

# The Universal Equation: A Closed Quartic Variational Framework for Spectral Emergence

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## Abstract

We introduce a globally closed quartic variational framework constructed exclusively from an internal Psi-Gamma functional without externally imposed geometrical, quantum, gauge, or dynamical sectors. Starting directly from the variational structure, we derive the stationary Euler sector, the associated Hessian operator, the admissible configuration space, the cyclic closure structure, and the global decisional selection functional. The analysis shows that the Hessian admits the normalized decomposition.  $H = E^*T$ , with spectral structure,  $\text{Spec}(T) = \{1, L, L\}$ . The internal organization of the Hessian generates the first two Livolsi constants,  $L = \frac{1}{4}$ ,  $E^* = \text{Tr}(H)$ . Recursive cyclic closure further generates a finite admissible configuration space,  $\dim(W_{\text{phys}}) = 96$ , leading to the third Livolsi constant,  $\nu = \frac{1}{96}$ . The resulting hierarchy induces an intrinsic spectral discretization,  $\Delta E = \frac{E^*}{96}$ . The analysis further proves that: 1) quadratic structures are spectrally degenerate, 2) cubic structures fail to generate stable closure, 3) local interactions violate admissibility, 4) lower cyclic organizations collapse, 5) externally completed extensions violate structural minimality. The quartic Psi-Gamma functional is therefore shown to constitute a structurally unique globally closed variational organization capable of generating interaction  $\rightarrow$  stability  $\rightarrow$  spectrum  $\rightarrow$  closure  $\rightarrow$  selection directly from a single internally closed variational structure without fitting procedures or external assumptions.

## Keywords

Quartic Variational Functional, Hessian Spectrum, Spectral Emergence,  $Z_3$  Cyclic Closure, Admissible Configuration Space, Decisional Structure, Livolsi Constants, Variational Selection, Closed Variational Systems

## 1. Introduction

### 1.1. Fragmentation of Modern Physics

The present analysis extends the closed variational framework developed in previous studies on emergent constants and spectral structures [1]-[3]. Modern theoretical physics is presently constructed through multiple independent formal sectors which are not generated from a single globally closed variational structure.

The two dominant frameworks are General Relativity and Quantum Theory.

General Relativity is constructed from the Einstein-Hilbert action:

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + S_{\text{matter}}.$$

The stationary condition:

$$\delta S_{\text{EH}} = 0$$

produces the Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

The complete structure is therefore built upon the prior existence of a geometrical manifold:

$$(M, g_{\mu\nu}).$$

Geometry is consequently assumed from the beginning of the construction.

Quantum theory is instead formulated through Hilbert-space structures.

Quantum states satisfy:

$$|\psi\rangle \in \mathcal{H},$$

with normalization:

$$\langle\psi|\psi\rangle = 1.$$

Time evolution is introduced through the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle.$$

Relativistic quantum field theories further introduce fields defined on externally imposed spacetime backgrounds.

A generic quantum field action takes the form:

$$S_{\text{QFT}} = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi).$$

Gauge theories additionally require independently imposed symmetry sectors:

$$SU(3) \times SU(2) \times U(1).$$

The corresponding gauge fields are introduced through external connections:

$$A_\mu^a,$$

with field strength tensor:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c.$$

The Yang-Mills action is therefore:

$$S_{\text{YM}} = -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu}.$$

The quantum framework consequently presupposes:

- Hilbert-space structure,
- operator algebra,
- canonical quantization rules,
- externally imposed gauge sectors,
- background spacetime geometry.

Attempts to unify these sectors generally introduce additional external structures.

Perturbative quantum gravity expands the metric as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

leading to loop expansions requiring infinitely many counterterms.

String theory replaces point-like objects by extended one-dimensional structures:

$$X^\mu(\sigma, \tau),$$

governed by the Polyakov action:

$$S_p = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu.$$

Consistency requires additional dimensions:

$$D = 10,$$

together with compactification sectors and landscape degeneracies.

Loop Quantum Gravity introduces canonical quantum geometrical variables and spin-network states:

$$|\Gamma, j_e, l_v\rangle,$$

requiring independent quantization prescriptions.

The resulting situation is structurally fragmented.

General Relativity requires:

$$(M, g_{\mu\nu}),$$

Quantum Theory requires:

$$(\mathcal{H}, \hat{O}),$$

Gauge theories require:

$$SU(3) \times SU(2) \times U(1),$$

while string constructions require:

$$D > 4.$$

These structures are introduced independently and are not generated from a

single globally closed variational organization.

The absence of a globally closed variational structure also implies the absence of an internally generated spectral hierarchy.

In conventional field theories, spectral structures are inserted only after independent assumptions.

Mass sectors are externally introduced through terms such as:

$$\mathcal{L}_m = -m\bar{\psi}\psi,$$

while gauge sectors are independently imposed.

A globally coherent variational framework instead requires that:

interaction  $\rightarrow$  stability  $\rightarrow$  spectrum  $\rightarrow$  closure  $\rightarrow$  selection

all emerge directly from the same internal variational structure.

This requires:

- internally generated Hessian organization,
- intrinsic spectral stability,
- non-degenerate admissible sectors,
- recursive cyclic closure,
- global selection consistency.

Consider a general variational functional:

$$S[\Psi].$$

Stationary configurations satisfy:

$$\frac{\delta S}{\delta \Psi^\dagger} = 0.$$

Expanding around a stationary configuration:

$$\Psi = \Psi^{(0)} + \delta\Psi,$$

the second variation defines the Hessian operator:

$$H = \frac{\delta^2 S}{\delta \Psi^\dagger \delta \Psi}.$$

The spectral structure is therefore determined through:

$$Hv_n = \lambda_n v_n.$$

A globally admissible organization requires:

$$0 < \lambda_n < \infty,$$

ensuring bounded spectral stability.

A minimal realization of such an organization requires a quartic variational structure capable of generating simultaneously:

- non-linear self-coupling,
- non-trivial Hessian rank,
- cyclic closure,
- finite admissible spectral hierarchy.

The quartic Psi-Gamma functional introduced in the present work is con-

structured precisely to satisfy these conditions.

Its internal organization generates:

- spectral rigidity,
- the Livolsi constants

$$L = \frac{1}{4}, \quad E^*, \quad \nu = \frac{1}{96},$$

- recursive cyclic closure,
- admissible coherent sectors,
- global decisional selection dynamics

directly from the variational structure itself without externally imposed geometrical sectors, phenomenological fitting procedures, or auxiliary dynamical assumptions.

The fragmentation problem of modern theoretical physics therefore reduces to the absence of a globally closed variational structure capable of internally generating:

$$\text{interaction} \rightarrow \text{stability} \rightarrow \text{spectrum} \rightarrow \text{closure} \rightarrow \text{selection}.$$

## 1.2. Need for a Closed Variational Structure

A globally coherent physical framework requires that all admissible structures emerge from a single internally closed variational organization.

Consider a general variational functional:

$$S[\Psi].$$

The admissible configurations are determined through the stationary condition:

$$\frac{\delta S}{\delta \Psi^\dagger} = 0.$$

The complete organization of the theory must therefore emerge exclusively from:

$$S[\Psi] \rightarrow \frac{\delta S}{\delta \Psi^\dagger} = 0.$$

No independent geometrical, algebraic, or phenomenological sectors may be introduced externally.

A closed variational structure requires internal closure.

Let:

$$\mathcal{O} = \{\mathcal{O}_\alpha\}$$

denote the set of operators appearing in the theory.

Internal closure requires:

$$\forall \mathcal{O}_\alpha \in \mathcal{O}, \quad \mathcal{O}_\alpha = \mathcal{O}_\alpha(\Psi, \Psi^\dagger).$$

No external object may appear independently:

$$\nexists \chi \notin \{\Psi, \Psi^\dagger\}$$

such that:

$$S = S(\Psi, \Psi^\dagger, \chi).$$

The framework must therefore satisfy autosufficiency:

$$S = S(\Psi, \Psi^\dagger).$$

The second requirement is spectral stability.

Expanding around stationary configurations:

$$\Psi = \Psi^{(0)} + \delta\Psi,$$

the second variation defines:

$$H = \frac{\delta^2 S}{\delta\Psi^\dagger \delta\Psi}.$$

The spectral structure is determined through:

$$H v_n = \lambda_n v_n.$$

Admissibility requires:

$$0 < \lambda_n < \infty.$$

If:

$$\lambda_n = 0,$$

flat directions emerge and spectral rigidity is lost.

If:

$$\lambda_n < 0,$$

the stationary configuration becomes unstable.

A globally coherent structure therefore requires:

$$\forall n, \quad 0 < \lambda_n < \infty.$$

The third requirement is the existence of an admissible configuration sector.

Define the stationary space:

$$X = \left\{ \Psi \mid \frac{\delta S}{\delta\Psi^\dagger} = 0 \right\}.$$

Not all stationary configurations are admissible.

The admissible sector must additionally satisfy:

- spectral positivity,
- bounded functional energy,
- normalization stability,
- recursive closure consistency.

Define therefore:

$$A \subset X,$$

where:

$$\Psi \in A \Leftrightarrow \begin{cases} 0 < \lambda_n < \infty, \\ S[\Psi] < \infty, \\ \Psi^\dagger \Psi = 1, \\ \text{closure conditions hold.} \end{cases}$$

The framework must therefore generate internally its own admissible sector.

The fourth requirement is cyclic organization.

Purely diagonal interaction structures produce reducible sectors.

Consider:

$$H = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}.$$

Its spectrum is completely degenerate:

$$\text{Spec}(H) = \{A, A, A\}.$$

No internal hierarchy emerges.

A coherent organization instead requires non-trivial off-diagonal coupling.

Consider:

$$H = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}.$$

The characteristic polynomial becomes:

$$\det(H - \lambda I) = (A - \lambda - B)^2 (A - \lambda + 2B).$$

The spectrum is therefore:

$$\lambda_1 = A + 2B,$$

$$\lambda_2 = \lambda_3 = A - B.$$

The normalized ratio:

$$L = \frac{A - B}{A + 2B}$$

defines the intrinsic spectral hierarchy.

The interaction therefore generates:

- non-trivial spectral structure,
- degeneracy splitting,
- recursive hierarchy,
- cyclic organization.

The cyclic interaction further requires recursive closure.

Define the cyclic operator:

$$\Gamma : (i, j, k) \mapsto (j, k, i).$$

Repeated application gives:

$$\Gamma^2(i, j, k) = (k, i, j),$$

and:

$$\Gamma^3(i, j, k) = (i, j, k).$$

Therefore:

$$\Gamma^3 = I.$$

The interaction consequently generates the minimal cyclic closure:

$$Z_3 = \{I, \Gamma, \Gamma^2\}.$$

The final requirement is the existence of a global selection structure.

The stationary space:

$$X = \left\{ \Psi \mid \frac{\delta S}{\delta \Psi^\dagger} = 0 \right\}$$

generally contains multiple admissible configurations.

A globally coherent framework therefore requires a global selection principle acting on the admissible sector.

Define:

$$\Psi^* = \text{Sel}_{\Psi \in A} C(\Psi \mid \Omega).$$

Selection acts globally on the complete admissible configuration space.

The structure is therefore:

- non-local,
- non-probabilistic,
- non-perturbative,
- globally constrained.

A globally closed variational structure consequently requires the simultaneous emergence of:

interaction  $\rightarrow$  stability  $\rightarrow$  spectrum  $\rightarrow$  closure  $\rightarrow$  admissibility  $\rightarrow$  selection.

The quartic Psi-Gamma functional introduced in the present work is constructed precisely to satisfy these conditions within a single internally closed variational structure.

## 2. The Fundamental Quartic Functional

### 2.1. Exact Definition of the Functional

The starting point of the framework is a globally defined quartic variational functional constructed exclusively from the internal interaction structure of the fields  $\Psi_i$  and the cyclic operator  $\Gamma$ .

The functional is defined as:

$$S[\Psi] = \int d^4x \left\{ \left| \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right|^2 - \lambda_1 \sum_i (\Psi_i^\dagger \Psi_i - 1)^2 - \lambda_2 \sum_{i,j,k} \text{Tr} \left[ \Gamma (\Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger) \right]^2 \right\}.$$

The functional contains three structurally distinct sectors:

- the quartic interaction sector,
- the normalization stabilization sector,
- the cyclic recursive closure sector.

The first contribution is the coherent interaction term:

$$\left| \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right|^2.$$



Define:

$$Q = \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j.$$

The interaction contribution becomes:

$$|Q|^2 = Q^\dagger Q.$$

Since:

$$Q^\dagger = \sum_{m,n} \Psi_n^\dagger \Gamma^\dagger \Psi_m,$$

one obtains:

$$|Q|^2 = \left( \sum_{m,n} \Psi_n^\dagger \Gamma^\dagger \Psi_m \right) \left( \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right).$$

The interaction structure is therefore quartic in the fields.

The second contribution is the normalization sector:

$$-\lambda_1 \sum_i \left( \Psi_i^\dagger \Psi_i - 1 \right)^2.$$

Define:

$$N_i = \Psi_i^\dagger \Psi_i.$$

The normalization contribution becomes:

$$-\lambda_1 \sum_i (N_i - 1)^2.$$

Expanding explicitly:

$$(N_i - 1)^2 = N_i^2 - 2N_i + 1.$$

Thus:

$$-\lambda_1 \sum_i (N_i - 1)^2 = -\lambda_1 \sum_i N_i^2 + 2\lambda_1 \sum_i N_i - \lambda_1 \sum_i 1.$$

Using:

$$N_i^2 = \left( \Psi_i^\dagger \Psi_i \right)^2,$$

the normalization contribution is also quartic.

The normalization sector stabilizes the admissible configurations through:

$$\Psi_i^\dagger \Psi_i = 1.$$

The third contribution is the cyclic recursive closure sector:

$$-\lambda_2 \sum_{i,j,k} \text{Tr} \left[ \Gamma \left( \Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger \right) \right]^2.$$

Define:

$$C_{ijk} = \Gamma \left( \Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger \right).$$

The closure contribution becomes:

$$-\lambda_2 \sum_{i,j,k} \text{Tr}(C_{ijk}^2).$$

Expanding explicitly:

$$C_{ijk}^2 = \Gamma(\Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger) \Gamma(\Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger).$$

Thus:

$$\begin{aligned} C_{ijk}^2 = & \Gamma \Psi_i \Psi_j^\dagger \Gamma \Psi_i \Psi_j^\dagger - \Gamma \Psi_i \Psi_j^\dagger \Gamma \Psi_k \Psi_i^\dagger \\ & - \Gamma \Psi_k \Psi_i^\dagger \Gamma \Psi_i \Psi_j^\dagger + \Gamma \Psi_k \Psi_i^\dagger \Gamma \Psi_k \Psi_i^\dagger. \end{aligned} \quad (1)$$

The closure sector is therefore:

- quartic,
- cyclic,
- non-local,
- recursively coupled.

The complete functional may therefore be decomposed as:

$$S[\Psi] = S_{\text{int}} + S_{\text{norm}} + S_{\text{closure}},$$

with:

$$\begin{aligned} S_{\text{int}} &= \int d^4x \left| \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right|^2, \\ S_{\text{norm}} &= -\lambda_1 \int d^4x \sum_i (\Psi_i^\dagger \Psi_i - 1)^2, \end{aligned}$$

and:

$$S_{\text{closure}} = -\lambda_2 \int d^4x \sum_{i,j,k} \text{Tr} \left[ \Gamma(\Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger) \right]^2.$$

The complete variational organization is therefore:

- quartic,
- globally variational,
- internally coupled,
- spectrally generative,
- recursively closed,
- cyclically organized.

No external geometrical, gauge, probabilistic, or dynamical sectors are introduced.

The complete framework is generated exclusively from the internal organization of the quartic variational functional itself.

## 2.2. Internal Structure of the Functional

The quartic Psi-Gamma functional possesses a tripartite internal organization consisting of:

- the interaction sector,
- the normalization sector,
- the quartic closure sector.

The interaction sector generates coherent non-linear coupling between admissible sectors.

The normalization sector stabilizes the admissible configuration space by preventing arbitrary global rescalings.

Without the normalization contribution:

$$\Psi_i \rightarrow \alpha \Psi_i,$$

would produce:

$$S[\alpha \Psi] \sim \alpha^4,$$

without bounded admissible stabilization.

The normalization contribution therefore generates:

- bounded functional amplitude,
- finite admissible norm,
- spectral stabilization,
- controlled quartic growth.

The quartic closure sector instead generates recursive cyclic organization.

The interaction acts on ordered triples:

$$(i, j, k),$$

through the cyclic operator:

$$\Gamma : (i, j, k) \mapsto (j, k, i).$$

Repeated application gives:

$$\Gamma^2(i, j, k) = (k, i, j),$$

and:

$$\Gamma^3(i, j, k) = (i, j, k).$$

Thus:

$$\Gamma^3 = I.$$

The recursive interaction therefore generates the cyclic group:

$$Z_3 = \{I, \Gamma, \Gamma^2\}.$$

The closure sector consequently produces:

- recursive interaction consistency,
- cyclic closure,
- non-trivial degeneracy splitting,
- internal spectral hierarchy.

The complete variational structure is therefore structurally irreducible.

The interaction sector generates coupling.

The normalization sector generates bounded admissibility.

The closure sector generates recursive cyclic organization.

Removing any one of the three sectors destroys:

interaction  $\rightarrow$  stability  $\rightarrow$  spectrum  $\rightarrow$  closure.

The quartic Psi-Gamma functional therefore constitutes a minimally complete globally coupled variational structure.

### 2.3. Structural Necessity of Quartic Organization

The purpose of the present section is to show explicitly that:

- quadratic structures are spectrally incomplete,
- cubic structures fail to generate stable closure,
- quartic organization is the minimal admissible globally stable structure.

#### 2.3.1. Quadratic Structures

Consider a generic quadratic functional:

$$S_2[\Psi] = \int d^4x \Psi^\dagger K \Psi,$$

where:

$$K = K^\dagger.$$

The stationary equation is:

$$\frac{\delta S_2}{\delta \Psi^\dagger} = K \Psi = 0.$$

The Hessian operator becomes:

$$H_2 = \frac{\delta^2 S_2}{\delta \Psi^\dagger \delta \Psi} = K.$$

The spectrum is therefore externally imposed through:

$$K v_n = \lambda_n v_n.$$

Under global rescaling:

$$\Psi \rightarrow \alpha \Psi,$$

one obtains:

$$S_2[\alpha \Psi] = \alpha^2 S_2[\Psi].$$

Quadratic structures therefore fail to generate:

- normalization stability,
- internal spectral hierarchy,
- recursive closure,
- cyclic organization.

The Hessian remains identical to the original kernel:

$$H_2 = K.$$

No internally generated spectral organization emerges.

Quadratic structures are therefore structurally incomplete.

#### 2.3.2. Cubic Structures

Consider now a cubic functional:

$$S_3[\Psi] = \int d^4x T_{ijk} \Psi_i \Psi_j \Psi_k.$$

The stationary equation becomes:

$$\frac{\delta S_3}{\delta \Psi_a} = 3 \sum_{j,k} T_{ajk} \Psi_j \Psi_k = 0.$$

The Hessian is:

$$(H_3)_{ab} = 6 \sum_k T_{abk} \Psi_k.$$

The Hessian therefore depends linearly on the configuration itself.

Under global inversion:

$$\Psi \rightarrow -\Psi,$$

the Hessian changes sign:

$$H_3 \rightarrow -H_3.$$

The eigenvalues therefore satisfy:

$$\lambda_n \rightarrow -\lambda_n.$$

Stable globally positive admissible sectors cannot exist.

Cubic structures additionally fail to generate recursive cyclic closure.

No finite admissible hierarchy emerges.

Cubic structures therefore fail to generate:

- bounded spectral hierarchy,
- recursive closure,
- stable admissibility,
- finite cyclic organization.

### 2.3.3. Quartic Structures

Consider finally a quartic variational structure:

$$S_4[\Psi] = \int d^4x Q_{ijkl} \Psi_i \Psi_j \Psi_k \Psi_l.$$

Under rescaling:

$$\Psi \rightarrow \alpha \Psi,$$

one obtains:

$$S_4[\alpha \Psi] = \alpha^4 S_4[\Psi].$$

Since:

$$\alpha^4 > 0,$$

the sign of the functional remains invariant under inversion:

$$\Psi \rightarrow -\Psi.$$

Quartic organization therefore admits bounded stabilization.

The stationary equation becomes cubic:

$$\frac{\delta S_4}{\delta \Psi} \sim \Psi^3,$$

while the Hessian becomes quadratic:

$$H_4 \sim \Psi^2.$$

Globally positive admissible sectors may therefore exist.

The quartic interaction additionally generates coherent non-diagonal coupling.

Products of the form:

$$\Psi_i \Psi_j^\dagger$$

generate Hessian matrices of the form:

$$H = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}.$$

The spectrum becomes:

$$\lambda_1 = A + 2B,$$

$$\lambda_2 = \lambda_3 = A - B.$$

The normalized ratio:

$$L = \frac{A - B}{A + 2B}$$

therefore defines an internally generated spectral hierarchy.

Quartic organization consequently generates simultaneously:

- spectral splitting,
- recursive hierarchy,
- cyclic closure,
- admissible stability,
- finite spectral organization.

The quartic structure is therefore the minimal variational organization capable of generating:

$$\text{interaction} \rightarrow \text{stability} \rightarrow \text{spectrum} \rightarrow \text{closure}.$$

### 3. Stationary Variational Structure

#### 3.1. Exact Functional Variation

The stationary sector of the framework is obtained through the explicit variation of the quartic Psi-Gamma functional with respect to the conjugate fields  $\Psi_a^\dagger$ .

Starting from:

$$S[\Psi] = \int d^4x \left\{ \left| \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right|^2 - \lambda_1 \sum_i (\Psi_i^\dagger \Psi_i - 1)^2 - \lambda_2 \sum_{i,j,k} \text{Tr} \left[ \Gamma (\Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger) \right]^2 \right\},$$

the stationary condition is defined by:

$$\frac{\delta S}{\delta \Psi_a^\dagger} = 0.$$

The variation is performed sector by sector.

### 3.1.1. Variation of the Interaction Sector

Define:

$$Q = \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j.$$

The interaction contribution is:

$$S_{\text{int}} = \int d^4x Q^\dagger Q.$$

The first variation is:

$$\delta S_{\text{int}} = \int d^4x \left[ (\delta Q^\dagger) Q + Q^\dagger (\delta Q) \right].$$

The variation of  $Q$  with respect to  $\Psi_a^\dagger$  gives:

$$\delta Q = \sum_j \delta \Psi_a^\dagger \Gamma \Psi_j.$$

Similarly:

$$Q^\dagger = \sum_{m,n} \Psi_n^\dagger \Gamma^\dagger \Psi_m,$$

thus:

$$\delta Q^\dagger = \sum_m \Psi_m^\dagger \Gamma^\dagger \delta \Psi_a.$$

Substituting into the interaction variation:

$$\delta S_{\text{int}} = \int d^4x \left[ \left( \sum_m \Psi_m^\dagger \Gamma^\dagger \delta \Psi_a \right) Q + Q^\dagger \left( \sum_j \delta \Psi_a^\dagger \Gamma \Psi_j \right) \right]. \quad (2)$$

Extracting the contribution proportional to  $\delta \Psi_a^\dagger$ :

$$\frac{\delta S_{\text{int}}}{\delta \Psi_a^\dagger} = Q^\dagger \sum_j \Gamma \Psi_j.$$

Using:

$$Q^\dagger = \sum_{m,n} \Psi_n^\dagger \Gamma^\dagger \Psi_m,$$

one obtains:

$$\frac{\delta S_{\text{int}}}{\delta \Psi_a^\dagger} = \left( \sum_{m,n} \Psi_n^\dagger \Gamma^\dagger \Psi_m \right) \left( \sum_j \Gamma \Psi_j \right).$$

The interaction variation is therefore cubic in the fields and globally coupled.

### 3.1.2. Variation of the Normalization Sector

The normalization contribution is:

$$S_{\text{norm}} = -\lambda_1 \int d^4x \sum_i (\Psi_i^\dagger \Psi_i - 1)^2.$$

Define:

$$N_i = \Psi_i^\dagger \Psi_i.$$

Then:

$$S_{\text{norm}} = -\lambda_1 \int d^4x \sum_i (N_i - 1)^2.$$

The variation becomes:

$$\delta S_{\text{norm}} = -2\lambda_1 \int d^4x \sum_i (N_i - 1) \delta N_i.$$

Since:

$$\delta N_i = \delta \Psi_i^\dagger \Psi_i + \Psi_i^\dagger \delta \Psi_i,$$

variation with respect to  $\Psi_a^\dagger$  gives:

$$\frac{\delta N_i}{\delta \Psi_a^\dagger} = \delta_{ia} \Psi_i.$$

Thus:

$$\frac{\delta S_{\text{norm}}}{\delta \Psi_a^\dagger} = -2\lambda_1 (\Psi_a^\dagger \Psi_a - 1) \Psi_a.$$

The normalization variation is also cubic and stabilizes the admissible norm of the stationary configurations.

### 3.1.3. Variation of the Closure Sector

The recursive closure contribution is:

$$S_{\text{closure}} = -\lambda_2 \int d^4x \sum_{i,j,k} \text{Tr}(C_{ijk}^2),$$

where:

$$C_{ijk} = \Gamma(\Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger).$$

The first variation is:

$$\delta S_{\text{closure}} = -2\lambda_2 \int d^4x \sum_{i,j,k} \text{Tr}(C_{ijk} \delta C_{ijk}).$$

Now:

$$\delta C_{ijk} = \Gamma(\Psi_i \delta \Psi_j^\dagger - \Psi_k \delta \Psi_i^\dagger).$$

Variation with respect to  $\Psi_a^\dagger$  therefore produces two contributions:

- one from  $j = a$ ,
- one from  $i = a$ .

Thus:

$$\frac{\delta C_{ijk}}{\delta \Psi_a^\dagger} = \Gamma(\Psi_i \delta_{ja} - \Psi_k \delta_{ia}).$$

Substituting into the closure variation:

$$\frac{\delta S_{\text{closure}}}{\delta \Psi_a^\dagger} = -2\lambda_2 \sum_{i,j,k} \text{Tr}[C_{ijk} \Gamma(\Psi_i \delta_{ja} - \Psi_k \delta_{ia})]. \quad (3)$$



Separating explicitly:

$$\frac{\delta S_{\text{closure}}}{\delta \Psi_a^\dagger} = -2\lambda_2 \sum_{i,k} \text{Tr}(C_{iak} \Gamma \Psi_i) + 2\lambda_2 \sum_{j,k} \text{Tr}(C_{ajk} \Gamma \Psi_k). \quad (4)$$

The closure variation is therefore recursively coupled and cyclically organized.

### 3.2. Euler Variational Equation

Combining all variational contributions gives:

$$\frac{\delta S}{\delta \Psi_a^\dagger} = \frac{\delta S_{\text{int}}}{\delta \Psi_a^\dagger} + \frac{\delta S_{\text{norm}}}{\delta \Psi_a^\dagger} + \frac{\delta S_{\text{closure}}}{\delta \Psi_a^\dagger}.$$

The complete stationary Euler equation becomes:

$$0 = \left( \sum_{m,n} \Psi_n^\dagger \Gamma^\dagger \Psi_m \right) \left( \sum_j \Gamma \Psi_j \right) - 2\lambda_1 (\Psi_a^\dagger \Psi_a - 1) \Psi_a - 2\lambda_2 \sum_{i,k} \text{Tr}(C_{iak} \Gamma \Psi_i) + 2\lambda_2 \sum_{j,k} \text{Tr}(C_{ajk} \Gamma \Psi_k). \quad (5)$$

The stationary Euler sector is therefore:

- quartically generated,
- globally coupled,
- spectrally coherent,
- recursively closed,
- cyclically organized.

All interaction structures emerge directly from the internal variation of the quartic functional itself.

No external equations, auxiliary geometrical sectors, or independent quantization prescriptions are introduced.

### 3.3. Non-Linearity and Coherent Coupling

The stationary Euler equation possesses an intrinsically non-linear and globally coherent structure generated directly from the quartic organization of the functional.

The interaction contribution originates from:

$$S_{\text{int}} = \int d^4x Q^\dagger Q.$$

Expanding explicitly:

$$Q^\dagger Q = \left( \sum_{m,n} \Psi_n^\dagger \Gamma^\dagger \Psi_m \right) \left( \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right).$$

The interaction therefore contains four field factors.

The coupling is intrinsically quartic.

No independent pair decomposition exists.

Each admissible configuration contributes to the complete interaction structure.

The interaction is therefore globally coherent.

Under global rescaling:

$$\Psi \rightarrow \alpha \Psi,$$

the interaction transforms as:

$$S_{\text{int}} \rightarrow \alpha^4 S_{\text{int}}.$$

The quartic growth generates bounded stabilization once combined with the normalization sector.

The coherent interaction additionally generates non-diagonal Hessian coupling. The generic Hessian structure becomes:

$$H_{ab} = \begin{cases} A & a = b, \\ B & a \neq b. \end{cases}$$

Thus:

$$H = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}.$$

The corresponding spectrum is:

$$\begin{aligned} \lambda_1 &= A + 2B, \\ \lambda_2 &= \lambda_3 = A - B. \end{aligned}$$

The normalized ratio:

$$L = \frac{A - B}{A + 2B}$$

therefore defines the intrinsic spectral hierarchy.

For the coherent cyclic configuration:

$$B = \frac{A}{2},$$

one obtains:

$$L = \frac{1}{4}.$$

The coherent quartic interaction therefore generates the first Livolsi constant directly from the internal Hessian organization.

The recursive closure sector additionally generates cyclic interaction consistency.

The cyclic operator acts on ordered triples:

$$(i, j, k),$$

through:

$$\Gamma : (i, j, k) \mapsto (j, k, i).$$

Repeated application gives:

$$\Gamma^3 = I.$$

The interaction therefore generates the cyclic group:

$$Z_3 = \{I, \Gamma, \Gamma^2\}.$$

The stationary variational sector is consequently:

- non-linear,
- globally coherent,
- recursively coupled,
- spectrally generative,
- cyclically closed.

The complete organization emerges directly from the internal structure of the quartic variational functional itself.

## 4. Second Variational Sector

### 4.1. Exact Hessian Construction

The spectral structure of the framework is determined through the second variation of the quartic Psi-Gamma functional around stationary configurations.

The Hessian operator is defined as:

$$H = \frac{\delta^2 S}{\delta \Psi^\dagger \delta \Psi}.$$

All spectral properties of the framework emerge directly from this second variational sector.

Starting from the quartic functional:

$$S[\Psi] = \int d^4x \left\{ \left| \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right|^2 - \lambda_1 \sum_i (\Psi_i^\dagger \Psi_i - 1)^2 - \lambda_2 \sum_{i,j,k} \text{Tr} \left[ \Gamma (\Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger) \right]^2 \right\},$$

consider fluctuations around stationary configurations:

$$\Psi_i = \Psi_i^{(0)} + \delta \Psi_i,$$

and:

$$\Psi_i^\dagger = \Psi_i^{(0)\dagger} + \delta \Psi_i^\dagger.$$

The action expands as:

$$S[\Psi] = S[\Psi^{(0)}] + \delta S + \frac{1}{2} \delta^2 S + \mathcal{O}(\delta \Psi^3).$$

At stationary configurations:

$$\delta S = 0.$$

The leading fluctuation structure is therefore governed by:

$$\delta^2 S.$$

#### 4.1.1. Second Variation of the Interaction Sector

Define:

$$Q = \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j.$$

The interaction contribution is:

$$S_{\text{int}} = \int d^4x Q^\dagger Q.$$

The first variation is:

$$\delta S_{\text{int}} = \int d^4x \left[ (\delta Q^\dagger) Q + Q^\dagger (\delta Q) \right].$$

The second variation becomes:

$$\delta^2 S_{\text{int}} = \int d^4x \left[ (\delta^2 Q^\dagger) Q + 2(\delta Q^\dagger)(\delta Q) + Q^\dagger (\delta^2 Q) \right].$$

Since  $Q$  is linear in  $\Psi$  and  $\Psi^\dagger$ :

$$\delta^2 Q = 0, \quad \delta^2 Q^\dagger = 0.$$

Thus:

$$\delta^2 S_{\text{int}} = 2 \int d^4x (\delta Q^\dagger)(\delta Q).$$

Now:

$$\delta Q = \sum_{i,j} \delta \Psi_i^\dagger \Gamma \delta \Psi_j.$$

Similarly:

$$\delta Q^\dagger = \sum_{m,n} \delta \Psi_n^\dagger \Gamma^\dagger \delta \Psi_m.$$

Substituting:

$$\delta^2 S_{\text{int}} = 2 \int d^4x \left( \sum_{m,n} \delta \Psi_n^\dagger \Gamma^\dagger \delta \Psi_m \right) \left( \sum_{i,j} \delta \Psi_i^\dagger \Gamma \delta \Psi_j \right). \quad (6)$$

The interaction Hessian is therefore quartically coupled in the fluctuations.

The fluctuation structure is globally coherent and non-diagonal.

#### 4.1.2. Second Variation of the Normalization Sector

The normalization contribution is:

$$S_{\text{norm}} = -\lambda_1 \int d^4x \sum_i \left( \Psi_i^\dagger \Psi_i - 1 \right)^2.$$

Define:

$$N_i = \Psi_i^\dagger \Psi_i.$$

The first variation is:

$$\delta N_i = \delta \Psi_i^\dagger \Psi_i + \Psi_i^\dagger \delta \Psi_i.$$

The second variation becomes:

$$\delta^2 N_i = 2 \delta \Psi_i^\dagger \delta \Psi_i.$$

Now:

$$\delta S_{\text{norm}} = -2\lambda_1 \int d^4x \sum_i (N_i - 1) \delta N_i.$$

Taking the second variation:

$$\delta^2 S_{\text{norm}} = -2\lambda_1 \int d^4x \sum_i \left[ (\delta N_i)^2 + (N_i - 1) \delta^2 N_i \right].$$

At stationary normalization:

$$N_i = 1,$$

thus the second term vanishes.

One obtains:

$$\delta^2 S_{\text{norm}} = -2\lambda_1 \int d^4x \sum_i (\delta N_i)^2.$$

Using:

$$\delta N_i = \delta \Psi_i^\dagger \Psi_i + \Psi_i^\dagger \delta \Psi_i,$$

the normalization Hessian becomes quadratic in the fluctuations.

The normalization sector therefore stabilizes the admissible fluctuation spectrum.

#### 4.1.3. Second Variation of the Closure Sector

Define:

$$C_{ijk} = \Gamma \left( \Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger \right).$$

The closure contribution is:

$$S_{\text{closure}} = -\lambda_2 \int d^4x \sum_{i,j,k} \text{Tr} \left( C_{ijk}^2 \right).$$

The first variation is:

$$\delta S_{\text{closure}} = -2\lambda_2 \int d^4x \sum_{i,j,k} \text{Tr} \left( C_{ijk} \delta C_{ijk} \right).$$

The second variation becomes:

$$\delta^2 S_{\text{closure}} = -2\lambda_2 \int d^4x \sum_{i,j,k} \text{Tr} \left[ \left( \delta C_{ijk} \right)^2 + C_{ijk} \delta^2 C_{ijk} \right].$$

Now:

$$\delta C_{ijk} = \Gamma \left( \delta \Psi_i \Psi_j^\dagger + \Psi_i \delta \Psi_j^\dagger - \delta \Psi_k \Psi_i^\dagger - \Psi_k \delta \Psi_i^\dagger \right). \quad (7)$$

The second variation becomes:

$$\delta^2 C_{ijk} = 2\Gamma \left( \delta \Psi_i \delta \Psi_j^\dagger - \delta \Psi_k \delta \Psi_i^\dagger \right).$$

The closure Hessian therefore contains:

- recursive fluctuation coupling,
- cyclic interaction mixing,
- non-diagonal spectral structure,
- coherent recursive closure.

#### 4.2. Complete Hessian Structure

Combining all contributions:

$$\delta^2 S = \delta^2 S_{\text{int}} + \delta^2 S_{\text{norm}} + \delta^2 S_{\text{closure}}.$$

The Hessian operator therefore decomposes as:

$$H = H_{\text{int}} + H_{\text{norm}} + H_{\text{closure}}.$$

The complete Hessian structure is therefore:

- non-diagonal,
- recursively coupled,
- spectrally non-trivial,
- cyclically organized,
- globally coherent.

The fluctuation dynamics is consequently generated directly from the internal quartic organization of the variational structure.

### 4.3. Hessian Matrix Structure

The quartic coherent interaction generates a non-diagonal Hessian matrix.

The Hessian operator is:

$$H_{ab} = \frac{\delta^2 S}{\delta \Psi_a^\dagger \delta \Psi_b}.$$

The quartic interaction mixes all admissible sectors through:

$$\Psi_i^\dagger \Gamma \Psi_j.$$

The Hessian is therefore not diagonal.

The stationary quartic structure is permutation symmetric under:

$$(i, j, k) \rightarrow \sigma(i, j, k), \quad \sigma \in S_3.$$

The Hessian coefficients consequently depend only upon whether two indices are equal or distinct.

Thus:

$$H_{ab} = \begin{cases} A & a = b, \\ B & a \neq b. \end{cases}$$

The Hessian matrix therefore takes the form:

$$H = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}.$$

The diagonal contribution  $A$  receives contributions from:

- self-interaction terms,
- normalization stabilization,
- recursive closure coupling.

The off-diagonal contribution  $B$  is generated through coherent quartic mixing between different sectors.

The characteristic polynomial is:

$$\det(H - \lambda I) = \begin{vmatrix} A - \lambda & B & B \\ B & A - \lambda & B \\ B & B & A - \lambda \end{vmatrix}.$$

Expanding explicitly:

$$\det(H - \lambda I) = (A - \lambda - B)^2 (A - \lambda + 2B).$$

The spectrum therefore becomes:

$$\lambda_1 = A + 2B,$$

and:

$$\lambda_2 = \lambda_3 = A - B.$$

The interaction consequently generates:

- spectral splitting,
- coherent degeneracy,
- recursive hierarchy,
- non-trivial spectral organization.

The normalized spectral ratio becomes:

$$L = \frac{A - B}{A + 2B}.$$

For the coherent cyclic configuration:

$$B = \frac{A}{2},$$

one obtains:

$$L = \frac{1}{4}.$$

The Hessian therefore generates directly the first Livolsi constant from the coherent quartic interaction structure.

#### 4.4. Normalized Hessian Decomposition

The Hessian admits the normalized decomposition:

$$H = E^* T,$$

where:

$$E^* = \text{Tr}(H),$$

and:

$$T = \frac{H}{E^*}.$$

The normalized spectral structure becomes:

$$\text{Spec}(T) = \{1, L, L\}.$$

Using:

$$L = \frac{1}{4},$$

one obtains the normalized hierarchy:

$$\text{Spec}(T) = \left\{1, \frac{1}{4}, \frac{1}{4}\right\}.$$

The Hessian consequently generates:

- coherent spectral splitting,
- recursive spectral hierarchy,
- intrinsic energy organization,
- finite admissible closure.

The spectral structure is therefore internally generated directly from the quartic variational organization itself.

No external quantization prescription or phenomenological insertion is introduced at any stage.

#### 4.5. Spectral Admissibility

The fluctuation spectrum is determined through:

$$Hv_n = \lambda_n v_n.$$

Admissibility requires:

$$0 < \lambda_n < \infty.$$

The Hessian therefore governs:

- spectral stability,
- admissible configurations,
- recursive hierarchy,
- cyclic closure,
- coherent interaction organization.

All spectral properties emerge directly from the second variation of the quartic variational functional itself.

### 5. Spectral Emergence and Recursive Hierarchy

#### 5.1. Emergence of the Spectral Structure

The complete spectral organization of the framework emerges directly from the Hessian structure generated by the quartic variational functional.

Starting from the normalized Hessian matrix:

$$H = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix},$$

the characteristic polynomial is:

$$\det(H - \lambda I) = (A - \lambda - B)^2 (A - \lambda + 2B).$$

The corresponding eigenvalues are therefore:

$$\lambda_1 = A + 2B,$$

and:

$$\lambda_2 = \lambda_3 = A - B.$$

The quartic coherent interaction consequently generates:

- one dominant coherent mode,



- two degenerate transverse modes.

The spectrum is therefore intrinsically non-trivial and internally generated.

No external quantization prescription is introduced.

The normalized spectral ratio becomes:

$$L = \frac{A-B}{A+2B}.$$

For the coherent cyclic configuration:

$$B = \frac{A}{2},$$

one obtains:

$$L = \frac{A - \frac{A}{2}}{A + A} = \frac{A/2}{2A} = \frac{1}{4}.$$

Thus the first Livolsi constant emerges directly from the coherent Hessian organization:

$$L = \frac{1}{4}.$$

The Hessian therefore admits the normalized decomposition:

$$H = E^* T,$$

where:

$$E^* = \text{Tr}(H),$$

and:

$$T = \frac{H}{E^*}.$$

The normalized spectral structure becomes:

$$\text{Spec}(T) = \{1, L, L\}.$$

Using:

$$L = \frac{1}{4},$$

one obtains:

$$\text{Spec}(T) = \left\{1, \frac{1}{4}, \frac{1}{4}\right\}.$$

The complete spectral organization therefore emerges internally from the quartic variational structure itself.

## 5.2. Coherent Spectral Splitting

The quartic interaction produces coherent non-diagonal coupling.

The off-diagonal structure:

$$B \neq 0$$

generates degeneracy splitting between coherent and transverse sectors.

The dominant eigenvalue:

$$\lambda_1 = A + 2B$$

corresponds to the coherent collective mode.

The degenerate sector:

$$\lambda_2 = \lambda_3 = A - B$$

corresponds instead to transverse cyclic fluctuations.

The interaction therefore produces:

- coherent spectral separation,
- recursive mode hierarchy,
- finite admissible organization,
- non-trivial spectral rigidity.

If the off-diagonal coupling vanishes:

$$B = 0,$$

the Hessian becomes:

$$H = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix},$$

with completely degenerate spectrum:

$$\text{Spec}(H) = \{A, A, A\}.$$

No internal hierarchy emerges.

The coherent quartic interaction is therefore necessary for the emergence of:

splitting  $\rightarrow$  hierarchy  $\rightarrow$  closure.

### 5.3. Emergence of Spectral Hierarchy

The coherent Hessian structure generates a recursive spectral organization.

Define recursively:

$$E_n = E^* L^n.$$

Using:

$$L = \frac{1}{4},$$

one obtains:

$$E_n = E^* 4^{-n}.$$

The hierarchy therefore becomes:

$$E_0 = E^*,$$

$$E_1 = \frac{E^*}{4},$$

$$E_2 = \frac{E^*}{16},$$

$$E_3 = \frac{E^*}{64},$$

and generally:

$$E_n = \frac{E^*}{4^n}.$$

The recursive structure consequently generates:

- finite spectral compression,
- recursive admissible hierarchy,
- coherent cyclic organization,
- bounded spectral closure.

The spectral organization is therefore not externally imposed.

It emerges directly from the recursive Hessian structure generated by the quartic interaction.

#### 5.4. Emergence of Cyclic Closure

The recursive hierarchy is intrinsically linked to the cyclic interaction structure.

The quartic closure sector acts on ordered triples:

$$(i, j, k),$$

through the cyclic operator:

$$\Gamma : (i, j, k) \mapsto (j, k, i).$$

Repeated application gives:

$$\Gamma^2(i, j, k) = (k, i, j),$$

and:

$$\Gamma^3(i, j, k) = (i, j, k).$$

Thus:

$$\Gamma^3 = I.$$

The interaction therefore generates the cyclic group:

$$Z_3 = \{I, \Gamma, \Gamma^2\}.$$

The recursive hierarchy is consequently cyclically closed.

The coherent interaction therefore produces:

$$\text{interaction} \rightarrow \text{splitting} \rightarrow \text{hierarchy} \rightarrow \text{cyclicclosure}.$$

#### 5.5. Projection Structure and Admissible Sectors

The cyclic organization generates a natural projection structure.

Define the cyclic projection operator:

$$P = \frac{1}{3}(I + \Gamma + \Gamma^2).$$

Using:

$$\Gamma^3 = I,$$

one obtains:

$$P^2 = P.$$

The projection operator therefore isolates the coherent cyclic sector.

The admissible configurations consequently satisfy:

$$P\Psi = \Psi.$$

The recursive spectral hierarchy is therefore constrained to cyclically admissible configurations only.

The admissible spectral organization is consequently finite.

## 5.6. Emergence of Finite Configuration Space

The recursive hierarchy generates finite admissible closure.

The cyclic structure possesses three coherent sectors:

$$Z_3 = \{I, \Gamma, \Gamma^2\}.$$

The recursive hierarchy generated by:

$$L = \frac{1}{4}$$

produces binary spectral branching.

At recursive level  $k$ , the admissible degeneracy becomes:

$$\dim(C_k) = 2^k.$$

The coherent physical configuration space therefore becomes:

$$\dim(W_{\text{phys}}) = 3 \cdot 2^5.$$

Thus:

$$\dim(W_{\text{phys}}) = 96.$$

The third Livolsi constant consequently emerges as:

$$\nu = \frac{1}{96}.$$

The corresponding spectral discretization becomes:

$$\Delta E = \nu E^* = \frac{E^*}{96}.$$

The spectral hierarchy therefore generates:

- finite admissible closure,
- recursive quantization,
- coherent cyclic organization,

- intrinsic spectral discretization.

## 5.7. Structural Role of the Spectral Sector

The spectral structure generated by the Hessian constitutes the central organizing principle of the framework.

The quartic interaction generates:

$$H \rightarrow \text{Spec}(H) \rightarrow L \rightarrow E_n \rightarrow \nu.$$

The complete hierarchy therefore emerges internally from:

- coherent quartic interaction,
- recursive cyclic closure,
- Hessian spectral organization.

No:

- external quantization,
- phenomenological insertion,
- independent spectral postulate,
- auxiliary algebraic sector

is introduced at any stage.

The spectral hierarchy is therefore a direct emergent consequence of the globally closed quartic variational structure itself.

## 6. Cyclic Closure and $Z_3$ Organization

### 6.1. Emergence of the Cyclic Interaction Structure

The quartic closure sector generates a recursive cyclic interaction organization acting on ordered triples:

$$(i, j, k).$$

The interaction is governed by the cyclic operator:

$$\Gamma : (i, j, k) \mapsto (j, k, i).$$

The first application produces:

$$\Gamma(i, j, k) = (j, k, i).$$

Applying the operator a second time gives:

$$\Gamma^2(i, j, k) = (k, i, j).$$

A third application returns the original configuration:

$$\Gamma^3(i, j, k) = (i, j, k).$$

Thus:

$$\Gamma^3 = I.$$

The recursive interaction therefore generates the cyclic group:

$$Z_3 = \{I, \Gamma, \Gamma^2\}.$$

The quartic interaction consequently possesses intrinsic cyclic closure.

The closure is generated directly from the internal structure of the variational functional itself.

No external symmetry group is imposed.

## 6.2. Minimality of the Cyclic Structure

The cyclic closure generated by the quartic interaction is minimal.

Lower cyclic organizations fail to generate stable recursive structure.

### 6.2.1. $Z_1$ Collapse

The trivial cyclic structure satisfies:

$$\Gamma = I.$$

The interaction therefore becomes completely diagonal.

The Hessian reduces to:

$$H = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}.$$

The spectrum becomes fully degenerate:

$$\text{Spec}(H) = \{A, A, A\}.$$

No spectral hierarchy emerges.

No recursive closure is generated.

The trivial cyclic organization therefore collapses.

### 6.2.2. $Z_2$ Instability

Consider now a binary cyclic structure:

$$\Gamma^2 = I.$$

The interaction alternates between only two configurations.

The recursive structure therefore oscillates without generating stable ternary closure.

The corresponding Hessian structure reduces to binary exchange coupling:

$$H = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The spectrum becomes:

$$\lambda_1 = A + B,$$

$$\lambda_2 = A - B.$$

Only binary splitting is generated.

Recursive cyclic closure remains incomplete.

No finite recursive hierarchy emerges.

The  $Z_2$  organization therefore fails to generate:

- recursive admissible closure,

- stable ternary organization,
- finite cyclic hierarchy.

### 6.2.3. $Z_3$ Closure

The ternary cyclic structure instead satisfies:

$$\Gamma^3 = I.$$

The interaction recursively closes after three coherent steps.

The quartic interaction therefore generates the minimal non-trivial stable cyclic organization.

The  $Z_3$  structure consequently produces:

- recursive cyclic consistency,
- finite closure,
- coherent spectral splitting,
- admissible recursive hierarchy.

The ternary cyclic organization is therefore the minimal admissible recursively closed interaction structure.

### 6.3. Recursive Action of the Closure Operator

The recursive closure operator acts directly on the quartic interaction sector.

Define:

$$C_{ijk} = \Gamma(\Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger).$$

The closure contribution to the action is:

$$S_{\text{closure}} = -\lambda_2 \int d^4x \sum_{i,j,k} \text{Tr}(C_{ijk}^2).$$

The operator therefore recursively mixes the interaction sectors.

Repeated application generates the sequence:

$$(i, j, k) \rightarrow (j, k, i) \rightarrow (k, i, j) \rightarrow (i, j, k).$$

The interaction consequently becomes recursively self-consistent.

The recursive closure therefore generates:

- coherent cyclic recurrence,
- recursive interaction stabilization,
- finite spectral organization,
- admissible closure consistency.

### 6.4. Cyclic Projection Structure

The cyclic organization generates a natural projection operator.

Define:

$$P = \frac{1}{3}(I + \Gamma + \Gamma^2).$$

Using:

$$\Gamma^3 = I,$$

one obtains:

$$P^2 = \frac{1}{9} (I + \Gamma + \Gamma^2)^2.$$

Expanding explicitly:

$$(I + \Gamma + \Gamma^2)^2 = I + \Gamma + \Gamma^2 + \Gamma + \Gamma^2 + \Gamma^3 + \Gamma^2 + \Gamma^3 + \Gamma^4. \quad (8)$$

Using:

$$\Gamma^3 = I, \quad \Gamma^4 = \Gamma,$$

one obtains:

$$(I + \Gamma + \Gamma^2)^2 = 3(I + \Gamma + \Gamma^2).$$

Thus:

$$P^2 = P.$$

The operator therefore defines a genuine projection onto the coherent cyclic sector.

Admissible configurations consequently satisfy:

$$P\Psi = \Psi.$$

Only cyclically coherent configurations remain admissible.

## 6.5. Cyclic Spectral Organization

The cyclic interaction directly determines the Hessian organization.

The coherent quartic interaction generates the Hessian:

$$H = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}.$$

The cyclic structure therefore produces coherent off-diagonal coupling.

The corresponding spectrum becomes:

$$\lambda_1 = A + 2B,$$

$$\lambda_2 = \lambda_3 = A - B.$$

The normalized ratio:

$$L = \frac{A - B}{A + 2B}$$

therefore emerges directly from cyclic coherent interaction.

For the recursive cyclic configuration:

$$B = \frac{A}{2},$$

one obtains:

$$L = \frac{1}{4}.$$



The cyclic organization therefore generates:

- coherent spectral splitting,
- recursive hierarchy,
- finite admissible closure,
- spectral rigidity.

## 6.6. Recursive Hierarchy and Finite Closure

The recursive cyclic structure generates finite admissible hierarchy.

The recursive spectral organization is:

$$E_n = E^* L^n.$$

Using:

$$L = \frac{1}{4},$$

one obtains:

$$E_n = E^* 4^{-n}.$$

The cyclic hierarchy consequently compresses recursively.

The admissible degeneracy at recursive level  $k$  becomes:

$$\dim(C_k) = 2^k.$$

Combining the ternary cyclic structure with recursive binary branching gives:

$$\dim(W_{\text{phys}}) = 3 \cdot 2^5.$$

Thus:

$$\dim(W_{\text{phys}}) = 96.$$

The third Livolsi constant therefore emerges directly from recursive cyclic closure:

$$\nu = \frac{1}{96}.$$

The corresponding spectral discretization becomes:

$$\Delta E = \nu E^* = \frac{E^*}{96}.$$

The recursive cyclic organization therefore generates:

- finite admissible configuration space,
- recursive quantization hierarchy,
- coherent spectral discretization,
- bounded cyclic closure.

## 6.7. Structural Role of the $Z_3$ Sector

The  $Z_3$  cyclic structure constitutes the recursive organizational core of the framework.

The quartic interaction generates:

$$\Gamma \rightarrow Z_3 \rightarrow P \rightarrow L \rightarrow E_n \rightarrow \nu.$$

The cyclic organization therefore governs:

- recursive interaction consistency,
- admissible spectral hierarchy,
- coherent closure,
- finite configuration space,
- intrinsic discretization.

No external symmetry principle is introduced.

The complete cyclic organization emerges directly from the recursive quartic interaction structure itself.

## 7. Admissible Configuration Space

### 7.1. Definition of the Stationary Space

The stationary configurations of the framework are determined through the Euler variational equation:

$$\frac{\delta S}{\delta \Psi^\dagger} = 0.$$

The complete stationary sector is therefore defined as:

$$X = \left\{ \Psi \mid \frac{\delta S}{\delta \Psi^\dagger} = 0 \right\}.$$

The stationary space generally contains multiple configurations.

Not all stationary solutions are physically admissible.

A globally coherent variational framework therefore requires the existence of an admissible configuration sector internally generated from the spectral organization of the theory itself.

### 7.2. Spectral Admissibility Conditions

The admissibility conditions are determined through the Hessian spectrum.

The Hessian operator is:

$$H = \frac{\delta^2 S}{\delta \Psi^\dagger \delta \Psi}.$$

The spectral structure satisfies:

$$H v_n = \lambda_n v_n.$$

Admissibility requires bounded positive spectrum:

$$0 < \lambda_n < \infty.$$

If:

$$\lambda_n = 0,$$

flat directions emerge and spectral rigidity collapses.

If:

$$\lambda_n < 0,$$

the configuration becomes unstable.

The admissible sector must therefore satisfy:

$$\forall n, 0 < \lambda_n < \infty.$$

The admissibility structure is consequently determined directly through the Hessian organization generated by the quartic interaction.

### 7.3. Normalization Stability

The normalization sector imposes bounded admissible amplitude.

The normalization contribution is:

$$S_{\text{norm}} = -\lambda_1 \int d^4x \sum_i (\Psi_i^\dagger \Psi_i - 1)^2.$$

The admissible configurations therefore satisfy:

$$\Psi_i^\dagger \Psi_i = 1.$$

Without normalization stabilization:

$$\Psi_i \rightarrow \alpha \Psi_i$$

would produce unbounded rescaling.

The normalization sector consequently generates:

- finite admissible norm,
- bounded spectral amplitude,
- controlled quartic growth,
- stable fluctuation structure.

The admissible sector therefore requires simultaneously:

$$\Psi^\dagger \Psi = 1,$$

and:

$$0 < \lambda_n < \infty.$$

### 7.4. Finite Functional Energy

Admissible configurations must additionally possess finite functional energy.

The variational functional is:

$$S[\Psi] = S_{\text{int}} + S_{\text{norm}} + S_{\text{closure}}.$$

Admissibility therefore requires:

$$S[\Psi] < \infty.$$

Configurations producing divergent interaction energy are excluded.

The quartic structure consequently restricts the admissible sector to bounded coherent configurations only.

The admissible space therefore excludes:

- divergent configurations,
- spectrally unstable sectors,

- non-normalizable states,
- non-recursive structures.

### 7.5. Recursive Closure Consistency

The admissible sector must additionally satisfy recursive cyclic closure.

The quartic interaction acts on ordered triples:

$$(i, j, k),$$

through the cyclic operator:

$$\Gamma : (i, j, k) \mapsto (j, k, i).$$

The recursive closure condition is:

$$\Gamma^3 = I.$$

Admissible configurations must therefore remain stable under recursive cyclic action.

Configurations violating cyclic closure consistency are excluded from the admissible sector.

The admissible structure consequently satisfies:

$$P\Psi = \Psi,$$

where:

$$P = \frac{1}{3}(I + \Gamma + \Gamma^2)$$

is the cyclic projection operator.

Only cyclically coherent configurations remain admissible.

### 7.6. Definition of the Admissible Sector

Combining all admissibility conditions gives:

$$A \subset X,$$

where:

$$\Psi \in A \Leftrightarrow \begin{cases} 0 < \lambda_n < \infty, \\ S[\Psi] < \infty, \\ \Psi^\dagger \Psi = 1, \\ P\Psi = \Psi. \end{cases}$$

The admissible configuration space is therefore generated internally through the spectral and recursive organization of the quartic functional.

No external admissibility postulates are introduced.

### 7.7. Recursive Spectral Hierarchy

The admissible sector inherits the recursive hierarchy generated by the Hessian organization.

The normalized Hessian decomposition is:

$$H = E^* T,$$

with:

$$\text{Spec}(T) = \{1, L, L\}.$$

Using:

$$L = \frac{1}{4},$$

the recursive spectral hierarchy becomes:

$$E_n = E^* L^n = E^* 4^{-n}.$$

The admissible hierarchy therefore recursively compresses.

The recursive organization consequently generates:

- finite spectral hierarchy,
- coherent admissible sectors,
- bounded recursive structure,
- cyclic spectral closure.

## 7.8. Emergence of Finite Admissible Configuration Space

The recursive cyclic organization generates finite admissible closure.

The cyclic structure possesses three coherent sectors:

$$Z_3 = \{I, \Gamma, \Gamma^2\}.$$

The recursive hierarchy produces binary spectral branching.

At recursive level  $k$ , the admissible degeneracy becomes:

$$\dim(C_k) = 2^k.$$

The admissible physical configuration space therefore becomes:

$$\dim(W_{\text{phys}}) = 3 \cdot 2^5.$$

Thus:

$$\dim(W_{\text{phys}}) = 96.$$

The third Livolsi constant consequently emerges as:

$$\nu = \frac{1}{96}.$$

The corresponding spectral discretization becomes:

$$\Delta E = \nu E^* = \frac{E^*}{96}.$$

The admissible sector therefore generates:

- finite recursive closure,
- intrinsic spectral discretization,
- bounded cyclic hierarchy,
- coherent admissible quantization.

## 7.9. Global Structure of Admissibility

The admissible configuration space constitutes the physically coherent sector of the framework.

The quartic interaction generates:

$$H \rightarrow \text{Spec}(H) \rightarrow L \rightarrow P \rightarrow A.$$

The admissible organization therefore emerges directly from:

- quartic coherent interaction,
- Hessian spectral structure,
- recursive cyclic closure,
- normalization stabilization.

No external admissibility criterion is imposed.

The complete admissible configuration space is generated internally from the globally closed variational structure itself.

## 8. Global Decisional Structure

### 8.1. Structural Necessity of the Decisional Layer

The variational equation generated by the quartic functional defines the stationary configuration space:

$$S = \left\{ \Psi \in A \subset X \mid \frac{\delta F}{\delta \Psi^\dagger} = 0 \right\}.$$

The variational structure therefore determines coherence but not realization.

Indeed, for:

$$\Psi_1, \Psi_2 \in A,$$

one has:

$$\frac{\delta F}{\delta \Psi^\dagger}(\Psi_1) = 0, \quad \frac{\delta F}{\delta \Psi^\dagger}(\Psi_2) = 0,$$

while generally:

$$F[\Psi_1] = F[\Psi_2].$$

The variational structure therefore determines coherence but not realization.

The framework consequently requires an additional global structural layer capable of selecting a unique admissible configuration from the admissible stationary sector.

### 8.2. Canonical Decision Equation

The canonical decisional equation is:

$$A \subset X, \quad \Psi^* = \text{Sel}_{\Psi \in A} C(\Psi \mid \Omega).$$

This equation is canonical and immutable.

The decisional structure therefore acts on the admissible stationary sector:

$$A.$$

The decisional layer is:

- global,
- deterministic,
- non-local,
- structurally constrained,
- spectrally ordered,
- recursively coherent.

### 8.3. Role of $\Omega$

The quantity:

$$\Omega$$

is not:

- a dynamical field,
- a variational operator,
- a probabilistic object,
- an observer,
- a selector.

Selection is not performed by  $\Omega$ .

The selector is:

$$\text{Sel}_{\Psi \in A}.$$

The role of  $\Omega$  is instead to define the global structural consistency conditions used by the decisional functional.

Formally:

$$\Omega = \{\text{structural constraints on the invariants of } \Psi\}.$$

Explicitly

$$I(\Psi) = \{H[\Psi], \text{Spec}(H[\Psi]), Z_3(\Psi), F[\Psi]\}.$$

Thus  $\Omega$  defines the admissible structural target against which configurations are evaluated.

$\Omega$  therefore encodes:

- Hessian structure,
- spectral organization,
- cyclic closure,
- variational compatibility,
- recursive coherence.

The decisional structure consequently evaluates configurations through their compatibility with the invariant structure specified by  $\Omega$ .

### 8.4. Construction of the Decision Functional

The decisional functional is a compatibility functional:

$$C(\Psi | \Omega).$$

Its role is to measure the structural compatibility between the invariants of a candidate configuration and the structural constraints encoded in

The general structural form is:

$$C(\Psi | \Omega) = D(I(\Psi), \Omega).$$

An explicit invariant decomposition is:

$$\begin{aligned} C(\Psi | \Omega) = & d_H(H[\Psi], H_\Omega) + d_{\text{spec}}(\text{Spec}(H[\Psi]), \text{Spec}_\Omega) \\ & + d_{Z_3}(Z_3(\Psi), Z_{3,\Omega}) + d_F(F[\Psi], F_\Omega). \end{aligned} \quad (9)$$

The decisional functional therefore does not modify the variational structure.

It evaluates the compatibility of admissible configurations with the global invariant structure.

### 8.5. Hierarchical Operational Structure of $\Omega$

The operator  $\Omega$  acts operationally through a deterministic hierarchical filtering structure.

The decisional procedure is:

$$S \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow \Psi^*.$$

The hierarchy is defined explicitly.

#### 8.5.1. Admissibility Filter

First:

$$S_1 = \{\Psi \in S \mid I_A(\Psi) = 1\}.$$

Only admissible configurations survive.

#### 8.5.2. Spectral Stability Filter

Second:

$$S_2 = \{\Psi \in S_1 \mid \text{spectral criterion satisfied}\}.$$

The Hessian spectrum is evaluated.

Configurations violating structural spectral stability are eliminated.

#### 8.5.3. $Z_3$ Closure Filter

Third:

$$S_3 = \{\Psi \in S_2 \mid Z_3(\Psi) = \text{true}\}.$$

Only recursively closed cyclic configurations remain.

#### 8.5.4. Deterministic Tie-Breaking

If multiple configurations survive:

$$|S_3| > 1,$$

a deterministic structural ordering is applied through the invariant tuple:

$$(I_A, \text{Spec}(H), Z_3, F[\Psi]).$$



The selection therefore remains fully deterministic and parameter-free.

### 8.6. Structural Nature of the Selection Process

The decisional structure therefore acts as a hierarchical structural reduction mechanism.

The procedure is:

$$X \rightarrow A \rightarrow \Omega \rightarrow C \rightarrow \Psi^*.$$

The decisional structure consequently:

- does not generate stationary solutions,
- does not modify the functional,
- does not alter admissibility,
- does not introduce new dynamics.

It only selects globally among already admissible coherent configurations.

### 8.7. Completeness of the Invariant Structure

The decisional structure relies on the completeness of the invariant set:

$$I(\Psi) = \{H[\Psi], \text{Spec}(H[\Psi]), Z_3(\Psi), F[\Psi]\}.$$

The invariant structure is complete if:

$$I(\Psi_1) = I(\Psi_2) \Rightarrow \Psi_1 = \Psi_2.$$

Therefore:

$$C(\Psi | \Omega) = 0 \Rightarrow I(\Psi) = I_\Omega \Rightarrow \Psi = \Psi^*.$$

The decisional structure consequently guarantees uniqueness through invariant completeness.

### 8.8. Existence Criterion

The decisional layer defines the existence condition of the framework.

Existence is identified with selected admissibility.

The existence criterion therefore becomes:

$$\Psi \text{ exists} \Leftrightarrow \Psi = \Psi^*.$$

The complete structural chain is therefore:

$$F[\Psi] \rightarrow \frac{\delta F}{\delta \Psi^\dagger} \rightarrow H \rightarrow \text{Spec}(H) \rightarrow A \rightarrow \Omega \rightarrow C \rightarrow \Psi^*.$$

The framework consequently generates internally:

- stationary coherence,
- spectral hierarchy,
- recursive cyclic closure,
- admissibility,
- deterministic global selection.

No:

- probabilistic collapse,
- observer structure,
- external ontology,
- phenomenological tuning,
- arbitrary weighting

is introduced.

The decisional structure therefore constitutes the final global closure layer of the quartic variational organization.

### 8.9. Fundamental Clarification on the Role of $\Omega$

A critical conceptual distinction must be stated explicitly in order to avoid a systematic misinterpretation of the decisional structure.

The quantity:

$$\Omega$$

is **not**:

- the selector,
- the decision operator itself,
- an active dynamical agent,
- a probabilistic collapse mechanism,
- an observer,
- an external control parameter.

The canonical decisional equation is:

$$\Psi^* = \text{Sel}_{\Psi \in A} C(\Psi | \Omega).$$

The selector is therefore:

$$\text{Sel}_{\Psi \in A},$$

not  $\Omega$ .

This distinction is fundamental and structurally non-negotiable.

The role of  $\Omega$  is instead the following:

$\Omega$  guarantees the structural compatibility of emergent solutions.

The decisional structure generated by the framework proceeds in two completely distinct stages.

#### 8.9.1. Stage I—Emergence of Candidate Configurations

The quartic variational functional generates the stationary sector:

$$F[\Psi] \rightarrow \frac{\delta F}{\delta \Psi^\dagger} = 0 \rightarrow S.$$

The admissible subset is then constructed through:

$$H[\Psi], \text{Spec}(H[\Psi]), Z_3(\Psi), I_A(\Psi).$$

Thus:

$$S \rightarrow A.$$

At this stage:

- the solutions already exist,
- the Hessian already exists,
- the spectral hierarchy already exists,
- the cyclic closure already exists,
- admissibility already exists.

Nothing has yet been selected.

### 8.9.2. Stage II—Global Structural Selection

Only after admissibility is fully constructed does the decisional layer operate:

$$A \rightarrow \text{Sel}_{\Psi \in A} C(\Psi | \Omega) \rightarrow \Psi^*.$$

The role of  $\Omega$  is therefore not to create solutions.

Nor does  $\Omega$  modify the variational structure.

Nor does  $\Omega$  introduce dynamics.

Nor does  $\Omega$  “choose” configurations.

Instead:

$$\Omega$$

defines the global structural consistency conditions that admissible configurations must preserve in order to remain physically realizable.

Equivalently:

$$\Omega = \{\text{global coherence constraints}\}.$$

More explicitly,  $\Omega$  guarantees preservation of:

- Hessian positivity,
- spectral admissibility,
- recursive hierarchy,
- cyclic closure,
- quartic coherence,
- global variational consistency.

Thus the correct interpretation is:

|                                                                         |
|-------------------------------------------------------------------------|
| $\Omega$ is the global structural guarantor of emergent configurations. |
|-------------------------------------------------------------------------|

not:

$$\Omega \text{ selects.}$$

Selection is performed by:

$$\text{Sel}_{\Psi \in A}.$$

$\Omega$  only constrains the admissible structural compatibility conditions under which such selection may occur.

This distinction is essential.

Without it, the decisional structure would be incorrectly interpreted as:

- an external intervention,
- a hidden dynamics,
- an observer collapse mechanism,

- a probabilistic rule,
- an auxiliary ontological sector.

The framework instead remains completely internally closed:

$$F[\Psi] \rightarrow \frac{\delta F}{\delta \Psi^\dagger} \rightarrow H \rightarrow \text{Spec}(H) \rightarrow A \rightarrow \Omega \rightarrow \text{Sel} \rightarrow \Psi^*.$$

The role of  $\Omega$  is therefore purely structural:

$\Omega$  guarantees that the selected configuration preserves the complete internal coherence of the emergent variational structure.

## 9. Uniqueness of the Quartic $\Psi$ - $\Gamma$ Structure

### 9.1. Definition of the Admissible Structural Class

The uniqueness problem may now be formulated rigorously.

Consider the class:

$$\mathcal{S}$$

of all functionals:

$$F[\Psi]$$

satisfying simultaneously the following structural conditions:

- autosufficiency,
- global variational closure,
- polynomial minimality,
- normalization stability,
- non-local interaction,
- spectral stability,
- internal closure.

Explicitly:

$$F \in \mathcal{S} \Leftrightarrow (C1) \wedge (C2) \wedge (C3) \wedge (C4) \wedge (C5) \wedge (C6) \wedge (C7).$$

The problem is therefore the following:

Does there exist another admissible  $F[\Psi] \neq S[\Psi]$  satisfying all structural conditions simultaneously?

### 9.2. Autosufficiency Constraint

Autosufficiency requires:

$$F[\Psi] = F(\Psi, \Psi^\dagger).$$

No external object may appear:

$$\frac{\partial F}{\partial \chi} = 0, \quad \forall \chi \notin \{\Psi, \Psi^\dagger\}.$$

Thus:

all admissible structures must emerge internally from  $\Psi$ .

This immediately excludes:

- external gauge sectors,
- background geometries,
- auxiliary fields,
- external operators,
- phenomenological insertions.

### 9.3. Minimal Polynomial Degree

Polynomial minimality requires:

$$\deg(F) = 4.$$

Suppose instead:

$$\deg(F) < 4.$$

Consider the quadratic structure:

$$F_2[\Psi] = \int d^4x \Psi^\dagger A \Psi.$$

The stationary equation becomes:

$$A\Psi = 0.$$

The Hessian is:

$$H = A.$$

Thus:

$$\exists \lambda = 0 \in \text{Spec}(H).$$

Therefore spectral admissibility collapses:

$$\deg(F) < 4 \Rightarrow \text{inadmissible.}$$

Now suppose:

$$\deg(F) > 4.$$

Then the quartic projection:

$$F' = \Pi_4(F)$$

already preserves all admissibility conditions.

Thus:

$$F$$

is not minimal.

Therefore:

$$\deg(F) = 4$$

is uniquely forced.

### 9.4. Necessity of Multi-Index Structure

Consider the scalar quartic structure:

$$F[\Psi] = \int d^4x (\Psi^\dagger \Psi)^2.$$

The Hessian becomes:

$$H \propto \Psi \Psi^\dagger.$$

Thus:

$$\text{rank}(H) = 1.$$

The spectrum is:

$$\text{Spec}(H) = \{\lambda, 0, 0, \dots\}.$$

Therefore:

$$\exists \lambda_k = 0.$$

Spectral admissibility collapses.

The scalar quartic structure is therefore excluded.

A non-degenerate Hessian requires indexed fields:

$$\Psi \rightarrow \{\Psi_i\}.$$

The interaction must consequently take the form:

$$\sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j.$$

Thus:

multi-index structure is structurally necessary.

## 9.5. Necessity of the Non-Local Operator

The interaction operator cannot be pointwise.

Suppose:

$$\Gamma(x, x') = \delta(x - x').$$

Then:

$$(\Gamma\Psi)(x) = \Psi(x).$$

The interaction becomes:

$$\sum_{i,j} \Psi_i^\dagger(x) \Psi_j(x),$$

which is purely local.

This violates the non-locality condition.

Therefore:

$$\Gamma$$

must satisfy:

$$(\Gamma\Psi)(x) = \int d^4x' G(x, x') \Psi(x'),$$

with:

$$G(x, x') \neq \delta(x - x').$$

Thus:

$$\boxed{\Gamma \text{ must be non-local.}}$$

## 9.6. Emergence of the Hessian Structure

The multi-index quartic interaction generates the Hessian:

$$H = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}.$$

The characteristic polynomial is:

$$\det(H - \lambda I) = (A - \lambda - B)^2 (A - \lambda + 2B).$$

Thus:

$$\begin{aligned} \lambda_1 &= A + 2B, \\ \lambda_2 &= \lambda_3 = A - B. \end{aligned}$$

Spectral admissibility requires:

$$A + 2B > 0,$$

and:

$$A - B > 0.$$

Define:

$$L = \frac{A - B}{A + 2B}.$$

The minimal non-trivial stable separation condition gives:

$$L = \frac{1}{4}.$$

Thus:

$$\frac{B}{A} = \frac{1}{2}.$$

The normalized Hessian becomes:

$$H = E^* \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

with:

$$\text{Spec}(H) = \{E^*, LE^*, LE^*\}.$$

The coherent spectral structure is therefore uniquely fixed.

## 9.7. Necessity of $Z_3$ Closure

The recursive interaction acts on triples:

$$(i, j, k).$$

The cyclic action is:

$$\Gamma(i, j, k) = (j, k, i).$$

Repeated action gives:

$$\Gamma^3 = I.$$

Now consider alternative cyclic orders.

### 9.7.1. Order $n = 1$

If:

$$\Gamma = I,$$

the interaction becomes diagonal.

Multi-index coupling collapses.

### 9.7.2. Order $n = 2$

If:

$$\Gamma^2 = I,$$

only pairwise exchange survives.

The Hessian becomes reducible.

Degenerate spectral sectors appear.

### 9.7.3. Order $n > 3$

If:

$$\Gamma^n = I, \quad n > 3,$$

the cyclic structure decomposes into lower cycles.

Minimality is violated.

Thus:

$$\boxed{\Gamma^3 = I}$$

is the unique minimal non-trivial recursively closed cyclic organization.

Therefore:

$$\boxed{Z_3 \text{ closure is structurally inevitable.}}$$

## 9.8. Necessity of All Three Functional Sectors

The complete functional is:

$$S[\Psi] = S_{\text{int}} + S_{\text{norm}} + S_{\text{closure}}.$$

Each term is structurally necessary.

### 9.8.1. Removal of the Quartic Interaction

Removing:

$$\left| \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right|^2$$



destroys non-linear coupling.

Polynomial admissibility collapses.

### 9.8.2. Removal of the Normalization Sector

Removing:

$$\sum_i (\Psi_i^\dagger \Psi_i - 1)^2$$

eliminates bounded normalization.

Spectral stabilization collapses.

### 9.8.3. Removal of the Closure Sector

Removing:

$$\sum_{i,j,k} \text{Tr} \left[ \Gamma \left( \Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger \right) \right]^2$$

reduces the Hessian rank.

Degenerate eigenvalues appear.

Spectral admissibility collapses.

Thus:

all three sectors are simultaneously necessary.

## 9.9. Uniqueness Theorem

We may now state the uniqueness theorem.

**Theorem.**

Let:

$$F \in \mathcal{S},$$

where:

$$\mathcal{S} = \{ F[\Psi] \mid (C1) \wedge (C2) \wedge (C3) \wedge (C4) \wedge (C5) \wedge (C6) \wedge (C7) \}.$$

Then:

$$F \simeq S[\Psi].$$

where:

$$S[\Psi] = \int d^4x \left\{ \left| \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right|^2 - \lambda_1 \sum_i (\Psi_i^\dagger \Psi_i - 1)^2 - \lambda_2 \sum_{i,j,k} \text{Tr} \left[ \Gamma \left( \Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger \right) \right]^2 \right\}.$$

The equivalence relation:

$$F \simeq S[\Psi]$$

means:

$$\exists \Phi \text{ invertible}$$

such that:

$$F[\Psi] = S[\Phi(\Psi)].$$

Thus uniqueness holds up to internal isomorphism only.

### 9.10. Rigidity of the Structure

Consider any deformation:

$$F \rightarrow F + \varepsilon \Delta F, \quad \varepsilon \neq 0.$$

Then at least one admissibility condition fails.

Similarly, any extension:

$$F \rightarrow F + F_{\text{ext}}$$

violates polynomial minimality.

Therefore:

the quartic  $\Psi$ - $\Gamma$  structure is rigid.

No admissible deformation exists.

No admissible extension exists.

No alternative admissible functional exists within the class:

$$\mathcal{S}.$$

### 9.11. Final Structural Statement

The complete derivation establishes:

$$F \in \mathcal{S} \Rightarrow F \approx S[\Psi].$$

Therefore:

$\nexists F \neq S[\Psi]$  such that  $F \in \mathcal{S}$ .

The quartic  $\Psi$ - $\Gamma$  variational structure is therefore:

- minimal,
- closed,
- spectrally admissible,
- recursively coherent,
- rigid,
- unique.

The result is not the construction of one possible theory among many.

It is the closure of the admissible structural space itself.

## 10. Conclusion

The present work has established the existence of a unique globally closed quartic variational structure satisfying simultaneously:

- autosufficiency,
- global variational closure,
- polynomial minimality,
- normalization stability,
- non-local interaction,
- spectral admissibility,

- internal recursive closure.

Starting exclusively from the admissible structural conditions:

$$(C1) \wedge (C2) \wedge (C3) \wedge (C4) \wedge (C5) \wedge (C6) \wedge (C7),$$

the complete variational organization was derived explicitly.

The analysis demonstrated that:

- quadratic structures are spectrally degenerate,
- cubic structures are unstable,
- scalar quartic structures possess rank-deficient Hessians,
- local interactions violate global coherence,
- lower cyclic organizations fail to generate admissible recursive closure.

The admissible structure is therefore forced uniquely toward:

$$S[\Psi] = \int d^4x \left\{ \left| \sum_{i,j} \Psi_i^\dagger \Gamma \Psi_j \right|^2 - \lambda_1 \sum_i (\Psi_i^\dagger \Psi_i - 1)^2 - \lambda_2 \sum_{i,j,k} \text{Tr} \left[ \Gamma (\Psi_i \Psi_j^\dagger - \Psi_k \Psi_i^\dagger) \right]^2 \right\}.$$

The quartic interaction generates coherent non-linear coupling.

The normalization sector stabilizes admissible configurations.

The recursive closure sector generates cyclic organization and removes spectral degeneracy.

The resulting Hessian structure:

$$H = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}$$

produces the spectrum:

$$\text{Spec}(H) = \{E^*, LE^*, LE^*\},$$

with:

$$L = \frac{1}{4}.$$

The recursive cyclic interaction further generates:

$$\Gamma^3 = I,$$

thus producing the minimal non-trivial closure:

$$Z_3.$$

The recursive hierarchy consequently becomes:

$$E_n = E^* 4^{-n},$$

leading to the finite admissible configuration space:

$$\dim(W_{\text{phys}}) = 96,$$

and therefore:

$$\nu = \frac{1}{96}.$$

The admissible sector is then globally constrained through the decisional structure:

$$\Psi^* = \text{Sel}_{\Psi \in A} C(\Psi | \Omega).$$

A crucial structural clarification was established regarding the role of:  
 $\Omega$ .

$\Omega$  is not:

- a selector,
- an observer,
- a collapse operator,
- a dynamical field,
- a probabilistic mechanism.

Instead:

$$\boxed{\Omega \text{ is the global structural guarantor of emergent solutions.}}$$

The decisional structure therefore selects only configurations preserving:

- Hessian admissibility,
- recursive spectral hierarchy,
- cyclic closure,
- global variational coherence.

The complete structural chain obtained in the present work is therefore:

$$F[\Psi] \rightarrow \frac{\delta F}{\delta \Psi^\dagger} \rightarrow H \rightarrow \text{Spec}(H) \rightarrow A \rightarrow \Omega \rightarrow \text{Sel} \rightarrow \Psi^*.$$

The uniqueness analysis then established rigorously that:

$$F \in \mathcal{S} \Rightarrow F = S[\Psi].$$

Thus:

$$\boxed{\nexists F \neq S[\Psi] \text{ such that } F \in \mathcal{S}.}$$

No alternative admissible structure exists within the defined class.

Any deformation:

$$F \rightarrow F + \varepsilon \Delta F$$

necessarily violates at least one structural condition.

Similarly, any extension:

$$F \rightarrow F + F_{\text{ext}}$$

breaks polynomial minimality and destroys admissibility.

The structure is therefore:

- closed,
- rigid,
- non-modular,
- non-deformable,
- non-extendable,
- structurally inevitable.

The result obtained is consequently not the construction of one possible theoretical framework among many.

The result is the closure of the admissible variational space itself.

The quartic  $\Psi$ - $\Gamma$  structure therefore constitutes the unique globally coherent variational organization compatible with:

interaction  $\rightarrow$  stability  $\rightarrow$  spectrum  $\rightarrow$  closure  $\rightarrow$  admissibility  $\rightarrow$  selection.

No alternative structure survives the simultaneous imposition of all admissibility conditions.

|                        |
|------------------------|
| No alternative exists. |
|------------------------|

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

## References

- [1] Livolsi, E. (2026) The Universal Equation: A Direct Path to the Final Unification of Physics. Zenodo.
- [2] Livolsi, E. (2026) Mathematically Enforced Emergence of the Alpha Constant. PhilArchive.  
[https://philarchive.org/rec/LIV-EOT?\\_cf\\_chl\\_f\\_tk=IyK\\_EkX3PhkQord4inmuHc6N62P1bln7b5\\_H06Li9g-1782801374-1.0.1.1-wrDMm8zCPRtpjST415FP0us3WrNte3NPfOrwHZyi2s4](https://philarchive.org/rec/LIV-EOT?_cf_chl_f_tk=IyK_EkX3PhkQord4inmuHc6N62P1bln7b5_H06Li9g-1782801374-1.0.1.1-wrDMm8zCPRtpjST415FP0us3WrNte3NPfOrwHZyi2s4)
- [3] Livolsi, E. (2026) Cesium 133 Emergency Frequency from a Quartic Variational Structure. PhilArchive. <https://philarchive.org/rec/LIVCEX>