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Finite Time Domain Dynamics of Photonic Fields

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Abstract

Here we consider the case of a photonic field in a finite time interval. We expand the corresponding field in annihilation and creation operators, derive the corresponding Hamiltonian and path integrate. So, we obtain the photonic field's finite time interval Green function. As an application we consider the generation of photons from currents.

Keywords

Photon, Finite Time Domain, Time Evolution, Path Integrals

1. Introduction

QFT constitutes an important area with a variety of applications [1]-[4]. It deals with the dynamics of particles characterized by their spin, charge and mass. It is composed by a variety of approaches including lattice quantum field theory [5] [6] and numerical techniques [7] [8].

In previous work we have considered the finite time evolution of systems which include Dirac fields, scalar fields and massive or massless vector fields [9]-[12], in contrast to the infinite time in/out scattering study. The whole approach can be applied in the study of the time evolution of appropriately prepared systems or their possible decay. Here we develop further our techniques and study the finite time interval generation and evolution of photonic fields via the path integral evaluation of the Green function.

We start from the standard Lagrangian and Hamiltonian densities, expand appropriately the field in term of annihilation and creation operators, derive the corresponding Hamiltonian and path integrate. So, for vacuum-to-vacuum transitions we get the corresponding Green function. It gives full dynamical information of the photonic field. It is used in the study of the generation of photons

from currents and their possible evolution in spacetime.

The present paper develops as follows. In Section 2 we give the Lagrangian and the Hamiltonian of the present photonic system and expand in terms of annihilation and creation operators. In Section 3 we calculate the path integral within a finite time interval and derive the transition amplitude from vacuum to vacuum. In Section 4 we extract the Green function while in Section 5 we use that Green function in the study of the generation and evolution of photons. Finally in Section 6 we give our conclusions.

Here we set $c=\hbar=1$. Then $e^2=\alpha\simeq\frac{1}{137}$. α is the fine structure constant. Throughout the paper Greek indices take values from 0 to 3 while Latin ones take values from 1 to 3.

2. System and Quantization

Here at first we quantize the photonic field. We proceed in real time and use the metric $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$ where the first component corresponds to time. We represent the spacetime with $\{x_0\equiv t,\boldsymbol{x}\}$, $\boldsymbol{x}\in R^3$, and the photon field with $A_\mu=\{A_0,\boldsymbol{A}\}$. We set $F_{\mu\nu}(t,\boldsymbol{x})$ to be the field strength tensor. Then the Lagrangian of the photon field couple with a real current J_μ is given as

$$L=-\frac{1}{4}F_{\mu\nu}(t,\boldsymbol{x})F^{\mu\nu}(t,\boldsymbol{x})-\frac{1}{2}\lambda\left(\partial_\mu A^\mu(t,\boldsymbol{x})\right)^2-J_\mu(t,\boldsymbol{x})A^\mu(t,\boldsymbol{x}) \tag{1}$$

We introduce a gauge fixing term and λ is a Lagrange multiplier [1] [2]. Further

$$F^{\mu\nu}(t,\boldsymbol{x})=\partial^\mu A^\nu(t,\boldsymbol{x})-\partial^\nu A^\mu(t,\boldsymbol{x}) \tag{2}$$

On applying variational techniques, the photonic field satisfies the equation

$$\Box A^\nu-(1-\lambda)\partial^\nu\left(\partial_\mu A^\mu\right)=J^\nu \tag{3}$$

where $\Box$ is the d'Alembertian. Now we act on the homogeneous form of Equation (3) with the divergence and get $\Box\left(\partial_\mu A^\mu\right)=0$. Then we act on the homogeneous form of Equation (3) with the d'Alembertian and finally we get

$$\Box\Box A^\mu=0 \tag{4}$$

The conjugate momenta are given as

$$\pi^\nu(t,\boldsymbol{x})=-F^{0\nu}(t,\boldsymbol{x})-\lambda g^{0\nu}\left(\partial_\mu A^\mu\right) \tag{5}$$

Then we get $\pi^0=-\lambda\partial_\rho A^\rho$ which is nonzero. Further, the electric field is given as $E^i=\pi^i=-F^{0i}=-\partial^0A^i+\partial^iA^0$. So, the Hamiltonian becomes

$$\begin{aligned}H(E,A)&=\pi^\mu\partial_0A_\mu-L=\pi^i\left(-\pi_i+\partial_iA_0\right)-\lambda\partial_\rho A^\rho\partial_0A_0-L\\&=\frac{1}{2}\boldsymbol{E}^2+\frac{1}{4}F^{ij}F_{ij}-\left(\partial_i\pi^i\right)A^0+\frac{1}{2}\lambda\left(\partial_\rho A^\rho\right)^2\\&\quad-\lambda\partial_\rho A^\rho\partial_0A^0+J^0A^0-\boldsymbol{J}\cdot\boldsymbol{A}\\&=\frac{1}{2}\boldsymbol{E}^2+\frac{1}{2}\boldsymbol{B}^2-(\nabla\cdot\boldsymbol{E})A^0-\frac{1}{2}\lambda\left[\left(\partial_0A^0\right)^2-\left(\partial_iA^i\right)^2\right]\\&\quad+J^0A^0-\boldsymbol{J}\cdot\boldsymbol{A}\end{aligned} \tag{6}$$

We have used $\partial^0 A^i = F^{0i} + \partial^i A^0 = -\pi^i + \partial^i A^0$ and integrated by parts to set the replacement $\pi^i \partial_i A_0 \to \left(\partial_i \pi^i\right) A^0$. Further $\partial_i \pi^i = \nabla \cdot \boldsymbol{E}$. We notice that the magnetic field has the form $B^i = \frac{1}{2}\varepsilon^{ijk} F^{jk}$. ε^{ijk} is the 3-dimensional totally antisymmetric tensor with $\varepsilon^{123} = 1$.

An alternative form of the Hamiltonian is

$$\begin{aligned} H = &\frac{1}{2}\sum_{i=1}^{3}\left[\left(\partial_0 A^i\right)^2 - \left(\partial_i A^0\right)^2\right] + \frac{1}{2}\sum_{i>j}\left(\partial_i A^j - \partial_j A^i\right)^2 \\ &-\frac{1}{2}\lambda\left[\left(\partial_0 A^0\right)^2 - \left(\partial_i A^i\right)^2\right] + J^0 A^0 - \boldsymbol{J}\cdot\boldsymbol{A} \end{aligned} \tag{7}$$

Further, the following canonical commutation relations of equal time are valid

$$\left[A^\mu(t,\boldsymbol{x}), \pi^\nu(t,\boldsymbol{y})\right] = ig^{\mu\nu}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{8}$$

$$\left[\pi^\mu(t,\boldsymbol{x}), \pi^\nu(t,\boldsymbol{y})\right] = \left[A^\mu(t,\boldsymbol{x}), A^\nu(t,\boldsymbol{y})\right] = 0 \tag{9}$$

$$\left[\dot{A}^0(t,\boldsymbol{x}), \dot{A}^0(t,\boldsymbol{y})\right] = \left[\dot{A}^i(t,\boldsymbol{x}), \dot{A}^j(t,\boldsymbol{y})\right] = 0 \tag{10}$$

$$\left[\dot{A}^\mu(t,\boldsymbol{x}), A^\nu(t,\boldsymbol{y})\right] = ig^{\mu\nu}\left[1+\left(\frac{1}{\lambda}-1\right)g^{\mu 0}\right]\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{11}$$

$$\left[\dot{A}^0(t,\boldsymbol{x}), \dot{A}_j(t,\boldsymbol{y})\right] = i\left(1-\frac{1}{\lambda}\right)\partial_j^x\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{12}$$

Now we proceed to the solution of Equation (4). We restrict to expressions of the form $\varphi(t)\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}$. Then the general solution can be obtained as a superposition of such solutions. $\varphi(t)$ must satisfy the ordinary differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + p^2\right)^2 \varphi = 0 \tag{13}$$

It has the general solution

$$\varphi(t) = (a+bt)\mathrm{e}^{ipt} + (c+dt)\mathrm{e}^{-ipt} \tag{14}$$

We notice that in the above solution the time in the two polynomials can be interpreted as an appropriate derivative on p. So, on expanding in terms of annihilation and creation operators the general solution of Equation (4) is expected to have the following form (notice that $x_0 \equiv t$),

$$\begin{aligned} A^\mu(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega_{\boldsymbol{k}}} \sum_{\kappa=0}^{3}\Big[&\varepsilon^{\mu *}(\boldsymbol{k},\kappa)\left(a^{(\kappa)+}(\boldsymbol{k}) + b^{(\kappa)+}(\boldsymbol{k})x_0\right)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \\ &+\varepsilon^{\mu}(\boldsymbol{k},\kappa)\left(a^{(\kappa)}(\boldsymbol{k}) + b^{(\kappa)}(\boldsymbol{k})x_0\right)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\Big] \end{aligned} \tag{15}$$

where the sum on κ is on the polarization directions and $\varepsilon^\mu(\boldsymbol{k},\kappa)$ is the μ component of the κ polarization vector. Further in the present massless case

$$\omega_{\boldsymbol{k}} = |\boldsymbol{k}| = k_0 \tag{16}$$

and

$$k^\nu k_\nu = 0 \tag{17}$$

Now we substitute Equation (15) in the homogeneous form of Equation (3) and conclude that

$$\varepsilon^{\mu}\left(\boldsymbol{k},\kappa\right)b^{(\kappa)}\left(\boldsymbol{k}\right)=i\frac{\lambda-1}{\lambda+1}\frac{k^{\mu}k_{\nu}}{k_0}\varepsilon^{\nu}\left(\boldsymbol{k},\kappa\right)a^{(\kappa)}\left(\boldsymbol{k}\right) \tag{18}$$

$$\varepsilon^{\mu *}\left(\boldsymbol{k},\kappa\right)b^{(\kappa)+}\left(\boldsymbol{k}\right)=-i\frac{\lambda-1}{\lambda+1}\frac{k^{\mu}k_{\nu}}{k_0}\varepsilon^{\nu *}\left(\boldsymbol{k},\kappa\right)a^{(\kappa)+}\left(\boldsymbol{k}\right) \tag{19}$$

The free annihilation and creation operators obey the following equations

$$a^{(\kappa)+}\left(t,\boldsymbol{k}\right)=a^{(\kappa)+}\left(0,\boldsymbol{k}\right)\mathrm{e}^{i\omega_{\boldsymbol{k}}t} \tag{20}$$

$$a^{(\kappa)}\left(t,\boldsymbol{k}\right)=a^{(\kappa)}\left(0,\boldsymbol{k}\right)\mathrm{e}^{-i\omega_{\boldsymbol{k}}t} \tag{21}$$

$$b^{(\kappa)+}\left(t,\boldsymbol{k}\right)=b^{(\kappa)+}\left(0,\boldsymbol{k}\right)\mathrm{e}^{i\omega_{\boldsymbol{k}}t} \tag{22}$$

$$b^{(\kappa)}\left(t,\boldsymbol{k}\right)=b^{(\kappa)}\left(0,\boldsymbol{k}\right)\mathrm{e}^{-i\omega_{\boldsymbol{k}}t} \tag{23}$$

To make the above equations consistent the following commutation rules of the a, a^{+} must be valid

$$\begin{aligned}&\left[a^{(\kappa)}\left(\boldsymbol{k}\right),a^{(\kappa')+}\left(\boldsymbol{k}'\right)\right]\\&=2k^{0}\left(2\pi\right)^{3}\delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{k}'\right)\left(-g_{\kappa\kappa'}+\frac{\lambda-1}{2\lambda k^{0}}\left(g_{\kappa 0}k_{\kappa'}+g_{\kappa' 0}k_{\kappa}-\frac{k_{\kappa}k_{\kappa'}}{k^{0}}\right)\right)\end{aligned} \tag{24}$$

All other such commutation relations are zero. Therefore

$$\left[a^{(i)}\left(\boldsymbol{k}\right),a^{(j)+}\left(\boldsymbol{k}'\right)\right]=2k^{0}\left(2\pi\right)^{3}\delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{k}'\right)\left[\delta_{ij}+\frac{1-\lambda}{2\lambda}\frac{k_{i}k_{j}}{\left|\boldsymbol{k}\right|^{2}}\right] \tag{25}$$

$$\left[a^{(0)}\left(\boldsymbol{k}\right),a^{(i)+}\left(\boldsymbol{k}'\right)\right]=2k_{i}'\left(2\pi\right)^{3}\delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{k}'\right)\frac{1-\lambda}{2\lambda} \tag{26}$$

$$\left[a^{(i)}\left(\boldsymbol{k}\right),a^{(0)+}\left(\boldsymbol{k}'\right)\right]=2k_{i}\left(2\pi\right)^{3}\delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{k}'\right)\frac{1-\lambda}{2\lambda} \tag{27}$$

$$\left[a^{(0)}\left(\boldsymbol{k}\right),a^{(0)+}\left(\boldsymbol{k}'\right)\right]=-2k^{0}\left(2\pi\right)^{3}\delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{k}'\right)\frac{\lambda+1}{2\lambda} \tag{28}$$

The minus sign in the $\kappa=\kappa'=0$ commutation relation is an indication of the indefinite metric we use to confront the gauge dependence of A_{μ}. In fact, although an indefinite norm Fock space is used to maintain the locality properties, physical interpretation is obtained only for the vectors in a physical positive norm subspace [13] [14].

Finally, we obtain the following Hamiltonian in normal order form

$$\begin{aligned}&I\left(a^{(q)}\left(\boldsymbol{k}\right),a^{(q)+}\left(\boldsymbol{k}\right),a^{(0)}\left(\boldsymbol{k}\right),a^{(0)+}\left(\boldsymbol{k}\right),t\right)=\int\mathrm{d}^{3}xH\left(E,A\right)\\&=\int\frac{\mathrm{d}^{3}k}{\left(2\pi\right)^{3}2\omega_{\boldsymbol{k}}}\sum_{q=1}^{3}\left[\omega_{\boldsymbol{k}}a^{(q)+}\left(\boldsymbol{k}\right)a^{(q)}\left(\boldsymbol{k}\right)+j^{(q)*}\left(t,\boldsymbol{k}\right)a^{(q)}\left(\boldsymbol{k}\right)+j^{(q)}\left(t,\boldsymbol{k}\right)a^{(q)+}\left(\boldsymbol{k}\right)\right]\\&+\int\frac{\mathrm{d}^{3}k}{\left(2\pi\right)^{3}2\omega_{\boldsymbol{k}}}\left[-\omega_{\boldsymbol{k}}a^{(0)+}\left(\boldsymbol{k}\right)a^{(0)}\left(\boldsymbol{k}\right)+j^{(0)*}\left(t,\boldsymbol{k}\right)a^{(0)}\left(\boldsymbol{k}\right)+j^{(0)}\left(t,\boldsymbol{k}\right)a^{(0)+}\left(\boldsymbol{k}\right)\right]\end{aligned} \tag{29}$$

Notice the change of the sign in front of the frequency in the last term. We have set

$$j^{(\kappa)}(t,\boldsymbol{k})=\left[\varepsilon^{\mu *}(\boldsymbol{k},\kappa)-i\varepsilon^{\nu *}(\boldsymbol{k},\kappa)\frac{\lambda-1}{\lambda+1}\frac{k^{\mu}k_{\nu}}{k_0}x_0\right]J_{\mu}(t,\boldsymbol{k})\qquad 0\le\kappa\le 3 \tag{30}$$

and

$$J_{\mu}(t,\boldsymbol{k})=\int \mathrm{d}^3x\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}J_{\mu}(t,\boldsymbol{x}) \tag{31}$$

We notice that the polarization vectors satisfy the identities

$$\sum_{\kappa,\kappa'=0}^{3}g^{\kappa\kappa'}\varepsilon^{\mu}(\boldsymbol{k},\kappa)\varepsilon^{\nu *}(\boldsymbol{k},\kappa')=g^{\mu\nu} \tag{32}$$

3. Path Integration

Now we proceed to the path integration of the above system [15]-[18]. At first, we consider a single mode of the form

$$H_0=\omega a^{+}a+j^{*}a+ja^{+} \tag{33}$$

Such modes appear in Equation (29).

We construct a path integral representation of the time evolution operator of the above Hamiltonian

$$U_0(T)=\mathrm{e}^{-i\int_0^T H_0\mathrm{d}\tau} \tag{34}$$

We consider coherent states. The Hamiltonian H_0 obeys the expression

$$\left\langle\alpha^{*}\left|H_0\right|\alpha'\right\rangle=\omega\alpha^{*}\alpha'+j^{*}\alpha'+j\alpha^{*} \tag{35}$$

At first, on the one hand we suppose a small time t to get

$$\begin{aligned}\left\langle\alpha^{*}\left|U_0(t)\right|\alpha'\right\rangle&=\left[1-it\left(\omega\alpha^{*}\alpha'+j^{*}\alpha'+j\alpha^{*}\right)+O\left(t^2\right)\right]\mathrm{e}^{\alpha^{*}\alpha'}\\&=\mathrm{e}^{\alpha^{*}\alpha'(1-it\omega)-it\left(\alpha^{*}j+j^{*}\alpha'\right)}+O\left(t^2\right)\end{aligned} \tag{36}$$

and on the other according to the over-completeness relation of the coherent states, we have the identity

$$\left\langle\alpha^{*}\left|U_2U_1\right|\alpha'\right\rangle=\int\frac{\mathrm{d}^2\alpha''}{\pi}\left\langle\alpha^{*}\left|U_2\right|\alpha''\right\rangle\mathrm{e}^{-\alpha''^{*}\alpha''}\left\langle\alpha''^{*}\left|U_1\right|\alpha'\right\rangle \tag{37}$$

On applying repeatedly Equations (36) (37) the finite time evolution operator takes the following path integral form

$$\left\langle\alpha_f^{*}\left|U_0(T)\right|\alpha_i\right\rangle=\lim_{N\to\infty}\int\prod_{p=1}^{N-1}\frac{\mathrm{d}^2\alpha_p}{\pi}\exp\left[iS_0\left(\alpha_p^{*},\alpha_p\right)\right] \tag{38}$$

where

$$\begin{aligned}S_0\left(\alpha_p^{*},\alpha_p\right)=&i\sum_{p=1}^{N-1}\alpha_p^{*}\left(\alpha_p-\alpha_{p-1}\right)-i\alpha_N^{*}\alpha_{N-1}-\omega\varepsilon\sum_{p=1}^{N}\alpha_p^{*}\alpha_{p-1}\\&-\varepsilon\sum_{p=1}^{N}\left(\alpha_p^{*}j_p+j_{p-1}^{*}\alpha_{p-1}\right)\end{aligned} \tag{39}$$

$$\varepsilon=\frac{T}{N} \tag{40}$$

and

$$\alpha_0=\alpha_i,\qquad \alpha_N^{*}=\alpha_f^{*} \tag{41}$$

Now on letting $N \to \infty$ we get the path integral representation of the matrix elements of $U_0(T)$. They are

$$\left\langle \alpha_f^* \left| U_0(T) \right| \alpha_i \right\rangle = \int_{\alpha(0)=\alpha_i}^{\alpha^*(T)=\alpha_f^*} \frac{D\alpha^*(\tau) D\alpha(\tau)}{\pi} \exp\left[iS_0\left(\alpha^*(\tau), \alpha(\tau)\right)\right] \tag{42}$$

with the action

$$\begin{aligned} &S_0\left(\alpha^*(\tau), \alpha(\tau)\right) \\ &= -i\alpha_f^* \alpha(T) + \int_0^T \mathrm{d}\tau \left\{ \alpha^*(\tau)\left[i\dot{\alpha}(\tau) - \omega\alpha(\tau)\right] - \alpha^*(\tau) j(\tau) - j^*(\tau)\alpha(\tau)\right\} \end{aligned} \tag{43}$$

The path integral (42) under the action (43) can be evaluated exactly as it is Gaussian. By varying $\alpha^*(\tau)$ the following saddle point relation can be extracted

$$i\dot{\alpha}(\tau) - \omega\alpha(\tau) - j(\tau) = 0 \tag{44}$$

It has the solution

$$\alpha(t) = \mathrm{e}^{-i\omega t}\alpha_i - i\int_0^t \mathrm{d}\tau\, \mathrm{e}^{-i\omega(t-\tau)} j(\tau) \tag{45}$$

In the above solution the boundary conditions (41) have been taken into account.

Now the action (43), under the differential equation (44), takes the form

$$S_0\left(\alpha^*(\tau), \alpha(\tau)\right) = -i\alpha_f^*\alpha(T) - \int_0^T \mathrm{d}\tau\, j^*(\tau)\alpha(\tau) \tag{46}$$

Finally, we get

$$\left\langle \alpha_f^* \left| U_0(T) \right| \alpha_i \right\rangle = \exp\left[iS_0\left(\alpha^*(\tau), \alpha(\tau)\right)\right] \tag{47}$$

So, on using Equations (45) (46) (47) we get the propagator of the Hamiltonian (33) in the form

$$\begin{aligned} K\left(\alpha_i, \alpha_f^*; T\right) &= \left\langle \alpha_f^* \left| U_0(T) \right| \alpha_i \right\rangle \\ &= \exp\left\{ \alpha_f^* \mathrm{e}^{-i\omega T}\alpha_i - i\int_0^T \mathrm{d}t \left(\alpha_f^* \mathrm{e}^{-i\omega(T-t)} j(t) + j^*(t)\mathrm{e}^{-i\omega t}\alpha_i\right)\right. \\ &\left. - \int_0^T \int_0^t j^*(t)\mathrm{e}^{-i\omega(t-t')} j(t')\mathrm{d}t'\mathrm{d}t \right\} \end{aligned} \tag{48}$$

Moreover, the possible transition amplitude A from a state $|\psi\rangle$ at time zero to the state $|\chi\rangle$ at time T has the diagonal form [19] [20]

$$A(\psi \to \chi) = \int \frac{\mathrm{d}^2\alpha}{\pi} \langle \chi | \alpha \rangle K\left(\alpha, \alpha^*; T\right) \langle \alpha | \psi \rangle \tag{49}$$

Below we consider transition amplitudes from vacuum to vacuum. Then $|\chi\rangle = |\psi\rangle = |0\rangle$.

In fact, in Equation (29) there appear Hamiltonians of the form of Equation (33). So, on applying Equation (48) on each $\boldsymbol{k}$ -term of Equation (29) we get the propagator

$$
\begin{aligned}
&K\left(\alpha_i^{(q)}(\boldsymbol{k}),\alpha_f^{(q)*}(\boldsymbol{k}),\alpha_i^{(0)}(\boldsymbol{k}),\alpha_f^{(0)*}(\boldsymbol{k});T\right)\\
&=\exp\left\{\sum_{q=1}^{3}\left[\alpha_f^{(q)*}(\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}\,\alpha_i^{(q)}(\boldsymbol{k})\right.\right.\\
&\quad -i\int_0^T \mathrm{d}t\left(\alpha_f^{(q)*}(\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}(T-t)}\,j^{(q)}(t,\boldsymbol{k})+j^{(q)*}(t,\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}t}\,\alpha_i^{(q)}(\boldsymbol{k})\right)\\
&\quad \left.-\int_0^T\int_0^t j^{(q)*}(t,\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t-t')}\,j^{(q)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\right]+\alpha_f^{(0)*}(\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}T}\,\alpha_i^{(0)}(\boldsymbol{k})\\
&\quad -i\int_0^T \mathrm{d}t\left(\alpha_f^{(0)*}(\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}(T-t)}\,j^{(0)}(t,\boldsymbol{k})+j^{(0)*}(t,\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}t}\,\alpha_i^{(0)}(\boldsymbol{k})\right)\\
&\quad \left.-\int_0^T\int_0^t j^{(0)*}(t,\boldsymbol{k})\mathrm{e}^{i\omega_{\boldsymbol{k}}(t-t')}\,j^{(0)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\right\}
\end{aligned}
\tag{50}
$$

Take notice of the different sign on the exponentials of the terms which include superscript (0).

In order to evaluate the correlation functions generating functional of the $\boldsymbol{k}$-term, we have to integrate diagonally from a vacuum at time zero to a vacuum at time T in the same way as in Equation (49) but with four oscillators. Then we can apply Equation (50) and get the corresponding amplitude, *i.e.*, the following expression is valid

$$
\begin{aligned}
Z_{\boldsymbol{k}}(J)=&\prod_{\eta=0}^{3}\left[\int\frac{\mathrm{d}^2\alpha^{(\eta)}(\boldsymbol{k})}{\pi}\right]\prod_{\eta=0}^{3}\left[\left\langle 0\middle|\alpha^{(\eta)}(\boldsymbol{k})\right\rangle\right]\\
&\times K\left(\alpha^{(q)}(\boldsymbol{k}),\alpha^{(q)*}(\boldsymbol{k}),\alpha^{(0)}(\boldsymbol{k}),\alpha^{(0)*}(\boldsymbol{k});T\right)\prod_{\eta=0}^{3}\left[\left\langle\alpha^{(\eta)}(\boldsymbol{k})\middle|0\right\rangle\right]
\end{aligned}
\tag{51}
$$

The initial and final vacuum states have the form

$$
\left\langle 0\middle|\alpha^{(\eta)}(\boldsymbol{k})\right\rangle=\exp\left[-\frac{1}{2}\left|\alpha^{(\eta)}(\boldsymbol{k})\right|^2\right] \tag{52}
$$

and

$$
\left\langle\alpha^{(\eta)}(\boldsymbol{k})\middle|0\right\rangle=\exp\left[-\frac{1}{2}\left|\alpha^{(\eta)}(\boldsymbol{k})\right|^2\right] \tag{53}
$$

We notice that accidentally the above expression looks like a trace. So, we get

$$
\begin{aligned}
Z_{\boldsymbol{k}}(J)=\exp&\left\{-\sum_{q=1}^{3}\ln\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}\right)-\ln\left(1-\mathrm{e}^{i\omega_{\boldsymbol{k}}T}\right)\right.\\
&-i\frac{1}{2}\sum_{q=1}^{3}\int_0^T\int_0^T j^{(q)*}(t,\boldsymbol{k})\frac{\cos\left(\omega_{\boldsymbol{k}}\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)}j^{(q)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\\
&\left.+i\frac{1}{2}\int_0^T\int_0^T j^{(0)*}(t,\boldsymbol{k})\frac{\cos\left(\omega_{\boldsymbol{k}}\left|t-t'\right|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)}j^{(0)}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\right\}
\end{aligned}
\tag{54}
$$

or equivalently

$$Z_{\boldsymbol{k}}(J)=\exp\left\{-\sum_{q=1}^{3}\ln\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}\right)-\ln\left(1-\mathrm{e}^{i\omega_{\boldsymbol{k}}T}\right)\right.$$
$$\left.+i\frac{1}{2}\sum_{\eta,\eta'=0}^{3}g_{\eta\eta'}\int_{0}^{T}\int_{0}^{T}j^{(\eta)*}(t,\boldsymbol{k})\frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)}j^{(\eta')}(t',\boldsymbol{k})\mathrm{d}t'\mathrm{d}t\right\} \tag{55}$$

In the present problem $j^{(\eta)}(t,\boldsymbol{k})$ is given by Equation (30).

Now we consider the expression

$$j^{(\eta)*}(t,\boldsymbol{k})\frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)}j^{(\eta')}(t',\boldsymbol{k})$$
$$=\left(J_{\mu}(t,\boldsymbol{k})\right)^{*}J_{\mu'}(t',\boldsymbol{k})\left[\varepsilon^{\mu}(\boldsymbol{k},\eta)+ix_{0}\frac{\lambda-1}{\lambda+1}\frac{k^{\mu}k_{\nu}}{k_{0}}\varepsilon^{\nu}(\boldsymbol{k},\eta)\right] \tag{56}$$
$$\times\frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)}\left[\varepsilon^{\mu'*}(\boldsymbol{k},\eta')-i\frac{\lambda-1}{\lambda+1}\frac{k^{\mu'}k_{\nu'}}{k_{0}}\varepsilon^{\nu'*}(\boldsymbol{k},\eta')x_{0}\right]$$

appearing in Equation (55) and on the one hand we use Equation (32) and the identity $k^{\nu}k_{\nu}=0$ and on the other we interpret x_0 as an operator of the form $-i\frac{\partial}{\partial\omega_{\boldsymbol{k}}}$ (see the discussion below Equation (14) as well) and write the symmetric difference there as a commutator of x_0 with the rest of the expression. On evaluating we get a derivative of that expression. Moreover, we take into account the commutation relation (28) and get the relation

$$\sum_{\eta,\eta'=0}^{3}g_{\eta\eta'}j^{(\eta)*}(t,\boldsymbol{k})\frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)}j^{(\eta')}(t',\boldsymbol{k})$$
$$=\left(J_{\mu}(t,\boldsymbol{k})\right)^{*}J_{\nu}(t',\boldsymbol{k})\left[g^{\mu\nu}-\frac{\lambda-1}{2\lambda}\frac{\partial}{\partial\omega_{\boldsymbol{k}}}\frac{k^{\mu}k^{\nu}}{k_{0}}\right]\frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'|-\frac{\omega_{\boldsymbol{k}}T}{2}\right)}{\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)} \tag{57}$$

which we substitute in Equation (55).

So, according to the above considerations the finite time domain generating functional of the photonic field's correlation functions can be extracted. It has the form

$$Z(J)=\prod_{k}Z_{k}(J)$$

$$=\exp\left\{-\int\frac{\mathrm{d}^{3}k}{(2\pi)^{3}2\omega_{k}}\sum_{q=1}^{3}\ln\left(1-\mathrm{e}^{-i\omega_{k}T}\right)-\int\frac{\mathrm{d}^{3}k}{(2\pi)^{3}2\omega_{k}}\ln\left(1-\mathrm{e}^{i\omega_{k}T}\right)+i\frac{1}{2}\int\frac{\mathrm{d}^{3}k}{(2\pi)^{3}2\omega_{k}}\right. \quad (58)$$

$$\left.\times\int_{0}^{T}\int_{0}^{T}\left(J_{\mu}(t,\boldsymbol{k})\right)^{*}J_{\nu}(t',\boldsymbol{k})\left[g^{\mu\nu}-\frac{\lambda-1}{2\lambda}\frac{\partial}{\partial\omega_{k}}\frac{k^{\mu}k^{\nu}}{k_{0}}\right]\frac{\cos\left(\omega_{k}\left|t-t'\right|-\frac{\omega_{k}T}{2}\right)}{\sin\left(\frac{\omega_{k}T}{2}\right)}\mathrm{d}t'\mathrm{d}t\right\}$$

Upon using the above expression, we derive below the corresponding Green function.

4. Green Function

In the present section we procced to the evaluation of the finite time interval Green function of the lagrangian of photonic fields. We give it in a series of possible representations.

According to standard results and the previous discussion the Green function has the form

$$G^{\mu\nu}(\boldsymbol{x}-\boldsymbol{x}',t-t';T)=\left\langle 0\left|T\left(A^{\mu}(t,\boldsymbol{x})A^{\nu}(t',\boldsymbol{x}')\right)\right|0\right\rangle$$
$$=-\frac{1}{Z(0)}\frac{\delta^{2}}{\delta J_{\mu}\delta J_{\nu}}Z(J)\Big|_{J=0} \quad (59)$$

Upon evaluating the functional derivatives from Equation (58) we obtain the photon Green function $G^{\mu\nu}(\boldsymbol{x},\tau;T)$ in the following components' form

$$G^{i0}(\boldsymbol{x},\tau;T)=G^{0i}(\boldsymbol{x},\tau;T)$$
$$=-i\int\frac{\mathrm{d}^{3}k}{(2\pi)^{3}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}k^{i}\frac{\lambda-1}{\lambda}\frac{1}{4|\boldsymbol{k}|^{2}\sin^{2}\left(\frac{|\boldsymbol{k}|T}{2}\right)} \quad (60)$$
$$\times\left[|\tau||\boldsymbol{k}|\sin\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)+\frac{T}{2}|\boldsymbol{k}|\cos\left(|\boldsymbol{k}||\tau|\right)\right]$$

$$G^{ij}(\boldsymbol{x},\tau;T)$$
$$=i\int\frac{\mathrm{d}^{3}k}{(2\pi)^{3}}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left\{\delta^{ij}\frac{\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)}{2|\boldsymbol{k}|\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)}-\frac{\lambda-1}{\lambda}k^{i}k^{j}\frac{1}{4|\boldsymbol{k}|^{3}\sin^{2}\left(\frac{|\boldsymbol{k}|T}{2}\right)}\right.$$
$$\times\left[|\tau||\boldsymbol{k}|\sin\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)+\frac{T}{2}|\boldsymbol{k}|\cos\left(|\boldsymbol{k}||\tau|\right)\right. \quad (61)$$
$$\left.\left.+\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)\right]\right\}$$

and

$$
\begin{aligned}
&G^{00}\left(\boldsymbol{x},\tau;T\right)\\
&=-i\int\frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left\{\frac{\cos\left(\left|\boldsymbol{k}\right|\left|\tau\right|-\frac{\left|\boldsymbol{k}\right|T}{2}\right)}{2\left|\boldsymbol{k}\right|\sin\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)}+\frac{\lambda-1}{\lambda}\frac{1}{4\left|\boldsymbol{k}\right|\sin^2\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)}\right.\\
&\times\left[\left|\tau\right|\left|\boldsymbol{k}\right|\sin\left(\left|\boldsymbol{k}\right|\left|\tau\right|-\frac{\left|\boldsymbol{k}\right|T}{2}\right)\sin\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)+\frac{T}{2}\left|\boldsymbol{k}\right|\cos\left(\left|\boldsymbol{k}\right|\left|\tau\right|\right)\right.\\
&\left.\left.-\cos\left(\left|\boldsymbol{k}\right|\left|\tau\right|-\frac{\left|\boldsymbol{k}\right|T}{2}\right)\sin\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)\right]\right\}
\end{aligned}
\tag{62}
$$

Now we use standard results to expand equations (60) (61) on fixed tensors with coefficients which are scalars. So, on the one hand we get the expression

$$
G^{0i}\left(\boldsymbol{x},\tau;T\right)=G^{i0}\left(\boldsymbol{x},\tau;T\right)=A\left(\boldsymbol{x}^2,\tau;T\right)x^i \tag{63}
$$

where

$$
\begin{aligned}
A\left(\boldsymbol{x}^2,\tau;T\right)&=-i\frac{1}{\boldsymbol{x}^2}\int\frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left(\boldsymbol{k}\cdot\boldsymbol{x}\right)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\lambda-1}{\lambda}\frac{1}{4\left|\boldsymbol{k}\right|^2\sin^2\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)}\\
&\times\left[\left|\tau\right|\left|\boldsymbol{k}\right|\sin\left(\left|\boldsymbol{k}\right|\left|\tau\right|-\frac{\left|\boldsymbol{k}\right|T}{2}\right)\sin\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)+\frac{T}{2}\left|\boldsymbol{k}\right|\cos\left(\left|\boldsymbol{k}\right|\left|\tau\right|\right)\right]
\end{aligned}
\tag{64}
$$

and the other hand the expression

$$
G^{ij}\left(\boldsymbol{x},\tau;T\right)=\left[B\left(\boldsymbol{x}^2,\tau;T\right)+C\left(\boldsymbol{x}^2,\tau;T\right)\right]\delta^{ij}+D\left(\boldsymbol{x}^2,\tau;T\right)x^ix^j \tag{65}
$$

where

$$
\begin{aligned}
&B\left(\boldsymbol{x}^2,\tau;T\right)\\
&=-i\frac{1}{2}\frac{1}{\boldsymbol{x}^2}\int\frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left[-\left(\boldsymbol{k}\cdot\boldsymbol{x}\right)^2+\boldsymbol{k}^2\boldsymbol{x}^2\right]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\lambda-1}{\lambda}\frac{1}{4\left|\boldsymbol{k}\right|^3\sin^2\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)}\\
&\times\left[\left|\tau\right|\left|\boldsymbol{k}\right|\sin\left(\left|\boldsymbol{k}\right|\left|\tau\right|-\frac{\left|\boldsymbol{k}\right|T}{2}\right)\sin\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)+\frac{T}{2}\left|\boldsymbol{k}\right|\cos\left(\left|\boldsymbol{k}\right|\left|\tau\right|\right)\right.\\
&\left.+\cos\left(\left|\boldsymbol{k}\right|\left|\tau\right|-\frac{\left|\boldsymbol{k}\right|T}{2}\right)\sin\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)\right]
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
&D\left(\boldsymbol{x}^2,\tau;T\right)\\
&=-i\frac{1}{2}\frac{1}{\left(\boldsymbol{x}^2\right)^2}\int\frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\left[3\left(\boldsymbol{k}\cdot\boldsymbol{x}\right)^2-\boldsymbol{k}^2\boldsymbol{x}^2\right]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\lambda-1}{\lambda}\frac{1}{4\left|\boldsymbol{k}\right|^3\sin^2\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)}\\
&\times\left[\left|\tau\right|\left|\boldsymbol{k}\right|\sin\left(\left|\boldsymbol{k}\right|\left|\tau\right|-\frac{\left|\boldsymbol{k}\right|T}{2}\right)\sin\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)+\frac{T}{2}\left|\boldsymbol{k}\right|\cos\left(\left|\boldsymbol{k}\right|\left|\tau\right|\right)\right.\\
&\left.+\cos\left(\left|\boldsymbol{k}\right|\left|\tau\right|-\frac{\left|\boldsymbol{k}\right|T}{2}\right)\sin\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)\right]
\end{aligned}
\tag{67}
$$

and

$$C\left(\boldsymbol{x}^2,\tau;T\right)=i\int\frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\cos\left(\left|\boldsymbol{k}\right|\left|\tau\right|-\frac{\left|\boldsymbol{k}\right|T}{2}\right)}{2\left|\boldsymbol{k}\right|\sin\left(\frac{\left|\boldsymbol{k}\right|T}{2}\right)} \tag{68}$$

The present final photonic field's Green function is the same with the photonic field's Green function extracted in ref. [12] with alternative methods.

Finally, we notice that if we let $T\to\infty$ we get

$$G^{i0}\left(\boldsymbol{x},\tau\right)=G^{0i}\left(\boldsymbol{x},\tau\right)=i\int\frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}k^i\frac{\lambda-1}{\lambda}\frac{1}{4\left|\boldsymbol{k}\right|}\left|\tau\right|\mathrm{e}^{-i\left|\boldsymbol{k}\right|\left|\tau\right|} \tag{69}$$

$$G^{ij}\left(\boldsymbol{x},\tau\right)=-\int\frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[\frac{\delta^{ij}}{2\left|\boldsymbol{k}\right|}\mathrm{e}^{-i\left|\boldsymbol{k}\right|\left|\tau\right|}-\frac{\lambda-1}{\lambda}\frac{k^ik^j}{4\left|\boldsymbol{k}\right|^3}\left(i\left|\tau\right|\left|\boldsymbol{k}\right|+1\right)\mathrm{e}^{-i\left|\boldsymbol{k}\right|\left|\tau\right|}\right] \tag{70}$$

$$G^{00}\left(\boldsymbol{x},\tau\right)=\int\frac{\mathrm{d}^3k}{\left(2\pi\right)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[\frac{1}{2\left|\boldsymbol{k}\right|}\mathrm{e}^{-i\left|\boldsymbol{k}\right|\left|\tau\right|}+\frac{\lambda-1}{\lambda}\frac{1}{4\left|\boldsymbol{k}\right|}\left(i\left|\tau\right|\left|\boldsymbol{k}\right|-1\right)\mathrm{e}^{-i\left|\boldsymbol{k}\right|\left|\tau\right|}\right] \tag{71}$$

The above components can be summarized in the standard expression

$$G^{\mu\nu}\left(\boldsymbol{x},\tau\right)=-i\int\frac{\mathrm{d}^4k}{\left(2\pi\right)^4}\mathrm{e}^{-ik\cdot x}\left[\frac{g^{\mu\nu}}{k^2+i0}-\frac{\lambda-1}{\lambda}\frac{k^\mu k^\nu}{\left(k^2+i0\right)^2}\right] \tag{72}$$

5. Applications

Now we proceed to the generation of photonic fields from currents. We set the current as $J_\nu\left(\boldsymbol{x},t\right)$. Then

$$A^\mu\left(t,\boldsymbol{x}\right)=-i\int_0^T\mathrm{d}t'\int\mathrm{d}^3\boldsymbol{x}'G^{\mu\nu}\left(\boldsymbol{x}-\boldsymbol{x}',t-t';T\right)J_\nu\left(\boldsymbol{x}',t'\right) \tag{73}$$

We suppose a charged particle following a trajectory $\boldsymbol{y}\left(t\right)$. Then, the current 4-vector is given as

$$J_\nu\left(\boldsymbol{x},t\right)=Q\mathrm{v}_\nu\left(t\right)\delta^{(3)}\left(\boldsymbol{x}-\boldsymbol{y}\left(t\right)\right) \tag{74}$$

$\mathrm{v}_\nu\left(t\right)$ $\nu=0,1,2,3$ is the velocity and Q the charge. Additionally let

$$\mathrm{v}=\left(1,\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3\right)=\left(1,\mathbf{v}\right) \tag{75}$$

The above current is conserved as

$$\begin{aligned}\partial_\nu J^\nu\left(\boldsymbol{x},t\right)&=\partial^0J^0\left(\boldsymbol{x},t\right)-\partial^iJ^i\left(\boldsymbol{x},t\right)\\&=Q\partial^0y^i\left(t\right)\partial^{x_i}\left[\delta^{(3)}\left(\boldsymbol{x}-\boldsymbol{y}\left(t\right)\right)\right]-Q\mathrm{v}^i\left(t\right)\partial^{x_i}\left[\delta^{(3)}\left(\boldsymbol{x}-\boldsymbol{y}\left(t\right)\right)\right]\\&=0\end{aligned} \tag{76}$$

We have used the equation $\partial^0\boldsymbol{y}\left(t\right)=\mathbf{v}$.

So, the above theory can be applied to get

$$A^0(t,\boldsymbol{x}) = -iQ\int_0^T \left\{ G^{00}\left(\boldsymbol{x}-\boldsymbol{y}(t'),t-t';T\right) - A\left(\left(\boldsymbol{x}-\boldsymbol{y}(t')\right)^2,t-t';T\right)\left(\boldsymbol{x}-\boldsymbol{y}(t')\right)^j \mathrm{v}_j(t') \right\} \mathrm{d}t' \quad (77)$$

and

$$\begin{aligned} A^i(t,\boldsymbol{x}) = -iQ\int_0^T \Big\{ & A\left(\left(\boldsymbol{x}-\boldsymbol{y}(t')\right)^2,t-t';T\right)\left(\boldsymbol{x}-\boldsymbol{y}(t')\right)^i \\ & -\left[B\left(\left(\boldsymbol{x}-\boldsymbol{y}(t')\right)^2,t-t';T\right) + C\left(\left(\boldsymbol{x}-\boldsymbol{y}(t')\right)^2,t-t';T\right)\right]\mathrm{v}^i(t') \\ & - D\left(\left(\boldsymbol{x}-\boldsymbol{y}(t')\right)^2,t-t';T\right)\left(\boldsymbol{x}-\boldsymbol{y}(t')\right)^i\left(\boldsymbol{x}-\boldsymbol{y}(t')\right)^j \mathrm{v}_j(t') \Big\} \mathrm{d}t' \end{aligned} \quad (78)$$

Source-generated potentials could be gauge dependent but physical results must not be affected by the choice of the gauge. So, they should be independent of λ.

When $\lambda = 1$ we get the Feynman gauge. The limiting $\lambda \to \infty$ corresponds to the Landau gauge. Here we consider the Feynman gauge. Then

$$\begin{aligned} G^{\mu\nu}(\boldsymbol{x},\tau;T) &= -ig^{\mu\nu}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)}{2|\boldsymbol{k}|\sin\left(\frac{|\boldsymbol{k}|T}{2}\right)} \\ &= g^{\mu\nu}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\cos\left(|\boldsymbol{k}||\tau|-\frac{|\boldsymbol{k}|T}{2}\right)}{|\boldsymbol{k}|\left(1-\mathrm{e}^{-i|\boldsymbol{k}|T}\right)}\mathrm{e}^{-i\frac{|\boldsymbol{k}|T}{2}} \\ &= g^{\mu\nu}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{2|\boldsymbol{k}|}\left[\mathrm{e}^{i|\boldsymbol{k}||\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)|\boldsymbol{k}|T}+\mathrm{e}^{-i|\boldsymbol{k}||\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in|\boldsymbol{k}|T}\right] \\ &= g^{\mu\nu}\frac{1}{4\pi^2}\int_0^{\infty}\mathrm{d}k\frac{\sin\left(k|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\left[\mathrm{e}^{ik|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)kT}+\mathrm{e}^{-ik|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-inkT}\right] \\ &= g^{\mu\nu}\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{|\boldsymbol{x}|^2-\left(|\tau|+nT\right)^2} = g^{\mu\nu}\sum_{n=-\infty}^{\infty}F\left(\boldsymbol{x},|\tau|+nT\right) \end{aligned} \quad (79)$$

where

$$F(\boldsymbol{x},\tau) = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-ik\cdot x}\frac{1}{k^2+i0} = -\frac{1}{4\pi^2}\frac{1}{x^2-i0} \quad (80)$$

Moreover, as we can derive from the above expressions

$$G^{\mu\nu}(\boldsymbol{x},\tau;T) = G^{\mu\nu}(\boldsymbol{x},T-\tau;T) \quad 0 \le \tau \le T \quad (81)$$

Therefore, $G^{\mu\nu}(\boldsymbol{x},\tau;T)$ obeys periodic boundary conditions with respect the time.

We notice that due to the periodicity of the Green function and the equations (73) and (79) the potentials satisfy appropriate periodic endpoint matching conditions.

Further for an arbitrary current of the form of Equation (74) we get

$$\begin{aligned} A^{\mu}(t,\boldsymbol{x}) &= -iQ\int_0^T G^{\mu\nu}\left(\boldsymbol{x}-\boldsymbol{y}(t'),t-t';T\right)\mathrm{v}_{\nu}(t')\,\mathrm{d}t' \\ &= -iQ\int_0^T \sum_{n=-\infty}^{\infty} F\left(\boldsymbol{x}-\boldsymbol{y}(t'),|t-t'|+nT\right)g^{\mu\nu}\mathrm{v}_{\nu}(t')\,\mathrm{d}t' \\ &= -iQ\sum_{n=-\infty}^{\infty}\int_0^T F\left(\boldsymbol{x}-\boldsymbol{y}(t'),t-t'+nT\right)g^{\mu\nu}\mathrm{v}_{\nu}(t')\,\mathrm{d}t' \end{aligned} \tag{82}$$

In the last equality we distinguish the cases $t \geq t'$ and $t < t'$. In the case of the latter inequality, we replace $n \to -n$. Then the two cases become identical.

We observe that if the current is periodic with period T then we get

$$A^{\mu}(t,\boldsymbol{x}) = -iQ\int_{-\infty}^{\infty} F\left(\boldsymbol{x}-\boldsymbol{y}(t'),t-t'\right)g^{\mu\nu}\mathrm{v}_{\nu}(t')\,\mathrm{d}t' \tag{83}$$

The period T is still hidden in the periodic position and velocity vectors of the current.

More particularly in the case of a charge moving uniformly along the x-direction with velocity v_0, Equation (75) becomes

$$\mathrm{v} = (1,\mathrm{v}_0,0,0) \tag{84}$$

Then

$$\boldsymbol{y}(t) = (\mathrm{v}_0 t,0,0) \tag{85}$$

If $\mathrm{v}_0 = 0$ the particle becomes stationary. Then the potential within a finite time domain is

$$\begin{aligned} A^{0}(\rho,\boldsymbol{x}) &= -iQ\int_0^T G^{00}\left(\boldsymbol{x},\rho-\tau;T\right)\mathrm{d}\tau \\ &= -iQ\frac{1}{4\pi^2}\int_0^T \mathrm{d}\tau\sum_{n=-\infty}^{\infty}\frac{1}{|\boldsymbol{x}|^2-\left(|\rho-\tau|+nT\right)^2} \\ &= iQ\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}s\frac{1}{s^2-|\boldsymbol{x}|^2+i0} = Q\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}|} \end{aligned} \tag{86}$$

In the last equality we have performed a Wick rotation and we have integrated. We observe that $A^{0}(\rho,\boldsymbol{x})$ is independent of ρ. The above result is the Coulomb potential of a single point charge in a three-dimensional space with a finite time domain.

Further since $g^{i0} = 0$ we get

$$A^{i}(\rho,\boldsymbol{x}) = 0 \tag{87}$$

In general, we should expect the distinct physical fact that stationary currents give time independent potentials. In ref. [12] we have studied the similar case of the value of the potential A^{μ} generated by a point charge at the end of a finite time interval.

Generally, the calculated potentials represent a quantity defined by the chosen transition amplitude. This interpretation follows from the Green function's vacuum to vacuum boundary prescription. We use vacuum as in quantum field the-

ories the vacuum state is the ground state of the relevant theory. We notice that the Green function can be defined up to a solution of the relevant homogeneous equation that we study.

Finally, we notice that finite-time or transient physical phenomena in currents can occur before the system under study stabilizes into a steady-state regime. *I.e.* we have a short-lived time-dependent behavior. Such processes are characterized by interactions and redistributions within the system.

6. Conclusions

Here we considered the dynamics of photonic fields in a finite-time interval. We used path-integral techniques as well as the Hamiltonian quantization approach. We integrated according to standard methods and derived finite time-domain Green functions. Integral representations were given as well. We applied the derived expressions to source-generated potentials in a four-dimensional spacetime. We can use these results in the derivation of finite time interval predictions.

Further we can observe the gauge invariance under local gauge transformations. In fact, gauge transformations should leave invariant both the Lagrangian density and the field strength tensor. Generally, a gauge transformation should not have an effect on any physical process. Therefore, we can conclude that the potentials are not themselves observables. In fact, any physical observable is expected to be gauge invariants. Moreover, to impose the gauge condition we use the method of Lagrange multipliers and add to the Lagrangian a term which vanishes for configurations which satisfy the same condition (gauge fixing). Further the gauge transformation terms are expected to obey the same periodic temporal boundary conditions as the potentials.

We notice that the present method is an alternative compared to the numerical integration of equations or to the use of lattice field theories.

In subsequent publications we intend to consider other fields, interacting or free, and study them.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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