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Non-Inclusive 1-Good-Neighbor Diagnosability of BubbleSort Star Graph

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Abstract

Diagnosability serves a vital function in assessing the reliability of interconnection networks. A non-inclusive 1-good-neighbor faulty set is defined so that any two such faulty sets are mutually non-inclusive, while every fault-free vertex keeps at least one fault-free neighbor. This paper determines the non-inclusive 1-good-neighbor diagnosability of the bubble-sort star graph BS_n, which is shown to be $6n-13$ when $n \geq 6$ and $8n-21$ under the PMC model when $n \geq 4$.

Keywords

Bubble-Sort Star Graph, Non-Inclusive 1-Good-Neighbor Diagnosability, MM* Model, PMC Model, Interconnection Network

1. Introduction

With the widespread popularization and rapid development of computers, the era of big data has arrived, and the importance of fault diagnosis for complex network systems has become increasingly prominent; the study of diagnosability has emerged as one of the key topics. For complex network multiprocessor systems, processor failures occur from time to time, and faulty processors can disrupt system operation, and in severe cases, may cause irreversible damage. Consequently, locating defective processors via diagnosis and substituting them keeps the system running smoothly, while the diagnostic capability of multiprocessor systems embedded in complex networks acts as a key measure of overall reliability. A multiprocessor system is commonly modeled as an undirected multigraph, in which processors correspond to vertices and the links among them correspond to edges. To conduct precise fault diagnosis, scholars have successively proposed the classic fault diagnosis models: the PMC model [1] and the MM* model [2], which remain

the two most frequently adopted models in this field.

In the PMC model, any two adjacent nodes test one another. To be specific, suppose that u and v are adjacent and that u tests v; the outcome is written as $\sigma(u,v)$. When the tester u is free of faults, $\sigma(u,v)=0$ if v is free of faults as well, whereas $\sigma(u,v)=1$ if v is fault. Should u itself be fault, the outcome cannot be trusted.

The MM* model, by contrast, builds on the notion of a comparator node. Concretely, a comparator node w runs simultaneous tests on two of its neighbors u and v and then compares the two outcomes, yielding a comparison result $\sigma(u,v;w)$. If the comparator node w is free of faults and both u and v are free of faults, then $\sigma(u,v;w)=0$; if w is free of faults and either u or v is fault, then $\sigma(u,v;w)=1$. When w is fault, the comparison outcome becomes unreliable. The MM* model requires that u, v, and w must undergo the comparison test.

For a multiprocessor system, all test results are recorded as the syndrome σ, which is a function from E to $\{0,1\}$.

Along with the PMC model, Preparata *et al.* introduced the concept of t-diagnosability: a system is said to be t-diagnosable provided that any faulty set comprising at most t faulty nodes in the system can be identified and diagnosed in a unique manner; the largest t for which this holds is called the diagnosability t(G) of the system.

Lai *et al.* proposed conditional diagnosability [3], introducing a rationality constraint on faulty sets: all neighbors of any vertex are not entirely contained in the faulty set. The conditional diagnosabilities of hypercubes, k-ary n-cubes, augmented cubes, matching composition networks and split-star as well as (n,k)-star networks have been investigated [4]-[9]. Peng *et al.* then introduced the g-good-neighbor conditional diagnosability [10], under which every fault-free node is required to possess no fewer than g neighbors that are fault-free. There exist research results on the g-good-neighbor conditional diagnosabilities of 3-ary n-cubes, (n,k)-star graphs, exchanged crossed cubes, twisted hypercubes, crossed cubes, locally exchanged twisted cubes, and arrangement graphs [11]-[17]. In 2022, the non-inclusive *g*-good-neighbor diagnosability was proposed [18]; the non-inclusive 1-good-neighbor diagnosability of the star graph S_n have been studied [19]. In this paper, we establish that the non-inclusive 1-good-neighbor diagnosability of the bubble-sort star graph BS_n equals $6n-13$ under the MM* model when $n\geq 6$, whereas it equals $8n-21$ under the PMC model when $n\geq 4$.

2. Preliminaries

A multiprocessor system may be described in abstract terms by a multigraph $G(V,E)$. $G(V,E)$ serves as an abstract description of a multiprocessor system, in which V gathers all the vertices and E gathers all the edges. For two vertex sets S and T, let $S-T$ denote the difference obtained by deleting from S

every vertex contained in T. Let F be the set of all faulty vertices in G, which is a subset of V. Therefore $V-F$ denotes the set obtained by removing the faulty vertices contained in subset F from the vertex set V; so, $V-F$ is what remains after the faulty vertices lying in F are deleted from V, so it consists exactly of the fault-free vertices of V. For any vertex $v \in V$, $N(v)$ denotes the collection of all vertices that are adjacent to v. Let A be a collection of V;then we define $N(A)$ of collection A refers to the union of the neighbor collections of all vertices v_i in A, with those vertices already contained in A excluded. A faulty set is said to be non-inclusive 1-good-neighbor provided that any two faulty sets of this type are mutually non-inclusive and, moreover, every fault-free vertex retains at least one neighbor that is also free of faults [18]. Let $[a,b]=\{x \mid a \le x \le b\}$. Define a permutation operation " $\star$ " in the following way: $u=v\star(i,j)$ when $u=x_1x_2\cdots x_i\cdots x_j\cdots x_n$, and $v=x_1x_2\cdots x_j\cdots x_i\cdots x_n$, where $x_i \in \{1,2,\cdots,n\}$, $x_i \neq x_j$ whenever $i \neq j$, and $i,j \in [1,n]$ [20]. The graph BS_n possesses $n!$ vertices, each taking the form $u=x_1x_2\cdots x_n$, where $x_i \in \{1,2,\cdots,n\}$ and $x_i \neq x_j$ for $i \neq j$. Two vertices u,v in $V(BS_n)$ are joined by an edge exactly when $v=u\star(1,i)$ for some i in $[2,n]$ or $v=u\star(i-1,i)$ for some i in $[3,n]$.

BS_n is a $(2n-3)$-regular graph and bipartite (see **Figure 1**). It could be partitioned into n subgraphs $BS_n^1, BS_n^2, \cdots, BS_n$, where each BS_n^i contains vertices with $x_n=i$. For $u \in V(BS_n^i)$, let $u^+=u(1,n)$, $u^-=u(n-1,n)$, and $N_u^+=\{u^+,u^-\}$. Denote $E_{i,j}(BS_n)=E_{BS_n}(V(BS_n^i),V(BS_n^j))$. Note that $BS_n^i \cong BS_{n-1}$ for $i \in [1,n]$.

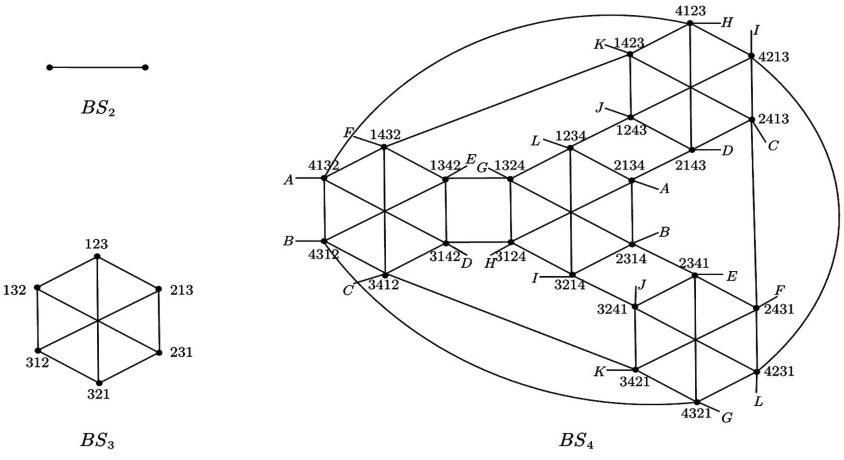

Figure 1. BS_2, BS_3 and BS_4.

3. Non-Inclusive 1-Good-Neighbor Diagnosability under the MM* Model

We first present lemmas for the main result, followed by upper and lower bounds of $t_{N_1}(BS_n)$.

3.1. Upper Bound

Lemma 1 ([2]). For two distinct subsets $F_1, F_2 \subseteq V(G)$, the pair F_1 and F_2 is not indistinguishable exactly when the following alternative holds(see **Figure 2**):

1) Some $u, v \in F_1 - F_2$ and some $w \in V(G) - (F_1 \cup F_2)$ satisfy $(u, v; w) \in E$;

2) Some $u, v \in F_2 - F_1$ and some $w \in V(G) - (F_1 \cup F_2)$ satisfy $(u, v; w) \in E$;

3) Some $\exists v \in F_1 \triangle F_2$ and some $u, w \in V(G) - (F_1 \cup F_2)$ satisfy $(u, v; w) \in E$.

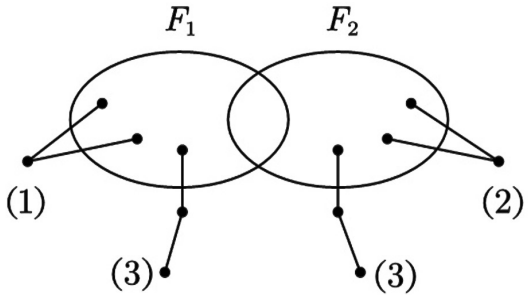

Figure 2. Illustration of Lemma 3.1.

Lemma 2 ([21]) For $n \geq 4$:

1) $\left|E_{i,j}(BS_n)\right| = 2(n-2)!$;

2) For $u, v \in V(BS_n^i)$, $N_u^+ \cap N_v^+ = \varnothing$;

3) $cn(BS_n) \leq 3$;

4) $\kappa(BS_n) = 2n-3$.

Lemma 3 ([21]). Let $F \subset V(BS_n)$ with $|F| \leq 4n-9$. If $BS_n - F$ is disconnected, then it satisfies one of the following:

1) Two components, one being an isolated vertex;

2) Three components, two being isolated vertices.

Lemma 4. $t_{N_1}(BS_n) \leq 6n-13$ for $n \geq 6$.

Proof. Let $P_3 = u_1u_2u_3$ be a 3-path in BS_n, $P = \{u_1, u_2, u_3\}$ (see **Figure 3**). Define $F_1 = N_{BS_n}(P) \cup \{u_1\}$, $F_2 = N_{BS_n}(P) \cup \{u_3\}$.

Claim 1. F_1 and F_2 are not distinguishable. Since $|F_1 - F_2| = |F_2 - F_1| = 1$, conditions (1) and (2) of Lemma 3.1 fail. For $F_1 \Delta F_2 = \{u_1, u_3\}$, we have $N_{BS_n}(u_i) - \{u_2\} \subset F_1 \cup F_2$, $i = \{1,3\}$, $N_{BS_n}(u_2) \subseteq F_1 \cup F_2$, so condition (3) also fails. By Lemma 3.1, F_1 and F_2 are not distinguishable.

Claim 2. F_1 and F_2 are not inclusive 1-good-neighbor faulty sets. We verify $\left|N_{BS_n}(u) \cap (BS_n - F_i)\right| \geq 1$ for all $u \in BS_n - F_i$ ($i = 1,2$).

Case 1: $u \in P$. Then $u = u_2$. We have $N_{BS_n}(u) \cap (BS_n - F_1) = \{u_3\}$, $N_{BS_n}(u) \cap (BS_n - F_2) = \{u_1\}$.

Case 2: $u \notin P \cup N_{BS_n}(P)$. Then $N_{BS_n}(u) \cap P = \varnothing$ and $N_{BS_n}(u) \cap N_{BS_n}(P) = \varnothing$, so $N_{BS_n}(u) \cap (F_1 \cup F_2) = \varnothing$. Since $2n-3 \geq 1$ for $n \geq 6$, we get $\left|N_{BS_n}(u) \cap (BS_n - F_1 \cup F_2)\right| \geq 1$. Above all,

$\left|N_{BS_n}(u)\cap\left(BS_n-F_i\right)\right|\geq 1$ for all $u\in BS_n-F_i$.

Claim 3. $|F_1|=|F_2|=6n-12$. By Lemma 3.2 (3), u_1 and u_3 have exactly one common neighbor u_2, while u_1,u_2 and u_2,u_3 have no common neighbor. We consider the number of nodes in $N_{BS_n}(P)$ under extreme cases to determine the range of diagnosability, $\left|N_{BS_n}(P)\right|\leq(2n-4)+(2n-5)+(2n-4)=6n-13$, $|F_1|=|F_2|=\left|N_{BS_n}(P)\right|+1=6n-13+1=6n-12$. Since BS_n is not $(6n-12)$-diagnosable, $t_{N_1}(BS_n)\leq 6n-13$ for $n\geq 6$. □

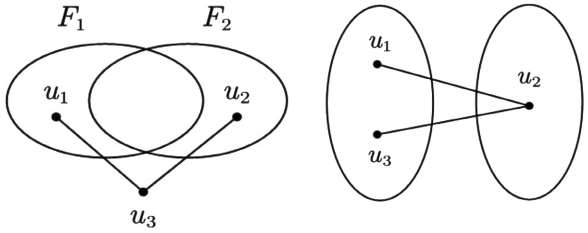

Figure 3. P_3 in BS_n.

3.2. Lower Bound

Lemma 5. Let F be a 1-good-neighbor faulty collection of BS_n with $|F|\leq 6n-14$ ($n\geq 6$). Then BS_n-F contains no isolated vertices, and no more than one of its components is isomorphic to K_2.

Proof. Because F is a 1-good-neighbor faulty set, there are no isolated vertices in BS_n-F. Assume for contradiction that there exist two K_2 components $S_1=\{v_1,v_2\}$, $S_2=\{u_1,u_2\}$. By Lemma 3.2 (3) and bipartiteness, $\left|N_{BS_n}(S_1)\cap N_{BS_n}(S_2)\right|\leq 6$. Also $\left|N_{BS_n}(S_1)\right|=\left|N_{BS_n}(S_2)\right|=2(2n-3)-2=4n-8$. $|F|\geq 2(4n-8)-6=8n-22>6n-14$, $n\geq 6$, contradiction.

Lemma 6. Let F be a 1-good-neighbor faulty set of BS_n with $|F|\leq 6n-14$ ($n\geq 6$). If $BS_n^j-F_j$ and $BS_n^k-F_k$ are connected, then $BS_n\left[V\left(BS_n^j-F_j\right)\cup V\left(BS_n^k-F_k\right)\right]$ is connected.

Proof. By Lemma 3.2 (1), $\left|E_{j,k}(BS_n)\right|=2(n-2)!$ for $n\geq 6$, $2(n-2)!>6n-14$, so at least one edge connects them, hence the union is connected. □

Lemma 7 ([21]). For $u\in V\left(BS_n^{[1,2]}\right)$, $u^+\in V\left(BS_n^{[3,n]}\right)$ or $u^-\in V\left(BS_n^{[3,n]}\right)$.

Lemma 8. Let F be a 1-good-neighbor faulty set of BS_n when $|F|\leq 6n-14$ for $n\geq 6$. Then BS_n-F falls into one of the two cases below:

1) BS_n-F remains connected;

2) BS_n-F splits into exactly two parts, one of which is isomorphic to a K_2.

Proof. Let F be a 1-good-neighbor faulty set of BS_n with $|F|\leq 6n-14$ and $F_i=F\cap V\left(BS_n^i\right)$ for $i\in[1,n]$. We assume $|F_1|\geq|F_2|\geq\cdots\geq|F_n|$. Suppose that BS_n-F is not connected. By Lemma 3.2(iv), $|F|\geq 2n-3$ □

Claim 4. $\left|F_3 \cup F_4 \cup \cdots \cup F_n\right| \geq 2$ and $\left|F_3\right| \leq 2n-5$.

Suppose that $\left|F_3 \cup F_4 \cup \cdots \cup F_n\right| < 2$. Then $\left|F_3\right| \leq 1$ and $F_4 = F_5 = \cdots = F_n = \varnothing$. Then $BS_n^i - F_i$ with $i \in [3,n]$ stays connected by Lemma 3.2(4). Lemma 3.6 forces $BS_n^{[3,n]} - F^{[3,n]}$ to be connected. If $F_3 = \varnothing$, then $BS_n - F$ stays connected by Lemma 3.7, a contradiction. Now we consider the case $\left|F_3\right| = 1$. In view of Lemmas 3.2(2) and 3.7, we may assume that there is exactly one vertex $u \in V\left(BS_n^{[1,2]} - F^{[1,2]}\right)$ (such that $N_u^+ \cap V\left(BS_n^{[3,n]}\right) = F_3$. F is a conditional faulty set, so there exists some $v \in V\left(BS_n^{[1,2]} - F^{[1,2]}\right)$) satisfy $(u,v) \in E\left(BS_n\right)$. Since $N_v^+ \cap F_3 = \varnothing$, $BS_n - F$ keep connected by Lemma 3.7, a contradiction. If $\left|F_3\right| > 2n-5$, then $\left|F_1\right| \geq \left|F_2\right| \geq \left|F_3\right| \geq 2n-4$, which implies $\left|F\right| \geq 3(2n-4) = 6n-12 > 6n-14$, a contradiction. By Claim 4 and Lemma 3.2(4), $BS_n^i - F_i$ is connected for every $i \in [3,n]$. Thus, by Lemma 3.6, $BS_n^{[3,n]} - F^{[3,n]}$ is connected. Since $BS_n - F$ is disconnected, at least one of $BS_n^i - F_i$ for $i \in [1,2]$ is disconnected, which leads to the following cases.

Case 1: Both $BS_n^1 - F_1$ and $BS_n^2 - F_2$ are disconnected.

In this case, we have $\left|F_2\right| \geq 2n-3$ and $\left|F_1\right| \leq (6n-14) - (2n-3) - 2 = 4n-13$. We can also get $\left|F_2\right| \leq 4n-13$ otherwise $\left|F\right| > 2(4n-13) = 8n-26 \geq 6n-14$ for $n \geq 6$, a contradiction. By Lemma 3.3, we consider three subcases:

Subcase 1.1: Both $BS_n^1 - F_1$ and $BS_n^2 - F_2$ consist of a component and two isolated vertices;

Let v_1, v_2, S_1 (resp., u_1, u_2, S_2) be the two isolated vertices and the other component of $BS_n^1 - F_1$ (resp., $BS_n^2 - F_2$)), respectively. Since $\left|E_{1,n}\left(BS_n\right)\right| = 2(n-2)!$ and every vertex in BS_n^1 has at most one neighbor in BS_n^n, we get $\left|E_{BS_n}\left(S_1, V\left(BS_n^n - F_n\right)\right)\right| \geq 2(n-2)! - (6n-14) > 1$ for $n \geq 6$, which implies $BS_n\left[V\left(BS_n^{[3,n]} - F^{[3,n]}\right) \cup S_1\right]$ is connected. Similarly, $BS_n\left[V\left(BS_n^{[3,n]} - F^{[3,n]}\right) \cup S_2\right]$ is connected. Since F is a 1-good-neighbor faulty set, $N_{BS_n}(u) - F \neq \varnothing$ for $u \in \{u_1, u_2, v_1, v_2\}$. By Lemma 3.5 and $BS_n - F$ being disconnected, there exist $u \in u_1, u_2$ and $v \in v_1, v_2$ such that $((u,v) \in E\left(BS_n - F\right)$ and the other two vertices are connected to $\left(BS_n^{[3,n]} - F^{[3,n]}\right) \cup S_1 \cup S_2$. Hence $BS_n - F$ satisfies (2).

Subcase 1.2: Only one of $BS_n^1 - F_1$ and $BS_n^2 - F_2$ consists of a component and two isolated vertices;

Without loss of generality, assume $BS_n^1 - F_1$ consists of a component S_1 and two isolated vertices v_1, v_2, while $BS_n^2 - F_2$ consists of a component S_2 and one isolated vertex u_1 (the case where $BS_n^2 - F_2$ has two isolated vertices and $BS_n^1 - F_1$ has one is symmetric and follows the same logic). By Lemma 3.2(1),

$\left|E_{1,n}\left(BS_n\right)\right|=2(n-2)!$ and every vertex in BS_n^1 has at most one neighbor in BS_n^n For $n\geq 6$, $2(n-2)!-(6n-14)>1$, so $\left|E_{BS_n}\left(S_1,V\left(BS_n^n-F_n\right)\right)\right|\geq 2(n-2)!-(6n-14)>1$. This implies $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup S_1\right]$ is connected. Similarly, $\left|E_{2,n}\left(BS_n\right)\right|=2(n-2)!$, so $\left|E_{BS_n}\left(S_2,V\left(BS_n^n-F_n\right)\right)\right|\geq 2(n-2)!-(6n-14)>1$ for $n\geq 6$. Thus, $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup S_1\cup S_2\right]$ is connected. Since F is a conditional faulty set, $N_{BS_n}(v)-F\neq\varnothing$ for all $v\in\{v_1,v_2,u_1\}$. By Lemma 3.5, BS_n-F contains no P_3 and at most one K_2; combined with BS_n-F being disconnected, the isolated vertices cannot all connect to $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup S_1\cup S_2\right]$, otherwise, BS_n-F would be connected. There must exist two isolated vertices v_1 and u_1 such that $(v_1,u_1)\in E\left(BS_n-F\right)$, forming a K_2, while the remaining isolated vertex v_2 connects to $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup S_1\cup S_2\right]$. This is because if v_2 also failed to connect, it would need to form another K_2 with another vertex, contradicting Lemma 3.5. Hence, BS_n-F has two components: one is the connected $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup S_1\cup S_2\cup v_2\right]$, and the other is the K_2 formed by v_1 and u_1. Thus, BS_n-F satisfies (2).

Subcase 1.3: Both $BS_n^1-F_1$ and $BS_n^2-F_2$ consist of a component and an isolated vertices.

Let S_1 be the non-isolated component and v_1 be the isolated vertex of $BS_n^1-F_1$. Let S_2 be the nonisolated component and u_1 be the isolated vertex of $BS_n^2-F_2$. By Lemma 3.2(1), $\left|E_{1,n}\left(BS_n\right)\right|=\left|E_{2,n}\left(BS_n\right)\right|=2(n-2)!$. For $n\geq 6$, $2(n-2)!-(6n-14)>1$, so $\left|E_{BS_n}\left(S_1,V\left(BS_n^n-F_n\right)\right)\right|>1$ and $\left|E_{BS_n}\left(S_2,V\left(BS_n^n-F_n\right)\right)\right|>1$. This implies $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup S_1\right]$ is connected, as both S_1 and S_2 connect to $BS_n^{[3,n]}-F^{[3,n]}$ Since F is a conditional faulty set, $N_{BS_n}(v_1)-F\neq\varnothing$ and $N_{BS_n}(u_1)-F\neq\varnothing$. By Lemma 3.5, BS_n-F has no isolated vertices and at most one K_2, combined with BS_n-F being disconnected, v_1 and u_1 cannot both connect to $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup S_1\cup S_2\right]$, otherwise, BS_n-F would be connected. Thus, v_1 and u_1 must be adjacent in BS_n-F, forming a K_2. The rest of the graph $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup S_1\cup S_2\right]$ remains connected. Hence, BS_n-F has two components: one connected component and one K_2, satisfying (2).

Case 2: Only $BS_n^2-F_2$ is disconnected. By Lemmas 3.6 and 3.7, $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup V\left(BS_n^1-F_1\right)\right]$ is connected. Since $|F_1|\geq|F_2|$ and $|F|\geq 2|F_2|$, $|F_2|\leq 4n-13$. By Lemma 3.3, $BS_n^2-F_2$ has a component S and at most two isolated vertices. Then $|S|\geq(n-1)!-|F_2|-2$ |−2. By the same argu-

ment as Case 1, $BS_n\left[V\left(BS_n^{[3,n]}-F^{[3,n]}\right)\cup V\left(BS_n^1-F_1\right)\cup S\right]$ is connected. By the definition of conditional faulty sets, the isolated vertices must be adjacent to $V\left(BS_n\right)-\left(F\cup V\left(BS_n^2\right)\right)$, implying BS_n-F is connected, a contradiction.

Case 3: Only $BS_n^1-F_1$ is disconnected. By Lemma 3.6, $BS_n^{[2,n]}-F^{[2,n]}$ is connected. Assume $BS_n^1-F_1$ has k components $L_1,L_2,\cdots,L_k\ (k\geq 2)$ with $\left|V\left(L_1\right)\right|\geq\left|V\left(L_2\right)\right|\geq\cdots\geq\left|V\left(L_k\right)\right|$.

Subcase 3.1: $\left|L_1\right|\leq 3$. By Lemma 3.5 and the property of conditional faulty sets, all $L_i\left(1\leq i\leq k\right)$ except at most one K_2 are connected to $BS_n^{[2,n]}-F^{[2,n]}$. Since BS_n-F is disconnected, $BS_n^1-F_1$ contains a K_2 with $N_{BS_n}\left(V\left(K_2\right)\right)\subseteq F$, so (2) holds.

Subcase 3.2: There exists $m\in[1,k]$ such that $\left|L_1\right|\geq\left|L_2\right|\geq\cdots\geq\left|L_m\right|\geq 4$. Let $l_j=\left|V\left(L_j\right)\right|$ for $1\leq j\leq m$. We use proof by contradiction to derive a contradiction in the number of vertices. First, we show $BS_n\left[V\left(BS_n^{[2,n]}-F^{[2,n]}\right)\cup V\left(L_j\right)\right]$ is connected for each $j\in[1,m]$. Suppose not, then $N_{BS_n^1}\left(L_j\right)\cup\left(U_{u\in V\left(L_j\right)}N_u^+\right)\subseteq F$. Since $|F|\leq 6n-14$, $\left|\bigcup_{u\in V\left(L_j\right)}N_u^+\right|=2l_j\leq 6n-14$, we have $l_j\leq 3n-7$. As $BS_n\left[L_j\right]$ is bipartite, $|F|\geq l_j\left(2n-3-l_j/2\right)$. For $4\leq l_j\leq 3n-7$, $|F|\geq\min\left\{8n-20,\left(3n-7\right)\left(n/2+1/2\right)\right\}>6n-16\ \left(n\geq 6\right)$, contradiction. Thus $BS_n\left[V\left(BS_n^{[2,n]}-F^{[2,n]}\right)\cup\left(U_{j\in[1,m]}V\left(L_j\right)\right)\right]$ is connected. Since BS_n-F is not connected, $m<k$. By Lemma 3.5, all $L_i\left(m<i\leq k\right)$ except at most one K_2 are connected to $BS_n^{[2,n]}-F^{[2,n]}$, so (2) holds. □

Lemma 9. Suppose that F_1 and F_2 denote two non-inclusive 1-good-neighbor faulty sets of BS_n when $\left|F_1\right|\leq 6n-13$, $\left|F_2\right|\leq 6n-13$ when $n\geq 6$. If F_1 and F_2 are not distinguishable, then $BS_n-F_1\cup F_2$ contains no isolated vertex.

Proof. Since $\left|F_1\right|,\left|F_2\right|\leq 6n-13$, we have $\left|F_1\cup F_2\right|\leq\left|F_1\right|+\left|F_2\right|\leq 6n-14<n!=\left|V\left(BS_n\right)\right|$ for $n\geq 6$. Thus, $F_1\cup F_2\neq V\left(BS_n\right)$. Let W be the set of isolated vertices of $BS_n-F_1\cup F_2$, and $H=BS_n-\left(F_1\cup F_2\cup W\right)$. Suppose that W. Let w be an isolated vertex of W, then $N_{BS_n}\left(w\right)-F_1\subseteq F_2-F_1$. Since F_1 is a 1-good-neighbor faulty set of BS_n, the vertex w has at least one neighbor in $V\left(BS_n\right)-F_1$. And since F_1 and F_2 are indistinguishable, by Lemma 3.1, $\left|N_{BS_n}\left(w\right)\cap\left(F_2-F_1\right)\right|=1$. Similarly, $N_{BS_n}\left(w\right)-F_2\subseteq F_1-F_2$ and $\left|N_{BS_n}\left(w\right)\cap\left(F_1-F_2\right)\right|=1$. Thus, $\left|N_{BS_n}\left(w\right)\cap\left(F_2\cap F_1\right)\right|=2n-5$. Hence, we have

$$\begin{aligned}\sum_{w\in W}\left|N_{F_1\cap F_2}\left(w\right)\right|&=|W|\left(2n-5\right)\leq\left|F_1\cap F_2\right|\\&\leq\left|F_1\cap F_2\right|\left(2n-3\right)\leq\left(\left|F_2\right|-1\right)\left(2n-3\right)\\&=\left(6n-14\right)\left(2n-3\right)\end{aligned}$$

Thus, $|W| \le \frac{(6n-14)(2n-3)}{2n-5} \le 2(6n-14) = 12n-28$.

Case 1: $V(H) = \varnothing$.

If $V(H) = \varnothing$, then we have

$n! = |F_1 \cup F_2| + |W| \le |F_1| + |F_2| + |W| \le 2(6n-13) + 12n - 28 = 24n - 54$. It is a contradiction for $n \ge 6$.

Case 2: $V(H)$

Since F_1 and F_2 are indistinguishable, $N_{BS_n}(F_1 \Delta F_2 \cup W) \subseteq F_1 \cap F_2$. And since F_1 and F_2 are noninclusive, $|F_1 - F_2| \ge 1$ $|F_2 - F_1| \ge 1$. Thus, $|F_1 \cap F_2| \le 6n-14$ -14. By Lemma 3.8, we know that $BS_n - F_1 \cap F_2$ consists of a large component and a K_2. Let G_1 be a component of $BS_n - F_1 \cap F_2$, which satisfies $u_1, w, u_2 \subseteq G_1$ where $u_1 = N_{BS_n}(w) \cap (F_1 - F_2)$ and $u_2 = N_{BS_n}(w) \cap (F_2 - F_1)$, then $|V(G_1)| \ge 3$. So the other component is a K_2. However, $|V(BS_n)| - |F_1 \cup F_2| - |W| \ge n! - 12n + 26 - 12n + 28 = n! - 24n + 54 > 2$ for $n \ge 6$. Thus, $BS_n - F_1 \cup F_2$ has no isolated vertex. □

Lemma 10. For any two non-inclusive 1-good-neighbor faulty sets F_1, F_2 when $|F_1| \le 6n-13$ and $|F_2| \le 6n-13$ where $n \ge 6$. F_1 and F_2 are not indistinguishable in BS_n under the MM*.

Proof. Assume, for contradiction, that F_1 and F_2 cannot be distinguished. Let $|F_1| \ge |F_2|$. By Lemma 3.9, $F_1 \cup F_2 \ne V(BS_n)$, and there are no isolated vertices in $BS_n - F_1 \cup F_2$. This follows from the fact that F_1 and F_2 are non-inclusive 1-good-neighbor faulty sets that cannot be distinguished from one another, no edge joins $F_1 \Delta F_2$ to $V(BS_n - F_1 \cup F_2)$, and $\delta(F_1 - F_2) \ge 2$, $\delta(F_2 - F_1) \ge 2$. Thus, $|F_1 \Delta F_2| \ge 4$. Furthermore, $N_{BS_n}(F_1 \Delta F_2) \subseteq F_1 \cap F_2$. Thus, $|F_1 \cap F_2| \le 6n-15 < 6n-14$. By Lemma 3.8, $BS_n - F_1 \cap F_2$ consists of a single large component together with a K_2. Since $|BS_n - F_1 \cup F_2| \ge |BS_n| - |F_1 \cup F_2| \ge n! - (12n-26) > 2$. we have $|F_1 \Delta F_2| \le 2$. However, $|F_1 \Delta F_2| \ge 4$, contradiction. Therefore, F_1 and F_2 are not indistinguishable in BS_n, and $t_{N1}(BS_n) \ge 6n-13$. □

Theorem 11. We have $t_{N1}(BS_n) = 6n-13$ for $n \ge 6$ under the MM* model.

4. Non-Inclusive 1-Good-Neighbor of BS_n under the PMC

In the part, we prove the upper bound and lower bounds of $t_{N_1}(BS_n)$.

Lemma 12 ([3]). Two distinct subsets F_1, $F_2 \subseteq V(G)$ are not indistinguishable if and only if at least one of the following holds (see **Figure 4**):

1) Some $v \in F_1 - F_2$ and some $u \in V(G) - (F_1 \cup F_2)$ satisfy $(u,v) \in E$:

2) Some $v \in F_2 - F_1$ and some $u \in V(G) - (F_1 \cup F_2)$ satisfy $(u,v) \in E$;

Lemma 13. $t_{N_1}(BS_n) \le 8n-21$ *with* $n \ge 5$.

Proof. Let $C_4 = u_1u_2u_3u_4u_1$ be a 4-cycle in BS_n, $C = \{u_1, u_2, u_3, u_4\}$, $F_1 = N_{BS_n}(C) \cup \{u_1, u_2\}$, and $F_2 = N_{BS_n}(C) \cup \{u_3, u_4\}$ (see **Figure 5**).

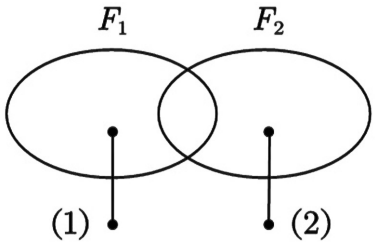

Figure 4. 4 illustration of lemma 4.1.

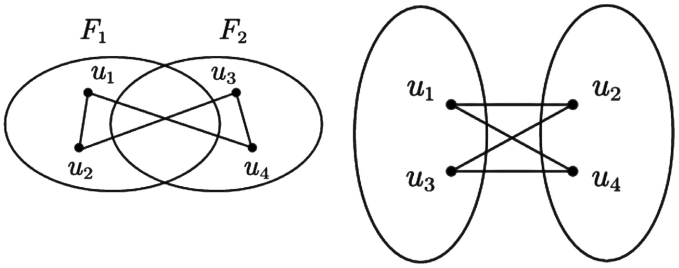

Figure 5. C_4 in BS_n.

Claim 5. $\left|F_3 \cup F_4 \cup \cdots \cup F_n\right| \geq 2$ and F_1 and F_2 are indistinguishable. $u \in V\left(BS_n\right) - F_1 \cup F_2$ is not adjacent to $v \in C$, otherwise $u \in N_{BS_n}(C)$. Since $F_1 \Delta F_2 = \{u_1, u_2, u_3, u_4\}$, there exists no vertex $u \in V(G) - (F_1 \cup F_2)$ adjacent to $v \in F_1 \Delta F_2$. By Lemma 4.1, F_1 and F_2 are indistinguishable.

Claim 6. $\left|F_3 \cup F_4 \cup \cdots \cup F_n\right| \geq 2$ and F_1 and F_2 are non-inclusive 1-good-neighbor faulty sets. since $\left|F_1 - F_2\right| = \left|F_2 - F_1\right| = 2$, F_1 and F_2 are non-inclusive. We verify $\left|N_{BS_n}(u) \cap \left(BS_n - F_i\right)\right| \geq 1$ for all $u \in BS_n - F_i (i = 1, 2)$:

Case 1: $u \in C$

For F_2, u_1 and u_2 are fault-free vertices, and $u_1 - u_2 \in E\left(BS_n\right)$. If $u \in P \cap \left(BS_n - F_2\right), u \in \{u_1, u_2\}$. $N_{BS_n}(u_1) \cap \left(BS_n - F_2\right) = u_2$ and $N_{BS_n}(u_2) \cap \left(BS_n - F_2\right) = u_1$ Similarly, if $u \in P \cap \left(BS_n - F_1\right)$, $u \in u_3, u_4$ $N_{BS_n}(u_3) \cap \left(BS_n - F_1\right) = u_4$ and $N_{BS_n}(u_4) \cap \left(BS_n - F_1\right) = u_3$.

Case 2: $u \notin C$

If $u \notin C \cup N_{BS_n}(C)$ $N_{BS_n}(u) \cap P = \varnothing$ and $N_{BS_n}(u) \cap N_{BS_n}(C) = \varnothing$. We have $N_{BS_n}(u) \cap \left(F_1 \cup F_2\right) = \varnothing$, where $2n - 3 \geq 1$ for $n \geq 4$, so $\left|N_{BS_n}(u) \cap \left(BS_n - F_1 \cup F_2\right)\right| \geq 1$. Above all, $\left|N_{BS_n}(u) \cap \left(BS_n - F_i\right)\right| \geq 1$ for all $u \in BS_n - F_i (i = 1, 2)$.

Claim 7. $\left|F_3 \cup F_4 \cup \cdots \cup F_n\right| \geq 2$ and $\left|F_1\right| = \left|F_2\right| = 8n - 20$. By Lemma 3.2, u_1 and u_3 (u_2 and u_4) have 2 common neighbor which are u_2 and (u_1 and u_3),while u_1 and u_2 or u_3 and u_4 have no common neighbor. We consider the number of nodes in $N_{BS_n}(P)$ under extreme cases to determine the range of diagnosability. Thus, $\left|N_{BS_n}(C)\right| = 4(2n - 5) - 2 = 8n - 22$, so $\left|F_1\right| = \left|F_2\right| = \left|N_{BS_n}(C)\right| + 2 = 8n - 22 + 2 = 8n - 20$. Since BS_n is not $(8n - 20)$-di-

agnosable, $t_{N_1}(BS_n) \leq 8n-20$ for $n \geq 4$ □

Lemma 14. Let F_1 and F_2 both non-inclusive 1-good-neighbor faulty sets, be given, when $|F_1|$ and $|F_2| \leq 8n-21$, $n \geq 4$. Then F_1, F_2 are not indistinguishable under PMC.

Proof. Suppose indistinguishable, then $N_{BS_n}(F_1 \Delta F_2) \subseteq F_1 \cap F_2$. Non-inclusive 1-good-neighbor requires $|F_1 - F_2| \geq 2$, $|F_2 - F_1| \geq 2$, so $|F_1 \Delta F_2| \geq 4$. $|F_1 \cap F_2| \geq 4(2n-5)-2 = 8n-22 \Rightarrow |F_1| \geq |F_1 \cap F_2| + 2 \geq 8n-20$, contradicts $|F_1| \leq 8n-21$. Hence distinguishable, $t_{N_1}(BS_n) \geq 8n-21$. □

Theorem 15. The $t_{N1}(BS_n) = 8n-21$ when $n \geq 4$ under PMC.

5 Conclusion

This paper investigates the non-inclusive 1-good-neighbor diagnosability of the bubble-sort star graph BS_n equals $6n-13$ under the MM* when $n \geq 6$, whereas it equals $8n-21$ under the PMC when $n \geq 4$. Compared with the study by Tian et al. on the non-inclusive g-good-neighbor diagnosability of star graphs, the bubble-sort star graph has a more complex construction, extending the research from general star graphs to more complex star graph variants. Future work will focus on the non-inclusive 2-good-neighbor diagnosability of bubble-sort star graphs under the PMC.

Author Contributions

Yu Dan conceived the study, proved the results, and wrote the paper. Lyu Shengmei revised this paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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