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Local Besov Regularity of Weak Solutions to Non-Uniformly Elliptic Obstacle Problems

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Abstract

This paper studies a class of elliptic obstacle problems in Besov spaces. By constructing appropriate admissible functions and using the finite difference theory, local regularity estimates for weak solutions of nonlinear elliptic obstacle problems are obtained.

Keywords

Finite Difference, Besov Space, Elliptic Obstacle Problem, Weak Solution

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 2$. Let $\psi \in W^{1,p}(\Omega)$ be an obstacle function, and introduce the admissible set

$$\mathcal{K}_\psi(\Omega) := \left\{ v \in W^{1,p}(\Omega) : v \geq \psi \text{ a.e. in } \Omega \right\}.$$

This paper mainly considers the nonlinear elliptic obstacle problem of the following form

$$\int_\Omega \left\langle \mathbf{A}(x, Du), D(\varphi - u) \right\rangle \mathrm{d}x \geq 0, \quad \varphi \in \mathcal{K}_\psi(\Omega), \tag{1}$$

where the vector function $\mathbf{A} = \mathbf{A}(x,\xi) : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ is measurable with respect to $x (x \in \Omega)$ differentiable with respect to the gradient variable $\xi \neq 0$, and for a given $p > 2$, satisfies the following conditions for almost every $x \in \Omega$ and all $\xi \in \mathbf{R}^n$:

$$\left\langle \mathbf{A}(x,\xi) - \mathbf{A}(x,\eta), \xi - \eta \right\rangle \geq \nu |\xi - \eta|^2 \left(\mu^2 + |\xi|^2 + |\eta|^2 \right)^{\frac{p-2}{2}}, \tag{2}$$

$$\left| \mathbf{A}(x,\xi) - \mathbf{A}(x,\eta) \right| \leq L |\xi - \eta| \left(\mu^2 + |\xi|^2 + |\eta|^2 \right)^{\frac{p-2}{2}}, \tag{3}$$

$$\left|\mathbf{A}(x,\xi)\right| \le l\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}}, \tag{4}$$

where the constants $\nu, L, l > 0$ and $0 \le \mu \le 1$.

The elliptic obstacle problem is an important field in mathematical research, whose purpose is to find solutions subject to given obstacle constraints under certain conditions. Moreover, it is a typical class of variational inequality problems, and the regularity of its solutions is one of the core issues in the theory of partial differential equations. Obstacle problems are divided into unilateral obstacle problems and bilateral obstacle problems. This paper mainly studies the unilateral elliptic obstacle problem. As is well known, the solutions of obstacle problems cannot be of class C^2, and this is independent of the regularity of the obstacle. Furthermore, this phenomenon gave rise to the origin of the concept of weak solutions and promoted the development of the theory of variational inequalities. The obstacle problem was first proposed by Stampacchia in [1], and its theory has important applications in fields such as elasticity, optimal control, and mathematical finance. Fichera [2] solved the Signorini problem in elastostatics. Lions and Stampacchia [3] studied the so-called variational inequality problems. It is worth mentioning that the regularity of solutions to obstacle problems is related to the obstacle function. For example, in linear obstacle problems, the obstacle function and the solution have the same regularity (see [4]-[6]), while this conclusion does not hold for nonlinear obstacle problems. Therefore, research on the regularity of solutions to nonlinear obstacle problems has been very active, and a series of related results have been obtained, see [7] [8] and the references therein. In summary, the in-depth development of obstacle problems has further enriched the theoretical system of partial differential equations and the calculus of variations.

Over the past decade or so, the regularity theory of obstacle problems has consistently been a focus of widespread attention and in-depth research among many scholars. Ok [9] studied the regularity properties of solutions to a class of obstacle problems with non-standard growth under appropriate assumptions on the variable exponent $p(x)$ and the obstacle function. De Filippis [10] investigated the regularity of solutions to some non-autonomous obstacle problems with (p, q)-growth. Byun and Namkyeong [11] studied the higher-order differentiability of weak solutions to a class of nonlinear elliptic obstacle problems with Orlicz growth. For more studies on elliptic obstacle problems, see references [12]-[16] and the references therein.

In recent years, research on Besov spaces has been increasingly growing. In 2017, Baisón *et al.* [17] considered the local Besov regularity of weak solutions to non-uniformly elliptic systems. In 2019, Clop *et al.* [18] extended the results in [17] and obtained higher-order fractional differentiability of the gradients of solutions to harmonic-type equations. In 2021, Zhang and Zheng [14] proved the regularity of the gradients of solutions to non-uniformly elliptic obstacle problems in Besov spaces. In the same year, Ma and Yao [19] proved local Besov estimates for weak solutions to homogeneous and non-homogeneous elliptic equations.

Subsequently, Zhao and Zheng [20] also proved the Besov regularity for elliptic obstacle problems with double-phase Orlicz growth. For more studies on Besov spaces, see references [21]-[24].

At present, the Besov regularity estimates for weak solutions to the nonlinear elliptic obstacle problem (1) have not yet been studied, and therefore this paper considers it. This paper is mainly inspired by the ideas in Eleuteri and Passarelli di Napoli [25] and Yao [26] regarding the higher-order differentiability of weak solutions to homogeneous elliptic obstacle problems and the Besov estimates for elliptic equations. By means of Hölder's inequality and Young's inequality, and using finite difference theory, we obtain the local Besov regularity of weak solutions to the non-uniformly elliptic obstacle problem (1).

To complete the proof of the main theorem, we will use the following assumption (A1).

Assumption (A1). For $0<\alpha<1$, there exists a measurable non-negative function $g \in L^{n/\alpha}_{loc}(\Omega)$, such that

$$\left|\mathbf{A}(x,\xi)-\mathbf{A}(y,\xi)\right| \le |x-y|^{\alpha}\left(g(x)+g(y)\right)\left(\mu^2+|\xi|^2\right)^{\frac{p-1}{2}}$$

for any $\xi \in \mathbf{R}^n$ and a.e. $x, y \in \Omega$.

Define the auxiliary function

$$V_p(\xi) := \left(\mu^2+|\xi|^2\right)^{\frac{p-2}{4}}\xi, \ \forall \xi \in \mathbf{R}^n.$$

Under the above auxiliary functions, using the finite difference method, Young's inequality and Hölder's inequality, we obtain the local Besov regularity of solutions to a class of homogeneous variational obstacle problems on bounded open domains. The main results of this paper differ from those in reference [14] in the following respects: reference [14] studies the Besov regularity of elliptic obstacle problems with double-phase growth, whereas this paper considers the regularity of solutions to a class of homogeneous elliptic obstacle problems satisfying condition (2)-(4), and the conditions used in the proof differ from those employed in reference [14]. Under the main regular assumptions (A1) imposed on $p(x)$ and $\mathbf{A}(x,\xi)$, we prove that the nonlinearity of gradients for the solution u inherits a higher fractional differentiability result provided that the gradient of obstacle $\nabla\psi$ satisfy a suitable Besov regularity. More precisely, our results are the following:

Theorem 1.1 Let $0<\alpha<1$. Suppose $\mathbf{A}$ satisfies conditions (2)-(4) and assumption (A1). If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of the elliptic obstacle problem (1), then for any $0<\alpha<\beta<1$, we have

$$D\psi \in B^{\beta}_{p,\infty,\mathrm{loc}}(\Omega) \Rightarrow V_p(Du) \in B^{\alpha}_{2,\infty,\mathrm{loc}}(\Omega).$$

The rest of this article is organized as follows. In Section 2, we present some classical notations, some basic definitions, and some auxiliary lemmas. In Section 3, we devote ourselves to the proofs of the main theorem, where we employ the

finite difference method to establish the higher fractional differentiability results.

2. Preliminaries

The constant C may vary from line to line in this paper and merely denotes a dependence on the relevant parameters. For any given $x_0 \in \Omega$, and constant $R>0$, we define $B_R(x_0) := B(x_0,R) := \{x : |x-x_0| < R\}$ to denote the open ball centered at x_0 with radius R. Without causing confusion, the center x_0 will be omitted. First, we give the definition of Besov spaces.

Definition 2.1 [19] For $0<\alpha<1$, $1 \le p,q < \infty$ and $h \in \Omega$. we say that $u : \mathbf{R}^n \to \mathbf{R}$ belongs to the Besov space $B^{\alpha}_{p,q}(\Omega)$ if $u \in L^p(\Omega)$ and

$$\|u\|_{B^{\alpha}_{p,q}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{B^{\alpha}_{p,q}(\Omega)} < \infty,$$

where

$$\|u\|_{L^p(\Omega)} := \left(\int_{\Omega} |u(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$$

and

$$[u]_{B^{\alpha}_{p,q}(\Omega)} := \left(\int_{\Omega}\left(\int_{\Omega} \frac{|\Delta_h u(x)|^p}{|h|^{\alpha p}} \mathrm{d}x\right)^{\frac{q}{p}} \frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}} < \infty,$$

where the finite difference operator $\Delta_h u(x) = \chi_{\Omega}(x+h)\left[u(x+h)-u(x)\right]$.

Similarly, we say that $u \in B^{\alpha}_{p,\infty}(\Omega)$ if $u \in L^p(\Omega)$ and $\dfrac{\Delta_h u}{|h|^{\alpha}} \in L^{\infty}\left(\dfrac{\mathrm{d}h}{|h|^n}; L^p(\Omega)\right)$ with the norm

$$\|u\|_{B^{\alpha}_{p,\infty}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{B^{\alpha}_{p,\infty}(\Omega)} < \infty,$$

where

$$[u]_{B^{\alpha}_{p,\infty}(\Omega)} := \sup_{h \in \mathbf{R}^n}\left(\int_{\Omega} \frac{|\Delta_h u(x)|^p}{|h|^{\alpha p}} \mathrm{d}x\right)^{\frac{1}{p}}. \tag{5}$$

The definition and properties of the finite difference operator are given below. $\forall f : \Omega \subset \mathbf{R}^n \to \mathbf{R}$, the finite difference operator is defined as

$$\Delta_h f(x) := f(x+h) - f(x).$$

We consider a compact subset inside Ω:

$$\Omega_{|h|} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > |h|\}.$$

Lemma 2.2 [25] Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \ge 1$, then

(1) $\Delta_h F \in W^{1,p}\left(\Omega_{|h|}\right)$ and $D_i\left(\Delta_h F\right) = \Delta_h\left(D_i F\right)$.

(2) If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then

$$\int_{\Omega} F\Delta_h G\mathrm{d}x = \int_{\Omega} G\Delta_{-h} F\mathrm{d}x.$$

(3) We have

$$\Delta_h\left(FG\right)\left(x\right) = F\left(x+h\right)\Delta_h G\left(x\right) + G\left(x\right)\Delta_h F\left(x\right).$$

Lemma 2.3 [14] If $0<\rho<R$, $|h|<\frac{R-\rho}{2}$, $1<p<+\infty$, and $F, DF \in L^p\left(B_R\right)$ then

$$\int_{B_\rho}\left|\Delta_h F\left(x\right)\right|^p \mathrm{d}x \le c\left(n,p\right)|h|^p \int_{B_R}\left|DF\left(x\right)\right|^p \mathrm{d}x.$$

Moreover,

$$\int_{B_\rho}\left|F\left(x+h\right)\right|^p \mathrm{d}x \le \int_{B_R}\left|F\left(x\right)\right|^p \mathrm{d}x.$$

Lemma 2.4 [15] [27] Let $U\subset\mathbf{R}^n$ be a bounded domain, for $0<\beta<\alpha$ we have the following conclusions:

(1) if $1<p<+\infty$ and $1\le s\le t\le+\infty$, then $B^{\alpha}_{p,s}\left(U\right)\subset B^{\alpha}_{p,t}\left(U\right)$;

(2) if $1<p<+\infty$ and $1\le s,\, t\le+\infty$, then $B^{\alpha}_{p,s}\left(U\right)\subset B^{\beta}_{p,t}\left(U\right)$.

Lemma 2.5 [25] Let $F\in L^2\left(B_R\right)$. Suppose that there exist $\rho\in\left(0,R\right)$, $\alpha\in\left(0,1\right)$ and $M>0$ such that

$$\sum_{s=1}^{n}\int_{B_\rho}\left|\Delta_{s,h}F\left(x\right)\right|^2 \mathrm{d}x \le M^2|h|^{2\alpha}$$

for $|h|<\frac{R-\rho}{2}$. Then $F\in L^{\frac{2n}{n-2\beta}}\left(B_\rho\right)$ for every $\beta\in\left(0,\alpha\right)$ and

$$\|F\|_{L^{\frac{2n}{n-2\beta}}\left(B_\rho\right)} \le c\left(M+\|F\|_{L^2\left(B_R\right)}\right)$$

with $c=c\left(n,N,p,\rho,R,\alpha,\beta\right)$.

Lemma 2.6 [25] Let $1<p<\infty$. There exists a constant $c=c\left(n,p\right)>0$ such that

$$c^{-1}\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}} \le \frac{\left|V_p\left(\xi\right)-V_p\left(\eta\right)\right|^2}{|\xi-\eta|^2} \le c\left(\mu^2+|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}$$

for any $\xi,\eta\in\mathbf{R}^n$.

The following lemma plays a crucial role in the proof of Theorem 1.1.

Lemma 2.7 [25] If $\mathbf{A}$ satisfies (2)-(4) and assumption (A1), then $\mathbf{A}$ is locally uniformly in VMO (vanishing mean oscillation).

From Lemma 2.7 and [28] ($p\left(x\right)=p$), the Calderón-Zygmund type estimate for the obstacle problem (1) can be obtained (see Lemma 2.8).

Lemma 2.8 [25] Let $2\le p\le n$, and $q>p$. Suppose that assumptions (2)-(4) hold and that $x\mapsto\mathbf{A}\left(x,\xi\right)$ is locally uniformly in VMO. Let $u\in\mathcal{K}_\psi\left(\Omega\right)$ be a weak solution of the obstacle problem (1). Then the following relation holds:

$$D\psi\in L^q_{\mathrm{loc}}\left(\Omega\right)\Rightarrow Du\in L^q_{\mathrm{loc}}\left(\Omega\right).$$

3. Proof of the Main Theorem

This part is mainly devoted to the proofs of Theorem 1.1. More precisely, under suitable assumptions on the operator $A(x,\xi)$ and the gradient of the obstacle $D\psi$, and based on Calderón-Zygmund type estimates and Besov space embedding relations, we established the higher fractional differentiability of the gradient of weak solutions to the elliptic obstacle problem (1) by employing the finite difference method.

Proof of Theorem 1.1: Let u be a weak solution of the variational inequality (1). For any $B_{2R} \subset\subset \Omega$, take a cut-off function $\eta \in C_0^\infty(\Omega)$, satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{R/2}$, $\eta \equiv 0$ on $\Omega \setminus B_R$ and $|D\eta| \le \frac{C}{R}$.

We assume $\varphi = u - \frac{1}{2}\Delta_{-h}\left(\eta^2 \Delta_h (u-\psi)\right)$, where $|h|$ is chosen sufficiently small to ensure $x+h \in \Omega$. We can verify that $\varphi \in \mathcal{K}_\psi(\Omega)$ through the following computation

$$\begin{aligned}
\varphi - \psi &= u - \frac{1}{2}\Delta_{-h}\left(\eta^2 \Delta_h (u-\psi)\right) - \psi \\
&= u - \psi + \frac{1}{2}\eta^2(x)\left(u(x+h) - \psi(x+h)\right) - \frac{1}{2}\eta^2(x)\left(u(x) - \psi(x)\right) \\
&\quad - \frac{1}{2}\eta^2(x-h)\left(u(x) - \psi(x)\right) + \frac{1}{2}\eta^2(x-h)\left(u(x-h) - \psi(x-h)\right) \\
&= \frac{1}{2}\eta^2(x)\left(u(x+h) - \psi(x+h)\right) + \frac{1}{2}\eta^2(x-h)\left(u(x-h) - \psi(x-h)\right) \\
&\quad + \left(\frac{1}{2} - \frac{1}{2}\eta^2(x)\right)\left(u(x) - \psi(x)\right) + \left(\frac{1}{2} - \frac{1}{2}\eta^2(x-h)\right)\left(u(x) - \psi(x)\right) \\
&\ge 0,
\end{aligned}$$

where we have used the $u \ge \psi$ and $0 \le \eta \le 1$ in the last inequality. Thus $\varphi(x)$ can be used as the admissible function of the variational inequality (1), we obtain

$$\int_\Omega \left\langle \mathbf{A}(x, Du), D\left(-\frac{1}{2}\Delta_{-h}\left(\eta^2 \Delta_h (u-\psi)\right)\right)\right\rangle \mathrm{d}x \ge 0,$$

By Lemma 2.2, we obtain

$$\int_{B_R} \left\langle \Delta_h \mathbf{A}(x, Du), D\left(\eta^2 \Delta_h (u-\psi)\right)\right\rangle \mathrm{d}x \le 0.$$

Rearranging the above expression yields

$$\begin{aligned}
I_1 &= \int_{B_R} \left\langle \mathbf{A}\left(x+h, Du(x+h)\right) - \mathbf{A}\left(x+h, Du(x)\right), \eta^2 D\left(\Delta_h u\right)\right\rangle \mathrm{d}x \\
&\le \int_{B_R} \left\langle \mathbf{A}\left(x+h, Du(x+h)\right) - \mathbf{A}\left(x+h, Du(x)\right), \eta^2 D\left(\Delta_h \psi\right)\right\rangle \mathrm{d}x \\
&\quad - 2\int_{B_R} \left\langle \mathbf{A}\left(x+h, Du(x+h)\right) - \mathbf{A}\left(x+h, Du(x)\right), \eta D\eta \Delta_h (u-\psi)\right\rangle \mathrm{d}x \\
&\quad + \int_{B_R} \left\langle \mathbf{A}\left(x, Du(x)\right) - \mathbf{A}\left(x+h, Du(x)\right), \eta^2 \Delta_h \left(Du - D\psi\right)\right\rangle \mathrm{d}x \\
&\quad + 2\int_{B_R} \left\langle \mathbf{A}\left(x, Du(x)\right) - \mathbf{A}\left(x+h, Du(x)\right), \eta D\eta \Delta_h (u-\psi)\right\rangle \mathrm{d}x \\
&= I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Estimate of I_1. By condition (2), we have

$$I_1 \geq \nu \int_{B_R} \left(\mu^2 + \left|Du(x)\right|^2 + \left|Du(x+h)\right|^2\right)^{\frac{p-2}{2}} \left|\Delta_h Du(x)\right|^2 \eta^2 \mathrm{d}x.$$

Estimate of I_2. By condition (3), Young's inequality, Lemma 2.3, Hölder's inequality and the properties of η, we have

$$\begin{aligned}
\left|I_2\right| &\leq \int_{B_R} \left|\mathbf{A}\left(x+h, Du(x+h)\right) - \mathbf{A}\left(x+h, Du(x)\right)\right| \eta^2 \left|\Delta_h D\psi\right| \mathrm{d}x \\
&\leq L \int_{B_R} \left(\mu^2 + \left|Du(x+h)\right|^2 + \left|Du(x)\right|^2\right)^{\frac{p-2}{2}} \left|\Delta_h Du\right| \eta^2 \left|\Delta_h D\psi\right| \mathrm{d}x \\
&\leq \varepsilon \int_{B_R} \left(\mu^2 + \left|Du(x+h)\right|^2 + \left|Du(x)\right|^2\right)^{\frac{p-2}{2}} \left|\Delta_h Du\right|^2 \eta^2 \mathrm{d}x \\
&\quad + C(\varepsilon, L) \int_{B_R} \left(\mu^2 + \left|Du(x+h)\right|^2 + \left|Du(x)\right|^2\right)^{\frac{p-2}{2}} \left|\Delta_h D\psi\right|^2 \eta^2 \mathrm{d}x \\
&\leq \varepsilon \int_{B_R} \left(\mu^2 + \left|Du(x+h)\right|^2 + \left|Du(x)\right|^2\right)^{\frac{p-2}{2}} \left|\Delta_h Du\right|^2 \eta^2 \mathrm{d}x \\
&\quad + C(\varepsilon, L) \left(\int_{B_{2R}} \left(\mu^p + \left|Du(x)\right|^p\right) \mathrm{d}x\right)^{\frac{p-2}{p}} \left(\int_{B_R} \left|\Delta_h D\psi\right|^p \mathrm{d}x\right)^{\frac{2}{p}}.
\end{aligned}$$

Estimate of I_3. Similar to the estimate for I_2, we get

$$\begin{aligned}
\left|I_3\right| &\leq 2 \int_{B_R} \left|\mathbf{A}\left(x+h, Du(x+h)\right) - \mathbf{A}\left(x+h, Du(x)\right)\right| \eta D\eta \left|\Delta_h (u-\psi)\right| \mathrm{d}x \\
&\leq L \int_{B_R} \left(\mu^2 + \left|Du(x+h)\right|^2 + \left|Du(x)\right|^2\right)^{\frac{p-2}{2}} \left|\Delta_h Du\right| \eta D\eta \left|\Delta_h (u-\psi)\right| \mathrm{d}x \\
&\leq \varepsilon \int_{B_R} \left(\mu^2 + \left|Du(x+h)\right|^2 + \left|Du(x)\right|^2\right)^{\frac{p-2}{2}} \left|\Delta_h Du\right|^2 \eta^2 \mathrm{d}x \\
&\quad + C(\varepsilon, L) \int_{B_R} \left(\mu^2 + \left|Du(x+h)\right|^2 + \left|Du(x)\right|^2\right)^{\frac{p-2}{2}} \left|D\eta\right|^2 \left|\Delta_h (u-\psi)\right|^2 \mathrm{d}x \\
&\leq \varepsilon \int_{B_R} \left(\mu^2 + \left|Du(x+h)\right|^2 + \left|Du(x)\right|^2\right)^{\frac{p-2}{2}} \left|\Delta_h Du\right|^2 \eta^2 \mathrm{d}x \\
&\quad + \frac{C}{R^2} |h|^2 \left(\int_{B_{2R}} \left(\mu^p + \left|Du(x)\right|^p\right) \mathrm{d}x\right)^{\frac{p-2}{p}} \left(\int_{B_{2R}} \left(\left|Du\right|^p + \left|D\psi\right|^p\right) \mathrm{d}x\right)^{\frac{2}{p}}.
\end{aligned}$$

Estimate of I_4. By assumption (A1) and the triangle inequality, we have

$$\begin{aligned}
\left|I_4\right| &\leq \int_{B_R} \left|\mathbf{A}\left(x, Du(x)\right) - \mathbf{A}\left(x+h, Du(x)\right)\right| \eta^2 \left|\Delta_h \left(Du - D\psi\right)\right| \mathrm{d}x \\
&\leq |h|^\alpha \int_{B_R} \left(g(x) + g(x+h)\right) \left(\mu^2 + \left|Du(x)\right|^2\right)^{\frac{p-1}{2}} \eta^2 \left|\Delta_h \left(Du - D\psi\right)\right| \mathrm{d}x \\
&= |h|^\alpha \int_{B_R} \left(g(x) + g(x+h)\right) \left(\mu^2 + \left|Du(x)\right|^2\right)^{\frac{p-1}{2}} \eta^2 \left|\Delta_h Du\right| \mathrm{d}x \qquad (6) \\
&\quad + |h|^\alpha \int_{B_R} \left(g(x) + g(x+h)\right) \left(\mu^2 + \left|Du(x)\right|^2\right)^{\frac{p-1}{2}} \eta^2 \left|\Delta_h D\psi\right| \mathrm{d}x \\
&=: \left|I_{41}\right| + \left|I_{42}\right|.
\end{aligned}$$

Next, we estimate I_{41} and I_{42} respectively. To estimate I_{41}, by Young's inequality, the properties of η, and Hölder's inequality, we obtain

$$\begin{aligned}|I_{41}| &\le \varepsilon\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\Delta_h Du|^2\,\mathrm{d}x\\ &\quad + C(\varepsilon)|h|^{2\alpha}\int_{B_R}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\left(g(x)+g(x+h)\right)^2\mathrm{d}x\\ &\le \varepsilon\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\Delta_h Du|^2\,\mathrm{d}x\\ &\quad + C(\varepsilon)|h|^{2\alpha}\left(\int_{B_R}\left(g(x)+g(x+h)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{2\alpha}{n}}\left(\int_{B_R}\mu^{\frac{np}{n-2\alpha}}+|Du(x)|^{\frac{np}{n-2\alpha}}\,\mathrm{d}x\right)^{\frac{n-2\alpha}{n}}.\end{aligned}\tag{7}$$

Estimate of I_{42}. By Hölder's inequality and the properties of η, we get

$$\begin{aligned}|I_{42}| &\le |h|^{\alpha}\left(\int_{B_R}\eta^2|\Delta_h D\psi|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{p}{p-1}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{p-1}{p}}\\ &\le |h|^{\alpha}\left(\int_{B_R}|\Delta_h D\psi|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{n}}\left(\int_{B_R}\left(\mu+|Du(x)|\right)^{\frac{np(p-1)}{n(p-1)-p\alpha}}\mathrm{d}x\right)^{\frac{n(p-1)-p\alpha}{np}}.\end{aligned}\tag{8}$$

Substituting Equations (7) and (8) into Equation (6), we obtain

$$\begin{aligned}|I_4| &\le \varepsilon\int_{B_R}\eta^2\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\Delta_h Du|^2\,\mathrm{d}x\\ &\quad + C(\varepsilon)|h|^{2\alpha}\left(\int_{B_R}\left(g(x)+g(x+h)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{2\alpha}{n}}\left(\int_{B_R}\mu^{\frac{np}{n-2\alpha}}+|Du(x)|^{\frac{np}{n-2\alpha}}\,\mathrm{d}x\right)^{\frac{n-2\alpha}{n}}\\ &\quad + |h|^{\alpha}\left(\int_{B_R}|\Delta_h D\psi|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{n}}\left(\int_{B_R}\left(\mu+|Du(x)|\right)^{\frac{np(p-1)}{n(p-1)-p\alpha}}\mathrm{d}x\right)^{\frac{n(p-1)-p\alpha}{np}}.\end{aligned}$$

Estimate of I_5. By assumption (A1), Hölder's inequality, the properties of η, and Lemma 2.3, we have

$$\begin{aligned}|I_5| &\le 2\int_{B_R}\left|\mathbf{A}(x,Du(x))-\mathbf{A}(x+h,Du(x))\right|\eta D\eta\left|\Delta_h(u-\psi)\right|\mathrm{d}x\\ &\le C|h|^{\alpha}\int_{B_R}\left(g(x)+g(x+h)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}\eta D\eta\left|\Delta_h(u-\psi)\right|\mathrm{d}x\\ &\le C|h|^{\alpha}\left(\int_{B_R}\eta^2|\Delta_h(u-\psi)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_R}|D\eta|^{\frac{p}{p-1}}\left(g(x+h)+g(x)\right)^{\frac{p}{p-1}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{p-1}{p}}\\ &\le \frac{C}{R}|h|^{\alpha+1}\left(\int_{B_{2R}}\left(|Du|+|D\psi|\right)^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{n}{\alpha}}dx\right)^{\frac{\alpha}{n}}\left(\int_{B_R}\left(\mu+|Du(x)|\right)^{\frac{np(p-1)}{n(p-1)-p\alpha}}\mathrm{d}x\right)^{\frac{n(p-1)-p\alpha}{np}}.\end{aligned}$$

Combining the estimate for $I_i\,(1\le i\le 5)$ and taking $\varepsilon=\frac{\nu}{8}$, we get

$$\begin{aligned}&\int_{B_R}\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\Delta_h Du(x)|^2\eta^2\mathrm{d}x\\ &\le C\left(\int_{B_{2R}}\left(\mu^p+|Du(x)|^p\right)\mathrm{d}x\right)^{\frac{p-2}{p}}\left(\int_{B_R}|\Delta_h D\psi|^p\,\mathrm{d}x\right)^{\frac{2}{p}}\\ &\quad + \frac{C}{R^2}|h|^2\left(\int_{B_{2R}}\left(\mu^p+|Du(x)|^p\right)\mathrm{d}x\right)^{\frac{p-2}{p}}\left(\int_{B_{2R}}\left(|Du|^p+|D\psi|^p\right)\mathrm{d}x\right)^{\frac{2}{p}}\end{aligned}$$

$$+C(\varepsilon)|h|^{2\alpha}\left(\int_{B_R}\left(g(x)+g(x+h)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{2\alpha}{n}}\left(\int_{B_R}\mu^{\frac{np}{n-2\alpha}}+|Du(x)|^{\frac{np}{n-2\alpha}}\,\mathrm{d}x\right)^{\frac{n-2\alpha}{n}}$$
$$+|h|^{\alpha}\left(\int_{B_R}|\Delta_h D\psi|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{n}}\left(\int_{B_R}\left(\mu+|Du(x)|\right)^{\frac{np(p-1)}{n(p-1)-p\alpha}}\mathrm{d}x\right)^{\frac{n(p-1)-p\alpha}{np}} \tag{9}$$
$$+\frac{C}{R}|h|^{\alpha+1}\left(\int_{B_{2R}}\left(|Du|+|D\psi|\right)^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{n}}\left(\int_{B_R}\left(\mu+|Du(x)|\right)^{\frac{np(p-1)}{n(p-1)-p\alpha}}\mathrm{d}x\right)^{\frac{n(p-1)-p\alpha}{np}}.$$

Next, we prove that $Du(x)\in L^{np/(n-2\alpha)}_{\mathrm{loc}}(\Omega)$. By Lemma 2.5, it follows that $|V_p(D\psi)|\in L^{2n/(n-2\alpha)}_{\mathrm{loc}}(\Omega)$ and by definition (5), we have $|D\psi|^p\le|V_p(D\psi)|^2$.

Moreover, since $\frac{np(p-1)}{n(p-1)-\alpha p}\le\frac{np}{n-2\alpha}(p\ge 2)$, we can obtain

$D\psi\in L^{np/(n-2\alpha)}_{\mathrm{loc}}(\Omega)\subset L^{np(p-1)/(n(p-1)-\alpha p)}_{\mathrm{loc}}(\Omega)$. By Lemma 2.8, we obtain

$Du\in L^{np/(n-2\alpha)}_{\mathrm{loc}}(\Omega)\subset L^{np(p-1)/(n(p-1)-\alpha p)}_{\mathrm{loc}}(\Omega)$. Hence, we have

$$\int_{B_R}|Du(x)|^{\frac{np(p-1)}{n(p-1)-p\alpha}}\mathrm{d}x\le C\quad\text{and}\quad\int_{B_R}|Du(x)|^{\frac{np}{n-2\alpha}}\mathrm{d}x\le C$$

Therefore, Equation (9) becomes

$$\int_{B_R}\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\Delta_h Du(x)|^2\eta^2\mathrm{d}x$$
$$\le C\left(\int_{B_R}|\Delta_h D\psi|^p\,\mathrm{d}x\right)^{\frac{2}{p}}+\frac{C}{R^2}|h|^2+C(\varepsilon)|h|^{2\alpha}\left(\int_{B_R}\left(g(x)+g(x+h)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{2\alpha}{n}} \tag{10}$$
$$+|h|^{\alpha}\left(\int_{B_R}|\Delta_h D\psi|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{n}}+\frac{C}{R}|h|^{2\alpha}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{n}},$$

where $C=C(n,p,\alpha,R)$ and we have also used the inequality $|h|^{\alpha+1}\le|h|^{2\alpha}$.

By Lemma 2.6 and Equation (10), we have

$$\int_{B_R}\left|\Delta_h\left(V_p(Du)\right)\right|^2\eta^2\mathrm{d}x$$
$$\le\int_{B_R}\left(\mu^2+|Du(x)|^2+|Du(x+h)|^2\right)^{\frac{p-2}{2}}|\Delta_h Du(x)|^2\eta^2\mathrm{d}x$$
$$\le C\left(\int_{B_R}|\Delta_h D\psi|^p\,\mathrm{d}x\right)^{\frac{2}{p}}+\frac{C}{R^2}|h|^2+C(\varepsilon)|h|^{2\alpha}\left(\int_{B_R}\left(g(x)+g(x+h)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{2\alpha}{n}} \tag{11}$$
$$+|h|^{\alpha}\left(\int_{B_R}|\Delta_h D\psi|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{n}}+\frac{C}{R}|h|^{2\alpha}\left(\int_{B_R}\left(g(x+h)+g(x)\right)^{\frac{n}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{n}}.$$

Dividing both sides of Equation (11) by $|h|^{2\alpha}$ and using the properties of η, we have

$$\int_{B_{R/2}}\frac{\left|\Delta_h\left(V_p(Du)\right)\right|^2}{|h|^{2\alpha}}\mathrm{d}x\le C\left(\int_{B_R}\frac{|\Delta_h D\psi|^p}{|h|^{\alpha p}}\mathrm{d}x\right)^{\frac{2}{p}}+\frac{C}{R^2}|h|^{2(1-\alpha)}+C\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\mathrm{d}x\right)^{\frac{2\alpha}{n}}$$
$$+C\left(\int_{B_R}\frac{|\Delta_h D\psi|^p}{|h|^{\alpha p}}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\mathrm{d}x\right)^{\frac{\alpha}{n}}+\frac{C}{R}\left(\int_{B_{2R}}g^{\frac{n}{\alpha}}(x)\mathrm{d}x\right)^{\frac{\alpha}{n}}, \tag{12}$$

where we have also used Lemma 2.3. Since Equation (12) holds for any h, taking the supremum over $h \in B(0,\delta)(\delta < R)$, we have

$$\begin{aligned}\sup_{|h|<\delta}\int_{B_{R/2}}\frac{\left|\Delta_h\left(V_p(Du)\right)\right|^2}{|h|^{2\alpha}}\mathrm{d}x &\le C\|D\psi\|_{B^{\alpha}_{p,\infty,\mathrm{loc}}(\Omega)}+\frac{C}{R^2}\delta^{2(1-\alpha)}+C\|g\|^2_{L^{\frac{n}{\alpha}}_{\mathrm{loc}}(B_{2R})}\\ &\quad +C\|D\psi\|_{B^{\alpha}_{p,\infty,\mathrm{loc}}(\Omega)}\|g\|_{L^{\frac{n}{\alpha}}_{\mathrm{loc}}(B_{2R})}+\frac{C}{R}\|g\|_{L^{\frac{n}{\alpha}}_{\mathrm{loc}}(B_{2R})}\\ &< +\infty,\end{aligned}$$

where we have also used $g \in L^{n/\alpha}_{\mathrm{loc}}(\Omega)$ and $D\psi \in B^{\beta}_{p,\infty,\mathrm{loc}}(\Omega) \subset B^{\alpha}_{p,\infty,\mathrm{loc}}(\Omega)$, by the fact that Lemma 2.4, so $V_p(Du) \in B^{\alpha}_{2,\infty,\mathrm{loc}}(\Omega)$. This completes the proof of Theorem 1.1.

4. Conclusion

This paper is mainly devoted to investigating the local Besov space regularity estimates for the gradients of weak solutions associated with a certain class of nonlinear elliptic obstacle problems defined within a bounded open domain. Throughout the whole research procedure, we construct a suitable admissible function, and carry out the derivation and rigorous argumentation by making full use of several fundamental analytical tools including Hölder's inequality, Young's inequality and the finite difference method. As a consequence, we successfully establish the local Besov-type regularity estimation results for the weak solutions corresponding to this family of nonlinear variational inequalities.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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