Scientific Research Publishing

Journal of High Energy Physics, Gravitation and Cosmology, 2026, 12(3), 1387-1408
https://www.scirp.org/journal/jhepgc
ISSN Online: 2380-4335
ISSN Print: 2380-4327

Mechanical Equations of a Particle Group in a General Reference Frame

Jinyong Liu

Independent Researcher, Zhucheng, China
Email: ljy2002001@163.com

How to cite this paper: Liu, J.Y. (2026) Mechanical Equations of a Particle Group in a General Reference Frame. *Journal of High Energy Physics, Gravitation and Cosmology*, **12**, 1387-1408.
https://doi.org/10.4236/jhepgc.2026.123070

Received: May 18, 2026
Accepted: June 15, 2026
Published: June 18, 2026

Abstract

In the framework of classical noninertial mechanics, the theoretical construction is predicated on the existence of a special reference frame, the "inertial frame", as a priori. Its essence is to extend the dynamical laws in the inertial frame to noninertial frames through coordinate transformation and formal modification, thus inherently relying on the definition and existence of the inertial frame, making it difficult to achieve a fundamental breakthrough in the inertial frame paradigm. This paper proposes a dynamic reconstruction path based on the intrinsic relations within the system, aiming to establish a new system of particle dynamics that is logically self-consistent and physically self-sufficient and does not presuppose an absolute space-time background. On the basis of Newton's laws of motion, in this paper, the equations of particle mechanics applicable to any translational reference frame (*i.e.*, a reference frame with an origin acceleration that can be any function of time) are derived. Through systematic analysis, it is proven that core physical quantities such as force, momentum, and the kinetic energy of a particle group all possess intrinsic properties independent of the choice of reference frame. By further introducing a particle subsystem grouping mechanism, this paper reveals the dynamical essence of internal interactions within the system and its coupling with the external environment, clarifying the physical origin of the energy hierarchy structure (internal kinetic energy and external kinetic energy) and its conservation conditions in general reference frames, thereby providing a solid theoretical foundation and an operational, verifiable mathematical framework for constructing a truly universal noninertial mechanics theory that does not require the introduction of fictitious inertial forces.

Keywords

Inertial Frame, Noninertial Frame, Grouped Mechanics, Internal Momentum, External Momentum, Internal Kinetic Energy, External Kinetic Energy

1. Introduction

The reference frame is a fundamental concept in classical mechanics. According to whether it satisfies Newton's first law (*i.e.*, whether a free particle with zero acceleration exists), it can be strictly divided into inertial and noninertial reference frames. According to the Galilean principle of relativity [1], the form of physical laws should remain unchanged in all inertial frames, and the motion state of an object can be described in any reference frame in principle. However, when this principle is extended to complex systems such as multibody, nonrigid, or strongly coupled systems, its theoretical implementation faces profound structural challenges, especially the tension between the form covariance of dynamical equations and physical interpretability.

In the classical treatment paradigm, the problem of noninertial frame dynamics is usually solved by introducing "inertial forces [2]", that is, by adding virtual force terms in noninertial frames to maintain the form of Newton's second law. This method is essentially a formal mapping—it does not reconstruct the fundamental logic of dynamics but rather "transplants" the physical laws of the inertial frame to accelerate reference frames through coordinate transformation. Under this framework, the total force acting on an object is decomposed into two components: one is the force arising from real physical interactions between substances (referred to as the "body force" in this paper to distinguish it from the inertial force), and the other is the "inertial force" solely caused by the acceleration of the reference frame and without a corresponding material source. If the two are treated equally, then the conservation of momentum, mechanical energy, and Newton's third law will no longer hold generally in noninertial frames; if one attempts to strictly distinguish between them, then it is necessary to establish independent, observable, and repeatable experimental criteria and operational definitions for inertial forces without presupposing an ideal inertial frame (for example, defining a reference frame with inertial force always zero as an inertial frame). From this, it can be clearly seen that the inertial force is essentially the geometric projection of the acceleration of the reference frame in the dynamical equation, lacking a reaction object, not participating in energy exchange, and not changing the total momentum of the system. This physical force originates from the objective interaction between substances, has a definite action-reaction pair, and can effectively change the momentum and energy state of the system. However, the existing theories have not yet provided a universal, observable, and operable criterion system to distinguish the two in any noninertial frame. Therefore, the assertion that the two are essentially different currently lacks sufficient empirical support and logical closure. The classical expansion path centered on "introducing inertial force" has fundamental paradigmatic limitations in constructing a noninertial frame mechanics theory that is truly independent of the inertial frame assumption and has physical self-sufficiency.

This paper is based on the fundamental idea of "generalization" of the reference frame, discarding the prior dependence on the inertial frame and instead taking the relative motion relationship among the particles within the system as the primary physical quantity to construct a new theoretical framework for mechanics. This framework aims to eliminate the conceptual redundancy, logical discontinuity, and physical unobservability caused by inertial forces, laying the foundation for the development of a self-consistent, universal, and physically meaningful mechanical system for noninertial frames. In this paper, such a reference frame that does not presuppose the state of motion and is applicable to any translational situation is defined as a "general reference frame", which is strictly expressed as follows:

Suppose that there is a reference frame S. From the perspective of reference frame S, an ideal inertial frame S' undergoes smooth, nonrotating, and nonjumping arbitrary translational motion relative to frame S (the velocity function of S' relative to S is continuously differentiable); then, reference frame S is defined as a general reference frame.

In mechanical modelling, a particle model is adopted in this paper to describe objects: When the evolution of their spatial positions and dynamic responses is considered, objects can be abstracted as particles with different masses and coordinates. All the interacting particles constitute a closed mechanical system [3]. In the following text, the dynamic behavior of a closed system composed of n particles in a general reference frame S is systematically studied, and its universal motion equations and conservation law expressions are derived.

2. Forces Acting on Particles in a General Reference Frame

Consider a closed mechanical system composed of n particles, with masses $m_1, m_2, \cdots, m_n$. Let S be a general reference frame and S' be an inertial frame, and let the translational velocity of S' relative to S be $\boldsymbol{v}$. Let the velocity of particle m_i in S be $\boldsymbol{v}_i$; then, its velocity in S' can be expressed as $\boldsymbol{v}_i' = \boldsymbol{v}_i - \boldsymbol{v}$. Let the external force acting on particle m_i in S' be $\boldsymbol{F}_i$. According to Newton's law of motion, we have the following:

$$\boldsymbol{F}_i = m_i \frac{\mathrm{d}\boldsymbol{v}_i'}{\mathrm{d}t} = m_i \frac{\mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}\right)}{\mathrm{d}t} \tag{1}$$

For two particles m_i and m_j, applying the above formula gives the following:

$$\frac{1}{m_i}\boldsymbol{F}_i - \frac{1}{m_j}\boldsymbol{F}_j = \frac{\mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right)}{\mathrm{d}t} \tag{2}$$

In the above equation, the translational velocity $\boldsymbol{v}$ of S' relative to S is eliminated. Formula (2) is the relationship between the force and velocity in a general reference frame, where the forces $\boldsymbol{F}_i$, $\boldsymbol{F}_j$, and S' have the same meaning as in the I frame and are true forces without inertial force components. Thus, the concept of force in an inertial frame is unambiguously transplanted to a general reference frame.

From Equation (2), we can obtain the following:

$$\frac{m_j}{m_i}\boldsymbol{F}_i - \boldsymbol{F}_j = m_j \frac{\mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right)}{\mathrm{d}t} \tag{3}$$

Equation (3) can be summarized as follows:

$$\sum_{j=1}^{n}\frac{m_j}{m_i}\boldsymbol{F}_i-\sum_{j=1}^{n}\boldsymbol{F}_j=\sum_{j=1}^{n}m_j\frac{\mathrm{d}\left(\boldsymbol{v}_i-\boldsymbol{v}_j\right)}{\mathrm{d}t} \tag{4}$$

If $\boldsymbol{F}_{ij}$ represents the force of m_j on m_i and $\boldsymbol{F}_{ij}=-\boldsymbol{F}_{ji}$, $\boldsymbol{F}_i=\sum_{j\neq i}\boldsymbol{F}_{ij}$ represents the sum of the forces of all the particles except m_i on m_i; then,

$$\sum_i\boldsymbol{F}_i=\sum_i\sum_{j\neq i}\boldsymbol{F}_{ij}=\sum_i\sum_{j<i}\boldsymbol{F}_{ij}+\sum_i\sum_{j>i}\boldsymbol{F}_{ij}=\sum_i\sum_{j<i}\boldsymbol{F}_{ij}+\sum_i\sum_{j<i}\boldsymbol{F}_{ji}=\sum_i\left(\sum_{j<i}\boldsymbol{F}_{ij}+\boldsymbol{F}_{ji}\right)=0$$

Therefore, we have $\sum_{i=1}^{n}\boldsymbol{F}_i=0$. Substituting into formula (4) and rearranging gives:

$$\boldsymbol{F}_i=\frac{m_i}{\sum_{j=1}^{n}m_j}\sum_{j=1}^{n}m_j\frac{\mathrm{d}\left(\boldsymbol{v}_i-\boldsymbol{v}_j\right)}{\mathrm{d}t} \tag{5}$$

This expression reveals the fundamental relationship between the force acting on a particle and its motion state in a general reference frame, constituting a universal form of Newton's second law in a noninertial reference frame. Whether it is gravitational force, electromagnetic force, or strong and weak interactions, formula (5) can describe the relationship between force and the motion of matter in the system. Further analysis reveals that the force acting on a particle can be uniquely determined by its relative motion state with that of other particles and that the magnitude of this force does not change with the choice of reference frame. This discovery breaks the traditional mechanics' dependence on inertial frames and "inertial forces", achieving theoretical deepening and expansion of the application scope within the framework of Newtonian mechanics.

3. Momentum of Particles in a General Reference Frame

From the relationship between force and momentum, integrating formula (5) gives the momentum of particle m_i as follows:

$$\boldsymbol{P}_i=\int\boldsymbol{F}_i\mathrm{d}t=\frac{m_i}{\sum_{j=1}^{n}m_j}\sum_{j=1}^{n}m_j\left(\boldsymbol{v}_i-\boldsymbol{v}_j\right)+\boldsymbol{C}_i \tag{6}$$

where $\boldsymbol{C}_i$ is the integration constant. Although $\boldsymbol{C}_i$ can take different constants, leading to different values of $\boldsymbol{P}_i$, its physical meaning is independent of the choice of reference frame. For this reason, when all the particles within the system are relatively stationary (*i.e.*, $\boldsymbol{v}_i=\boldsymbol{v}_j$, $i, j=1,2,\cdots,n$), the momentum is zero. On the basis of this condition, $\boldsymbol{C}_i=0$ can be obtained; thus, the momentum of particle m_i is defined as follows:

$$\boldsymbol{P}_i=\frac{m_i}{\sum_{j=1}^{n}m_j}\sum_{j=1}^{n}m_j\left(\boldsymbol{v}_i-\boldsymbol{v}_j\right) \tag{7}$$

Summing this equation yields $\sum_{i=1}^{n}\boldsymbol{P}_i = 0$, which indicates that in a general reference frame, the total momentum of the system satisfies the conservation law.

In contrast, in classical physics, the momentum value in an inertial frame varies with the choice of reference frame, which stems from the incompleteness of the initial condition setting for momentum. A brief analysis is as follows: From the general expression of force in an inertial frame $\boldsymbol{F} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$, integration gives momentum $\boldsymbol{P} = m\boldsymbol{v} + \boldsymbol{C}$. Given the initial condition $\begin{cases}\boldsymbol{v} = \boldsymbol{v}_0\\ \boldsymbol{P} = \boldsymbol{P}_0\end{cases}$, the integration constant momentum $\boldsymbol{C} = \boldsymbol{P}_0 - m\boldsymbol{v}_0$ can be determined, thereby yielding $\boldsymbol{P} = m\left(\boldsymbol{v} - \boldsymbol{v}_0\right) + \boldsymbol{P}_0$. This form of momentum expression is clearly independent of the reference frame. However, the traditional definition of momentum in an inertial frame is $\boldsymbol{P} = m\boldsymbol{v}$, which is actually a special case under specific initial conditions $\begin{cases}\boldsymbol{v}_0 = 0\\ \boldsymbol{P}_0 = 0\end{cases}$. Its more precise general form should be $\boldsymbol{P} = m\left(\boldsymbol{v} - \boldsymbol{v}_0\right)\big|_{\mathbf{v}_0=0}$. Although the two forms are similar, the latter better reflects the essence of momentum as a physical quantity that is independent of a specific reference basis. The different momenta observed for the same object in two different inertial frames are due to the use of different initial velocity benchmarks, and such comparisons have no substantive meaning in physics; the traditional definition obscures this essential property. Therefore, the true physical meaning of momentum lies in its invariance, which is determined by relative motion, rather than the simple product of apparent velocity and mass.

4. Kinetic Energy of a Group of Particles

To solve for the kinetic energy of a group of particles, let the displacement of particle m_i in the reference inertial frame S' be $\boldsymbol{r}_i'$. Since the velocity of particle m_i in frame S' is $\boldsymbol{v}_i' = \boldsymbol{v}_i - \boldsymbol{v}$, then $\mathrm{d}\boldsymbol{r}_i' = \boldsymbol{v}_i'\mathrm{d}t = \left(\boldsymbol{v}_i - \boldsymbol{v}\right)\mathrm{d}t$; thus,

$$\mathrm{d}T = \sum_{i=1}^{n}\boldsymbol{F}_i\mathrm{d}\boldsymbol{r}_i' = \sum_{i=1}^{n}\boldsymbol{F}_i\left(\boldsymbol{v}_i - \boldsymbol{v}\right)\mathrm{d}t = \sum_{i=1}^{n}\boldsymbol{F}_i\boldsymbol{v}_i\mathrm{d}t - \left(\sum_{i=1}^{n}\boldsymbol{F}_i\right)\boldsymbol{v}\mathrm{d}t$$

Because $\sum_{i=1}^{n}\boldsymbol{F}_i = 0$, it follows that:

$$\mathrm{d}T = \sum_{i=1}^{n}\boldsymbol{F}_i\boldsymbol{v}_i\mathrm{d}t \tag{8}$$

Substituting Equation (5) into Equation (8) gives:

$$\mathrm{d}T = \sum_{i=1}^{n}\left(\frac{m_i}{\sum_{j=1}^{n}m_j}\sum_{j=1}^{n}m_j\frac{\mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right)}{\mathrm{d}t}\right)\boldsymbol{v}_i\mathrm{d}t = \frac{1}{\sum_{j=1}^{n}m_j}\sum_{i=1}^{n}\sum_{j=1}^{n}m_im_j\boldsymbol{v}_i\mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right)$$

where

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \boldsymbol{v}_i \mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) = \sum_{j=1}^{n}\sum_{i=1}^{n} m_j m_i \boldsymbol{v}_j \mathrm{d}\left(\boldsymbol{v}_j - \boldsymbol{v}_i\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \boldsymbol{v}_i \mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) + \sum_{j=1}^{n}\sum_{i=1}^{n} m_j m_i \boldsymbol{v}_j \mathrm{d}\left(\boldsymbol{v}_j - \boldsymbol{v}_i\right)\right)$$

that is,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \boldsymbol{v}_i \mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) \mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right)$$

Therefore, we have:

$$dT = \frac{1}{2\sum_{j=1}^{n} m_j}\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) \mathrm{d}\left(\boldsymbol{v}_i - \boldsymbol{v}_j\right) \tag{9}$$

Integrating gives:

$$T = \frac{1}{4\sum_{j=1}^{n} m_j}\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \left(\boldsymbol{v}_i - \boldsymbol{v}_j\right)^2 + C \tag{10}$$

These results clearly indicate that the kinetic energy of a group of particles is completely determined by the relative motion of the particles within the system and that the integration constant C is independent of the choice of reference frame. If the kinetic energy is zero when all the particles within the system are relatively stationary, then $C = 0$ can be obtained. On this basis, the total kinetic energy of the group of particles can be defined as follows:

$$T = \frac{1}{4\sum_{j=1}^{n} m_j}\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \left(\boldsymbol{v}_i - \boldsymbol{v}_j\right)^2 \tag{11}$$

The kinetic energy of a group of particles is consistent with the total external force and total momentum of the particles and is independent of the reference frame. This finding indicates that kinetic energy, as a core physical quantity characterizing the internal energy state of the system, is essentially unaffected by the transformation of the external observation reference frame. In a general reference frame, the kinetic energy is a dynamic attribute of the system as a whole, resulting from the coordinated motion of all the particles and cannot be decomposed or attributed to a single particle. Therefore, the concept of independent kinetic energy does not apply to individual particles.

The above has provided formulas for force, momentum, and kinetic energy in the mechanics of particle group division. The modelling method of particle group division provides a new theoretical path and analytical paradigm for celestial mechanics, enabling the study of the dynamic behavior of complex multibody systems through structured division strategies; especially when the system can be divided into two subsystems, it can be analogized to the classical two-body problem to construct an analytical general solution, significantly enhancing the understanding and solution efficiency of the evolution process of multibody systems.

The mechanical behavior of the particles within each group is analysed below.

5. Mechanics of Particle Group Division

To elucidate the internal structure and dynamic-static behavior of complex systems, the n particles within the closed system are arbitrarily divided into k groups (a single particle can also be considered a special group). Each group is a subsystem, and we study the mechanical relationships between and within the subsystems.

Define the α group ($\alpha = 1, 2, \cdots, k$): This group contains particles with masses $m_{\alpha 1}, m_{\alpha 2}, \cdots, m_{\alpha i}, \cdots$ Several particles. Their velocity vectors in the general reference frame S are $\boldsymbol{v}_{\alpha 1}, \boldsymbol{v}_{\alpha 2}, \cdots, \boldsymbol{v}_{\alpha i}, \cdots$. Let the total mass of the group be defined as $M_\alpha = \sum_i m_{\alpha i}$ and its center of mass velocity be $\boldsymbol{V}_\alpha = \frac{1}{M_\alpha} \sum_i m_{\alpha i} \boldsymbol{v}_{\alpha i}$. On this basis, the fundamental force equation characterized by Equation (5) is applied to all the particles within the entire αth group:

$$\sum_i \boldsymbol{F}_{\alpha i} = \sum_i \left(\frac{m_{\alpha i}}{\sum_{j=1}^{n} m_j} \sum_{j=1}^{n} m_j \frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_j\right)}{\mathrm{d}t} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_i m_{\alpha i} \boldsymbol{v}_{\alpha i} - \sum_i m_{\alpha i} \frac{\sum_{j=1}^{n} m_j \boldsymbol{v}_j}{\sum_{j=1}^{n} m_j} \right) \tag{12}$$

The right side of the equation represents the sum of the resultant forces acting on all the particles within the group. From the definition of the center of mass velocity, we can obtain $\sum_i m_{\alpha i} \boldsymbol{v}_{\alpha i} = M_\alpha \boldsymbol{V}_\alpha$, $\sum_{j=1}^{n} m_j \boldsymbol{v}_j = \sum_{\beta=1}^{k} \sum_i m_{\beta i} \boldsymbol{v}_{\beta i} = \sum_{\beta=1}^{k} M_\beta \boldsymbol{V}_\beta$, and $\sum_{j=1}^{n} m_j = \sum_{\beta=1}^{k} \sum_i m_{\beta i} = \sum_{\beta=1}^{k} M_\beta$. Substituting these into (12) gives:

$$\sum_i \boldsymbol{F}_{\alpha i} = \frac{\mathrm{d}}{\mathrm{d}t} \left(M_\alpha \boldsymbol{V}_\alpha - M_\alpha \frac{\sum_{\beta=1}^{k} M_\beta \boldsymbol{V}_\beta}{\sum_{\beta=1}^{k} M_\beta} \right) = \frac{M_\alpha}{\sum_{\beta=1}^{k} M_\beta} \sum_{\beta=1}^{k} M_\beta \frac{\mathrm{d}\left(\boldsymbol{V}_\alpha - \boldsymbol{V}_\beta\right)}{\mathrm{d}t}$$

$\sum_i \boldsymbol{F}_{\alpha i}$ is the force acting on the αth particle group, denoted as $\boldsymbol{F}_\alpha$. Thus, we obtain the mechanical equation for the αth particle group as follows:

$$\boldsymbol{F}_\alpha = \frac{M_\alpha}{\sum_{\beta=1}^{k} M_\beta} \sum_{\beta=1}^{k} M_\beta \frac{\mathrm{d}\left(\boldsymbol{V}_\alpha - \boldsymbol{V}_\beta\right)}{\mathrm{d}t} \tag{13}$$

Similarly, by summing the momenta of all the particles within the group, we can obtain the momentum equation for the α th particle group as follows:

$$\boldsymbol{P}_\alpha = \sum_i \boldsymbol{P}_{\alpha i} = \frac{M_\alpha}{\sum_{\beta=1}^{k} M_\beta} \sum_{\beta=1}^{k} M_\beta \left(\boldsymbol{V}_\alpha - \boldsymbol{V}_\beta\right) \tag{14}$$

A comparison of (5) and (13) and (7) and (14) reveals that particle groups exhibit similar dynamic characteristics during interactions as individual particles do. On the basis of this characteristic, we can equivalently represent k particle groups as k new "particles" and, using Equation (11), perform a similar derivation for the external expression form of the total kinetic energy of the entire system, obtaining

$$\mathrm{d}T^{(ext)} = \sum_{\alpha=1}^{k} \boldsymbol{F}_\alpha \boldsymbol{V}_\alpha \mathrm{d}t = \sum_{\alpha=1}^{k} \left(\frac{M_\alpha}{\sum_{\beta=1}^{k} M_\beta} \sum_{\beta=1}^{k} M_\beta \frac{\mathrm{d}\left(\boldsymbol{V}_\alpha - \boldsymbol{V}_\beta\right)}{\mathrm{d}t} \right) \boldsymbol{V}_\alpha \mathrm{d}t$$

and ultimately:

$$T^{(ext)} = \frac{1}{4\sum_{\beta=1}^{k} M_\beta} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} M_\alpha M_\beta \left(\boldsymbol{V}_\alpha - \boldsymbol{V}_\beta\right)^2 \tag{15}$$

Equation (15) represents the sum of the macroscopic kinetic energy carried by the motion of all group centers of mass.

In this way, we obtain the force, momentum, and kinetic energy formulas for particle group partitioning mechanics. The particle group partitioning modelling method provides a new theoretical path and analytical paradigm for celestial mechanics, enabling the study of the dynamic behavior of complex multibody systems through structured partitioning strategies; especially when the system can be divided into two subsystems, it can be analogized to the classical two-body problem to construct an analytical general solution, significantly enhancing the understanding and efficiency of the evolution process of multibody systems.

By analysing a single particle within its "group" framework, we can clearly distinguish the internal and external sources of its force, momentum, and kinetic energy. The mechanical behavior of the particles within each group is analysed below.

5.1. External and Internal Forces

Consider a particle $m_{\alpha i}$ within the group. Its force is given by Equation (5) as follows:

$$\boldsymbol{F}_{\alpha i} = \frac{m_{\alpha i}}{\sum_{j=1}^{n} m_j} \sum_{j=1}^{n} m_j \frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_j\right)}{\mathrm{d}t} \tag{16}$$

On the other hand, the total external force on the entire group is given by Equation (13). Thus, from Equations (13) and (16), we have the following:

$$\boldsymbol{F}_{\alpha i} - \frac{m_{\alpha i}}{M_\alpha} \boldsymbol{F}_\alpha = \frac{m_{\alpha i}}{\sum_{j=1}^{n} m_j} \sum_{j=1}^{n} m_j \frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_j\right)}{\mathrm{d}t} - \frac{m_{\alpha i}}{\sum_{\beta=1}^{k} m_\beta} \sum_{\beta=1}^{k} M_\beta \frac{\mathrm{d}\left(\boldsymbol{V}_\alpha - \boldsymbol{V}_\beta\right)}{\mathrm{d}t}$$

Since $\sum_{\beta=1}^{k} M_\beta = \sum_{j=1}^{n} m_j$, the above equation can be transformed into:

$$\boldsymbol{F}_{\alpha i}-\frac{m_{\alpha i}}{M_\alpha}\boldsymbol{F}_\alpha=\frac{m_{\alpha i}}{\sum_{j=1}^{n}m_j}\left(\sum_{j=1}^{n}m_j\frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_j\right)}{\mathrm{d}t}-\sum_{\beta=1}^{k}M_\beta\frac{\mathrm{d}\left(\boldsymbol{V}_\alpha-\boldsymbol{V}_\beta\right)}{\mathrm{d}t}\right)\tag{17}$$

Because

$$\begin{aligned}&\sum_{j=1}^{n}m_j\frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_j\right)}{\mathrm{d}t}-\sum_{\beta=1}^{k}M_\beta\frac{\mathrm{d}\left(\boldsymbol{V}_\alpha-\boldsymbol{V}_\beta\right)}{\mathrm{d}t}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{n}m_j\boldsymbol{v}_{\alpha i}-\sum_{j=1}^{n}m_j\boldsymbol{v}_j-\sum_{\beta=1}^{k}M_\beta\boldsymbol{V}_\alpha+\sum_{\beta=1}^{k}M_\beta\boldsymbol{V}_\beta\right)\end{aligned}$$

where

$$\sum_{j=1}^{n}m_j\boldsymbol{v}_{\alpha i}-\sum_{\beta=1}^{k}M_\beta\boldsymbol{V}_\alpha=\boldsymbol{v}_{\alpha i}\sum_{j=1}^{n}m_j-\boldsymbol{V}_\alpha\sum_{j=1}^{n}m_j=\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{V}_\alpha\right)\sum_{j=1}^{n}m_j$$

$$-\sum_{j=1}^{n}m_j\boldsymbol{v}_j+\sum_{\beta=1}^{k}M_\beta\boldsymbol{V}_\beta=-\sum_{j=1}^{n}m_j\boldsymbol{v}_j+\sum_{j=1}^{n}m_j\boldsymbol{v}_j=0$$

Substituting the above results into Equation (17) gives:

$$\boldsymbol{F}_{\alpha i}-\frac{m_{\alpha i}}{M_\alpha}\boldsymbol{F}_\alpha=m_{\alpha i}\frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{V}_\alpha\right)}{\mathrm{d}t}\tag{18}$$

Substituting $\boldsymbol{V}_\alpha$ with $\boldsymbol{V}_\alpha=\frac{\sum_j m_{\alpha j}\boldsymbol{v}_{\alpha j}}{\sum_j m_{\alpha j}}$ in Equation (18) gives:

$$\boldsymbol{F}_{\alpha i}-\frac{m_{\alpha i}}{M_\alpha}\boldsymbol{F}_\alpha=m_{\alpha i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}_{\alpha i}-\frac{\sum_j m_{\alpha j}\boldsymbol{v}_{\alpha j}}{\sum_j m_{\alpha j}}\right)=\frac{m_{\alpha i}}{\sum_j m_{\alpha j}}\sum_j m_{\alpha j}\frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)}{\mathrm{d}t}$$

Thus, we obtain:

$$\boldsymbol{F}_{\alpha i}=\frac{m_{\alpha i}}{M_\alpha}\boldsymbol{F}_\alpha+\frac{m_{\alpha i}}{\sum_j m_{\alpha j}}\sum_j m_{\alpha j}\frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)}{\mathrm{d}t}\tag{19}$$

The above equation indicates that the resultant force $\boldsymbol{F}_{\alpha i}$ acting on particle $m_{\alpha i}$ can be clearly decomposed into two parts:

1) The external force component $\boldsymbol{F}_{\alpha i}^{(ext)}=\frac{m_{\alpha i}}{M_\alpha}\boldsymbol{F}_\alpha$ is outside the group. This term is proportional to the share of the particle in the total mass of the group. It represents how the external influence on the entire group is "distributed" to each internal particle according to "mass allocation".

2) The internal force $\boldsymbol{F}_{\alpha i}^{(int)}=\frac{m_{\alpha i}}{\sum_j m_{\alpha j}}\sum_j m_{\alpha j}\frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)}{\mathrm{d}t}$ from within the group. This term is entirely determined by the interaction forces between the particle and other particles within the group.

Therefore, the resultant force acting on any particle within a particle group is the vector sum of its external and internal forces. This decomposition method conforms to the principle of force superposition and provides a clear physical basis for distinguishing between internal and external action mechanisms within the system.

5.2. External and Internal Momenta

According to the superposition of momenta, the momentum of a particle can also be naturally decomposed. Analogous to the decomposition of external and internal forces, its momentum expression $\boldsymbol{P}_{\alpha i} = \frac{m_{\alpha i}}{\sum_{j=1}^{n} m_j} \sum_{j=1}^{n} m_j \left(\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_j \right)$ can be written as follows:

$$\boldsymbol{P}_{\alpha i} = \frac{m_{\alpha i}}{M_\alpha} \boldsymbol{P}_\alpha + \frac{m_{\alpha i}}{\sum_j m_{\alpha j}} \sum_j m_{\alpha j} \left(\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha j} \right) \tag{20}$$

Thus, we define the following:

1) External momentum $\boldsymbol{P}_{\alpha i}^{(ext)} = \frac{m_{\alpha i}}{M_\alpha} \boldsymbol{P}_\alpha$ represents the momentum share carried by particle $m_{\alpha i}$ because it "belongs to the group" and moves with the center of mass as a whole.

2) Internal momentum $\boldsymbol{P}_{\alpha i}^{(int)} = \frac{m_{\alpha i}}{\sum_j m_{\alpha j}} \sum_j m_{\alpha j} \left(\boldsymbol{v}_{\alpha i} - \boldsymbol{v}_{\alpha j} \right)$: the momentum obtained from the relative motion between particle $m_{\alpha i}$ and other particles within the group.

Total momentum $\boldsymbol{P}_{\alpha i} = \boldsymbol{P}_{\alpha i}^{(ext)} + \boldsymbol{P}_{\alpha i}^{(int)}$. In summary, the total momentum of the group is $\boldsymbol{P}_\alpha = \sum_i \boldsymbol{P}_{\alpha i}^{(ext)}$ (the sum of the internal momenta is zero).

5.3. External and Internal Kinetic Energies and Their Hierarchical Relationships

Compared with vectorial forces and momentum, kinetic energy, as a scalar quantity, offers richer connotations in its decomposition. When the relative motion and interactions among particles within a particle group are neglected and the entire group is abstracted as an equivalent particle, the kinetic energy possessed by this equivalent particle during interactions with other particles or groups is defined as the **external kinetic energy** of the particle group. From Equation (15), the external kinetic energy of the system after particle grouping is given by:

$$T^{(ext)} = \frac{1}{4\sum_{\beta=1}^{k} M_\beta} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} M_\alpha M_\beta \left(\boldsymbol{V}_\alpha - \boldsymbol{V}_\beta \right)^2 \tag{21}$$

Evidently, the system's external kinetic energy expressed in Equation (21) differs from the total kinetic energy in Equation (11), indicating that a portion of the kinetic energy remains confined within each particle group. To address this, the relationship between the two quantities requires further formalization.

Starting from the differential of total kinetic energy $\mathrm{d}T=\sum_{i=1}^{n}\boldsymbol{F}_i\boldsymbol{v}_i\mathrm{d}t$, we derive the following:

$$\begin{aligned}\mathrm{d}T&=\sum_{\alpha=1}^{k}\sum_{i}\boldsymbol{F}_{\alpha i}\boldsymbol{v}_{\alpha i}\mathrm{d}t=\sum_{\alpha=1}^{k}\sum_{i}\left(\boldsymbol{F}_{\alpha i}^{(ext)}+\boldsymbol{F}_{\alpha i}^{(int)}\right)\boldsymbol{v}_{\alpha i}\mathrm{d}t\\&=\sum_{\alpha=1}^{k}\sum_{i}\boldsymbol{F}_{\alpha i}^{(ext)}\boldsymbol{v}_{\alpha i}\mathrm{d}t+\sum_{\alpha=1}^{k}\sum_{i}\boldsymbol{F}_{\alpha i}^{(int)}\boldsymbol{v}_{\alpha i}\mathrm{d}t\end{aligned}\tag{22}$$

where

$$\sum_{\alpha=1}^{k}\sum_{i}\boldsymbol{F}_{\alpha i}^{(ext)}\boldsymbol{v}_{\alpha i}\mathrm{d}t=\sum_{\alpha=1}^{k}\sum_{i}\left(\frac{m_{\alpha i}}{M_\alpha}\boldsymbol{F}_\alpha\right)\boldsymbol{v}_{\alpha i}\mathrm{d}t=\sum_{\alpha=1}^{k}\boldsymbol{F}_\alpha\frac{\sum_i m_{\alpha i}\boldsymbol{v}_{\alpha i}}{M_\alpha}\mathrm{d}t=\sum_{\alpha=1}^{k}\boldsymbol{F}_\alpha\boldsymbol{V}_\alpha\mathrm{d}t=\mathrm{d}T^{(ext)}$$

is the differential of the kinetic energy generated by the external forces acting on the particle group, and its corresponding external kinetic energy is given by Equation (21). $\mathrm{d}T^{(int)}=\sum_{\alpha=1}^{k}\sum_{i}\boldsymbol{F}_{\alpha i}^{(int)}\boldsymbol{v}_{\alpha i}\mathrm{d}t$ is the sum of the differentials of the kinetic energy generated by the internal forces acting within each particle group. We call this the internal kinetic energy of the system, which is the sum of the internal kinetic energies of each particle group $T^{(int)}=\sum_{\alpha=1}^{k}T_\alpha^{(int)}$; thus, $\mathrm{d}T_\alpha^{(int)}=\sum_i\boldsymbol{F}_{\alpha i}^{(int)}\boldsymbol{v}_{\alpha i}\mathrm{d}t$. Substituting the formula for internal forces gives:

$$\begin{aligned}\mathrm{d}T_\alpha^{(int)}&=\sum_i\left(\frac{m_{\alpha i}}{\sum_j m_{\alpha j}}\sum_j m_{\alpha j}\frac{\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)}{\mathrm{d}t}\right)\boldsymbol{v}_{\alpha i}\mathrm{d}t\\&=\frac{1}{\sum_j m_{\alpha j}}\sum_i\sum_j m_{\alpha i}m_{\alpha j}\boldsymbol{v}_{\alpha i}\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)\end{aligned}$$

Substituting $\sum_i\sum_j m_{\alpha i}m_{\alpha j}\boldsymbol{v}_{\alpha i}\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)=\frac{1}{2}\sum_i\sum_j m_{\alpha i}m_{\alpha j}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)$ for $\sum_i\sum_j m_{\alpha i}m_{\alpha j}\boldsymbol{v}_{\alpha i}\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)$ in the above equation yields:

$$\mathrm{d}T_\alpha^{(int)}=\frac{1}{2\sum_j m_{\alpha j}}\sum_i\sum_j m_{\alpha i}m_{\alpha j}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)\mathrm{d}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)\tag{23}$$

After integration, we obtain

$$T_\alpha^{(int)}=\frac{1}{4\sum_j m_{\alpha j}}\sum_i\sum_j m_{\alpha i}m_{\alpha j}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)^2+C_\alpha$$

Considering that $T_\alpha^{(int)}=0$ holds when the particles within the αth particle group are relatively stationary, $C_\alpha=0$ can be derived. Therefore, the expression for the internal kinetic energy of the αth particle group is:

$$T_\alpha^{(int)}=\frac{1}{4\sum_j m_{\alpha j}}\sum_i\sum_j m_{\alpha i}m_{\alpha j}\left(\boldsymbol{v}_{\alpha i}-\boldsymbol{v}_{\alpha j}\right)^2\tag{24}$$

The total kinetic energy of a closed particle group is equal to the sum of the internal kinetic energy of its subsystems and the external kinetic energy of the system as a whole; that is,

$$T=\sum_{\alpha=1}^{k}T_{\alpha}^{(int)}+T^{(ext)} \tag{25}$$

Equation (25) is a profound physical statement: the total kinetic energy of a closed system is the superposition of the "overall translational kinetic energy" (external kinetic energy) of each substructure (*i.e.*, "particle group") and the "relative motion kinetic energy" (internal kinetic energy) of all the particles within these substructures.

The theory of internal and external kinetic energy of an object reveals the hierarchical structure of energy: when an object participates in external interactions as a whole, its macroscopic motion manifests as external kinetic energy, whereas as a composite system composed of microscopic particles, its internal motion is expressed as internal kinetic energy. External kinetic energy is caused mainly by work carried out by external forces, whereas internal kinetic energy stems from the action of internal forces and internal relative motion. When an external force is applied to a nonrigid object, it often simultaneously triggers adjustments in its internal structure, thereby leading to changes in internal kinetic energy. The divisibility of matter determines that energy has a multilevel organizational form. Many experiments have shown that between the internal and external kinetic energy of an object, a reduction in one form of energy is often accompanied by an increase in the other form of energy, and vice versa; this finding indicates that energy at different levels can be mutually converted and maintain the conservation of total energy under specific conditions.

The "layered" view of energy provides a unified theoretical framework for understanding energy storage and transfer methods from macroscopic celestial bodies and nonrigid mechanical structures to microscopic multiparticle systems. This finding reveals that the internal energy and macroscopic kinetic energy of an object are not isolated but exist in a hierarchical structure that can be mutually converted.

6. Returning to the Classical World—Physical Interpretation of Inertial Frames

In the previous sections, the dynamic laws of particles and particle groups in general reference frames have been systematically expounded. Next, we examine an important physical limit case: when the mass of a certain particle m_k in a particle group is much greater than the sum of the masses of all other particles, that is, $m_k \gg \sum_{j\neq k} m_j$. At this time, the system has an approximately "absolute" background reference frame.

6.1. Limit Form of the Force Equation

Equation (5) can then be transformed into:

$$\boldsymbol{F}_i=\frac{m_i}{\sum\limits_{j\neq k}m_j+m_k}\sum_{j\neq k}m_j\frac{\mathrm{d}\left(\boldsymbol{v}_i-\boldsymbol{v}_j\right)}{\mathrm{d}t}+\frac{m_i m_k}{\sum\limits_{j\neq k}m_j+m_k}\frac{\mathrm{d}\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)}{\mathrm{d}t}$$

For $m_k\gg\sum\limits_{j\neq k}m_j$, $\lambda=\sum\limits_{j\neq k}m_j/m_k\to 0$, so

$$\lim_{\lambda\to 0}\frac{m_i}{\sum\limits_{j\neq k}m_j+m_k}\sum_{j\neq k}m_j\frac{\mathrm{d}\left(\boldsymbol{v}_i-\boldsymbol{v}_j\right)}{\mathrm{d}t}=\lim_{\lambda\to 0}\frac{\lambda m_i}{\lambda+1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}_i-\frac{\sum\limits_{j\neq k}m_j\boldsymbol{v}_j}{\sum\limits_{j\neq k}m_j}\right)=0$$

$$\lim_{\lambda\to 0}\frac{m_i m_k}{\sum\limits_{j\neq k}m_j+m_k}\frac{\mathrm{d}\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)}{\mathrm{d}t}=\lim_{\lambda\to 0}\frac{m_i}{\lambda+1}\frac{\mathrm{d}\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)}{\mathrm{d}t}=m_i\frac{\mathrm{d}\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)}{\mathrm{d}t}$$

Thus, there is a limit:

$$\boldsymbol{F}_i=m_i\frac{\mathrm{d}\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)}{\mathrm{d}t}\quad\left(\lambda=\sum_{j\neq k}m_j/m_k\to 0\right)\tag{26}$$

This is the same as the standard form of Newton's second law in an inertial frame.

6.2. Limit Form of the Momentum Equation

Taking the limit of Equation (7) with respect to $\lambda=\sum\limits_{j\neq k}m_j/m_k\to 0$, similar to the derivation of the limit of force, we obtain the following:

$$\boldsymbol{P}_i=m_i\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)\quad\left(\lambda=\sum_{j\neq k}m_j/m_k\to 0\right)\tag{27}$$

This limit form is also the same as the expression of momentum for a particle in an inertial frame.

6.3. Limit Form of the Kinetic Energy Equation

Transforming the kinetic energy expression (11) into the form of

$$T=\frac{1}{4\sum\limits_{j=1}^{n}m_j}\sum_{\substack{i=1\\ i\neq k}}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n}m_i m_j\left(\boldsymbol{v}_i-\boldsymbol{v}_j\right)^2+\frac{m_k}{4\sum\limits_{j=1}^{n}m_j}\left(\sum_{i=1}^{n}m_i\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)^2+\sum_{j=1}^{n}m_j\left(\boldsymbol{v}_k-\boldsymbol{v}_j\right)^2\right)$$

Because $\sum\limits_{j=1}^{n}m_j\left(\boldsymbol{v}_k-\boldsymbol{v}_j\right)^2=\sum\limits_{i=1}^{n}m_i\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)^2$, the above equation becomes:

$$T=\frac{1}{4\sum\limits_{j=1}^{n}m_j}\sum_{\substack{i=1\\ i\neq k}}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n}m_i m_j\left(\boldsymbol{v}_i-\boldsymbol{v}_j\right)^2+\frac{m_k}{2\sum\limits_{j=1}^{n}m_j}\sum_{i=1}^{n}m_i\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)^2$$

Take the limit of $\lambda=\sum\limits_{j\neq k}m_j/m_k\to 0$ in the above equation, where

$$0\le\lim_{\lambda\to 0}\frac{1}{4\sum\limits_{j=1}^{n}m_j}\sum_{\substack{i=1\\ i\neq k}}^{n}\sum_{\substack{j=1\\ j\neq k}}^{n}m_i m_j\left(\boldsymbol{v}_i-\boldsymbol{v}_j\right)^2\le\lim_{\lambda\to 0}\frac{A^2\left(\sum\limits_{i=1,i\neq k}^{n}m_i\right)\left(\sum\limits_{j=1,j\neq k}^{n}m_j\right)}{4\left(\sum\limits_{j=1,j\neq k}^{n}m_j+m_k\right)}=\lim_{\lambda\to 0}\frac{\lambda A^2\sum\limits_{i=1,i\neq k}^{n}m_i}{4\left(\lambda+1\right)}=0$$

(where $A=\max\left\{\left|\boldsymbol{v}_i-\boldsymbol{v}_j\right|\right\}$)

$$\lim_{\lambda\to 0}\frac{m_k}{2\sum_{j=1}^{n}m_j}\sum_{i=1}^{n}m_i\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)^2=\lim_{\lambda\to 0}\frac{1}{2(\lambda+1)}\sum_{i=1}^{n}m_i\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)^2=\frac{1}{2}\sum_{i=1}^{n}m_i\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)^2$$

So,

$$T=\frac{1}{2}\sum_{i=1}^{n}m_i\left(\boldsymbol{v}_i-\boldsymbol{v}_k\right)^2\quad\left(\lambda=\sum_{j\neq k}m_j/m_k\to 0\right)\tag{28}$$

This limit also has the same form as the expression of the kinetic energy of a particle in an inertial frame. Through the above research, our main conclusion is that under the condition of “there being a central body with an absolutely dominant mass”, the mechanical equations of other particles in a general reference frame relative to this central body naturally converge to the standard equations in the inertial frame of Newtonian classical mechanics in the limit form. This indicates that a celestial body with a sufficiently large mass (such as the Sun in relation to its planetary system or the Earth in relation to macroscopic objects near its surface) provides a local and approximately ideal inertial reference environment in its vicinity. The classical theory and experiments of inertial frames can be regarded as special cases established under the “infinite mass background approximation” rather than the theoretical starting point. This also explains at a deeper level why classical mechanics has extremely high accuracy at the scale of the solar system, the Earth’s surface, etc.

7. Special “Inertial” Appearance in a General Reference Frame—The Center of the Mass Coordinate System

Now, we explore a special choice of reference frame: the center of mass coordinate system. This is a coordinate system in which the center of mass of the entire particle group is either at rest or moving at a constant velocity in a straight line.

Let the coordinates of the center of mass of the particle group be $\boldsymbol{R}_{CM}=\frac{1}{M}\sum_{i=1}^{n}m_i\boldsymbol{r}_i$ and its velocity be $\boldsymbol{V}_{CM}=\frac{\mathrm{d}\boldsymbol{R}_{CM}}{\mathrm{d}t}$. Afterward, the displacement $\boldsymbol{r}_i'=\boldsymbol{r}_i-\boldsymbol{R}_{CM}$ of the particle m_i in the center of the mass coordinate system is $\sum_{i=1}^{n}m_i\boldsymbol{r}_i'=0$, $\boldsymbol{v}_i'=\boldsymbol{v}_i-\boldsymbol{V}_{CM}$, and $\sum_{i=1}^{n}m_i\boldsymbol{v}_i'=0$.

7.1. Force Equations in the Center of the Mass Coordinate System

From $\boldsymbol{v}_i'=\boldsymbol{v}_i-\boldsymbol{V}_{CM}$, we obtain $\boldsymbol{v}_i-\boldsymbol{v}_j=\boldsymbol{v}_i'-\boldsymbol{v}_j'$, and substituting it into Equation (5) gives:

$$\boldsymbol{F}_i=\frac{m_i}{\sum_{j=1}^{n}m_j}\sum_{j=1}^{n}m_j\frac{\mathrm{d}\left(\boldsymbol{v}_i'-\boldsymbol{v}_j'\right)}{\mathrm{d}t}=m_i\frac{\mathrm{d}\boldsymbol{v}_i'}{\mathrm{d}t}-m_i\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{n}m_j\boldsymbol{v}_j'\Big/\sum_{j=1}^{n}m_j$$

Because $\sum_{j=1}^{n}m_j\boldsymbol{v}_j'=\sum_{i=1}^{n}m_i\boldsymbol{v}_i'=0$, therefore,

$$\boldsymbol{F}_i = m_i \frac{\mathrm{d}\boldsymbol{v}_i'}{\mathrm{d}t} \tag{29}$$

The above equation indicates that the force in the center-of-mass coordinate system has the same form as that in the inertial system; in other words, dynamically, the center-of-mass coordinate system is equivalent to the inertial system.

7.2. Momentum Equation in the Center of the Mass Coordinate System

Following the same principle as the derivation of the force equation, substituting $\boldsymbol{v}_i - \boldsymbol{v}_j = \boldsymbol{v}_i' - \boldsymbol{v}_j'$ and $\sum_{j=1}^{n} m_j \boldsymbol{v}_j' = 0$ into Equation (7) yields

$$\boldsymbol{P}_i = m_i \boldsymbol{v}_i' \tag{30}$$

The momentum equation also demonstrates the equivalence of the center-of-mass coordinate system and the inertial system.

7.3. Kinetic Energy in the Center of the Mass Coordinate System

Substituting $\boldsymbol{v}_i - \boldsymbol{v}_j = \boldsymbol{v}_i' - \boldsymbol{v}_j'$ into Equation (11) gives

$$T = \frac{1}{4\sum_{j=1}^{n} m_j} \sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \left(\boldsymbol{v}_i' - \boldsymbol{v}_j'\right)^2 = \frac{1}{4\sum_{j=1}^{n} m_j} \sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \left(\boldsymbol{v}_i'^2 - 2\boldsymbol{v}_i'\boldsymbol{v}_j' + \boldsymbol{v}_j'^2\right)$$

Among them

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \left(\boldsymbol{v}_i'^2 - 2\boldsymbol{v}_i'\boldsymbol{v}_j' + \boldsymbol{v}_j'^2\right) = \sum_{i=1}^{n} m_i \boldsymbol{v}_i'^2 \sum_{j=1}^{n} m_j - 2\sum_{i=1}^{n} m_i \boldsymbol{v}_i' \sum_{j=1}^{n} m_j \boldsymbol{v}_j' + \sum_{j=1}^{n} m_j \boldsymbol{v}_j'^2 \sum_{i=1}^{n} m_i$$

Substituting $\sum_{j=1}^{n} m_j \boldsymbol{v}_i'^2 \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} m_i \boldsymbol{v}_i'^2 \sum_{j=1}^{n} m_j$ and $\sum_{i=1}^{n} m_i \boldsymbol{v}_i' = 0$ into the above equation, we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \left(\boldsymbol{v}_i'^2 - 2\boldsymbol{v}_i'\boldsymbol{v}_j' + \boldsymbol{v}_j'^2\right) = 2\sum_{i=1}^{n} m_i \boldsymbol{v}_i'^2 \sum_{j=1}^{n} m_j$$

So,

$$T = \frac{1}{4\sum_{j=1}^{n} m_j} \left(2\sum_{i=1}^{n} m_i \boldsymbol{v}_i'^2 \sum_{j=1}^{n} m_j\right) = \frac{1}{2}\sum_{i=1}^{n} m_i \boldsymbol{v}_i'^2$$

The above equation indicates that T is the sum of n independent kinetic energy terms. On this basis, the kinetic energy of the i th particle can be defined as follows:

$$T_i = \frac{1}{2} m_i \boldsymbol{v}_i'^2 \tag{31}$$

The kinetic energy of a particle is completely consistent with the definition of kinetic energy in an inertial frame. This shows that in the center-of-mass coordinate system, the kinetic energy of a particle system can be decomposed into the kinetic energy of individual particles and has physical significance.

In the center-of-mass system, the mechanical equations can automatically be present in the standard form of a classical inertial frame and are unaffected by whether the original system is accelerating as a whole. This finding indicates that an inertial frame is not the theoretical foundation but rather a dynamic manifestation in a specific reference frame. For an isolated system, the center-of-mass system is a convenient description framework that can be defined by the internal mass distribution of the system and has no inertial forces. It is among the physical sources of an ideal inertial frame and is a special normative choice among general reference frames. It is also a powerful tool for analysing isolated or nearly isolated systems.

8. Conclusion and Prospects: More Universal Mechanical Paradigms

The "Mechanics of Particle Groups in General Reference Frames" constructed in this paper, although formally belonging to the category of noninertial system mechanics, systematically differs from classical noninertial system theory in terms of its theoretical origin, basic postulates, deductive logic, and ultimate goal. This work not only completes the formal derivation of key equations but also achieves a substantial increase in the philosophical foundation of mechanics—that is, a deep transformation from presupposing an absolute background to a theoretical paradigm that takes the internal relative motion relations of the system as the only observable basis.

8.1. Theoretical Significance of Paradigm Shift

- **Structural Limitations of Classical Non-Inertial System Theory:** This theory takes the idealized and unoperationalizable "inertial frame" as an unverifiable a priori postulate and relies on introducing "inertial forces" without corresponding physical sources as technical intermediaries to achieve mapping and solving within the inertial frame framework. This approach has two unavoidable theoretical tensions: First, it gives the inertial frame ontological priority, leading to a lack of theoretical self-sufficiency; second, inertial forces have neither independent dynamical origins nor direct observable carriers, making it impossible to obtain a unified and strict expression basis for the conservation of momentum, energy, and Newton's third law in systems containing such forces, exposing the incompleteness of its physical foundation.
- **Construction Logic of General Reference Frame Theory:** This paper establishes the relative motion between all pairs of particles within the system as the primary, directly measurable fundamental physical quantity. Although the derivation process temporarily borrows the language of inertial frames to ensure logical consistency, the core equations ultimately obtained—the fundamental equation of force (5), the momentum Equation (7), the expression for the total kinetic energy of the system (11), and the strictly derived decomposition rules for internal and external forces and the hierarchical structure of internal and

external energy (25)—have completely eliminated the dependence on specific parameters such as the position and velocity of the original inertial frame and do not contain any fictitious forces or virtual displacement terms. Thus, a logically self-consistent, physically self-sufficient, and universal dynamical system that does not require the assumption of an absolute rest reference frame is established.

8.2. Theoretical Deepening of Core Physical Concepts

On the basis of the general reference frame perspective for the systematic reconstruction of mechanical principles, this paper achieves ontological redefinitions of several fundamental physical concepts:

1) Ontological Reinterpretation of Force: Force is clearly defined as a function of the relative acceleration between pairs of particles within the system, an intrinsic physical quantity determined solely by the relative configuration and its temporal evolution between particles. This definition eliminates the dependence on external absolute space or the state of the observer, providing solid and universal theoretical support for its objective reality and reference frame independence.

2) The Relativistic Nature of Momentum: Momentum is no longer interpreted as an attribute of an object relative to an abstract "absolute space" but as the weighted integral of the relative motion between all pairs of particles within the system (Equation (7)). This definition not only naturally satisfies the conservation law in any translational inertial frame but also strictly guarantees its generalization consistency in any accelerating reference frame, thereby eliminating the conceptual ambiguity caused by reference frame selection in classical formulations.

3) Hierarchical Structure of Kinetic Energy: By strictly distinguishing between internal kinetic energy (representing the energy carried by the relative motion within subsystems) and external kinetic energy (representing the energy carried by the motion of the system's center of mass) (Equation (25)), this paper reveals the modular topological structure of energy storage and dissipation in composite systems. This structure provides a theoretical tool with clear physical meaning and computability for quantitatively analysing complex dynamic processes such as nonrigid body deformation dissipation, energy redistribution in flexible robots, multiscale cascades in turbulence, and angular momentum transport in gravitational multibody systems.

8.3. Theoretical Inclusiveness, Explanatory Power, and Application Prospects

1) Strict Inclusion of Classical Mechanics: Through two strictly implementable physical limit processes—the "infinite mass limit" and the "centre of mass reference frame"—classical Newtonian mechanics is perfectly reproduced as a special case within the new framework. This finding indicates that the new framework is not a replacement for existing theories but rather achieves a logical unification and fundamental reconstruction under a broader mathematical structure and more solid physical premises, fundamentally eliminating the absolute dependence of classical theories on an inertial background.

2) Modelling advantages for complex systems: The mechanics of particle groups have transcended the numerical strategy level of traditional multibody problems and have been elevated to a system cognition paradigm with profound physical connotations. It systematically decouples high-dimensional strongly coupled dynamical problems into multilevel subsystem interaction problems with clear physical boundaries, thereby providing a universal analytical paradigm with both physical fidelity and computational feasibility for astrophysics (such as predicting the relaxation timescale of star clusters and simulating angular momentum transfer during galaxy mergers), molecular dynamics (such as the competition mechanism between internal and external kinetic energy in protein folding pathways and quantifying conformational entropy changes induced by ligand binding), and even the modelling of emergent dynamics in complex network systems.

3) Application potential in frontier cross-disciplinary fields

"Mechanics of particle groups in general reference frames" not only constitutes a logically closed theoretical system but also, owing to the ontological universality of its conceptual basis and the structural advantages of its analytical tools, clearly enables paths in multiple frontier technological fields:

a) Precision space dynamics and deep space exploration

- **High-precision orbit modelling in complex gravitational fields:** In missions such as Earth-Moon transfer, long-term residence at Lagrange points, or interplanetary cruising under the perturbation of multiple small celestial bodies, the actual environment of spacecraft significantly deviates from the ideal two-body field. Traditional perturbation methods rely on numerous empirical correction terms and frequent coordinate transformations, which can easily introduce systematic modelling errors. This theory supports modelling the main gravitational body, perturbation sources (including the moon, planets, asteroid groups, and even the equivalent particle of solar radiation pressure), and the spacecraft itself as an independent particle group; analysing the energy-momentum coupling mechanism between the intergroup external dynamics (orbital evolution) and intragroup internal dynamics (attitude manoeuvre, propulsion dissipation, thermal deformation), submeter orbit prediction accuracy, probabilistic assessment of collision risks, and global energy-optimal solutions for multipulse manoeuvres can be achieved.
- **Cooperative control of spacecraft clusters:** The entire formation is regarded as the "main group," with its center-of-mass motion dominated by the orbital constraints of the mission (external kinetic energy dominant); each satellite constitutes a functional subgroup within the main group, and issues such as maintaining relative configurations, autonomous obstacle avoidance, and dynamic fuel allocation can all be strictly expressed as the process of subgroups

regulating internal momentum and internal kinetic energy through internal force work. This modelling approach provides verifiable theoretical constraints and performance boundaries for designing distributed controllers with physical consistency, anti-interference robustness, and online adaptability.

b) Advanced robotics and intelligent bionic systems

- **Dynamic modelling of flexible multibody systems:** For systems such as snake-like robots, continuum manipulators, and soft actuators that have distributed elasticity, viscous dissipation, and active drive coupling, traditional rigid body models cannot describe the real-time redistribution of energy between macroscopic motion and microscopic vibration modes. This theory supports the natural division of the system into several physical subgroups (such as joint segments, fluid chambers, and flexible segments) on the basis of structural function and the strength of dynamical coupling; the dynamics of each subgroup are not only constrained by the motion of the upper level (external force input) but also determined by its internal elastic potential energy, damping dissipation, and electromechanical conversion (internal force action). On the basis of the hierarchical analysis of internal and external kinetic energy, the dominant flow paths of energy between rigid motion, elastic energy storage, viscous dissipation, and electromechanical feedback channels can be quantitatively identified, thereby providing a quantifiable theoretical basis for the design of compliant control laws, wideband vibration suppression strategies, and cyclic energy recovery architectures.
- **Modelling of emergent behaviors in multiagent clusters:** Systems such as drone swarms, autonomous vehicle fleets, or collections of molecular motors can all be formalized as multiparty dynamical objects. By applying the framework of group mechanics, it is possible to strictly model how local interaction rules among individuals (internal force model) can collaboratively guide the overall system toward an energy-optimal state (minimization of the sum of external and internal kinetic energy) under macroscopic task constraints (such as formation geometry maintenance and maximization of area coverage efficiency) or spontaneously form collective motion patterns such as vortices, travelling waves, and synchronous oscillations. This approach not only reproduces the experimental observations of biological swarms but also provides computable theoretical criteria for the robust design and failure-tolerant mechanisms of artificial swarm systems.

c) Cross-scale physical simulation and complex system science

- **Astrophysical and cosmological simulations:** From star clusters and galactic disks to large-scale cosmic structures, the distribution of matter exhibits strict hierarchical nesting features. The recursive grouping strategy of this theory is highly consistent with this: for example, in galactic-scale simulations, the entire galaxy can be set as the first-level group, the spiral arms and halo as the second-level subgroups, and giant molecular clouds or globular clusters as the third-level subgroups. This structure not only reduces the computational complexity of N-body simulations from $O\left(N^2\right)$ to approximately $O\left(N\right)$ but also strictly tracks the cross-scale transport fluxes of angular momentum, kinetic energy, and gravitational potential energy between different scale structures (such as the interface between the stellar disk and the dark matter halo), providing falsifiable diagnostic variables for understanding key evolutionary issues such as the excitation threshold of bar structures and the feedback efficiency of galaxy mergers.

- **Mesoscopic-scale biophysical modelling:** Although Newtonian mechanics fails at the quantum scale, its ideas of "hierarchical interaction" and "internal/external energy partitioning" have profound heuristic value. In molecular dynamics simulations, the entire protein can be regarded as the main group of the system, with its structural domains (α-helices, β-sheets, loop regions) as physical subgroups; when a ligand (external group) binds, in addition to analysing the overall binding free energy change (external kinetic energy level), the amplitude of internal conformational fluctuations, the rate of hydrogen bond network breakage/reformation, and the activation intensity of low-frequency collective modes (redistribution of internal kinetic energy) in specific structural domains can also be quantitatively calculated, thereby establishing a causal link from molecular binding events to biological functional outputs at the mechanical level.

9. Conclusion: Towards a More Fundamental Mechanical Expression

The "mechanics of particle groups in general reference frames" proposed in this paper has academic value far beyond being a tool for solving specific engineering problems. Instead, it represents a systematic reexamination and theoretical elevation of the philosophical foundation of classical mechanics.

- **Fundamental shift in theoretical foundation:** This theory successfully shifts the logical starting point of mechanics from the unverifiable "inertial spacetime background" to the observable, operable, and falsifiable internal relationship structure uniquely determined by the relative motion among all the particles of the system. Together, the core Equations (5), (7), and (11) and the hierarchical energy decomposition formula (25) form a universal dynamic framework that does not rely on fictional physical quantities, is logically closed and covariantly self-consistent, and has clear experimental testability.
- **Dual manifestation of unity and extensibility:** On the one hand, rigorous asymptotic analysis demonstrates that this theory converges precisely to classical Newtonian mechanics (validated by extensive experiments) under "infinite mass limit" and "center-of-mass frame" conditions, confirming Newtonian mechanics as an elegant special case within a specific parameter space. On the other hand, the triple analytical paradigm it establishes—modular (grouping), hierarchical (internal/external), and relational (relative motion)—provides a computable and verifiable meta-theoretical framework for charac-

terizing the multiscale, strongly nonlinear, and open dissipative systems prevalent in modern science (e.g., cell signalling networks, global climate systems, socioeconomic game models, and deep neural network training dynamics). This paradigm reveals that the macroscopic order of complex systems arises fundamentally from the coordinated evolution and energy-information coupling balance of their microscopic components across nested hierarchical levels.

- **Future Research Directions:** This work marks the starting point of theoretical development, with several critical avenues requiring further exploration:

Integration with Relativistic Mechanics: How can the intrinsic principle of relative motion be reconciled with the Lorentz covariance of special relativity and the spacetime geometric perspective of general relativity? Does a tensor formulation satisfying general covariance exist?

Incorporation of dissipative systems and nonequilibrium thermodynamics: How can nonconservative forces and entropy production terms be rigorously introduced into this framework to construct hierarchical energy-entropy coupled dynamical equations for open systems? Is there a correspondence with Onsager's reciprocal relations?

Development of Novel Computational Paradigms: Can multiresolution numerical algorithms based on recursive particle group decomposition be developed to achieve an exponential reduction in computational complexity for large-scale molecular dynamics or cosmological simulations while preserving physical fidelity? Can their convergence and stability be rigorously proven?

In summary, "Mechanics of Particle Groups in General Reference Frames" not only breaks the constraints of the traditional paradigm anchored in a priori inertial frames but also involves the construction of a universal mechanical theory with clear physical intuition, a rigorous logical structure, and applicability spanning from microscopic particles to macroscopic galaxies. More profoundly, it establishes a new paradigm for understanding the dynamic world, where first principles are rooted in the system's internal relations and hierarchical structure. Under this paradigm, core concepts such as force, momentum, and energy are firmly anchored in the evolution of matter's relative configurations, providing a novel conceptual language, verifiable analytical tools, and extensible intellectual resources to address the complex dynamic challenges of contemporary science and technology—including noninertial, multiscale, and strongly coupled systems. This work thus embodies both profound theoretical originality and clear practical feasibility.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Santos, F.C., Soares, V. and Tort, A.C. (2010) A Note on the Conservation of Me-

chanical Energy and the Galilean Principle of Relativity. *European Journal of Physics*, **31**, 827-834. https://doi.org/10.1088/0143-0807/31/4/012

[2] Crida, A., Baruteau, C., Griveaud, P., Lega, E., Masset, F., Béthune, W., *et al.* (2025) On Inertial Forces (Indirect Terms) in Problems with a Central Body. *The Open Journal of Astrophysics*, **8**. https://doi.org/10.33232/001c.141682

[3] Schwertassek, R. and Roberson, R.E. (1984) A State-Space Dynamical Representation for Multibody Mechanical Systems Part II: Systems with Closed Loops. *Acta Mechanica*, **51**, 15-29. https://doi.org/10.1007/bf01176386