get

$$\begin{aligned} & \text{Cov}_t \left(\pi_t^M(\tau), m_t^M(\tau) \right) \\ &= \frac{2}{\tau} \left\{ \frac{\tau}{2} \text{Cov}_t \left(\pi_t^A(\tau), m_t^A(\tau) \right) + \left[E_t \left(m_t^M(\tau) \right) - E_t \left(m_t^A(\tau) \right) \right] \right. \\ & \quad + \frac{1}{2} \left[\text{Var}_t \left(m_t^M(\tau) \right) - \text{Var}_t \left(m_t^A(\tau) \right) \right] - \tau \left[E_t \left(\pi_t^M(\tau) \right) - E_t \left(\pi_t^A(\tau) \right) \right] \\ & \quad \left. + \frac{\tau^2}{2} \left[\text{Var}_t \left(\pi_t^M(\tau) \right) - \text{Var}_t \left(\pi_t^A(\tau) \right) \right] \right\}. \end{aligned} \quad (11)$$

Since

$$E_t \left(m_t^A(\tau) \right) + \text{Var}_t \left(m_t^A(\tau) \right) / 2 = E_t \left(M_t^A(\tau) \right) = \exp \left(-\tau y_t^A(\tau) \right),$$

and

$$\exp \left(-\tau y_t^M(\tau) \right) = \exp \left(\tau \eta_t^M(\tau) \right) \exp \left(-\tau \eta_t^A(\tau) - \tau y_t^A(\tau) \right),$$

we can rewrite (11) and estimate the desired IRP simply by

$$\begin{aligned}
& \widehat{\text{Cov}}_t(\pi_t^M(\tau), m_t^M(\tau)) \\
&= \frac{2}{\tau} \left\{ \frac{\tau}{2} \widehat{\text{Cov}}_t(\pi_t^A(\tau), m_t^A(\tau)) + \exp(\tau \hat{\eta}_t^M(\tau)) \exp(-\tau \eta_t^A(\tau) - \tau y_t^A(\tau)) \right. \\
&\quad \left. - \exp(-\tau y_t^A(\tau)) - \tau [\hat{E}_t(\pi_t^M(\tau)) - \hat{E}_t(\pi_t^A(\tau))] \right\} \\
&\quad + \frac{\tau^2}{2} [\widehat{\text{Var}}_t(\pi_t^M(\tau)) - \widehat{\text{Var}}_t(\pi_t^A(\tau))] \Bigg\}, \tag{12}
\end{aligned}$$

where $\widehat{\text{Cov}}_t(m_t^A(\tau), \pi_t^A(\tau))$ is the estimator of $\text{Cov}_t(m_t^A(\tau), \pi_t^A(\tau))$, which can be obtained as the residual of (6) when $\eta_t^A(\tau)$ is observed and the other terms can be estimated. Observe that one IRP is essentially equal to the other IRP adjusted by the price, expected inflation and convexity gaps.

Replacing (12) in (9), and collecting $\hat{\eta}_t^M(\tau)$ at the left hand side, we have

$$\begin{aligned}
& \hat{\eta}_t^M(\tau) - \frac{2}{\tau} \exp(\tau \hat{\eta}_t^M(\tau)) \exp(-\tau \eta_t^A(\tau) - \tau y_t^A(\tau)) \\
&= \hat{E}_t(\pi_t^M(\tau)) - \frac{\tau}{2} \widehat{\text{Var}}_t(\pi_t^M(\tau)) + \widehat{\text{Cov}}_t(\pi_t^A(\tau), m_t^A(\tau)) \\
&\quad - \frac{2}{\tau} \exp(-\tau y_t^A(\tau)) - 2 [\hat{E}_t(\pi_t^M(\tau)) - \hat{E}_t(\pi_t^A(\tau))] \\
&\quad + \tau [\widehat{\text{Var}}_t(\pi_t^M(\tau)) - \widehat{\text{Var}}_t(\pi_t^A(\tau))] - \hat{\ell}_t^M(\tau), \tag{13}
\end{aligned}$$

what can be solved analytically for $\hat{\eta}_t^M(\tau)$. To see this, denote the right hand side by c and let

$$a = \frac{2}{\tau} \exp(-\tau \eta_t^A(\tau) - \tau y_t^A(\tau)),$$

such that (13) can be written as

$$\hat{\eta}_t^M(\tau) - a \exp(\tau \hat{\eta}_t^M(\tau)) = c.$$

After some algebra, and multiplying both sides by $\exp(\tau c)$, this is equivalent to

$$(-\tau \hat{\eta}_t^M(\tau) + \tau c) \exp(-\tau \hat{\eta}_t^M(\tau) + \tau c) = -\tau a \exp(\tau c),$$

which has infinite solutions given by

$$W_k(-\tau a \exp(\tau c)) = -\tau \hat{\eta}_t^M(\tau) + \tau c, \quad k \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of integers and W_k are branches of the Lambert W function (see § 4.13 of [7], for instance).

Most of these branches are complex variables, *i.e.*, they have a nonzero imaginary part. However, since we need to estimate yields and premiums, we must consider only branches that are real variables, what leave us with only two real-valued solutions, W_0 and W_{-1} , respectively called the principal and the secondary branches of the Lambert W function.

Note that we do not have to choose between W_0 or W_{-1} , since both are feasible under the stated assumptions. In fact, we use these two branches to bound, from below and from above, the estimator of the break even inflation rate of the

less liquid asset. Consequently, we also bound the estimator of the yield of this asset.

Then, we have the following two estimators of $\eta_t^M(\tau)$:

$$\hat{\eta}_{k,t}^M(\tau) = c - \frac{W_k(-\tau a \exp(\tau c))}{\tau}, \quad k \in \{-1, 0\}.$$

Replacing this equation in (8), we arrive at two respective estimators of $\hat{y}_t^M(\tau)$, given by

$$\hat{y}_{k,t}^M(\tau) = y_t^A(\tau) - \left[c - \frac{W_k(-\tau a \exp(\tau c))}{\tau} - \eta_t^A(\tau) \right], \quad k \in \{-1, 0\}. \quad (14)$$

Observe that, since $\tau a \exp(\tau c) > 0$, then we have that

$$W_{-1}(-\tau a \exp(\tau c)) < W_0(-\tau a \exp(\tau c)) < 0 \quad \text{and, therefore,} \quad \hat{y}_{-1,t}^M(\tau) < \hat{y}_{0,t}^M(\tau).$$

Also, It is easy to see that $\hat{y}_{0,t}^M$ increases as $\hat{\ell}_t^M(\tau)$ increases, and that $\hat{y}_{-1,t}^M(\tau)$ decreases as $\hat{\ell}_t^M(\tau)$ increases. Then, $\hat{y}_t^M(\tau)$ cannot belong to the gap between $\hat{y}_{-1,t}^M(\tau)$ and $\hat{y}_{0,t}^M(\tau)$.

In other words, when $\hat{\ell}_t^M(\tau) = 0$, $\hat{y}_{0,t}^M(\tau)$ is a lower bound for one possible value of $\hat{y}_t^M(\tau)$ while $\hat{y}_{-1,t}^M(\tau)$ is an upper bound for another possible value of $\hat{y}_t^M(\tau)$.

Let $\underline{c}$ be the expression of c when $\hat{\ell}_t^M(\tau) = 0$. Then, replacing it in (14), we have that our proposed lower bound $\hat{y}_{L,t}^M(\tau)$ for $\hat{y}_t^M(\tau)$ becomes

$$\hat{y}_{L,t}^M(\tau) = \hat{y}_{0,t}^M(\tau) = y_t^A(\tau) - \left[\underline{c} - \frac{W_0(-\tau a \exp(\tau \underline{c}))}{\tau} - \eta_t^A(\tau) \right]. \quad (15)$$

Similarly, our proposed upper bound $\hat{y}_{U,t}^M(\tau)$ for $\hat{y}_t^M(\tau)$ becomes

$$\hat{y}_{U,t}^M(\tau) = \hat{y}_{-1,t}^M(\tau) = y_t^A(\tau) - \left[\underline{c} - \frac{W_{-1}(-\tau a \exp(\tau \underline{c}))}{\tau} - \eta_t^A(\tau) \right]. \quad (16)$$

If the information that $\hat{\ell}_t^A(\tau) \leq \hat{\ell}_t^M(\tau)$ is available, then we can Let $\underline{c}$ be the expression of c when $\hat{\ell}_t^M(\tau) = \hat{\ell}_t^A(\tau)$ and use the same Equations (15) and (16) to obtain possibly more informative bounds.

The fact that $\hat{y}_{L,t}^M(\tau) > \hat{y}_{U,t}^M(\tau)$, that the lower bound is bigger than the upper bound, can be confusing at first sight. However, we must remember that there is two possible values for $y_t^M(\tau)$ under the same set of information, and the decision on which one is the correct one cannot be decided based on just the theory used in the paper but, usually, based on some external information. As shown at Section 4, the lower bound can be very far apart from the upper bound. In general, the more reasonable bound seems to be the one closest to the yield of the liquid asset. On the other way round, it would be implicitly assumed that the expected inflation, convexity, IRP, liquidity premium, or a combination of these, is very different for the less liquid asset than for the more liquid one.

In practice, it can be easier to work with the following alternative writing:

$$\hat{y}_{k,t}^M(\tau) = y_t^A(\tau) - \left\{ \hat{E}_t(\pi_t^M(\tau)) + \left[-\frac{\tau}{2} \widehat{\text{Var}}_t(\pi_t^A(\tau)) + \widehat{\text{Cov}}_t(m_t^A(\tau), \pi_t^A(\tau)) \right] \hat{y}_{k,t}(\tau) - \eta_t^A(\tau) \right\}, \quad (17)$$

for $k \in \{-1, 0\}$ and where

$$\hat{y}_{k,t}(\tau) = \frac{\underline{c} - \tau^{-1} W_k(-\tau a \exp(\tau \underline{c})) - \hat{E}_t(\pi_t^M(\tau))}{-\tau \widehat{\text{Var}}_t(\pi_t^A(\tau)) / 2 + \widehat{\text{Cov}}_t(m_t^A(\tau), \pi_t^A(\tau))} \quad (18)$$

for $k \in \{-1, 0\}$ because, when $\ell_t^A(\tau) = 0$ and $\hat{y}_t(\tau) = \hat{y}$ constant for all t and τ , then we can write

$$\underline{y}_t^M(\tau) = y_t^A(\tau) - \left\{ \hat{E}_t(\pi_t^M(\tau)) + [\eta_t^A(\tau) - \pi_t^A(\tau)] \hat{y} - \eta_t^A(\tau) \right\}. \quad (19)$$

4. Application

In Brazil, the inflation rate is measured by two popular indices among others, namely the Extended National Consumer Price Index (IPCA) and the General Market Price Index (IGP-M).

The IPCA is released by the Brazilian Institute of Geography and Statistics (IBGE) and, in June 1999, as part of a major fiscal and monetary policy reform in Brazil, especially the end of the US Dollar-Brazilian Real parity, it became the official price index to Brazilian Central Bank's (BCB) Monetary Policy, used in its inflation-targeting system. The choice of IPCA is intrinsically linked to the end of the US Dollar-Brazilian Real parity, since the IPCA's basket of goods is an approximation of the average Brazilian family's consumption of selected retail products and services, surveyed in the main Brazilian metropolitan areas and cities.

On the other hand, the IGP-M is released by the Brazilian Institute of Economics of Fundação Getulio Vargas (FGV/IBRE), a nonprofit organization, and it is one of the oldest Brazilian price indices. It is the arithmetic weighted average of three other different price indices also compiled by the FGV/IBRE, being 60% due to the Producer Price Index (IPA). The IPA is an index created to monitor variations in the prices of agricultural and industrial products in inter-business transactions, that is, at the trade stages preceding final consumption, including raw materials used in many types of heavy industry. Also, a significant weight is given to mining, petroleum and agricultural commodity prices, usually quoted in US Dollars around the world. This is an important feature that makes the IGP-M highly responsive to international price changes and consequently more volatile than the IPCA.

Both price indices are considered in the formulas of the federal government inflation-linked coupon notes called National Treasury Notes-Series B (NTN-B) and Series C (NTN-C), respectively. On the pre-fixed rate side, the main domestic zero-coupon payment notes are the National Treasury Notes (LTN) and the semi-annual coupon payment note National Treasury Notes-Series F (NTN-F). For

details, see [8].

However, there are just a few traded prefixed federal government notes in the Brazilian market such that the risk-free term structures can be better represented by traded rates from future rate swaps contracts (DI—Interbank Daily rate $\times$ fixed rate) reported daily by the B3 Brazilian stock exchange. These data are used by the Brazilian Private Insurance Authority (Susep) to estimate a (nominal) pre-fixed yield curve [9].

The B3 stock exchange also report daily reference rates for swaps (DI $\times$ IGP-M rates) whose pricing methodology is not based in daily traded deals, but on reported from undisclosed qualified market agents. These rates are used by Susep to estimate a real IGP-M-linked yield curve. Finally, a real IPCA-linked yield curve is estimated daily by the the Brazilian Financial and Capital Markets Association (ANBIMA) with data coming from the daily traded yield to maturity NTN-B rates, complemented by data from qualified market agents [9].

These nominal and real yield curves, using data from the last trading day of each month, are published by Susep and are available for download from Susep's webpage.

As just exposed, the use of IPCA for BCB's inflation targeting monetary policy, reduced the IGP-M relevance in Brazilian fixed income securities market and, since 2007, the Brazilian government do not issue any NTN-C. Nowadays, only notes with a maturity date in 2031 are still traded, with a few deals per day. As a consequence, the reference rates based on this illiquid market are highly volatile as well as the estimated IGP-M real yield curve. Despite the reduced market relevance, pension funds still hold old plans that use the IGP-M, and its derived real yield curve, to update benefit payments and to price future cash flows of their liability technical provisions. For this and other reasons, Susep replaced its previous yield curve estimation method by the recent proposal of Signorelli *et al.* [10].

Another way to try to see the real picture hidden behind this illiquid market is by finding an estimate or, at least, a lower bound, based on some more liquid assets. In this way, we apply our method to obtain lower bounds for the Brazilian IGP-M real yield curve, based on the Brazilian IPCA real yield curve, and its break-even inflation rate, whose data come from a much more liquid and efficient market than the NTN-C one. During the application of our method, we ended-up replicating some results from Vicente and Graminho [2] [3] as follows.

In this work, the yield curves were obtained from the Svensson [11] model using the parameters published by Susep on its website at

<https://www.gov.br/susep/pt-br/assuntos/informacoes-ao-mercado/solvencia-superpervisao-prudencial/monitoramento-prudencial/capital-minimo-requerido-e-tap/modelo-de-interpolacao-e-extrapolacao-da-ettj> and links therein, which comprises monthly data from December 2009 to December 2025 onward. For comparison purposes, we retain data from 192 months, from December 2010 to December 2025, that intersect the period in Vicente and Graminho [2] [3] (December 2010 to September 2013) and include recent market conditions.

We use these yield curves and the Equations (1) and (2) to find the break-even inflation rates, respectively corresponding to IPCA and IGP-M, for 1, 2, 3, 4, 10 and 50 years, as shown at **Figure 1**. From December 2010 to September 2013, both the IPCA and IGP-M break-even inflation rates varied around 5.5% per year. Then, the rates increased to more than 10% (for 50 years) in January and February 2016, and began to decrease to 1% in April 2020 for 1 year IPCA, and to -0.3% in February 2020 for 1 year IGP-M. Since then, the rates fluctuate around 5%, more volatile than before September 2013.

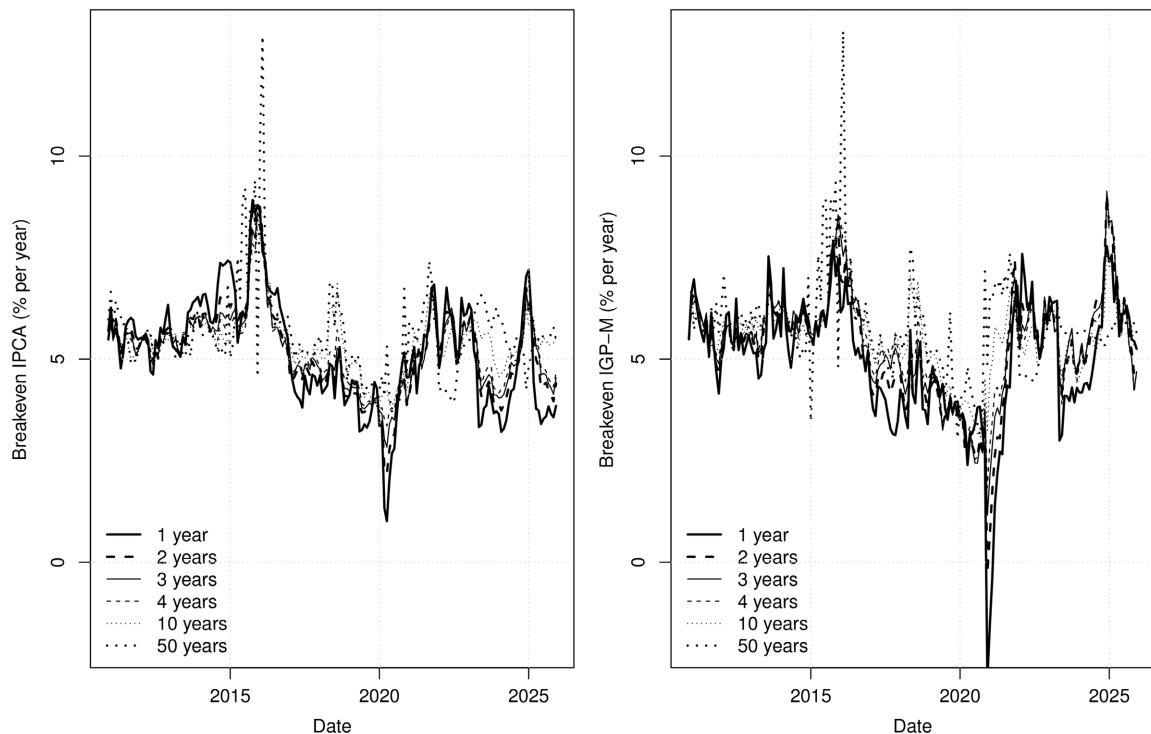


Figure 1. Brazilian break-even IPCA (left) and IGP-M (right) inflation rates for six different maturities. Monthly series from December 2010 to December 2025, in percent per year.

In order to apply our method we also need the expected IPCA and IGP-M inflation rates, as shown in Equations (6) and (7). The BCB Market Expectations System monitors market expectations about the main macroeconomic variables, providing important inputs for the monetary policy decision-making process.

Since May 1999, the BCB carries out the Focus Survey, a daily compilation of forecasts of about 160 banks, asset managers and other institutions in order to monitor the market expectations for several inflation indices, among other variables. The collection includes monthly, quarterly, and annual forecasts, depending on the variable and there is no mandatory provision of information. Summaries of these forecasts are released on its website for public download.

To foster the improvement of the predictive skills of the participants in the Focus Survey and to recognize their analytical effort, the BCB elaborates the Top 5 ranking. It ranks the institutions based on the accuracy of their short-, medium-

and long-term projections. The long-term Top 5 rankings for each variable are disclosed every January and consider the accuracy of the projections reported in each of the 12 months of the previous year.

In this work, following the choice in Vicente and Graminho [2] [3], the expected IPCA and IGP-M inflation rates were given by the median of the long-term Top 5 forecasters' IPCA and IGP-M data, published by BCB on its website at <https://www3.bcb.gov.br/expectativas2/#/consultaSeriesEstatisticas>, at the last trading day of the months from December 2010 to December 2025. These medians, for 1, 2, 3 and 4 years, are shown at Figure 2. At Figure 2 we also show expected inflation rates for horizons of 10 and 50 years. Since the Focus Survey provides reliable forecasts for just 4 years ahead, we had to decide on how to obtain estimates of the expected inflation rates for the remaining years. For the sake of simplicity, we decided to adopt the ubiquitous market practice of repeating the longer forecast for the remaining 6 and 46 years, respectively. The term structure for the expected IPCA inflation rates is decreasing most of the times, *i.e.*, the short-term expected inflation is bigger than the long-term one. The same happens for the expected IGP-M inflation rates. However, while the expected IPCA inflation rates increase until 2015, the expected IGP-M inflation rates decrease until the end of 2011, remain flat during 2012, increase during 2013 and decrease again until the end of 2014. Both series show, from 2015 onward, a roughly similar pattern, but more volatile for IGP-M.

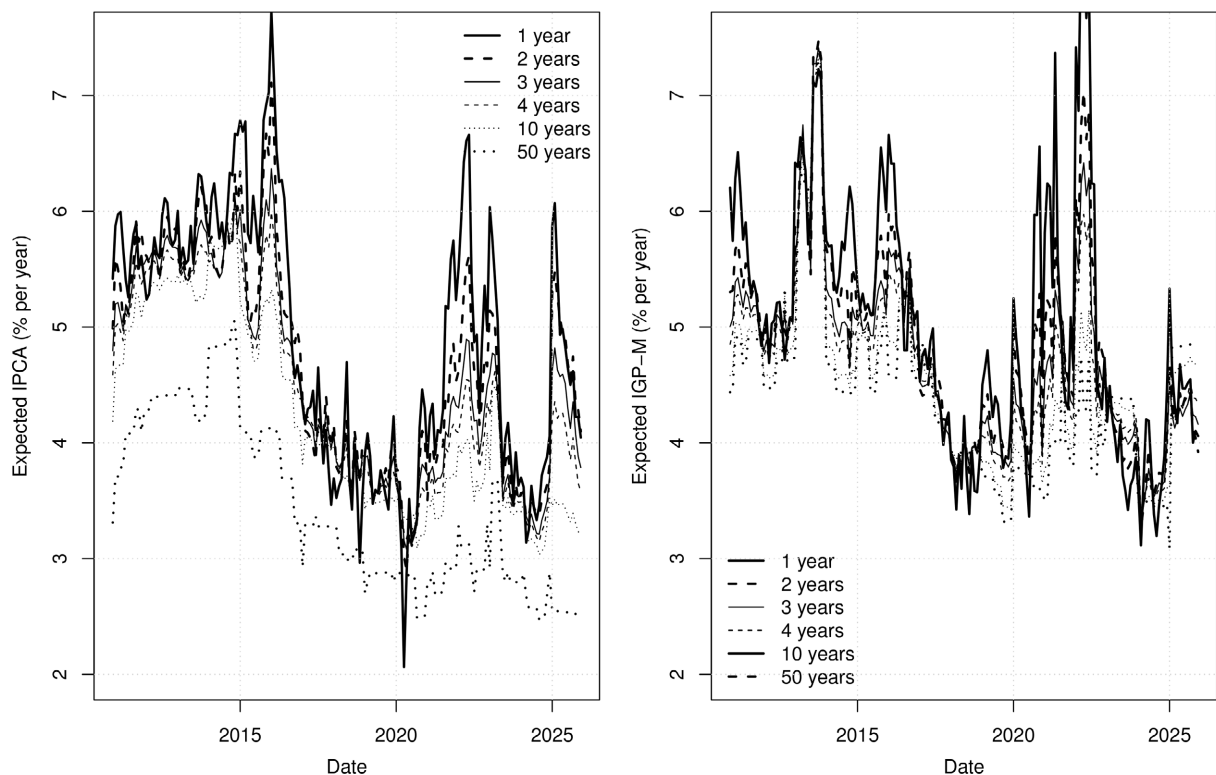


Figure 2. Median market expectation for IPCA (left) and IGP-M (right) inflation rates for six different maturities. Monthly series from December 2010 to December 2025, in percent per year.

On the other hand, the monthly observed IPCA and IGP-M indices were downloaded from Ipeadata, a website provided and managed by the Institute of Applied Economic Research (Ipea) of the Brazilian government. The continuously compounded monthly inflation rates were obtained by using the respective price indices in Equations (3) and (4), with $\tau = 1$. Since our data intersect those from Vicente and Graminho [2] [3], we followed them, did our own analysis, and modeled the IPCA inflation rate as an autoregressive process of order 1, an AR(1) process, given by

$$\pi_{t+1}^A(1) = \theta_0^A + \theta_1^A \pi_t^A(1) + a_{t+1}^A, \quad (20)$$

where a_{t+1}^A is a zero mean random variable with $\text{Var}_t(a_{t+1}^A) = \sigma_A^2$. Similarly, we have that the IGP-M inflation also follows an AR(1) process, as given by

$$\pi_{t+1}^M(1) = \theta_0^M + \theta_1^M \pi_t^M(1) + a_{t+1}^M, \quad (21)$$

where a_{t+1}^M is a zero mean random variable with $\text{Var}_t(a_{t+1}^M) = \sigma_M^2$.

Using the time series from December 2010 to December 2025, the estimates of θ_1^A and σ_A are equal to 0.54 and 0.0030, somewhat close to those estimated by Vicente and Graminho [2] [3]. The estimates of θ_1^M and σ_M , are equal to 0.66 and 0.0068, the latter being more than two times greater than its IPCA counterpart. Given these two standard deviation estimates, we used the formula (8) in Vicente and Graminho [2] [3] to obtain the convexity terms for maturities of 1, 2, 3, 4, 10 and 50 years. For IPCA, they are respectively 2.17, 2.37, 2.44, 2.47, 2.54 and 2.57 basis points while, for IGP-M, they are equal to 18.11, 21.24, 22.31, 22.83, 23.80 and 24.31 basis points.

For IPCA, Vicente and Graminho [2] [3] show that the liquidity premium is negligible and can be considered as zero, that is, $\hat{\ell}_t^A(\tau) = 0$. Thus, given the series of break-even and expected inflation rates and convexity from IPCA, we can obtain the inflation risk premium (IRP) by residual in Equation (6). This would give us the real IRP but, for comparison purposes, we evaluate the nominal IRP by adding the convexity in Equation (6) instead of subtracting it (for details, see [2] [3]). The time series of the nominal IRP for IPCA from December 2010 to December 2025, for maturities of 1, 2, 3, 4, 10 and 50 years, are shown at **Figure 3**. From December 2010 to September 2013, these time series are roughly the same as those from Vicente and Graminho [2] [3], except for the approximate period from July 2011 to January 2012, when they differ substantially on trend and magnitude. The mean of IRP for 1, 2, 3, 4, 10 and 50 years are respectively equal to 0.26%, 0.47%, 0.66%, 0.81%, 1.19% and 2.17%. During this period, negative IRP were more frequent for shorter horizons, since the respective medians are equal to 0.26%, 0.45%, 0.65%, 0.80%, 1.22% and 2.14%. This means that investors tolerate less returns to invest in nominal assets for short horizons, but do require more additional remuneration as longer the horizon. Notice that the IRP for IGP-M is not shown, since we consider the corresponding real yields to maturity highly volatile. In what follows, we apply our methods to obtain lower and upper bounds for these yields.

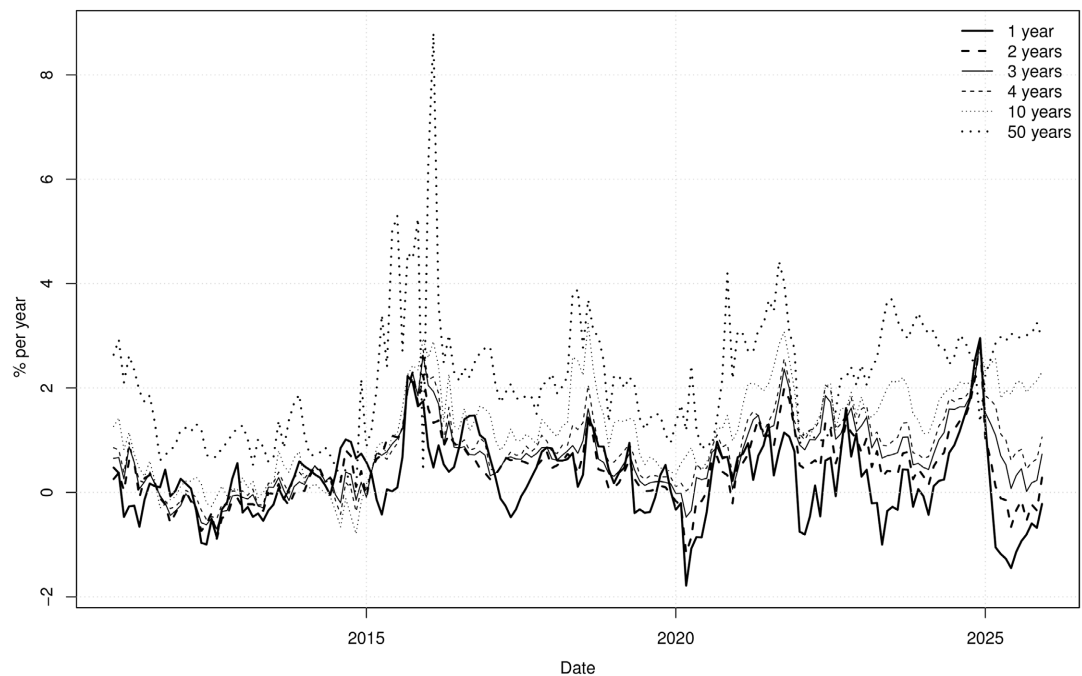


Figure 3. Nominal inflation risk premium (IRP) for IPCA inflation rates for six different maturities. Monthly series from December 2010 to December 2025, in percent per year.

We would like to stress that a negligible liquidity premium is desirable but neither assumed nor necessary to apply our method, as long as one can estimate the IRP of the more liquid asset which, by its turn, may require an estimate of its liquidity premium, maybe by using the technique presented at the Section 7 of [2] [3].

In order to estimate lower and upper bounds to $y_t^M(\tau)$, the continuously compounded zero-coupon IGP-M real yield to maturity, we first convert the previous nominal IRP to a real one by adding twice the respective convexity.

Now, we use Equations (15) and (16) to obtain $\hat{y}_{L,t}^M(\tau)$ and $\hat{y}_{U,t}^M(\tau)$, respectively for $\tau = 1, 2, 3, 4, 10, 50$ years, as shown at Figure 4. In this particular case, we just have $\hat{\ell}_t^M(\tau) = \hat{\ell}_t^A(\tau) = 0$. Lower bounds for 1, 2, 3 and 4 years are very big and are not shown. Bounds for 10 years, at some dates, specially in the first half of the series, could not be obtained because the argument of the Lambert W function was outside its domain.

During the period analyzed, our proposed upper bounds for 1, 2, 3 and 4 years, are greater or equal to the IPCA yields most of the time, the main exception being the period from 2020 to 2023, at all the presented maturities, when the break-even inflation rates were substantially smaller than the market expected values. This seems to happen for 10 years bounds, not considering that they could not be obtained for all the period. The exception occurs for the 50 years, where the IPCA yields remained the entire period above the upper bounds, and just below the lower bounds, probably as a result of the increasing IRP for longer horizons. Since IPCA is much less volatile than IGP-M, its associated IRP is expected to be smaller

than those associated to IGP-M and, thus it is expected IGP-M yields lower than IPCA ones, at least in the long run.

However, the IGP-M yields from Susep are sometimes between our lower and upper bounds, mainly from the middle of 2020 to the middle of 2022, and for almost the entire period for 10 and 50 year yields. Yields at these regions cannot be explained by our results. We did not investigate the reasons for this phenomenon, but suspect that it can be an undesirable side effect of the illiquid market affecting the estimation made by Susep, the result of the nonobservance of some of our assumptions, or caused by very poor forecasts data.

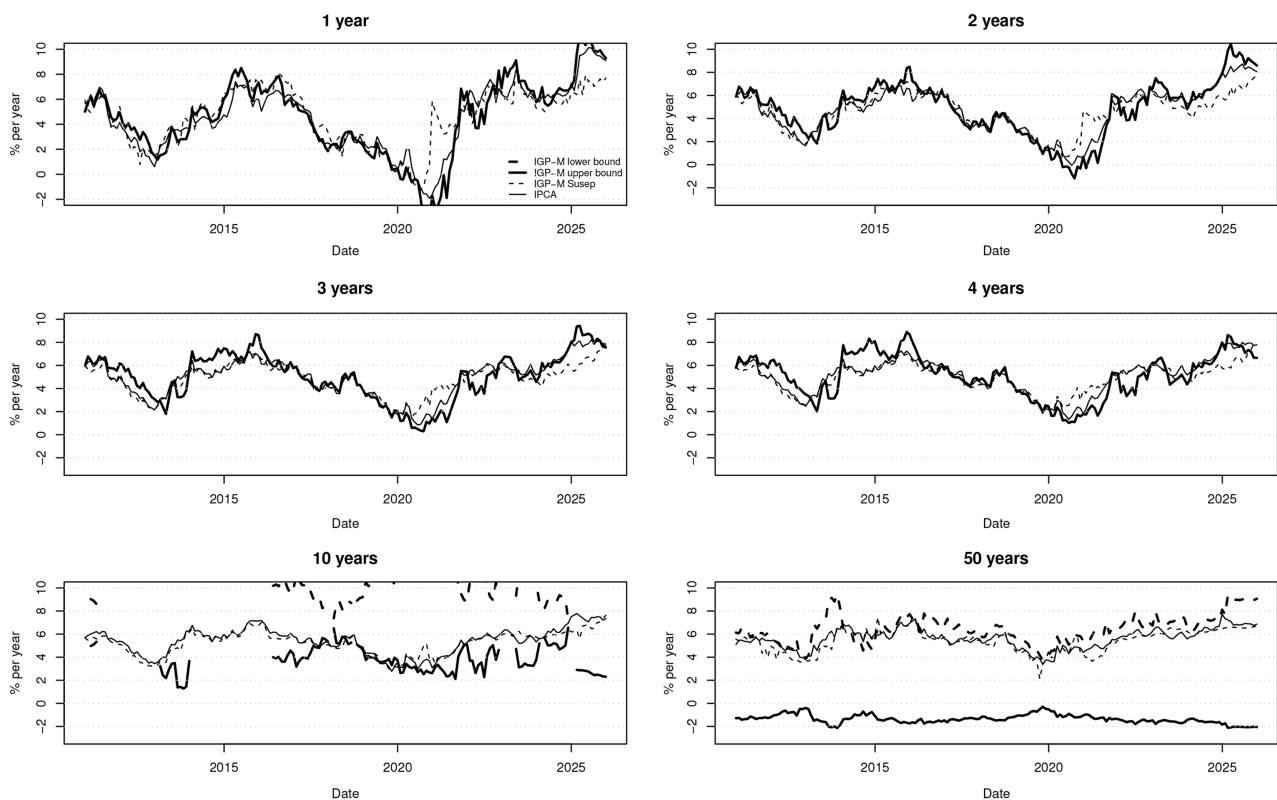


Figure 4. Continuously compounded zero-coupon IPCA and IGP-M real yields to maturity, for six different maturities. The IGP-M yields are provided by Susep and the respective lower and upper bounds are obtained from Equations (15) and (16), and using $\hat{\ell}_t^M(\tau) = \hat{\ell}_t^A(\tau) = 0$. Monthly series from December 2010 to December 2025, in percent per year.

5. Conclusions

In this article, we proposed a new method to obtain lower and upper bounds of a real yield curve. This formula is based on another real yield curve and a simple economic model of its break-even inflation rate that splits it in inflation expectation, convexity term, and liquidity and inflation risk premiums. Specifically, one real yield to maturity is equal to the other minus the gap between the respective break-even inflation rates.

Since the proposed lower bound is bigger than the upper bound, the result of our method is equivalent to an estimated impossibility space for a real yield curve.

Our method is specially useful when one of these curves is difficult to obtain but a respective inflation expectation series is available. Such difficulty may arise from reference rates based on an illiquid market, resulting in highly volatile or even biased yield curve. Our method is based on a few assumptions, namely a no arbitrage and a lognormal distribution. Our method may be useful for governments as well, since they can anticipate bounds for real yield curves based on planned future bonds.

We illustrate our method with yield curves from Brazil, since it issues sovereign bonds linked to IPCA and to IGP-M inflation indices. One nominal and two real yield curves, using data from the last trading day of each month, and published by the Brazilian Private Insurance Authority (Susep), were used to estimate lower bounds for the Brazilian IGP-M real yield curve, based on the Brazilian IPCA real yield curve, and its break-even inflation rate, whose data come from a much more liquid and efficient market than the source for Susep's own estimate.

During the period analyzed, our proposed lower bounds were usually lower than the IPCA yields, but the IGP-M yields from Susep were even lower. However, our method has characteristics that may make it useful for insurance, reinsurance, and pension companies that operate in illiquid markets, since it provides alternative and theory-based lower bounds on estimates of real yield maturity rates.

Finally, our method can be readily extended to more than one liquid real yield curve, since this implies more than one pair of bounds that can be combined or analyzed separately.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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