

Interaction between Portfolios under G2++ Model

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How to cite this paper: Uwaydah, L. and Jaffal, H. (2026) Interaction between Portfolios under G2++ Model. *Journal of Mathematical Finance*, 16, 143-164.
<https://doi.org/10.4236/jmf.2026.163008>

Received: January 12, 2026

Accepted: July 7, 2026

Published: July 10, 2026

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Abstract

The following paper will show an investigation regarding the hedging portfolios for the G2++ interest rate model which will be analyzed, in which a focus on the decomposition of its variations while using derived sensitivities from the instantaneous short-rate dynamics will also occur. Thus, this paper will provide the following information: a precise control of the associated approximation error, computed high-order sensitivities, an accurate polynomial approximation of the portfolio value variations with coefficients giving by the sensitivities. The central aim of this paper is to provide a comprehensive, portfolio oriented presentation of the hedging methodology. Its structure will begin with defining the composition of the portfolios, bonds and swaps, to derive their decomposition in terms of sensitivities. Later, a clear presentation of the optimized problem that underlies the hedging strategy will be made, to end the work with a range of numerical experiments to demonstrate the efficiency of the proposed approach. Plenty of hedging methods exist in literature, however, our approach is more optimal in the sense, due to the way the hedger is determined based on an upper bound on the potential loss which is induced by the hedging strategy itself.

Keywords

G2++ Model, CBB Portfolio, IRS Portfolio, Sensitivities, Hedging

1. Introduction

For the present work nowadays is solely devoted to enhancing and maintaining the problem of hedging a position that is sensitive to interest rate by means of a portfolio of bonds and swaps. Our methodology is based upon the solution approach rather than contributing to the exploration of the subject itself, which

would be already considered by many.

To begin with, it is widely understood that there is no consensual theoretical approach for the hedging of a position that is sensitive to interest rate when compared with what is happening in the equality area. Due to the interest rate being derived which thus compels at least two difficulties: Firstly, which focuses on the need to simultaneously consider various types of interest rates; Secondly, which focuses on a given interest rate itself that does not appear to be present aged as a tradable asset. Furthermore, when it comes to financial literature, it often revolves on hedging a single position by one or few instruments, regardless the fact the most practitioners tend to be faced with covering a portfolio position by another one. Hence, a lack of clear and common theoretical reference on this aspect is seen, and due to the constant focus on of the development on studies cases, then consequently lack of evidence based on particular situations is seen. Thus, in depth analysis of the general mechanism of portfolio oriented hedging is often resulted as missing. So, we contribute in this work to fill the upcoming lack and gap.

Although, in literature, hedging in the inter temporal framework is well considered. However, continuously making infinite number of trades of the underlying asset used in the hedging is actually untenable in practice as required in the theory, which thusly grants practitioners an interest and confidence on rolling one periods of hedging position. Based on this observation, this paper will focus on studying one period hedging.

The background of hedging is just to compensate for the loss affecting the position to hedge, by the gain collected from the hedging instrument. And this can be achieved by the cash flows matching approach (e.g.: [1] [2]), which some financial institutions used such means to manage the risk linked to yield curve risks affecting liabilities and assets. However, this approach is not always applicable and easily to put in place as the positions tend to be linked to very different characteristics. So, in order for the standard to be well established, the technique relies on matching the sensitivity of the position to hedge with those of the hedging instruments, rather than matching the cash flows. Such acquirement is possible once the two positions react on the same common risk factors.

Additionally, relying on the considered position and underlying risk factors, multiple kinds of sensitivities might be defined which consequently will lead to different hedging results. For instance, the Coupon Bearing Bond (CBB), is the well established one and second order sensitivities known under the names of MacAulay or Fisher-Weil, with duration and convexity [3] [4].

Although it is highly popular among academics and practitioners, the bond duration and convexity, it is important to shed a light on the use of these sensitivities which underlines the strong assumption that at the hedging time horizon, the interest rate curve has made a parallel shift movement. To further clarify, in such context, explicit formulas for the sensitivity are made available as well as the hedging operation, and the performance of the matching sensitivities for the positions to hedge and the hedging instrument are easily done. So, the enhancement of this classical duration convexity approach and the corresponding hedging has been

performed recently in [5]-[7].

Moreover, in general the parallel shift assumption appears to be unrealistic, since empirical observations show that at a given time horizon, the interest rate curve may have a very different shape. To elaborate further, it means that in order to perform a proper hedging strategy, a dynamic model for the interest rate must be introduced. This requirement has led to the introduction of stochastic duration [8] [9].

The final notion is theoretically considered superior to the Macaulay's or Fisher-Weil duration, which both require a yield curve to have made a parallel shift. Nevertheless, most empirical tests on bond immunization performance have not demonstrated any actual superiority of the stochastic duration when being compared to the simple classical duration [10].

Additionally, it is said to be unclear how the general portfolio and the extension of stochastic duration, as well as the convexity to interest rate which is based on instruments other than the CBBs, tend to be.

With respect to the shocks linked to the unobservable two uncertainty factors underlying the G2++ model, an analytical expression for the sensitivities for the coupon bearing bonds and interest rate swaps contracts are to be derived. As such, the hedging of a position sensitive to the interest rate, either by means of portfolio by bond or by swaps and in accordance with the market participants practice, it becomes easily transparent resulting from the equilibrium between the various involved sensitivities as seen in [11] [12]. Therefore, this paper's main objective is to fully document this hedging approach portfolio oriented mentioned, but not carried in [13].

To conclude, this paper will be assembled as follows. Starting with our main results that are solely started in Section 2, a recalling of the primer notions on the interest rate model G2++ will be provided. Later, a presentation of the formulation of the short rate as stochastic formula is given. To continue with, the results of our article in Section 3 providing the basic identity portfolio change and its decomposition in Subsection 3.1 which is the main key for the hedging approach in Subsection 3.2 where a presentation of the overall expression (hedging portfolio and hedging instruments) changes is performed. To last but not least, state in Subsection 3.3, the optimization problem, the last nonlinear and integer optimization problem, which is solved using metaheuristics methods (such as genetic algorithms or using the commercial solver CPLEX). This last is used in our numerical illustrations that are displayed in Section 4. In that section, a given examples of hedging a CBB position by a portfolio made by CBBs and IRSs. To finally conclude the paper in Section 5.

2. The G2++ Model

The two-additive-factor Gaussian model G2++ ([14]) describes the short rate r_t movement by the equation

$$r_t = \varphi(t) + x_{t,1} + x_{t,2} \quad (1)$$

where $t \mapsto \varphi(t)$ is a (deterministic) function which allows the model to fit the current observed interest rates. In (1), $x_{t;1}$ and $x_{t;2}$ may be viewed as state variables that the dynamics are assumed to be given by

$$dx_{t;1}(\cdot) = -\kappa_1 x_{t;1} dt + \sigma_1 dW_{t;1}(\cdot) \quad (2)$$

and

$$dx_{t;2}(\cdot) = -\kappa_2 x_{t;2} dt + \sigma_2 dW_{t;2}(\cdot). \quad (3)$$

All of these dynamics are given under the risk-adjusted risk-neutral measure. Here $W_{t;1}(\cdot)$ and $W_{t;2}(\cdot)$ are two correlated standard Brownian motions with a (constant) correlation $\rho \equiv \rho_{x_1, x_2}$, with $-1 < \rho < 1$. In (2) and (3), κ_1 , κ_2 , σ_1 and σ_2 are nonnegative real numbers which represent the model parameters.

3. Results

3.1. Portfolio Change

3.1.1. Portfolio Change of Coupon Bearing Bonds (CBB)

Let us denote by B_t the present time- t value of a portfolio made by bonds in long and/or short positions. So we assume that there are I^{**} types of bonds $B_{:,i^{**}}^{**}$ in long positions and I^* types of bonds $B_{:,i^*}^*$ in short positions inside the considered portfolio. Of course I^{**} and I^* stand for positive integer numbers. The time- t value of such a portfolio may be written as

$$B_t = \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} B_{t,i^{**}}^{**} - \sum_{i^*=1}^{I^*} n_{i^*}^* B_{t,i^*}^*. \quad (4)$$

Therefore there are $n_{i^{**}}^{**}$ bonds of type i^{**} each worth $B_{t,i^{**}}^{**}$, and $n_{i^*}^*$ bonds of type i^* each worth B_{t,i^*}^* .

The bond $B_{:,i^{**}}^{**}$, with a maturity

$$t_{M^{**}(i^{**})}^{**}(i^{**}),$$

is assumed to have coupons with the coupon rate

$$\text{rate_cpn}(i^{**})$$

paid at times

$$\mathcal{T}_{:,i^{**}}^{**} = \left(t_1^{**}(i^{**}), \dots, t_{j^{**}}^{**}(i^{**}), \dots, t_{M^{**}(i^{**})}^{**}(i^{**}) \right)$$

where $\mathcal{T}^{**} = \left(\mathcal{T}_{:,i^{**}}^{**} \right)_{i^{**} \in \{1, \dots, I^{**}\}}$ and has the face value $\text{face_value}(i^{**})$. Similarly

for the bond $B_{:,i^*}^*$, we have same notations but instead of the double star ** , we have only one star * .¹

¹The related notations for $B_{:,i^*}^*$ are $t_{M^*(i^*)}^*(i^*)$, $\text{rate_cpn}(i^*)$, $\text{face_value}(i^*)$ and

$$\mathcal{T}^* = \left(\mathcal{T}_{:,i^*}^* \right)_{i^* \in \{1, \dots, I^*\}}.$$

The future value $\mathcal{B}_t(\cdot)$ of the portfolio of CBB is not known at time 0, and should depend on the structure taken by the interest rate curve at the horizon time t . The idea we have explored either for the CBB portfolio in order to grasp the future value is to make use of the various sensitivities for the considered time horizon but determined at the present time 0. This have been led us to introduce a decomposition of the position change value. As detailed in [12], the change value portfolio bond during the time-period $(0, t)$ denoted by

change_value_port_Bond $_{0,t}(\cdot)$, under the G2++ model of the interest rate curve is approximated using the sentivities at k -th order. It appears here that the CBB portfolio is a linear combination of different CBB types. So the change value of the portfolio between the period $(0, t)$ depends on the variation of every single CBB.

$$\begin{aligned} & \text{change_value_port_Bond}_{0,t}(\cdot) \\ &= \mathcal{B}_t(\cdot) - \mathcal{B}_0(\cdot) \\ &= \sum_{i^{**}=1}^{I^{**}} n_{i^{**}} \left\{ \mathcal{B}_{t;i^{**}}^{**}(\cdot) - \mathcal{B}_{0;i^{**}}^{**} \right\} - \sum_{i^*=1}^{I^*} n_{i^*} \left\{ \mathcal{B}_{t;i^*}^*(\cdot) - \mathcal{B}_{0;i^*}^* \right\} \\ &\approx \text{Res}(t, \mathcal{B}; \tilde{Y}) + \sum_{k=1}^p \frac{(-1)^k}{k!} \text{Sens}'(k; t, \mathcal{B}; \tilde{Y}) \bullet \varepsilon^{[k]}(\cdot) \end{aligned} \quad (5)$$

where p in a non negative integer representing the big order of the sensitivities,

$\varepsilon = (\varepsilon_1, \varepsilon_2)$ the shock related to the movement of the yield curve under G2++, it belongs to special domain as detailed [11], $\tilde{Y}$ represent the G2++ parameters.²

The zero order sensitivities of the portfolio of CBB, also referred as a residual term, is defined by

$$\text{Res}(t, \mathcal{B}; \tilde{Y}) \equiv \sum_{i^{**}=1}^{I^{**}} n_{i^{**}} \text{Res_Bond}(t, \mathcal{T}_{:,i^{**}}^{**}; \tilde{Y}) - \sum_{i^*=1}^{I^*} n_{i^*} \text{Res_Bond}(t, \mathcal{T}_{:,i^*}^*; \tilde{Y}) \quad (6)$$

$$\begin{aligned} \text{Sens}(k; t, \mathcal{B}; \tilde{Y}) &\equiv \sum_{i^{**}=1}^{I^{**}} n_{i^{**}} \text{Sens_Bond}(k; t, \mathcal{T}_{:,i^{**}}^{**}; \tilde{Y}) \\ &\quad - \sum_{i^*=1}^{I^*} n_{i^*} \text{Sens_Bond}(k; t, \mathcal{T}_{:,i^*}^*; \tilde{Y}). \end{aligned} \quad (7)$$

The portfolio of CBB sensitivity denoted by $\text{Sens}(k; t, \mathcal{B}; \tilde{Y})$ is a $k+1$ -dimensional vector of real numbers. It may be useful to remind that

$$\begin{aligned} & \text{Sens_Bond}(k; t, \mathcal{T}_{:,i^{**}}^{**}; \tilde{Y}) \\ &\equiv f_{i^{**}}^{**} c_{i^{**}}^{**} \times \text{Sens_ZC}(k; t, t_1^{**}(i^{**}); \tilde{Y}) \times \tau_1^{**}(i^{**}) \\ &\quad + f_{i^{**}}^{**} c_{i^{**}}^{**} \times \sum_{j^{**}=2}^{M^{**}(i^{**})-1} \text{Sens_ZC}(k; t, t_{j^{**}}^{**}(i^{**}); \tilde{Y}) \times \tau_{j^{**}}^{**}(i^{**}) \\ &\quad + f_{i^{**}}^{**} \left(1 + c_{i^{**}}^{**} \tau_{M^{**}(i^{**})}^{**}(i^{**}) \right) \times \text{Sens_ZC}(k; t, t_{M^{**}(i^{**})}^{**}(i^{**}); \tilde{Y}). \end{aligned}$$

² $\tilde{Y} = (P^{mrkt}(0, T), P^{mrkt}(0, t), x_{0,1}, x_{0,2}, k_1, K_2, \sigma_1, \sigma_2, \rho)$ where T maturity of a zero coupon bond.

$$\begin{aligned} & \mathbf{Sens_ZC}(k; t, t_{j^{**}}^{**}(i^{**}); \tilde{Y}) \\ & \equiv \left(\Theta(t, t_{j^{**}}^{**}(i^{**})) \mathbf{c}(l-1, k) \left[\lambda_1(t, t_{j^{**}}^{**}(i^{**})) \right]^{k+1-l} \left[\lambda_2(t, t_{j^{**}}^{**}(i^{**})) \right]^{l-1} \right)_{l \in \{1, \dots, k+1\}}. \end{aligned}$$

The terms Θ , λ_1 and λ_2 are defined as follows

$$\begin{aligned} \Theta & \equiv \Theta(t, T; P^{mkt}(0, t), P^{mkt}(0, T); x_{0,1}, x_{0,2}; \tilde{Y}) \\ & = \exp \left[- \left\{ b(\kappa_1(T-t)) \mathcal{E}(t; \kappa_1) x_{0,1} + b(\kappa_2(T-t)) \mathcal{E}(t; \kappa_2) x_{0,2} - \frac{1}{T-t} c(t, T; \tilde{Y}) \right\} (T-t) \right] \quad (8) \end{aligned}$$

$$\begin{aligned} \lambda_1 & \equiv \lambda_1(t, T; \tilde{Y}) \\ & = \left\{ \sigma_1 b(\kappa_1(T-t)) \mathcal{F}^{\frac{1}{2}}(t; \kappa_1, \kappa_1) + \omega \sigma_2 b(\kappa_2(T-t)) \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2) \right\} (T-t) \quad (9) \end{aligned}$$

$$\lambda_2 \equiv \lambda_2(t, T; \kappa_2, \sigma_2, \rho) = \left\{ \sqrt{1 - \omega^2} \sigma_2 b(\kappa_2(T-t)) \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2) \right\} (T-t) \quad (10)$$

The sensitivities $\mathbf{Sens_Bond}(k; t, T_{j^*}^*; \tilde{Y})$ can be similarly defined by introducing

$$\mathbf{Sens_ZC}(k; t, t_{j^*}^*(i^*); \tilde{Y}) \text{ for all } j^*(i^*) \in \{1, 2, \dots, M^*(i^*)\}.$$

Therefore, the residual term $\mathbf{Res}(t, \mathcal{B}; \tilde{Y})$ and the k -th order sensitivity $\mathbf{Sens}(k, t, \mathcal{B}; \tilde{Y})$ for the portfolio of CBB appear to be actually as aggregations of various zero-coupons sensitivities

$$\mathbf{Res_ZC}(t, t_{j^{**}}^{**}(i^{**}); \tilde{Y}), \mathbf{Res_ZC}(t, t_{j^*}^*(i^*); \tilde{Y})$$

$$\mathbf{Sens_ZC}(k; t, t_{j^{**}}^{**}(i^{**}); \tilde{Y}), \mathbf{Sens_ZC}(k; t, t_{j^*}^*(i^*); \tilde{Y})$$

for all $j^{**} \in \{1, \dots, M^{**}(i^{**})\}$, $j^* \in \{1, \dots, M^*(i^*)\}$, $i^{**} \in \{1, \dots, I^{**}\}$ and $i^* \in \{1, \dots, I^*\}$.

It means that in the practical implementation, making use directly of the formula (8) and calling functions computing CBB sensitivities internally should not be the profitable way to compute the portfolio of CBB sensitivities. It may be convenient to pre-compute in advance the matrix of zero-coupon sensitivities and then to make appeal to the computed entries when they are really needed.

It should be emphasized that replacing the future change of the portfolio of CBB by a polynomial function (whose the coefficient is the initial portfolio sensitivities) is also interesting in the perspective of risk measurement and management. Indeed instead of re-evaluating the portfolio for each scenario of shocks considered (which is time and memory consuming) one has just to calculate the value of the polynomial function. Of course, this is meaningful whenever the remainder term is negligible. All the details for this decomposition and the remainder term are explained in details in [11].

3.1.2. Portfolio Change of Interest Rate Swaps (IRS)

Let us denote by S_t the time t value of a portfolio made by J^{**} types of payer swaps $S_{:,j^{**}}^{**}$ and J^* types of receiver swaps $S_{:,j^*}^*$. Of course J^{**} and J^* are positive integer numbers where $j^{**} \in \{1, \dots, J^{**}\}$ and $j^* \in \{1, \dots, J^*\}$.

The swap $S_{:,j^{**}}^{**}$ is assumed respectively to have the notional **notional** (j^{**}) ,

fair-rate prices **rate_Swap** (j^{**})

maturities $t_{M^{**}(j^{**})}^{**}(j^{**})$

and have the ordered payment times

$$\mathcal{T}_{:,j^{**}}^{**} = \left(t_1^{**}(j^{**}), \dots, t_{i^{**}}^{**}(j^{**}), \dots, t_{M^{**}(j^{**})}^{**}(j^{**}) \right)$$

where

$$\mathcal{T}^{**} = \left(\mathcal{T}_{:,j^{**}}^{**} \right)_{j^{**} \in \{1, \dots, J^{**}\}}^3$$

The future value $S_t(\cdot)$ of the portfolio of swaps is not known at time 0, and should depend on the structure taken by the interest rate curve at the horizon time t . The change value portfolio during the time-period $(0, t)$

change_value_port_swap $_{0,t}(\cdot)$, under the G2++ model of the interest rate curve is approximated using the sensitivities at k -th order (as written in [11]). Same as CBB portfolio, the swap portfolio is a linear combination of different swap types. So the change value of the portfolio between the period $(0, t)$ depends on the variation of every single swap. This leads us to write that the change value of the considered portfolio for the period $(0, t)$ is given by

$$\begin{aligned} & \text{change_value_port_Swap}_{0,t}(\cdot) \\ & \equiv S_t(\cdot) - S_0 \\ & = \sum_{j^{**}=1}^{J^{**}} n_{i^{**}}^{**} \left\{ S_{:,j^{**}}^{**}(\cdot) - S_{0,j^{**}}^{**} \right\} - \sum_{j^*=1}^{J^*} n_{i^*}^* \left\{ S_{:,j^*}^*(\cdot) - S_{0,j^*}^* \right\} \\ & \approx \text{Res}(t; \mathcal{S}; \tilde{Y}) + \sum_{k=1}^p \frac{(-1)^k}{k!} \text{Sens}'(k; t; \mathcal{S}; \tilde{Y}) \bullet \varepsilon^{[k]}(\cdot) \end{aligned} \quad (11)$$

Therefore, the portfolio sensitivity of order zero or residual term, measuring the passage of time when the yield curve remains unchanged, is given by

$$\begin{aligned} \text{Res}(t, \mathcal{S}; \tilde{Y}) & \equiv \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} \text{Res_Swap}(t, \mathcal{T}_{:,i^{**}}^{**}; \tilde{Y}) \\ & - \sum_{i^*=1}^{I^*} n_{i^*}^* \text{Res_Swap}(t, \mathcal{T}_{:,i^*}^*; \tilde{Y}). \end{aligned} \quad (12)$$

³resp. for the swap $S_{:,j^*}^*$ with a notional **notional** (j^*) , fair-rate prices **rate_Swap** (j^*) , maturities $t_{M^*(j^*)}^*(j^*)$ and the notation $\mathcal{T}_{:,j^*}^* = \left(t_1^*(j^*), \dots, t_{i^*}^*(j^*), \dots, t_{M^*(j^*)}^*(j^*) \right)$, $j^* \in \{1, 2, \dots, J^*\}$.

The zero order sensitivities for the i^{**} -th single swap position is

$$\begin{aligned} & \mathbf{Res_Swap}(t, \mathcal{T}_{:,i^{**}}^{**}; \tilde{Y}) \\ & \equiv N_{i^{**}}^{**} \times \left\{ -\left\{ y_{01}^{**}(i^{**}) - r_{i^{**}}^{**} \right\} P(t, t_1^{**}(i^{**})) t \right. \\ & \quad + \left(1 + \left\{ y_{01}^{**}(i^{**}) - r_{i^{**}}^{**} \right\} \tilde{\tau}_{j^{**}}^{**}(i^{**}) \right) \mathbf{Res_ZC}(t, t_1^{**}(i^{**}); \tilde{Y}) \\ & \quad - \mathbf{Res_ZC}(t, t_{M^{**}(i^{**})}^{**}(i^{**}); \tilde{Y}) \\ & \quad \left. - r_{i^{**}}^{**} \times \sum_{j^{**}=2}^{M^{**}(i^{**})} \mathbf{Res_ZC}(t, t_{j^{**}}^{**}(i^{**}); \tilde{Y}) \tau_{j^{**}}^{**} \right\}. \end{aligned} \quad (13)$$

$$\begin{aligned} \mathbf{Sens}(k; t, \mathcal{S}; \tilde{Y}) & \equiv \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} \mathbf{Sens_Swap}(k; t, \mathcal{T}_{:,i^{**}}^{**}; \tilde{Y}) \\ & \quad - \sum_{i^*=1}^{I^*} n_{i^*}^* \mathbf{Sens_Swap}(k; t, \mathcal{T}_{:,i^*}^*; \tilde{Y}). \end{aligned} \quad (14)$$

Here the k -th order sensitivity $\mathbf{Sens_Swap}(k; t, \mathcal{T}_{:,i^{**}}^{**}; \tilde{Y})$ for the i^{**} -th swap associated with the tenor $\mathcal{T}_{:,i^{**}}^{**}$ is defined by

$$\begin{aligned} & \mathbf{Sens_Swap}(k; t, \mathcal{T}_{:,i^{**}}^{**}; \tilde{Y}) \\ & = N_{i^{**}}^{**} \left\{ \left(1 + \left\{ y_{01}^{**}(i^{**}) - r_{i^{**}}^{**} \right\} \tilde{\tau}_1^{**}(i^{**}) \right) \mathbf{Sens_ZC}(k; t, t_1^{**}(i^{**}); \tilde{Y}) \right. \\ & \quad - \mathbf{Sens_ZC}(k; t, t_{M^{**}(i^{**})}^{**}(i^{**}); \tilde{Y}) \\ & \quad \left. - r_{i^{**}}^{**} \times \sum_{j^{**}=2}^{M^{**}(i^{**})} \mathbf{Sens_ZC}(k; t, t_{j^{**}}^{**}(i^{**}); \tilde{Y}) \times \tau_{j^{**}}^{**}(i^{**}) \right\}. \end{aligned} \quad (15)$$

The k -th sensitivity $\mathbf{Sens_ZC}(k; t, t_{j^{**}}^{**}(i^{**}); \tilde{Y})$ for the zero coupon with the maturity $t_{j^{**}}^{**}(i^{**})$ by

$$\begin{aligned} & \mathbf{Sens_ZC}(k; t, t_{j^{**}}^{**}(i^{**}); \tilde{Y}) \\ & \equiv \left(\Theta(t, t_{j^{**}}^{**}(i^{**})) \mathbf{c}(l-1, k) \left[\lambda_1(t, t_{j^{**}}^{**}(i^{**})) \right]^{k+1-l} \left[\lambda_2(t, t_{j^{**}}^{**}(i^{**})) \right]^{l-1} \right)_{l \in \{1, \dots, k+1\}} \end{aligned}$$

The expression for $\mathbf{Sens_Swap}(k; t, \mathcal{T}_{:,i^*}^*; \tilde{Y})$ is similarly defined by replacing each double star by a one star.

It should be emphasized that replacing the future change of the IRS portfolio by a polynomial function (whose the coefficient is the initial portfolio sensitivities) is also interesting in the perspective of risk measurement and management. Indeed, instead of re-evaluating the portfolio for each scenario of shocks considered (which is time and memory consuming) one has just to calculate the value of the

polynomial function. Of course, this is meaningful whenever the remainder term is negligible.

3.2. Hedging Formulation

Our main purpose in this Section is to show, when hedging a portfolio, how useful are the high order sensitivities, with respect to the shock factors underlying the structure of interest rate model taken into consideration.

First, we formulate the expression changes linked to both the portfolio to hedge and the hedging instruments. Next the involvement of the sensitivities and their offsetting effects are presented.

3.2.1. Profit & Loss for a Covered Portfolio

Let us denote by V_0 the present time 0-value of a portfolio assumed to be sensitive to the interest rate. At a future time horizon t , where t is some nonnegative real number, such a portfolio may suffer from a loss, in the sense that $V_t(\cdot) < V_0$. To try to maintain the (future) value $V_t(\cdot)$ to be close to V_0 , the portfolio manager has to put in place a hedging technique. Various approaches are known in theory and used in practice.

The idea underlying the hedging relies on using another portfolio, referred in the sequel as a hedging portfolio (or hedging instruments) such that, this last would lead to a nonnegative profit compensating the loss on the initial portfolio. Therefore instead of the absolute change

$$V_t(\cdot) - V_0 \equiv \mathbf{P \& L_naked_portfolio}_{0,t}(\cdot)$$

associated with the initial naked portfolio, at the horizon t the change for the covered portfolio is given by

$$\begin{aligned} & \mathbf{P \& L_covered_portfolio}_{0,t}(\cdot) \\ & \equiv \{V_t(\cdot) - V_0\} + \mathbf{P \& L_hedging_instrument}_{0,t}(\cdot). \end{aligned} \quad (16)$$

The hedging portfolio H is assumed at time-0 to have the value

$$H_0 = \sum_{i^{**}=1}^{I^{**}} H_{0;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} H_{0;i^*}^* n_{i^*}^*. \quad (17)$$

It means that H is made by I^{**} types of instruments $H_{i^{**}}^{**}$ in long positions and I^* types of instruments $H_{i^*}^*$ in short positions. For a given type i^{**} (resp. i^*), we make use of $n_{i^{**}}^{**}$ (resp. $n_{i^*}^*$) number of instruments $H_{i^{**}}^{**}$ (resp. $H_{i^*}^*$). The $\mathbf{P \& L_hedging_instrument}_{0,t}(\cdot)$ corresponding to the use of the hedging operation during the time-period $(0, t)$ is (roughly) given by

$$\mathbf{P \& L_hedging_instrument}_{0,t}(\cdot) = \{H_t(\cdot) - H_0\} - \mathbf{cost_H}_0 \quad (18)$$

such that

$$\begin{aligned} & \mathbf{P \& L_covered_portfolio}_{0,t}(\cdot) \\ & = V_t(\cdot) - V_0 + \sum_{i^{**}=1}^{I^{**}} \{H_{t;i^{**}}^{**}(\cdot) - H_{0;i^{**}}^{**}\} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \{H_{t;i^*}^*(\cdot) - H_{0;i^*}^*\} n_{i^*}^* - \mathbf{cost_H}_0 \end{aligned} \quad (19)$$

where

$$\text{cost}_{\mathbf{H}_0} = \left(\frac{1}{P(0,t)} - 1 \right) \left[\sum_{i=1}^{I^{**}} \left\{ \nu_0^{**} + \nu^{**} \left| h_{0;i}^{**} \right| \right\} \mathcal{N}_{i^{**}}^{**} n_{i^{**}}^{**} + \sum_{i=1}^{I^*} \left\{ \nu_0^* + \nu^* \left| h_{0;i}^* \right| \right\} \mathcal{N}_{i^*}^* n_{i^*}^* \right] \quad (20)$$

with ν_0^{**} , ν^{**} , ν_0^* and ν^* are fixed constants such that $0 \leq \nu_0^{**}, \nu_0^* < 1$ and $0 < \nu^{**}, \nu^* < 1$. Their numerical values depend on the market practice under consideration.

In (20), we have used the fact that for instance the instrument value $H_{0;i^{**}}^{**}$ is the product of its notional $\mathcal{N}_{i^{**}}^{**}$ and the value $h_{0;i^{**}}^{**}$. For an instrument with $h_{0;i^{**}}^{**} \neq 0$ during its life-time, as in the case of a (no-risky credit) bond for example, then the corresponding cost at time t is very often defined as $\nu^{**} H_{0;i^{**}}^{**}$; so here it is taken that $\nu_0^{**} = 0$. The introduction of ν_0^{**} and ν_0^* relies on the fact that for some instruments as a swap, one can have that the corresponding market value satisfies $H_{0;i^{**}}^{**} = 0$. In this case practitioners take as a base for the fee the corresponding notional $\mathcal{N}_{i^{**}}^{**}$ such that the cost is rather $\nu_0^{**} \mathcal{N}_{i^{**}}^{**}$ since the term $\nu_0^{**} H_{0;i^{**}}^{**}$ vanishes. The hedging problem for the initial portfolio V is reduced to suitably choose the financial instruments

$$H_{;1}^{**}, \dots, H_{;i^{**}}^{**}, \dots, H_{;I^{**}}^{**} \text{ and } H_{;1}^*, \dots, H_{;i^*}^*, \dots, H_{;I^*}^*$$

and the corresponding security numbers

$$n^{**} = (n_1^{**}, \dots, n_{i^{**}}^{**}, \dots, n_{I^{**}}^{**}) \text{ and } n^* = (n_1^*, \dots, n_{i^*}^*, \dots, n_{I^*}^*)$$

such that the value of

$$|\mathbf{P \& L_covered_portfolio}_{0,t}(\cdot)|$$

should be small as possible. The difficulty here is linked to the fact that the future values of the hedging instruments at time t are unknown at the present time 0 where the hedge strategy is built. The choice of the hedging instruments is dictated by the willing that the resultant effect of their change variations would roughly offset (*i.e.* going in the opposite direction) the change of the portfolio V to hedge. Then, the problem is reduced to a minimization problem in order to find a suitable allocation of the security numbers n^{**} and n^* . The full detail is now performed below.

3.2.2. Interaction between the Portfolio to Hedge and the Hedging Instruments

As presented in the next Section, under the G2++ model, the interest rate is governed by two uncertainty factors shocks, let us say $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$. Under the effect of these last, if $\mathcal{V}$ is one of V , $H_{;i^{**}}^{**}$ and $H_{;i^*}^*$, then we will see also in the sequel that it is reasonable to assume that for any nonnegative integer p

$$\mathcal{V}_t(\cdot) - \mathcal{V}_0 \approx \mathbf{Sens}(0, \mathcal{V}) + \sum_{k=1}^p \frac{(-1)^k}{k!} \sum_{l=0}^k \mathbf{Sens}(l, k-l, \mathcal{V}) \varepsilon_1^l(\cdot) \varepsilon_2^{k-l}(\cdot). \quad (21)$$

In this approximation, the shocks $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ have no reason to be small since they are realizations of standard normal random variables. A main point on the efficiency of (21) in the hedging operation relies on the suitable choice of the integer p such that the approximation-error

$$\mathcal{E}(\cdot) = \left| \left(\mathcal{V}_t(\cdot) - \mathcal{V}_0 \right) - \left(\mathbf{Sens}(0, \mathcal{V}) + \sum_{k=1}^p \frac{(-1)^k}{k!} \sum_{l=0}^k \mathbf{Sens}(l, k-l, \mathcal{V}) \varepsilon_1^l(\cdot) \varepsilon_2^{k-l}(\cdot) \right) \right| \quad (22)$$

is very small on the hedger's point of view, as $E(\cdot) \leq 10^{-12}$ for instance. Such a strong requirement may be useful since very often in practice one has to deal with a large position $n\mathcal{V}_0$ as $n=10^7$. Making use of the approximation (21) for $\mathcal{V}=V$, $\mathcal{V}=H^{**}$ and $\mathcal{V}=H^*$ then it arises that

$$\begin{aligned} & \mathbf{P \& L_covered_portfolio}_{0,t}(\cdot) \\ &= \left(\theta_0^V + \sum_{i^{**}=1}^{I^{**}} \theta_{0;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \theta_{0;i^*}^* n_{i^*}^* \right) \\ &+ \sum_{k=1}^p \frac{(-1)^k}{k!} \sum_{l=0}^k \left[\theta_{l,k}^V + \sum_{i^{**}=1}^{I^{**}} \theta_{l,k;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \theta_{l,k;i^*}^* n_{i^*}^* \right] \varepsilon_1^l(\cdot) \varepsilon_2^{k-l}(\cdot) \\ &+ \frac{1}{(p+1)!} \sum_{l=0}^{p+1} \left[\theta_{l,p+1}^V + \sum_{i^{**}=1}^{I^{**}} \theta_{l,p+1;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \theta_{l,p+1;i^*}^* n_{i^*}^* \right] \varepsilon_1^l(\cdot) \varepsilon_2^{p+1-l}(\cdot) \end{aligned} \quad (23)$$

where

$$\theta_0^V \equiv \mathbf{Sens}(0, V), \quad (24)$$

$$\theta_{0;i^{**}}^{**} \equiv \mathbf{Sens}\left(0, H_{:,i^{**}}^{**}\right) - \left(\frac{1}{P(t, t+\delta)} - 1 \right) \left(\mathcal{V}_0^{**} + \mathcal{V}^{**} \left| h_{t,i^{**}}^{**} \right| \right) \mathcal{N}_{i^{**}}^{**} n_{i^{**}}^{**}, \quad (25)$$

$$\theta_{0;i^*}^* \equiv \mathbf{Sens}\left(0, H_{:,i^*}^*\right) + \left(\frac{1}{P(t, t+\delta)} - 1 \right) \left(\mathcal{V}_0^* + \mathcal{V}^* \left| h_{t,i^*}^* \right| \right) \mathcal{N}_{i^*}^* n_{i^*}^*, \quad (26)$$

$$\theta_{l,k}^V \equiv \mathbf{Sens}(l, k-l, V), \quad (27)$$

$$\theta_{l,k;i^{**}}^{**} \equiv \mathbf{Sens}(l, k-l, H_{:,i^{**}}^{**}), \quad \theta_{l,k;i^*}^* \equiv \mathbf{Sens}(l, k-l, H_{:,i^*}^*), \quad (28)$$

$$\theta_{l,p+1}^V \equiv \mathbf{Sens}(l, p+1-l, V; \eta_1, \eta_2), \quad (29)$$

$$\theta_{l,p+1;i^{**}}^{**} \equiv \mathbf{Sens}(l, p+1-l, H_{:,i^{**}}^{**}; \eta_1, \eta_2), \quad (30)$$

$$\theta_{l,p+1;i^*}^* \equiv \mathbf{Sens}(l, p+1-l, H_{:,i^*}^*; \eta_1, \eta_2). \quad (31)$$

where $i^{**} \in \{1, \dots, I^{**}\}$; $i^* \in \{1, \dots, I^*\}$; $k, l \in \{1, \dots, p\}$ and η_1, η_2 ⁴ two real numbers depend on the suitable domain of ε_1 , ε_2 as defined in the sequel.

Under the view of

$$-\varepsilon_1^{\bullet} \leq \varepsilon_1(\cdot) \leq \varepsilon_1^{\bullet\bullet} \quad \text{and} \quad -\varepsilon_2^{\bullet} \leq \varepsilon_2(\cdot) \leq \varepsilon_2^{\bullet\bullet}$$

and under the domain of definition of these two for some nonnegative real num-

⁴Note that η_1, η_2 are different from those defined in the G2+ model.

bers ϵ_1^* , ϵ_1^{**} , ϵ_2^* and ϵ_2^{**} where

$$\epsilon_1 = \max\{\epsilon_1^*; \epsilon_1^{**}\} \text{ and } \epsilon_2 = \max\{\epsilon_2^*; \epsilon_2^{**}\}.$$

then the hedging problem previously presented above, is essentially driven by the minimization with respect to the objective function

$$\begin{aligned} F(n^{**}, n^*; \epsilon_1, \epsilon_2) = & \left| \theta_0^V + \sum_{i^{**}=1}^{I^{**}} \theta_{0;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \theta_{0;i^*}^* n_{i^*}^* \right| \\ & + \sum_{k=1}^p \frac{1}{k!} \sum_{l=0}^k \left| \theta_{l,k}^V + \sum_{i^{**}=1}^{I^{**}} \theta_{l,k;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \theta_{l,k;i^*}^* n_{i^*}^* \right| \epsilon_1^l \epsilon_2^{k-l} \\ & + \frac{1}{(p+1)!} \sum_{l=0}^{p+1} \left(\xi_{l,p+1}^V + \sum_{i^{**}=1}^{I^{**}} \xi_{l,p+1;i^{**}}^{**} n_{i^{**}}^{**} + \sum_{i^*=1}^{I^*} \xi_{l,p+1;i^*}^* n_{i^*}^* \right) \epsilon_1^l \epsilon_2^{p+1-l}. \end{aligned} \quad (32)$$

where $\xi_{l,p+1}^V$, is an upper bound of $\theta_{l,p+1}^V$ independent of η_1 , η_2 (resp. for $\xi_{l,p+1;i^{**}}^{**}$, $\xi_{l,p+1;i^*}^*$ and $\theta_{l,p+1;i^{**}}^{**}$, $\theta_{l,p+1;i^*}^*$).

Accurate statement and further development will be performed in the subsequent work. In this Section we can point the following facts:

1) An approximation as (21) plays an important role when performing hedging a given portfolio V by another portfolio H .

2) No matter is the exact nature of instruments $H_{:,i^*}^*$ and $H_{:,i^{**}}^{**}$, it appears that the challenge is just to get the corresponding approximations (21). This is performed in the sequel for the case of zero-coupon bonds, coupon-bearing bonds, interest rate swaps and their portfolios.

3) To introduce financial instrument sensitivities, with respect to the shock factors underlying the model used for the interest rate, as we do here seem to be not frequent in the financial literature. It has the advantage to not focus on a given asset to base the hedging. For instance, in the case of equity market, it is usual in stock option to take as a basis of reference for the hedging the underlying stock asset itself. A similar approach for the interest rate market is not so clear. So our approach about sensitivities with respect to the shock factors appear to be natural and interesting solution.

4) A main basic financial instrument in the interest rate market is the zero-coupon bond. Coupon-bearing bond and interest rate swap may be seen as linear combination of zero-coupon bonds. However, all of these instruments appear to be non-linear functions of the shock uncertainty factors underlying the interest rate model G2++ considered in this paper.

5) It is seen from this paper that considering just the first and second order sensitivities as the classical duration and convexity [4], are in general not sufficient to perform a good hedging operation. This is among the reasons why we consider here high order sensitivities.

6) Interest rate option, as the one on zero-coupon bond, appears to be a non-linear function of zero-coupon bonds. It is tempting to apply the approach, as (21), we have used here for instrument linear on zero-coupon bonds. Investigation on

this direction will be performed in one of the sequel works.

3.3. Optimization Problem

According to the above subsection, in a generic form, and by taking V the hedging portfolio and H the hedging instruments, then the hedging is reduced to the minimization problem

$$(P_0): (n^{**}, n^*) = \operatorname{argmin} \{ F(n^{**}, n^*) | (n^{**}, n^*) \in D_0 \} \quad (33)$$

with the constraint D_0 defined as the set of $n^{**} = (n_1^{**}, \dots, n_{I^{**}}^{**})$ and

$n^* = (n_1^*, \dots, n_{I^*}^*)$ such that

$$\left(\frac{1}{P(0, t)} - 1 \right) \left[\sum_{i^{**}=1}^{I^{**}} \mathcal{N}_{i^{**}}^{**} \left\{ \nu_0^{**} + \nu^{**} \left| h_{0,i^{**}}^{**} \right| \right\} n_{i^{**}}^{**} + \sum_{i^*=1}^{I^*} \mathcal{N}_{i^*}^* \left\{ \nu_0^* + \nu^* \left| h_{0,i^*}^* \right| \right\} n_{i^*}^* \right] \leq D \quad (34)$$

and where D is the amount allowed by the investor not to be exceed in the hedging operation.

For convenience we can deal with vectorial notations such that the constraint (34) may be written as

$$a^{**} \cdot n^{**} + a^* \cdot n^* \leq D \quad (35)$$

where

$$a^{**} = \left(\frac{1}{P(0, t)} - 1 \right) \left(\mathcal{N}_{1^{**}}^{**} \left\{ \nu^{**} \left| h_{0,1^{**}}^{**} \right| + \nu_0^{**} \right\} n_{1^{**}}^{**}, \dots, \mathcal{N}_{I^{**}}^{**} \left\{ \nu^{**} \left| h_{0,I^{**}}^{**} \right| + \nu_0^{**} \right\} n_{I^{**}}^{**} \right)$$

and

$$a^* = \left(\frac{1}{P(0, t)} - 1 \right) \left(\mathcal{N}_{1^*}^* \left\{ \nu^* \left| h_{0,1^*}^* \right| + \nu_0^* \right\} n_{1^*}^*, \dots, \mathcal{N}_{I^*}^* \left\{ \nu^* \left| h_{0,I^*}^* \right| + \nu_0^* \right\} n_{I^*}^* \right)$$

For the function F as defined in (32), (P_0) is an integer optimization problem defined by integer linear constraints. The objective function is both non-linear, non-convex and non-differentiable at the origin. To overcome these difficulties, we make use of a linearization technique which consists to replace the initial problem (P_0) by an equivalent linear problem (P_1) . Assuming that ϵ is a fixed given constant, we introduce the following function

$$G(x, n^{**}, n^*) = \sum_{k=0}^p \sum_{l=0}^k x_{l,k} + \frac{1}{(p+1)!} \sum_{l=0}^{p+1} \left\{ \xi_{l,p+1}^V + \sum_{i^{**}=1}^{I^{**}} \xi_{l,p+1;i^{**}}^{**} n_{i^{**}}^{**} + \sum_{i^*=1}^{I^*} \xi_{l,p+1;i^*}^* n_{i^*}^* \right\} \epsilon_1^l \epsilon_2^{p+1-l} \quad (36)$$

where the components $x_{l,k}$'s of x are real variables. Actually, our motivation here is to remove the non-linearity by setting for all $k \in \{0, 1, \dots, p\}$ and $l \in \{0, 1, \dots, k\}$

$$x_{l,k} = \frac{1}{k!} \left| \theta_{l,k}^V + \sum_{i^{**}=1}^{I^{**}} \theta_{l,k;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \theta_{l,k;i^*}^* n_{i^*}^* \right| \epsilon_1^l \epsilon_2^{k-l} \quad (37)$$

Therefore, we obtain the following result.

Proposition 1. The problem (P_0) is equivalent to the following minimization

problem

$$(P_1): (x, n^{**}, n^*) = \operatorname{argmin} \{ F(x, n^{**}, n^*) \mid (x, n^{**}, n^*) \in D_1 \} \quad (38)$$

where D_1 is defined as the set of triplets (x, n^{**}, n^*) satisfying the constraints

$$a^{**} \cdot n^{**} + a^* \cdot n^* \leq D \quad (39)$$

$$0 \leq x_{l,k} + \frac{1}{k!} \left\{ \theta_{l,k}^V + \sum_{i^{**}=1}^{I^{**}} \theta_{l,k;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \theta_{l,k;i^*}^* n_{i^*}^* \right\} \epsilon_1^l \epsilon_2^{k-l} \quad (40)$$

$$0 \leq x_{l,k} - \frac{1}{k!} \left\{ \theta_{l,k}^V + \sum_{i^{**}=1}^{I^{**}} \theta_{l,k;i^{**}}^{**} n_{i^{**}}^{**} - \sum_{i^*=1}^{I^*} \theta_{l,k;i^*}^* n_{i^*}^* \right\} \epsilon_1^l \epsilon_2^{k-l} \quad (41)$$

for all $k \in \{0, 1, \dots, p\}$ and $l \in \{0, 1, \dots, k\}$ with the restrictions that

$$0 \leq x = (x_{l,k})_{l,k} \quad (42)$$

$$n^{**} \in \mathbb{N}^{I^{**}} \text{ and } n^* \in \mathbb{N}^{I^*}. \quad (43)$$

The objective function G is defined in (36).

In this Proposition, by the equivalence between (P_0) and (P_1) we mean that if an optimal solution (n^{**}, n^*) to (P_0) does exist, then (P_1) admits an optimal solution (x, n^{**}, n^*) ⁵, and conversely if (x, n^{**}, n^*) is an optimal solution to (P_1) then (P_0) admits (n^{**}, n^*) as an optimal solution. Therefore, with the above result, we are led to solve problem (P_1) instead of (P_0) .

Observe that both the objective function and constraints associated with (P_1) are given by linear transformations, with mixed integer and real coefficients. The problem (P_1) , commonly referred as a Mixed Integer Linear Problem (MILP), is recognized as to be an NP-hard problem due to the non-convexity of the domain and the number of possible combinations of the variables. For small dimensions, MILP can be solved by exact methods that provide an exact optimal solution. In this case the most of available exact methods are Branch and Bound, Branch and Cut, Branch and Price [15] However the complexity of MILP exponentially increases with the number of variables and these mentioned methods can fail. To overcome this inconvenience, meta-heuristics methods, as Genetic Algorithm and Ant Colony Optimization [16] may be used. Usually there are various solvers which yield exact solution to the MILP for a moderate number of variables less than 1500 which is largely enough for our purpose. Therefore, we make use here the commercial CPLEX solver 9.0.

4. Numerical Illustrations

The transaction cost parameters in the numerical illustrations are chosen to be small but strictly positive, reflecting typical orders of magnitude observed for bid-ask spreads and trading fees in liquid markets. These values are not calibrated to

⁵With $x = (x_{l,k})_{l,k}$ and $x_{l,k}$ defined from n^* and n^{**} as in (37).

a specific market, but correspond to standard benchmark assumptions commonly used in the literature to illustrate the qualitative impact of transaction costs rather than to provide an exact empirical fit.

The portfolio to hedge is made by bonds in long and short positions, and there are both (payer and receiver) swaps and also bonds (in long and short positions) in the hedging portfolio. So our target here is to test how good or bad is our approach in such a situation.

First in **Table 1**, Yield curve we make use. For shortness we just examine the hedging behavior for the horizon 90 days and the calibration is given in the **Table 2**. This long-time horizon should be a good test for the approach robustness. All the features related to the hedging operation are summarized in **Table 3**. The characteristics of the bonds in the portfolio to hedge are presented in **Table 4**, with usually that each bond $\tilde{B}_i^{**}$ (resp. $\tilde{B}_j^{**}$) is in long (resp. short) position. The characteristics of the bonds inside the hedging portfolio are displayed in **Table 5**. Moreover in **Table 6** one can see the characteristics of the swaps included in the hedging portfolio.

The hedging allocation is displayed in **Table 7**, where it is seen the requirement of only 1267 bonds B_1^* in short position and 1007 payer swaps S_1^{**} .

The maximal amount D allowed to perform the hedging operation is supposed to be given as calculated in [11]. The hedging instruments cost is the sum of the costs related to the bonds and the swaps forming the hedging portfolio.

Table 1. Yield curve we make use.

maturity (years)	1	2	3	4	5	6	7	8	9	10
yield (%)	2.72	2.71	2.70	2.71	2.71	2.72	2.73	2.74	2.79	2.81

Table 2. Set calibration.

κ_1	1.11%
κ_2	3.51%
σ_1	1.13%
σ_2	2.81%
ρ	-7.21%

Table 3. Features of the hedging.

t	90 days
η	0.1%
λ	25%
$V_0^{**} = V_0^*$	0
$V^{**} = V^*$	1
$\varepsilon_1^* = \varepsilon_1^{**}$	5
$\varepsilon_2^* = \varepsilon_2^{**}$	5
p	8
γ	5%

Table 4. Characteristics of the CBB portfolio to hedge.

type	number	cpn	maturity	unit value
$\tilde{B}_1^{**}$	100,000	3%	15 years	600.18
$\tilde{B}_2^{**}$	150,000	5%	7 years	879.31
$\tilde{B}_3^{**}$	150,000	7%	5 years	997.07
$\tilde{B}_4^{**}$	175,000	4%	10 years	764.7
$\tilde{B}_5^{**}$	150,000	5%	12 years	809.38
$\tilde{B}_1^*$	1000	4%	8 years	803.72
$\tilde{B}_2^*$	9000	3.50%	3 years	913.71
$\tilde{B}_3^*$	1000	4.60%	4 years	920.21

Table 5. Characteristics of the hedging CBB instruments.

Type	number	face value	frequency	coupon	maturity	unit value
B_1^{**}	n_1^{**}	100	1 year	5.50%	7 years	90.62
B_1^*	n_1^*	100	1 year	3.50%	5 years	85.29

Table 6. Characteristics of the hedging IRS instruments.

type	number	notional	frequency	maturity	rate swap
S_1^{**}	m_1^{**}	1000,000	6 months	5 years	6.95%
S_1^*	m_1^*	1000,000	6 months	7 years	7.11%

Table 7. Hedging allocation for 90 days-horizon using calibration **Table 2**.

n_1^{**}	n_1^*	m_1^{**}	m_1^*	Max Profit or Loss	ratio
0	1267	1007	0	36,422,240	6.05%

$$\mathbf{D}_{port} = 598694057.04 \quad \mathbf{Cost_H}_0 = 2851458.57$$

5. Conclusions

To conclude, it's well established by practitioners and academics, throughout the years of methodology of hedging a portfolio, that it is a very severe issue. Multiple studies on hedging tend to essentially focus on its problems which relate to the usage of a single instrument, in which the management of the operations of a practitioners' (whom are considered as dealers), should be thought in terms of portfolios. Therefore, based on that case, the work is often devoted to the issues of hedging a sensitive position to interest rate by means of portfolio of the bonds or swaps.

Here, a portfolio oriented hedging approach is proposed, in which it relies on the G2++ model. It can be considered as an alternative approach of the practitioner methodology which is based on bumping some pillars of the interest rate curves.

Following, it has been recognized by many standard practical approaches, that plenty of drawbacks those involving a great sum of sensitivities and a consumed considerable time computed are viewed. Regardless of the sensitivities limitations to the first and second orders, a lack of insurance of the reliability of the hedging is attained, as the hedger cannot evaluate the possible loss magnitude which results from the usage of the methodology.

As a comparison, the approach we tackle is solely based on two uncertain factors which lead to the consideration of the high order sensitivities with almost immediate time computation. Also, the hedger has the potential to fix or acknowledge in advance the magnitude of maximal loss, based on the results from the use of the hedging as presented here.

Therefore, our approach could be viewed as an optimal one, especially when compared with logically well-founded alternative methodologies, for which the hedger lacks prior control of the loss that could incur from the hedging approach.

Nevertheless, it is important to note that a continuous risk from the underlying G2++ model used will remain, despite the possibility to control such a loss.

Moreover, the G2++ model is still the most accurate and right one to be used, regardless of the presence of a bad estimation of the parameters which could affect the span of a bad hedging. Thus, it remains unknown to acquire a true data generating process, for the state variables.

Further, the suitability of the underlying model, which is assured to be fulfilled in this work, is considered as another aspect and issue that our scope tackles.

Here, the sensitivities we used in the hedging approach, involve the unobservable state variables that are considered by the underlying G2++ model. To clarify, having such views on these variables, is not necessarily so intuitive for the hedger. Rather, it could be recalled that these variables used in our paper, tend to follow Gaussian Laws, so a conservative values for the shocks as -5 and 5 can be used in our numerical illustrations. Based on that, it is ought to be noted that the statistical considerations and numerical tests are means to help in forming views on extreme values for the shocks.

Henceforth, we have attempted to provide explicit formulas for all the expressions involved in our hedging approach, regarding the sensitivities for the individual position or for portfolios. However, as for the final instrument of hedging allocation, it's been deducted that the hedger needs a solver to solve the involved optimization problem.

In this paper, a usage of the CPLEX commercial solver is used, which may be a drawback if the hedger doesn't have at her disposal such a tool. So, we came up with another alternative solution which mainly depends on the hedger software system.

Multiple extension and various questions can be raised after the presented work.

The G2++ model is solely the underlying model of our hedging approach, so it relies only on the two factor uncertainties, which makes it more appealing to explore the situation, where an alternative interest rate model has more uncertainty

factors (such as, Arbitrage Free Dynamic Nelson-Siegel). Vaguely, it is considered as an open question of whether considering a general GATSM models as an underlying for the approach contributes to improve the hedging or if it simply implies only a complication from the practical point of view.

Henceforth, our work has been met with some limitation on the context, where just one interest rate curve is used. Nevertheless, most of the portfolios under consideration tend to generally contain instruments linked to multiple interest rate curves, as in the case of currency swaps, regardless if it's the one to hedge or the one to use for the hedging. Thus, this raises questions about the performance of the hedging operation in the multi-curves setting.

Can one extend his hedging approach to handle such situation?

In our presented paper, a focus on the implicit assumption of the instruments being considered inside the portfolios are only exposed to the interest rate risks. Even-though, a CBB or IRS are also sensitive to credit as a counterparty risk, or an equity and foreign exchange risk and so on. So, the uncertainty risk factors approach, like the G2++ model dealt with, can be extended to take into account at least both the risks linked to interest rate and credit. Such a question is important to be explored.

Furthermore, we have focused on the sensitivities of the portfolio, those made by CBBs and IRSs, with respect to shocks associated to the uncertainty factors which are underlined in the used G2++ model. So, it is previewed that the model parameters solely depend on the hedging results. As the value of these parameters depend on the method used to their inference, it could also be intriguing to analyze the parameters sensitivities and to decide if it is possible to include the hedging the model tends to risk, based on the parameter uncertainties or instability for the time-period hedging considered.

Thusly, when considering the previously mentioned constraints, we have avoided to use generated model like Zero Coupon process without the economical sense. Yet, it still remains and issue when considering the weakness of the G2++ model, since its calibration is done on an improper manner as not discarding these ZC prices higher than 1.

Therefore, in order to maintain the hedging perspective, two questions can be raised in the manner: 1) What could happen if the calibration is carried in a consistent manner? 2) How could one modify the standard G2++ model to attain a new one that is consistent with the possibility to deal with interest rates near zero or below?

In that case, this paper has focused on the fundamental aspect of the hedging portfolio oriented one, and the rolling hedging positions. While, keeping close track to the market reality, the hedging costs are continually kept being taken into consideration. To end with the notion that it seen sought to be interesting to perform empirical illustrations to assess the efficiency of our approach, especially when a long period of time is examined under rolling the hedging operations.

Hedging a portfolio is an issue which has been considered for a long date by

practitioners and academics, and this is always the case. Various available academic work on hedging are essentially focused on the hedging problems related to the use of one single instrument, though practitioners (as dealers for example) should manage their operations and think in term of portfolios. Considering this situation, the present work is devoted to the problem of hedging a position sensitive to interest rate by means of a portfolio of bonds and/or swaps.

Here we have proposed a hedging approach portfolio oriented which relies on the G2++ model. This can be seen as an alternative approach of the practitioner methodology based on bumping some pillars of the interest rate curves.

The standard practical approach has the drawback to involve a great number of sensitivities and to consume considerable time computation. More important, as these sensitivities are generally limited to the first and second orders, there is no insurance about the reliability of the hedging result as the hedger cannot a-priori evaluate the possible loss magnitude resulting from the use of the methodology.

For comparison, our approach is based only on two uncertainty factors and consequently high order sensitivities can be considered with almost immediate time computation. The hedger has the possibility to fix or know in advance the magnitude of maximal loss resulting from using the hedging results presented here.

Therefore, our approach may be seen as optimal when compared with alternative methodologies (possibly logically well-founded) but for which the hedger has no a-priori control of the loss which can incur from the hedging approach. Despite this possibility to control such a loss, it is important to note that there always remains the risk from the underlying G2++ model used. Indeed, if this last does not really well represent the market evolution then, the hedging approach based on it may behave badly. Moreover, even the G2++ is the right model to use, a bad estimates of the corresponding parameters can also span a bad hedging. Definitely, the true data generating process for the state variables remains unknown. The suitability of the underlying model (assumed to be fulfilled in this work) is another story and issue out of our scope here.

The sensitivities, used in our hedging approach, involve the unobservable state variables underlying the G2++ model considered. It means that having views on these variables as we largely used in this paper is not so intuitive for the hedger. But it would be recalled that these variables follow Gaussian laws, so a conservative values for the shocks as -5 and 5 can be used as we do over our numerical illustrations. We think that statistical considerations and numerical tests can help in forming views on extreme values for the shocks.

We have tried to provide explicit formulas of all of the expressions involved in our hedging approach, as the various sensitivities either for individual positions or for portfolios. But for the final instrument hedging allocation, the hedger needs a solver to solve the involved optimization problem. Here for the convenience, we have used the CPLEX commercial solver. This may be a drawback if the hedger does not have at her disposal such a tool. So another alternative solution has to be

found depending on the hedger software system. Various extensions and questions can be raised after this present work.

The G2++ model is the underlying model of our hedging approach. As this model relies just on the two-factor uncertainties, it would be interesting to explore the situation where an alternative interest rate model with more uncertainty factors as the Arbitrage Free Dynamic Nelson-Siegel. More generally it is an open question whether considering a general GATSM model as an underlying for the approach contributes to improve the hedging or it implies just a complication from the practical point of view.

Our work has been limited on the context where just one interest rate curve is used. However usually the portfolios under consideration (either the one to hedge or the other used for the hedging) generally contains instruments linked to various interest rate curves, as in the case of currency swaps. This raises naturally the question of performing the hedging operation in the multi-curves setting. Can we extend our hedging approach to handle this situation?

Our present paper is focused on the implicit assumption that the instruments under consideration inside the portfolios are only exposed to interest rate risks. However, a CBB or IRS is also sensitive to credit (or counterparty) risk, equity and foreign exchange risk and so on. Probably the uncertainty risk factors approach, as the G2++ model we deal with, can be extended in order to take into account at least both the risks linked to interest rate and credit. This is an important question which deserves to be explored.

Our present investigation has been focused on the sensitivities of the portfolio (made by CBBs, IRSs) with respect to shocks associated to the uncertainty factors underlying the used G2++ model. It is seen that the model parameters are very determining on the hedging result. As the value of these parameters depend on the method used to their inference, it may be interesting to analyze the parameters sensitivities and to decide whether it makes sense to include in the hedging the model risk, coming from the parameter uncertainties or instability for the time-period hedging considered.

When considering the above-mentioned constraints, we have avoided to use generated model Zero Coupon prices without economical sense. But actually the weakness of the G2++ model remains an issue, for at least due to the fact that the model calibration is done on an improper manner as not discarding these ZC prices higher than 1. Always staying in the hedging perspective, two questions can be raised:

- 1) what does happen if the calibration is done in a consistent manner?
- 2) how to modify the standard G2++ model in order to get a new one which is consistent with the possibility to deal with interest rates near zero or below?

The present work has focused on the fundamental aspect of the hedging portfolio oriented and for a one-period step, as conform to the situation of rolling hedging positions as many practitioners do. To stick close on the market reality, we have taken into consideration the hedging costs. Then it may be interesting to

perform empirical illustrations in order to assess the efficiency or not of our approach when a long time period is considered under rolling the hedging operations.

Acknowledgements

I would like to thank and acknowledge the American Community School of Beirut for providing an astounding educational environment that greatly nourishes talent. Additionally, I would like to thank Wolfram Alpha for providing the means to develop the figures utilized throughout the paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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