

Equatorial Circular and Spherical Photon Orbits around Kerr-Type Black Holes in Expanding Universe

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Abstract

This paper investigates the effects that may arise in null (light-like) geodesics in the vicinity of Kerr-type black holes. Exact solutions for circular equatorial and spherical photon orbits are obtained using the geodesic equations of the Kerr-de Sitter metric, which includes the cosmological constant λ . By solving the radial equation simultaneously with its first derivative, expressions for the photon angular momentum and Carter constant are derived. Based on these results, an equation for equatorial circular photon orbits around Kerr black holes is obtained, generalizing the well-known Bardeen-Press-Teukolsky formula to the case of a nonzero cosmological constant. An equation describing spherical photon orbits is also derived and solved, extending the known result of E. Teo to include the influence of the cosmological constant. A comparison between expressions with and without the cosmological constant makes it possible to estimate its potential influence and determine the threshold above which these effects become significant.

Keywords

Black Hole, Photon Orbit, Cosmological Constant, Null Geodesics

1. Introduction

Among the major discoveries and achievements of recent decades in astronomy and astrophysics, including the detection of gravitational waves, direct imaging of supermassive black holes Sgr A* and M87*, and the discovery of the accelerated expansion of the Universe, the latter occupies a special place. While the existence of gravitational waves had been predicted more than a century ago, the accelerated expansion of the Universe came as a complete surprise and challenged prevailing

cosmological concepts.

Previously, it had been assumed that the expansion of the Universe, as a consequence of the Big Bang, would either proceed at a constant rate determined by the Hubble parameter or gradually slow down, possibly leading eventually to a contraction back toward the primordial singularity. However, accelerated expansion was not predicted by any accepted cosmological model.

Explaining this phenomenon required the introduction of a new cosmological framework involving dark energy and dark matter, which are now believed to dominate the present-day Universe. Although this interpretation is widely accepted, many researchers prefer to regard the effect as an intrinsic property of the vacuum itself. This concept is currently the most widely accepted, to believe that it is simply a property of the vacuum itself. Regardless of the physical origin of the acceleration mechanism, a formal mathematical description in terms of the cosmological constant Λ had already been proposed by Willem de Sitter in 1916-1917 [1] [2]. Most modern cosmological models are based on this paradigm.

The motivation for introducing the cosmological constant was quite natural: A. Einstein and W. de Sitter [1] [2] both sought to restore symmetry within the laws of physics by allowing gravity and antigravity to coexist in a manner analogous to attraction and repulsion in electromagnetism. For most of the twentieth century, however, neither theoretical nor experimental evidence existed that could determine the value of this constant. Consequently, the majority of physicists believed that the only possible value of Λ , if it existed at all, was zero. This situation changed dramatically in 1998-1999; two independent collaborations concluded that the Universe is not merely expanding, but expanding at an accelerating rate [3] [4].

The current accepted value of the cosmological constant is

$$\Lambda = (1.090 \pm 0.029) \times 10^{-52} \text{ m}^{-2},$$

or equivalently $5.25 \times 10^{-10} \text{ J/m}^3$ in energy-density units [5].

Considerable attention has therefore been devoted to studying the influence of the cosmological constant on the geodesics of test particles and photons in various spacetime metrics. The number of related publications is extensive. Without attempting a complete review, we mention several important studies devoted to Kerr-(anti)-de Sitter (KdS) [6]-[9], Kerr-Newman-de Sitter (KNdS) [10]-[12], and Reissner-Nordström-de Sitter (RNdS) metrics [13], including analyses of both timelike geodesics for test particles and null geodesics for photons. In many of these works, solutions are expressed in terms of Weierstrass elliptic functions, Kleinian hyperelliptic functions, and Gauss hypergeometric functions [6]-[10].

Other widely discussed topics include the formation of black hole shadows [11] [12] [14], gravitational lensing, deflection of light from distant sources, and frame dragging effects such as the Lense-Thirring precession [7] [10].

Of practical interest is the shift in the radius of spherical photon orbits caused by a non-zero cosmological constant, obtained in the first (linear) approximation in Ref. [13].

Several studies have also investigated the critical value of the reduced cosmological constant λ (Eq. 1.13), determined from photon-orbit parameters. In Ref. [15]-[18], calculations were performed for values of the reduced cosmological constant in the range 10^{-5} - 10^{-2} .

Critical values of $\lambda = 0.0952$ for the KdS metric and $\lambda = 0.074$ for the RNdS metric were calculated. These limits may be exceeded only in a naked singularity [16].

A particularly interesting analysis was carried out in Ref. [14], where the balance between black hole horizons and cosmological horizons was studied. For the KdS metric, the corresponding critical value was found to be $\lambda = 0.0509$ ¹.

One of the motivations for the present work is to estimate the order of magnitude of the cosmological constant from photon orbit calculations and to determine whether its influence answer the question of whether its influences on photon motion near black holes could become observable.

The paper is organized as follows. In Sec. II, the equation for equatorial circular photon orbits is derived using the radial geodesic equation and its first derivative. In Sec. III, the equation governing spherical photon orbits is obtained from the angular geodesic equations and expressed in terms using angular geodesic equations, we derive the equation for a spherical orbits, expressed in terms of azimuthal shift per complete latitudinal revolution. The resulting solution generalizes the well-known result of E. Teo to the case of a nonzero cosmological constant. Section IV summarizes the results and presents conclusions.

2. Equatorial Circular Photon Orbits within the Kerr-de Sitter Metric

In terms of Boyer-Lindquist coordinates and in geometrical units $c = G = 1$, the Kerr-de Sitter (KdS) metric has the following form [19].

$$ds^2 = -\frac{\Delta_r}{\Sigma\Upsilon} \left[dt - a \sin^2 \theta d\varphi \right]^2 + \Sigma \left[\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right] + \frac{\Delta_\theta \sin^2 \theta}{\Sigma\Upsilon} \left[a dt - (r^2 + a^2) d\varphi \right]^2 \quad (1.1)$$

where M is the black hole mass, a is the spin parameter, and Λ is the cosmological constant. The following notations are also used here:

$$\Sigma = r^2 + a^2 \cos^2 \theta \quad (1.1a)$$

$$\Delta_r = (r^2 + a^2) \left(1 - \frac{\Lambda}{3} r^2 \right) - 2Mr \quad (1.2)$$

$$\Delta_\theta = 1 + \frac{a^2 \Lambda}{3} \cos^2 \theta \quad (1.3)$$

$$\Upsilon = 1 + \frac{\Lambda}{3} a^2 \quad (1.4)$$

Using the Hamilton-Jacobi transformation, the variables in the metric (1.1) are

¹Recalculated. Authors [14] used expression such as (1.13) without factor 1/3. So their original result is 3 times greater.

separated to form a system of 4 first-order geodesic differential equations [12] [14] [20]

$$\Sigma \dot{t} = \frac{a \sin^2 \theta (L - aE \sin^2 \theta)}{\Delta_\theta \sin^2 \theta} + \frac{(r^2 + a^2) [(r^2 + a^2) E - aL]}{\Delta_r} \quad (1.5)$$

$$\Sigma \dot{\phi} = \frac{L - a \sin^2 \theta E}{\Delta_\theta \sin^2 \theta} + \frac{a(r^2 + a^2) E - aL}{\Delta_r} \quad (1.6)$$

$$\Sigma^2 \dot{\theta}^2 = \Delta_\theta (Q + (L - aE)^2) - \frac{(a \sin^2 \theta E - L)^2}{\sin^2 \theta} \quad (1.7)$$

$$\Sigma^2 \dot{r}^2 = ((r^2 + a^2) E - aL)^2 - \Delta_r (Q + (L - aE)^2) \equiv R(r) \quad (1.8)$$

Here, we introduced angular momentum L , photon energy E , and the Carter constant Q . The first two concepts are used here not quite exactly, because spacetime at infinity is not, in this case, flat, but de Sitter space. All of them are integrals of motion.

Orbits in KdS spacetime are able to form two different families with constant radii: plane equatorial circular orbits and the spherical ones. Due to the constant orbital radii, Equation (1.8) and its derivative must be equal to zero: $R(r)$ and $dR/dr = 0$.

Two important dimensionless quantities are determined as a result of the simultaneous solution of these two equations:

$$\xi = L/ME \quad (1.9)$$

and

$$\eta = Q/M^2 E^2 \quad (1.10)$$

The results are as follows

$$\xi = -\frac{r^3 - 3r^2 + a^2 r + a^2 + \lambda a^2 r (r^2 + a^2)}{a(r - 1 - \lambda r (2r^2 + a^2))} \quad (1.11)$$

$$\eta = -\frac{r^3(r^3 - 6r^2 + 9r - 4a^2 + \lambda^2 a^4 r^3 + 2\lambda a^2 r^3 + 6\lambda a^2 r^2)}{a^2 (r - 1 - \lambda r (2r^2 + a^2))^2} \quad (1.12)$$

Here and below, to keep the notation simpler, we will use dimensionless variables referred to the common unit M . That means that we still write r , but it means r/M , a means a/M , and so on. The dimensionless reduced cosmological constant is now

$$\lambda = \frac{1}{3} \Lambda M^2 \quad (1.13)$$

Both dependencies $\xi(\lambda)$ and $\eta(\lambda)$ are shown in **Figure 1** for $r = 1.5$ (1), $r = 2$ (2) and $r = 3$ (3) and in **Figure 2** for $r = 2$ (1) and $r = 3$ (2), respectively. It is clearly visible that for $\lambda \leq 10^{-5}$ there is no dependence; it appears and increases starting from 10^{-3} and further. **Figure 3** and **Figure 4** show the dependences of the same quantities on orbital radii in the range $r = 1.5 - 4.0$ for $\lambda = 0$ curves

(1) and $\lambda = 0.01$ (2). All dependences are shown for rotation parameter $a = 1$. For other values of the spin parameter, the dependences are qualitatively the same.

Since the Carter constant is related to the longitudinal projection of the orbit, $\eta = 0$ denotes equatorial photon orbits. Therefore, to determine the equatorial orbits, it is necessary to solve the cubic equation from the numerator of Equation (1.12).

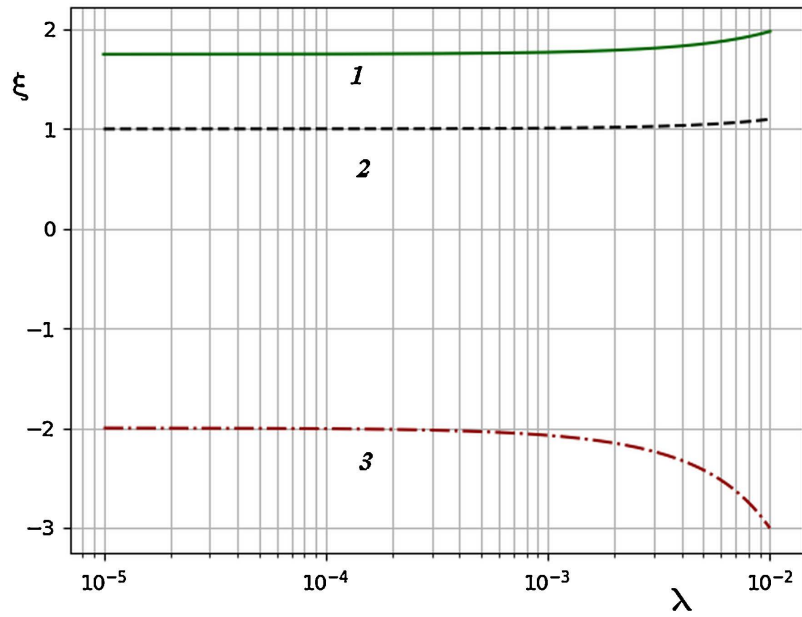


Figure 1. Plot of the dimensionless angular momentum ξ versus the reduced cosmological constant λ , for $r = 1.5$ (1); 2 (2) and 3 (3).

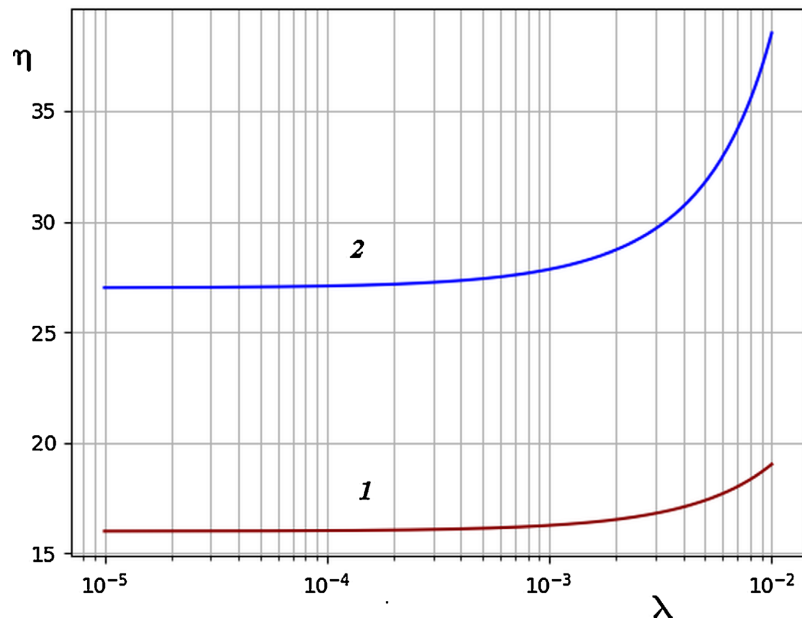


Figure 2. Plot of the dimensionless Cramer constant η versus the reduced cosmological constant λ , for $r = 2$ (1); 3 (2).

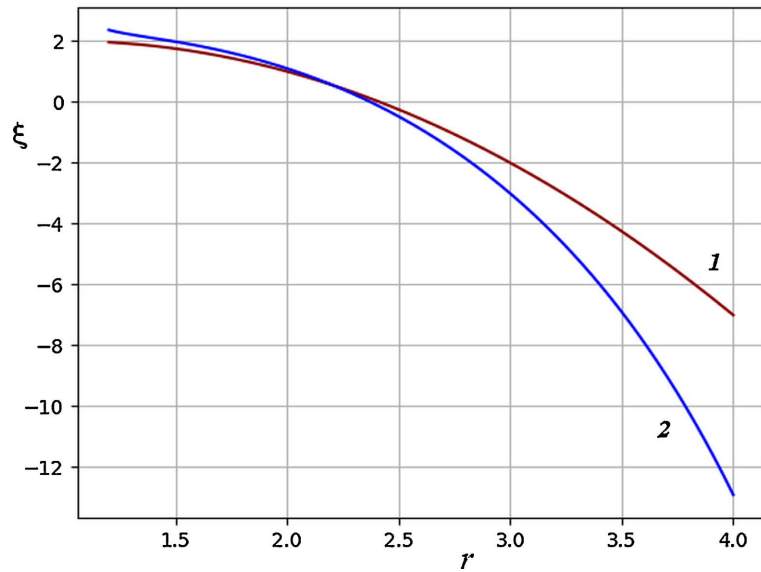


Figure 3. Plot of the dimensionless angular momentum versus orbital radius without the reduced cosmological constant $\lambda = 0$ (1) and with $\lambda = 0.01$ (2).

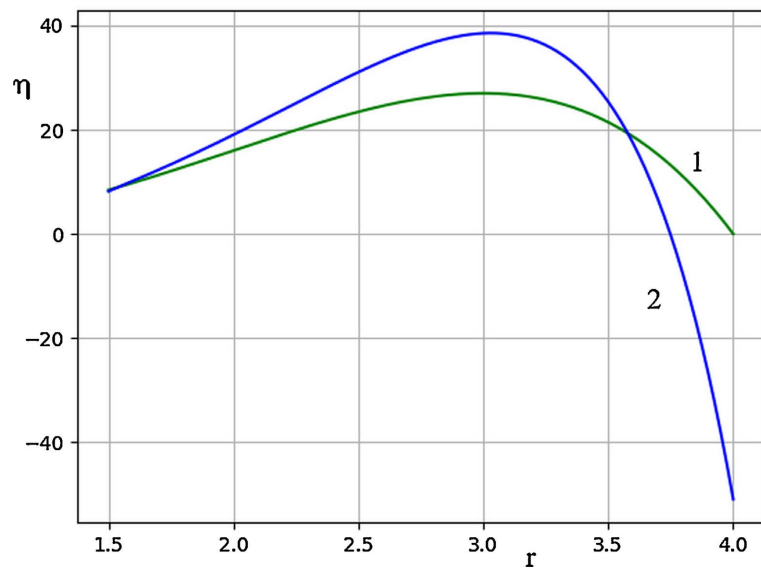


Figure 4. Plot of the dimensionless Cramer constant λ versus orbital radius for $\lambda = 0$ (1) and $\lambda = 0.01$ (2).

$$(1+\gamma)^2 r^3 - 6(1-\gamma)r^2 + 9r - 4a^2 = 0 \quad (1.14)$$

where indicated

$$\gamma = a^2 \lambda \quad (1.15)$$

Its solution is

$$r = \frac{2}{(1+\gamma)^2} \left[(1-\gamma) + \sqrt{1-14\gamma+\gamma^2} \cos \left[\frac{1}{3} \arccos \left[\frac{8(1-\gamma)^3 - 9(1-\gamma)(1+\gamma)^2 + 2a^2(1+\gamma)^4}{(1-14\gamma+\gamma^2)^{3/2}} \right] \right] \right] \quad (1.16)$$

This result is shown in **Figure 5**. For values of the cosmological constant $\gamma \leq 10^{-5}$ its presence has no effect on the equatorial circular photon orbits. With further increase of the cosmological constant, the radii of the equatorial orbits decrease. **Figure 5** shows, as examples, the cases $a^2 = 1$ (1) and $a^2 = 0.5$ (2).

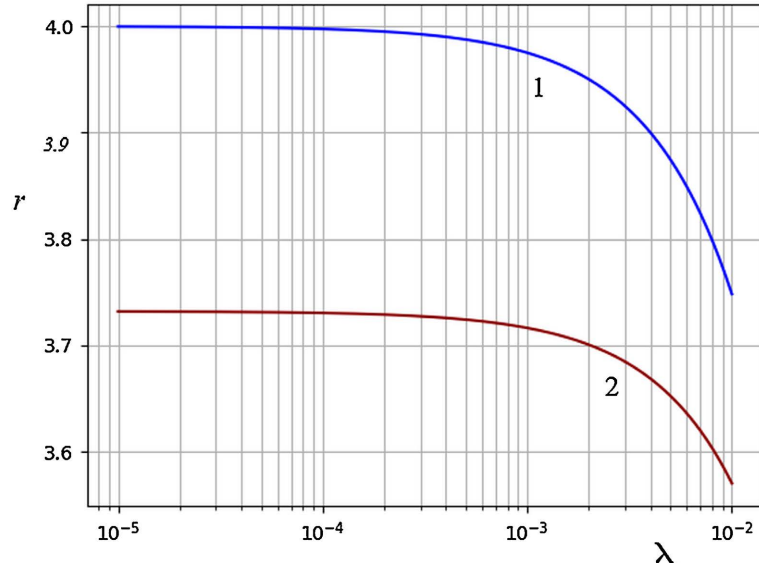


Figure 5. Radius of the equatorial circular orbit as a function of the spin parameter for $a^2 = 1$ (1) and $a^2 = 0.5$ (2).

Equation (1.16) indicates a possible upper limit for the reduced cosmological constant. For the maximal value $a^2 = 1$, the expression under the square root must be positive or zero, that means $\lambda \leq 1/14 = 0.07$. It is noteworthy that, in our calculations, we were unable to reach this boundary value.

It may arise an impression that Equation (1.14) contradicts the well-known formula for equatorial photon orbits [21].

However, we will show below that this impression is erroneous.

Let us consider the case where $\lambda \ll 1$ and neglect the small λ terms in (1.16). As a result, the latter will take the form

$$r = 2 \left(1 + \cos \left(\frac{1}{3} \arccos(2a^2 - 1) \right) \right) \quad (1.17)$$

The last step consists of using a trigonometric formula

$$\arccos(2a^2 - 1) = 2 \arccos(\pm a) \quad (1.18)$$

Finally, we have the well-known equation

$$r = 2 \left[1 + \cos \left(\frac{2}{3} \arccos(\pm |a|) \right) \right] \quad (1.19)$$

Thus, Equation (1.16) can be treated as the direct generalization of Equation (1.19) for the presence of a cosmological constant.

From Equation (1.16), an expression for the change in orbital radius can be de-

rived in the first (linear) approximation with respect to the cosmological constant. Retaining only terms of the first degree in λ , and after a series of transformations, we obtain in standard variables:

$$\delta r = -2\Lambda a M \left[a + 3a \cos(\phi) + \sin(\phi) \frac{(27M^2 - 25a^2)}{3\sqrt{M^2 - a^2}} \right] \quad (1.20)$$

where angle ϕ

$$\phi = \frac{2}{3} \arccos\left(\pm \frac{a}{M}\right) \quad (1.21)$$

3. Spherical Photon Orbits in Kerr-de Sitter Metric

To study spherical photon orbits, it is necessary to solve equations (1.6) and (1.7) together. By dividing them, we obtain

$$\frac{d\varphi}{d\theta} = \frac{\frac{\xi - a \sin^2 \theta}{\Delta_\theta \sin^2 \theta} + \frac{a(r^2 + a^2 - a\xi)}{\Delta_r}}{\sqrt{\Delta_\theta \left[\eta + (\xi - a)^2 \right] - \frac{(a \sin^2 \theta - \xi)^2}{\sin^2 \theta}}} \quad (2.1)$$

where

$$\Delta_r = (r^2 + a^2)(1 - \lambda r^2) - 2r \quad (2.2)$$

After replacing variables

$$w = \cos^2 \theta \text{ and } \Delta_\theta = 1 + \lambda a^2 w \quad (2.3)$$

we get the equation

$$\frac{d\varphi}{d\theta} = \frac{1}{2(1 + a^2 \lambda w) \sqrt{P_3(w)}} \left(\frac{\xi}{1 - w} - a \right) + \frac{a(r^2 + a^2 - a\xi)}{2\Delta_r P_3(w)} \quad (2.4)$$

where P_3 is the third-degree polynomial

$$P_3(w) = -(1 + \lambda b) a^2 w^3 + (\lambda b a^2 - \eta - \xi^2 + a^2) w^2 + \eta w \quad (2.5)$$

and we introduced the designation

$$b = \eta + (\xi - a)^2 \quad (2.6)$$

As a result, terms 2 and 3 in (2.4) can be integrated directly according to the formulas (3.137.6) and (3.131.5) in the handbook [22], respectively.

The first term in Equation (2.4) was divided into 2 similar integrals using the mathematical identity

$$\begin{aligned} \frac{1}{(1 + a^2 \lambda w)(1 - w)} &= \frac{1}{a^2 \lambda \left((a^2 \lambda)^{-1} + w \right) (1 - w)} \\ &= \left[\frac{1}{(a \lambda)^{-1} + w} + \frac{1}{1 - w} \right] \frac{1}{(1 + a^2 \lambda)} \end{aligned} \quad (2.7)$$

After all integrations, the equation of spherical orbits takes on its final form

$$\Delta\varphi = \frac{4}{\sqrt{(w_1 - w_2)(1 + \lambda b)}} \left[\frac{\xi}{a(1 - w_1)} \Pi(n_1, m) + \left(\frac{\lambda a^2 \xi}{1 + a^2} - \frac{1}{1 + \lambda a^2 w_1} \right) \Pi(n_2, m) + \frac{r^2 + a^2 - a\xi}{\Delta_r} K(m) \right] \quad (2.8)$$

where w_1 and w_2 are positive and negative roots of the cubic polynomial (2.5), respectively (the third one is 0), $K(m)$ and $\Pi(n, m)$ are the complete elliptic integrals of the first and third kind, respectively. In addition, designation were introduced:

$$n_1 = \frac{w_1}{1 - w_1}, n_2 = \frac{\lambda a^2 w_1}{1 + \lambda a^2 w_1}, m = \frac{w_1}{w_1 - w_2} \quad (2.9)$$

It is easy to verify that for $\lambda = 0$, the result (2.8) coincides with the one obtained by E. Teo [23].

Figure 6 and **Figure 7** show the results of calculations using Equation (2.8) for the dependence of azimuthal shift $\Delta\varphi(\lambda)$ for prograde ($\Delta\varphi > 0$) trajectories with radii $r = 1.2$ (1), $r = 1.5$ (2) and $r = 2$ (3) (**Figure 6**) and retrograde trajectories ($\Delta\varphi < 0$) with radii $r = 2.5$ (1), $r = 3$ (2) and $r = 3.5$ (3) (**Figure 7**). The azimuthal shift increases with the constant lambda for the former orbits, while for the latter, on the contrary, it decreases in absolute value.

The same results can be seen in **Figure 8**, which shows the dependence of the azimuthal shift for orbital radii between $1.2 \leq r \leq 4.0$ for $\lambda = 0$ (1) and $\lambda = 0.01$ (2). Up to the radius of separation of the directions of rotation $r_s = 1 + \sqrt{2}$ (for $a = 1$), the prograde orbits' azimuthal angles are larger for the curve with a non-zero cosmological constant (2), whereas for retrograde orbits these angles are smaller in absolute values.

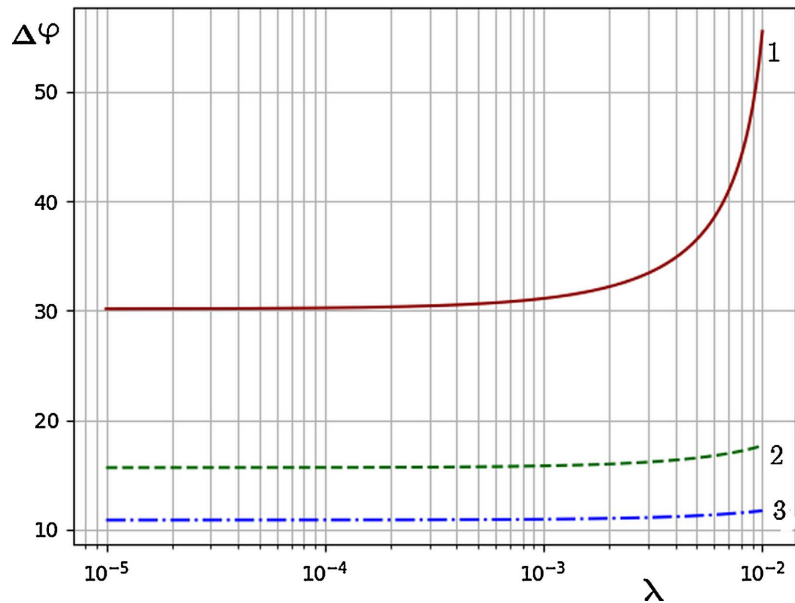


Figure 6. Plot of the dependence of the azimuthal shift $\Delta\varphi$ on the reduced parameter λ for spherical orbit radii $r = 1.2$ (1); 1.5 (2) and 2 (3).

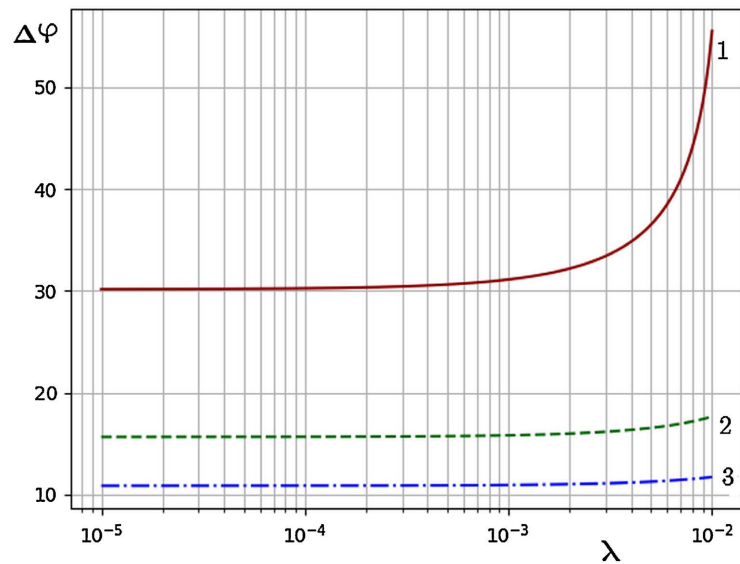


Figure 7. Plot of the dependence of the azimuthal shift $\Delta\varphi$ on the reduced parameter λ for spherical orbit radii $r = 2.5$ (1); 3 (2) and 3.5 (3).

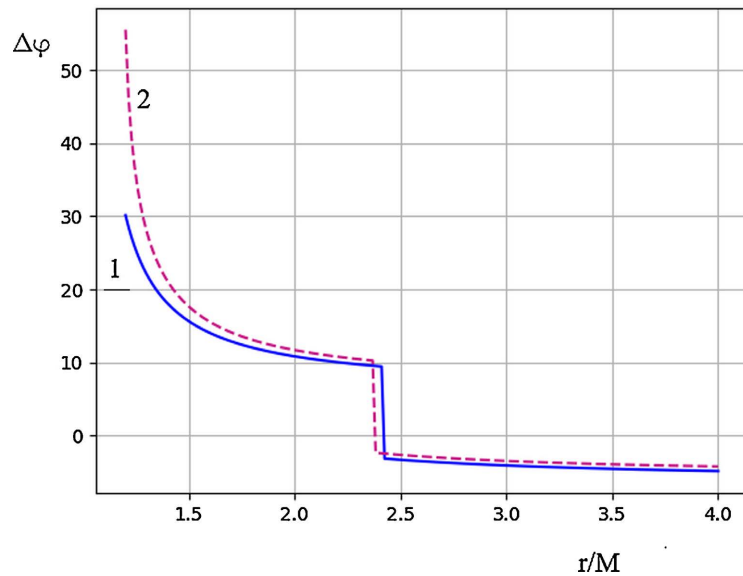


Figure 8. Plot of the dependence of the azimuthal shift $\Delta\varphi$ on the radius of a spherical orbit when $\lambda = 0$ (1) and $\lambda = 0.01$ (2).

4. Conclusions

The immediate results of this theoretical study are summarized as follows:

1) Expressions for the photon angular momentum and the Carter constant in the Kerr-de Sitter metric were obtained, taking into account the presence of the cosmological constant Λ .

2) An equation for the equatorial circular trajectories of photons around Kerr black holes in the presence of a cosmological constant was derived and solved. It was shown to represent a straightforward generalization of the well-known expression corresponding to the case without a cosmological constant.

3) It is found that the presence of the cosmological constant leads to a decrease in the radii of equatorial circular orbits.

4) The equation governing spherical photon trajectories around Kerr-type black holes was solved. The obtained result reduces directly to the expression previously derived by E. Teo in the absence of a cosmological constant.

5) It was shown that, with increasing cosmological constant, the angular shift along spherical photon orbits increases for prograde rotating photons and decreases for retrograde rotating photons.

6) Estimates for the magnitude of the reduced cosmological constant capable of producing observable effects in the orbital photon field were obtained. It's important to emphasize that no conclusions are made regarding the actual physical value of the cosmological constant. Rather, the analysis shows that if the cosmological constant is significantly smaller than the estimated values, its influence on the orbital motion of photons near black holes becomes negligible.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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