Journal of Applied Mathematics and Physics, 2026, 14(11), 4893-4905
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Four-Stage Fourth-Order Strong-Stability-Preserving Runge-Kutta Method for Nonlinear Autonomous Scalar Equation

Fuxing Hu

Department of Mathematics, Huizhou University, Huizhou, China
Email: fuxing hoo@163.com

How to cite this paper: Hu, F.X. (2026) Four-Stage Fourth-Order Strong-Stability-Preserving Runge-Kutta Method for Nonlinear Autonomous Scalar Equation. *Journal of Applied Mathematics and Physics*, **14**, 4893-4905.
https://doi.org/10.4236/jamp.2026.1411237

Received: October 26, 2026
Accepted: November 23, 2026
Published: November 26, 2026

Copyright © 2026 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/
Open Access

Abstract

Strong-stability-preserving (SSP) time discretizations based on Runge-Kutta (RK) methods have already been shown to be effective and nonlinearly stable for semidiscrete approximations of hyperbolic conservation laws. For general ordinary differential equations (ODEs), the explicit four-stage fourth-order SSPRK method has been proved to be nonexistent in the literatures. However, we find that such a method can be obtained for the nonlinear autonomous scalar case. Generally, the semidiscrete approximations of hyperbolic conservation laws generate decoupled nonlinear systems and one only needs to successively evolve each nonlinear autonomous scalar by SSPRK method. For simplicity, the standard RK method is rearranged into the some different form. Under this form, we can also easily analyse the strong stability property of method. And what is more, comparing the classical SSPRK method, this new form is more compact and leads to less constraint conditions which facilitates the solution of optimization problem. In addition, we can concisely prove the optimality of second- and third-order SSPRK methods under this form. In the end, the order of accuracy and SSP restrictions of four-stage fourth-order SSPRK method are numerically tested.

Keywords

Runge-Kutta Methods, Strong Stability, Four-Stage Fourth-Order Method

1. Introduction

For the hyperbolic conservation laws, the semidiscrete method (also called the method of lines) is especially useful in devising the methods with order of accu-

racy greater than 2, and it often generates autonomous scalar equations. This approach considers the spatial and temporal discretizations in two stages. Since hyperbolic partial differential equations easily generate discontinuous solutions, the stable spatial and temporal discretizations are preferred. Generally, we first discretize spatial variables with some suitable schemes (e.g. total variation diminishing (TVD) schemes [1], essentially non-oscillatory (ENO) schemes [2] [3], weighted ENO schemes [4] [5], and so on), leaving the problem continuous in time and this leads to a system of ODEs. In [6], the nonlinear instability has been shown numerically by method coupling the TVD spatial discretization with linearly stable temporal discretization. This suggests the necessity of nonlinearly stable temporal discretization.

The SSPRK methods for discretizing time variable are first developed in [7] [8] and previously named TVD-RK methods. The essence of SSPRK is to write the standard RK scheme into the convex combinations of forward Euler schemes. If the forward Euler scheme is strongly stable with a certain CFL restriction, then high-order SSPRK schemes with a modified CFL restriction can be also strongly stable. Under the assumption that the number of stage s is equal to the order p, the optimal SSPRK methods with order of accuracy of 1, 2 and 3 have been found. In [9] [10], with relax restriction $s > p$, a new class of optimal high-order SSPRK methods is developed. These new schemes have better CFL restrictions of strong stability than the schemes have been developed before.

In [7], a four-stage fourth-order scheme is given but with two negative coefficients. Then a slightly improved version is presented in [6] but the number of negative coefficient is still two. To make them keep the SSP property, two additional adjoint operators have to be introduced, leading to the increased requirement of both computational cost and memory storage. For the general ODEs, the four-stage fourth-order SSPRK method with all coefficients nonnegative has been proved to be unattainable, and the lower three-stage third-order SSPRK has been the often resort. Fortunately, for the nonlinear autonomous scalar ODE, we find out an optimal explicit four-stage fourth-order SSPRK method with all the coefficients non-negative. For simplicity, we rearrange the standard RK formula into a more compact form, and under such form, the strong stability of method is still easy to analyse. When comparing with classical SSPRK method [7] [8], we do not impose the requirement of convex combination of forward Euler method on each stage. This relaxation makes the number of undetermined coefficients largely decreased, and more importantly, it leads to the decrease of the number of the objective functions and renders the optimization problem solvable. In addition, if the classical SSPRK method can be rearranged into this compact form, then the CFL restriction of compact form will be equal to that of classical SSPRK method. Finally, under this compact form, we re-prove the optimality of two-stage second-order and three-stage third-order SSPRK methods. The proofs presented here are more approachable than the ones have given in [6]. In addition, the multi-step SSP schemes and relaxation Runge-Kutta methods can refer to [11] and [12].

The organization of this paper is as follows. In Section 2 SSP, we briefly review the SSPRK schemes and basic properties of the methods. In Section 3 of SSP22 SSP33, we transform the standard RK schemes into some king of equivalent forms, and re-prove the optimality of two-stage second-order and three-stage third-order SSPRK methods. The optimal four-stage fourth-order SSPRK method for autonomous scalar equation is shown in Section 4 SSP. Finally, in Section 5, we test numerically the order of accuracy of method by a nonlinear autonomous scalar ODE.

2. The Classical Explicit SSPRK Schemes

We consider the hyperbolic conservation laws

$$u_t + f(u)_x = 0, \tag{2.1}$$

where u may be a vector, and then $f(u)$ represents the flux vector function. With the method of lines, we first discrete the spatial variable x by suitable methods (e.g. TVD, ENO, WENO and so on), and get the ODEs

$$u_t = L(u). \tag{2.2}$$

A method is strongly stable with a given norm $\|\cdot\|$ provided the sequence $\{u^n\}$ satisfies $\|u^{n+1}\| \le \|u^n\|$ for all $n \ge 0$.

To ensure the strong stability of temporal discretizing methods, the standard explicit RK schemes

$$\begin{aligned} &u^{(0)} = u^n, \\ &u^{(i)} = u^{(0)} + \Delta t \sum_{j=0}^{i-1} a_{ij} L\left(u^{(j)}\right), \quad i = 1, \cdots, s, \\ &u^{n+1} = u^{(s)}, \end{aligned} \tag{2.3}$$

is rearranged into the classical form [7]

$$\begin{aligned} &u^{(0)} = u^n, \\ &u^{(i)} = \sum_{j=0}^{i-1} \left(\hat{\alpha}_{ij} u^{(j)} + \Delta t \hat{\beta}_{ij} L\left(u^{(j)}\right)\right), \quad i = 1, \cdots, s, \\ &u^{n+1} = u^{(s)}. \end{aligned} \tag{2.4}$$

The number of undetermined coefficients in (2.4) is more than that in (2.3). It is also noted that, each method of form (2.4) can be converted to a unique method of form (2.3), but the conversion from (2.3) to (2.4) is not unique. If all the coefficients satisfy

$$\hat{\alpha}_{ij} \ge 0, \hat{\beta}_{ij} \ge 0 \text{ and } \hat{\alpha}_{ij} = 0 \text{ only if } \hat{\beta}_{ij} = 0, \tag{2.5}$$

then the scheme (2.4) can be actually regarded as convex combinations of few forward Euler methods.

Theorem 2.2. *If the forward Euler methods is strongly stable under the CFL restriction* $\Delta t < \Delta t_{FE}$, *then RK methods* (2.4) *satisfying coefficient conditions*

(2.5) *is also strongly stable provided* $\Delta t < c\Delta t_{FE}$, *where* $c = \min_{i,j} \frac{\hat{\alpha}_{ij}}{\hat{\beta}_{ij}}$.

Under the optimal requirement of $\max \min_{i,j} \frac{\hat{\alpha}_{ij}}{\hat{\beta}_{ij}}$, the two-stage second-order and three-stage third-order SSPRK methods are constructed as follows

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L\left(u^n\right), \\ u^{n+1} &= \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t L\left(u^{(1)}\right), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L\left(u^n\right), \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L\left(u^{(1)}\right), \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L\left(u^{(2)}\right), \end{aligned} \tag{2.7}$$

where, for both methods, the CFL restriction $c = \max \min_{i,j} \frac{\alpha_{ij}}{\beta_{ij}} = 1$. For the four-stage fourth-order method, the best case found in [6] is

$$\begin{aligned} u^{(1)} &= u^n + \frac{1}{2}\Delta t L\left(u^n\right) \\ u^{(2)} &= \frac{649}{1600}u^n - \frac{10890423}{25193600}\Delta t \tilde{L}\left(u^n\right) + \frac{951}{1600}u^{(1)} + \frac{5000}{7873}\Delta t L\left(u^{(1)}\right) \\ u^{(3)} &= \frac{53989}{2500000}u^n - \frac{102261}{5000000}\Delta t \tilde{L}\left(u^n\right) + \frac{4806213}{20000000}u^{(1)} - \frac{5121}{20000}\Delta t \tilde{L}\left(u^{(1)}\right) \\ &\quad + \frac{23619}{32000}u^{(2)} + \frac{7873}{10000}\Delta t L\left(u^{(2)}\right) \\ u^{n+1} &= \frac{1}{5}u^n + \frac{1}{10}\Delta t L\left(u^n\right) + \frac{6127}{30000}u^{(1)} + \frac{1}{6}\Delta t L\left(u^{(1)}\right) + \frac{7873}{30000}u^{(2)} \\ &\quad + \frac{1}{3}u^{(3)} + \frac{1}{6}\Delta t L\left(u^{(3)}\right). \end{aligned} \tag{2.8}$$

The CFL restriction of this scheme is $c = 0.936$ and two additional adjoint function evaluations are introduced to preserve the SSP property.

3. Proof of Optimality of Two-Stage Second-Order and Three-Stage Third-Order SSPRK Methods

In order to prove the optimality of these two SSPRK methods, we rearrange the standard RK scheme (2.3) into some kind of different form and make the global optimization problem solvable. It is noted that all the results in this section can be extended to non-autonomous equations of form $u'(t) = f\left(t, u(t)\right)$, since, for two-stage second-order and three-stage third-order RK methods, the order conditions for $u'(t) = f\left(t, u(t)\right)$ are same as the ones for $u'(t) = f\left(u(t)\right)$. We first rearrange the s-stage standard RK method (2.3) into the following form

$$\begin{aligned} u^{(0)} &= u^n, \\ u^{(i)} &= \sum_{j=0}^{i-1} \alpha_{ij} u^{(j)} + \Delta t \beta_{i,i-1} L\left(u^{(i-1)}\right), \quad i = 1, \cdots, s, \\ u^{n+1} &= u^{(s)}, \end{aligned} \tag{3.1}$$

where $\sum_{j=0}^{i-1} \alpha_{ij} = 1$. It is clearly found that, for each $u^{(i)}$ there appears only one function valuation $L\left(u^{(i-1)}\right)$. The following theorem clarify the relationship between the methods (2.3) and (3.1).

Theorem 3.1. *For each standard RK method of the form* (2.3), *there is only one method of* (3.1) *which is equivalent to it, and the conversion is also true.*

Proof. We first express the standard RK method (2.3) as

$$\begin{pmatrix} u^{(1)} - u^{(0)} \\ u^{(2)} - u^{(0)} \\ \vdots \\ u^{(s)} - u^{(0)} \end{pmatrix} = A \begin{pmatrix} L\left(u^{(0)}\right) \\ L\left(u^{(1)}\right) \\ \vdots \\ L\left(u^{(s-1)}\right) \end{pmatrix},$$

where

$$A = \begin{pmatrix} a_{10} & & & \\ a_{20} & a_{21} & & \\ \vdots & \vdots & \ddots & \\ a_{s0} & a_{s1} & \cdots & a_{s,s-1} \end{pmatrix}.$$

Since $a_{i,i-1} \neq 0, i = 1, \cdots, s$, the lower triangular matrix A is nonsingular, and then we get

$$\begin{pmatrix} L\left(u^{(0)}\right) \\ L\left(u^{(1)}\right) \\ \vdots \\ L\left(u^{(s-1)}\right) \end{pmatrix} = A^{-1} \begin{pmatrix} u^{(1)} - u^{(0)} \\ u^{(2)} - u^{(0)} \\ \vdots \\ u^{(s)} - u^{(0)} \end{pmatrix}.$$

Denote $A^{-1} = \left(r_1, r_2, \cdots, r_s\right)^{\mathrm{T}}$, where r_i is the ith row vector of A^{-1}, then it gives

$$L\left(u^{(i)}\right) = r_{i+1}\left(u^{(1)} - u^{(0)}, u^{(2)} - u^{(0)}, \cdots, u^{(s)} - u^{(0)}\right)^{\mathrm{T}}, \ i = 0, \cdots, s-1.$$

For the ith stage ($i \geq 2$) of standard RK method (2.3), substituting the function evaluations $L\left(u^{(j)}\right)$ ($j = 0, \cdots, i-2$) by the above formulas, we finally obtain the RK method (3.1). The conversion from RK method (3.1) to standard RK method (2.3) can be proved similarly. □

Theorem 3.2. *If the forward Euler methods is strongly stable in norm* $\|\cdot\|$ u*nder the CFL restriction* $\Delta t < \Delta t_{FE}$, *then the compact RK methods* (3.1) *satisfying the following coefficient conditions*

$$\alpha_{ij} \geq 0, \quad j = 0, \cdots, i-2,$$
$$\alpha_{i,i-1} > 0, \quad \beta_{i,i-1} > 0,$$

is also strongly stable in the same norm provided $\Delta t \leq c\Delta t_{FE}$, *where* $c = \min_{1 \leq i \leq s} \frac{\alpha_{i,i-1}}{\beta_{i,i-1}}$.

Proof. For clearly, we express (3.1) in detail as

$$\begin{aligned}
&u^{(0)} = u^n, \\
&u^{(1)} = \alpha_{10}u^{(0)} + \Delta t\beta_{10}L\left(u^{(0)}\right), \\
&u^{(2)} = \alpha_{20}u^{(0)} + \alpha_{21}u^{(1)} + \Delta t\beta_{21}L\left(u^{(1)}\right), \\
&\quad \vdots \\
&u^{(s)} = \alpha_{s0}u^{(0)} + \cdots + \alpha_{s,s-2}u^{(s-2)} + \alpha_{s,s-1}u^{(s-1)} + \Delta t\beta_{s,s-1}L\left(u^{(s-1)}\right), \\
&u^{n+1} = u^{(s)}.
\end{aligned} \tag{3.2}$$

Since $\alpha_{10} = 1$, $\beta_{10} > 0$ and the forward Euler methods is strongly stable in norm $\|\cdot\|$,

$$\begin{aligned}
\left\|u^{(1)}\right\| &= \left\|\alpha_{10}u^{(0)} + \Delta t\beta_{10}L\left(u^{(0)}\right)\right\| \\
&= \alpha_{10}\left(u^{(0)} + \Delta t\frac{\beta_{10}}{\alpha_{10}}L\left(u^{(0)}\right)\right) \\
&\leq \left\|u^{(0)}\right\|,
\end{aligned}$$

and the last inequality is valid provided the CFL restriction $\Delta t \leq \frac{\alpha_{10}}{\beta_{10}}\Delta t_{FE}$. Analogously, $\left\|u^{(i)}\right\| \leq \left\|u^{(0)}\right\|$, $i = 2, \cdots, s-1$, provided $\Delta t \leq \min\left\{\frac{\alpha_{10}}{\beta_{10}}, \cdots, \frac{\alpha_{i,i-1}}{\beta_{i,i-1}}\right\}\Delta t_{FE}$. Finally,

$$\begin{aligned}
\left\|u^{n+1}\right\| &= \left\|u^{(s)}\right\| \\
&= \left\|\alpha_{s0}u^{(0)} + \cdots + \alpha_{s,s-2}u^{(s-2)} + \alpha_{s,s-1}u^{(s-1)} + \Delta t\beta_{s,s-1}L\left(u^{(s-1)}\right)\right\| \\
&= \alpha_{s0}u^{(0)} + \cdots + \alpha_{s,s-2}u^{(s-2)} + \alpha_{s,s-1}\left(u^{(s-1)} + \Delta t\frac{\beta_{s,s-1}}{\alpha_{s,s-1}}L\left(u^{(s-1)}\right)\right) \\
&= \left\|\alpha_{s0}u^{(0)}\right\| + \cdots + \left\|\alpha_{s,s-2}u^{(s-2)}\right\| + \alpha_{s,s-1}\left(u^{(s-1)} + \Delta t\frac{\beta_{s,s-1}}{\alpha_{s,s-1}}L\left(u^{(s-1)}\right)\right) \\
&= \alpha_{s0}\left\|u^{(0)}\right\| + \cdots + \alpha_{s,s-2}\left\|u^{(s-2)}\right\| + \alpha_{s,s-1}\left\|u^{(s-1)}\right\| \\
&= \left(\alpha_{s0} + \cdots + \alpha_{s,s-2} + \alpha_{s,s-1}\right)\left\|u^{(0)}\right\| \\
&= \left\|u^{(0)}\right\| = \left\|u^n\right\|
\end{aligned}$$

The second inequality requires $\Delta t \leq \frac{\alpha_{s,s-1}}{\beta_{s,s-1}}\Delta t_{FE}$ and the last inequality re-

quires $\Delta t \le \min_{1\le i\le s-1} \frac{\alpha_{i,i-1}}{\beta_{i,i-1}} \Delta t_{FE}$. Hence, for strong stability $\left\|u^{n+1}\right\| \le \left\|u^{n}\right\|$, the CFL coefficient must be chosen to be $c = \min_{1\le i\le s} \frac{\alpha_{i,i-1}}{\beta_{i,i-1}}$. □

It is noted that the compact SSPRK methods (3.1) can be regarded as the special cases of classical SSPRK methods (2.4) since the coefficient conditions in Theorem 3.2. also satisfy the coefficient conditions (2.5).

Theorem 3.3. *If an s-stage classical SSP-RK method of form* (2.4) *can be rearranged into the compact SSP-RK method of form* (3.1), *then* $\min_{i,j} \frac{\hat{\alpha}_{ij}}{\hat{\beta}_{ij}} \le \min_{i} \frac{\alpha_{i,i-1}}{\beta_{i,i-1}}$.

Proof. For the ith stage ($i = 1,\cdots,s$), by simple symbolic operation, we can obtain □

$$\alpha_{i,i-1} = \hat{\alpha}_{i,i-1} + \frac{\hat{\beta}_{i,i-2}}{\hat{\beta}_{i-1,i-2}},$$
$$\beta_{i,i-1} = \hat{\beta}_{i,i-1}, \quad i = 1,\cdots,s.$$

Then it gives

$$\frac{\alpha_{i,i-1}}{\beta_{i,i-1}} = \frac{\hat{\alpha}_{i,i-1}}{\hat{\beta}_{i,i-1}} + \frac{\hat{\beta}_{i,i-2}}{\hat{\beta}_{i-1,i-2}\hat{\beta}_{i,i-1}},$$

which indicates

$$\frac{\hat{\alpha}_{i,i-1}}{\hat{\beta}_{i,i-1}} \le \frac{\alpha_{i,i-1}}{\beta_{i,i-1}}.$$

Therefore, for the ith stage, we have

$$\min_{1\le j\le i-1} \frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}} \le \frac{\alpha_{i,i-1}}{\beta_{i,i-1}}.$$

Finally, we have

$$\min_{i,j} \frac{\hat{\alpha}_{ij}}{\hat{\beta}_{ij}} = \min_{1\le i\le s} \min_{1\le j\le i-1} \frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}} \le \min_{i} \frac{\alpha_{i,i-1}}{\beta_{i,i-1}}.$$

This theorem suggests that CFL restriction of the classical SSPRK method will be equal to that of the corresponding compact SSPRK method provided classical SSPRK method can be rearranged into the compact form.

Theorem 3.4. *The optimal two-stage second-order compact SSPRK method is*

$$u^{(0)} = u^{n},$$
$$u^{(1)} = u^{(0)} + \Delta t L\left(u^{(0)}\right),$$
$$u^{n+1} = \frac{1}{2}u^{(0)} + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t L\left(u^{(1)}\right),$$

which has a CFL coefficient of 1.

Proof. We first present the two-stage second-order compact method of form (3.1)

$$\begin{aligned} u^{(0)} &= u^n, \\ u^{(1)} &= \alpha_{10}u^{(0)} + \Delta t\beta_{10}L\left(u^{(0)}\right), \\ u^{n+1} &= \alpha_{20}u^{(0)} + \alpha_{21}u^{(1)} + \Delta t\beta_{21}L\left(u^{(1)}\right), \end{aligned}$$

where $\sum_{j=0}^{i-1}\alpha_{ij} = 1, i = 1,2$. To achieve the second order accuracy, the undetermined coefficients must conform to the order conditions

$$\begin{aligned} \alpha_{21}\beta_{10} + \beta_{21} &= 1, \\ \beta_{21}\beta_{10} &= \frac{1}{2}. \end{aligned}$$

According to the above two equations, we obtain $\frac{1}{\beta_{10}} + \frac{\alpha_{21}}{\beta_{21}} = 2$. Since the optimality of two-stage second-order SSPRK method is to find the maximum of $\min\left\{\frac{1}{\beta_{10}}, \frac{\alpha_{21}}{\beta_{21}}\right\}$, obviously, $\frac{1}{\beta_{10}} = \frac{\alpha_{21}}{\beta_{21}} = 1$ is the optimal situation. Coupling this optimal situation with the above order conditions, we will easily get □

$$\begin{aligned} &\alpha_{10} = 1, \quad \beta_{10} = 1, \\ &\alpha_{20} = \frac{1}{2}, \quad \alpha_{21} = \frac{1}{2}, \quad \beta_{21} = \frac{1}{2}. \end{aligned}$$

Theorem 3.5. *The optimal three-stage third-order compact SSPRK method is*

$$\begin{aligned} u^{(0)} &= u^n, \\ u^{(1)} &= u^{(0)} + \Delta tL\left(u^{(0)}\right), \\ u^{(2)} &= \frac{3}{4}u^{(0)} + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta tL\left(u^{(1)}\right), \\ u^{n+1} &= \frac{1}{3}u^{(0)} + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta tL\left(u^{(2)}\right), \end{aligned}$$

which has a CFL coefficient of 1.

Proof. As in the above theorem, we also first give the three-stage third-order compact SSPRK methods of form (3.1)

$$\begin{aligned} u^{(0)} &= u^n, \\ u^{(1)} &= \alpha_{10}u^{(0)} + \Delta t\beta_{10}L\left(u^{(0)}\right), \\ u^{(2)} &= \alpha_{20}u^{(0)} + \alpha_{21}u^{(1)} + \Delta t\beta_{21}L\left(u^{(1)}\right), \\ u^{n+1} &= \alpha_{30}u^{(0)} + \alpha_{31}u^{(1)} + \alpha_{32}u^{(2)} + \Delta t\beta_{32}L\left(u^{(2)}\right), \end{aligned}$$

where $\sum_{j=0}^{i-1}\alpha_{ij} = 1, i = 1,2,3$. Third order accuracy requires the following order conditions

$$\beta_{32}\left(\beta_{21}+\alpha_{21}\beta_{10}\right)^2+\alpha_{32}\beta_{21}\beta_{10}^2=\frac{1}{3},$$
$$\beta_{32}\beta_{21}\beta_{10}=\frac{1}{6},$$
$$\alpha_{21}\beta_{10}\beta_{32}+\alpha_{32}\beta_{10}\beta_{21}+\beta_{21}\beta_{32}=\frac{1}{2},$$
$$\alpha_{21}\alpha_{32}\beta_{10}+\alpha_{31}\beta_{10}+\alpha_{32}\beta_{21}+\beta_{32}=1.$$

Dividing the third equation by the second equation, we get

$$\frac{1}{\beta_{10}}+\frac{\alpha_{21}}{\beta_{21}}+\frac{\alpha_{32}}{\beta_{32}}=3$$

Also the maximum of $\min\left\{\frac{1}{\beta_{10}},\frac{\alpha_{21}}{\beta_{21}},\frac{\alpha_{32}}{\beta_{32}}\right\}$ can arrive 1 at most when $\frac{1}{\beta_{10}}=\frac{\alpha_{21}}{\beta_{21}}=\frac{\alpha_{32}}{\beta_{32}}=1$. Combining these information with the system of order conditions, we can easily find out the solution □

$$\alpha_{10}=1,\quad \beta_{10}=1,$$
$$\alpha_{20}=\frac{3}{4},\quad \alpha_{21}=\frac{1}{4},\quad \beta_{21}=\frac{1}{4},$$
$$\alpha_{30}=\frac{1}{3},\quad \alpha_{31}=0,\quad \alpha_{32}=\frac{2}{3},\quad \beta_{32}=\frac{2}{3}.$$

4. Optimal Four-Stage Fourth-Order SSPRK Scheme

The above two theorems show the optimal two-stage second-order and three-stage third-order compact SSPRK methods. It is expectable that they are exactly the same as the ones in [7] [8], but the proofs of optimality here are more straightforward when compared with the ones shown in [6]. For four-stage fourth-order SSPRK method with all coefficients non-negative, it has been always proved to be unattainable in the literatures [6]. This proof is processed under the order conditions of non-autonomous ODEs $u'(t)=f\left(t,u(t)\right)$. Fortunately, the order conditions of autonomous scalar ODE $u'(t)=f\left(u(t)\right)$ are not totally same as the ones of non-autonomous case.

To see it more clearly, we first give the order conditions of four-stage fourth-order RK method of form (3.1) for $u'(t)=f\left(t,u(t)\right)$,

$$\begin{aligned}
&\alpha_{41}c_2+\alpha_{42}c_3+\alpha_{43}c_4+\beta_{43}=1,\\
&\alpha_{43}\left(\alpha_{32}\beta_{21}c_2+\beta_{32}c_3\right)+\alpha_{42}\beta_{21}c_2+\beta_{43}c_4=1/2,\\
&\alpha_{43}\left(\alpha_{32}\beta_{21}c_2^2+\beta_{32}c_3^2\right)+\alpha_{42}\beta_{21}c_2^2+\beta_{43}c_4^2=1/3,\\
&\alpha_{43}\left(\alpha_{32}\beta_{21}c_2^3+\beta_{32}c_3^3\right)+\alpha_{42}\beta_{21}c_2^3+\beta_{43}c_4^3=1/4,\\
&\alpha_{43}\beta_{32}\beta_{21}c_2+\beta_{43}\left(\alpha_{32}\beta_{21}c_2+\beta_{32}c_3\right)=1/6,\\
&\alpha_{43}\beta_{32}\beta_{21}c_2c_3+\beta_{43}c_4\left(\beta_{32}c_3+\alpha_{32}\beta_{21}c_2\right)=1/8,\\
&\alpha_{43}\beta_{32}\beta_{21}c_2^2+\beta_{43}\left(\alpha_{32}\beta_{21}c_2^2+\beta_{32}c_3^2\right)=1/12,\\
&\beta_{43}\beta_{32}\beta_{21}\beta_{10}=1/24,
\end{aligned}\tag{4.1}$$

where $c_2=\beta_{10}$, $c_3=\alpha_{21}\beta_{10}+\beta_{21}$ and $c_4=\alpha_{31}\beta_{10}+\beta_{32}+\alpha_{32}\left(\alpha_{21}\beta_{10}+\beta_{21}\right)$. Next, we show the order conditions of four-stage fourth-order RK method of form (3.1) for $u'(t)=f\left(u(t)\right)$,

$$\begin{aligned}
&\alpha_{41}c_2+\alpha_{42}c_3+\alpha_{43}c_4+\beta_{43}=1,\\
&\alpha_{43}\left(\alpha_{32}\beta_{21}c_2+\beta_{32}c_3\right)+\alpha_{42}\beta_{21}c_2+\beta_{43}c_4=1/2,\\
&\alpha_{43}\left(\alpha_{32}\beta_{21}c_2^2+\beta_{32}c_3^2\right)+\alpha_{42}\beta_{21}c_2^2+\beta_{43}c_4^2=1/3,\\
&\alpha_{43}\left(\alpha_{32}\beta_{21}c_2^3+\beta_{32}c_3^3\right)+\alpha_{42}\beta_{21}c_2^3+\beta_{43}c_4^3=1/4,\\
&\alpha_{43}\beta_{32}\beta_{21}c_2+\beta_{43}\left(\alpha_{32}\beta_{21}c_2+\beta_{32}c_3\right)=1/6,\\
&\alpha_{43}\beta_{32}\beta_{21}c_2c_3+\beta_{43}c_4\left(\beta_{32}c_3+\alpha_{32}\beta_{21}c_2\right)+\frac{1}{2}\alpha_{43}\beta_{32}\beta_{21}c_2^2\\
&+\frac{1}{2}\beta_{43}\left(\alpha_{32}\beta_{21}c_2^2+\beta_{32}c_3^2\right)=1/6,\\
&\beta_{43}\beta_{32}\beta_{21}\beta_{10}=1/24,
\end{aligned}\tag{4.2}$$

where $c_2=\beta_{10}$, $c_3=\alpha_{21}\beta_{10}+\beta_{21}$ and $c_4=\alpha_{31}\beta_{10}+\beta_{32}+\alpha_{32}\left(\alpha_{21}\beta_{10}+\beta_{21}\right)$, and here the parameters c_2, c_3 and c_4 is just used to simplify the equations (4.2). The order conditions (4.2) are composed of seven equations not eight equations, which is also suggested by

$$u'=f,\ u''=f'f,\ u'''=f''f^2+f'^2f,\ u^{(4)}=f'''f^3+4f''f'f^2+f'^3f,$$

where we denote $f(u)$ by f. In addition, comparing these two formulas, it is found that the solution of (4.1) must be the solution of (4.2), but the converse from (4.2) to (4.1) is not true. Hence, although the four-stage fourth-order SSPRK method with all the coefficients nonnegative for non-autonomous ODEs is non-existent, the method for the autonomous scalar case is indeed attainable.

Theorem 4.1. *The optimal four-stage fourth-order SSPRK method for autonomous scalar ODE is*

$$\begin{aligned}
&u^{(0)}=u^n,\\
&u^{(i)}=\sum_{j=0}^{i-1}\alpha_{ij}u^{(j)}+\Delta t\beta_{i,i-1}L\left(u^{(i-1)}\right),\quad i=1,\cdots,4,\\
&u^{n+1}=u^{(4)},
\end{aligned}$$

where $\sum_{j=0}^{i-1}\alpha_{ij}=1, i=1,\cdots,4$, *and the undetermined coefficients are*

$$\begin{aligned}
&\alpha_{10}=1, && \beta_{10}=1.142778739745144974110597729166 81,\\
&\alpha_{20}=0.747166141507978283064869967791 27,\\
&\alpha_{21}=0.252833858492021716935130032208 73, && \beta_{21}=0.288933158172414898180062895087 28,\\
&\alpha_{30}=0.562863591761406800892117936962 56,\\
&\alpha_{31}=0.157339644422601249708289442601 13,\\
&\alpha_{32}=0.279796763815991949399592620436 31, && \beta_{32}=0.203515211481327094143866970707 83,\\
&\alpha_{40}=0.311653502455800230167305170104 16,\\
&\alpha_{41}=0,\\
&\alpha_{42}=0.145758486086502958974002082852 18,\\
&\alpha_{43}=0.542588011457696810858692747043 66, && \beta_{43}=0.620058043934451043107784598822 54,
\end{aligned}$$

and it has a CFL coefficient of 0.875.

In (4.2), similar to the proofs in Theorems 3.4 and 3.5, dividing the fifth equation by seventh equation, we can also have

$$\frac{1}{\beta_{10}}+\frac{\alpha_{21}}{\beta_{21}}+\frac{\alpha_{32}}{\beta_{32}}+\frac{\alpha_{43}}{\beta_{43}}=4.$$

If we wish $\max\min\left\{\frac{1}{\beta_{10}},\frac{\alpha_{21}}{\beta_{21}},\frac{\alpha_{32}}{\beta_{32}},\frac{\alpha_{43}}{\beta_{43}}\right\}=1$, then

$$\frac{1}{\beta_{10}}=\frac{\alpha_{21}}{\beta_{21}}=\frac{\alpha_{32}}{\beta_{32}}=\frac{\alpha_{43}}{\beta_{43}}=1.$$

Combining these equations with the order conditions (4.2), we find all the coefficients are non-negative but $\alpha_{41}=-\frac{1}{14}$. Therefore, we should not expect the four-stage fourth-order SSPRK methods to have a CFL restriction of 1. Without the graceful condition $\frac{1}{\beta_{10}}=\frac{\alpha_{21}}{\beta_{21}}=\frac{\alpha_{32}}{\beta_{32}}=\frac{\alpha_{43}}{\beta_{43}}=1$, it is very hard to exactly find optimal SSPRK method. A compromise way is to approximately find the optimal SSPRK method by solving a mathematical optimization problem,

$$\max\min\left\{\frac{1}{\beta_{10}},\frac{\alpha_{21}}{\beta_{21}},\frac{\alpha_{32}}{\beta_{32}},\frac{\alpha_{43}}{\beta_{43}}\right\},$$

subject to *the order conditions* (4.2),

$$\alpha_{ij}\geq 0,\ \alpha_{i,i-1}>0,\ \beta_{i,i-1}>0,\ i=1,\cdots,4;\ j=0,\cdots,i-2.$$

This optimization problem can be solved by Matlab Optimization Toolbox and the results have been given in Theorem 4.1. Actually, to find the maximal CFL number is a complex optimization problem. That is, under condition (4.2) and limitation to be non-negative for coefficients, by Matlab Optimization Toolbox, we find the maximal CFL number is 0.875. It is noted that there exist no rigorous mathematical derivation.

5. Numerical Tests

In Theorem 4.1, for autonomous scalar ODE, an optimal four-stage SSPRK method with all coefficients non-negative is presented. The still open problem is to numerically verify the order of accuracy of this method.

Consider the nonlinear ODE of first order

$$u'=u^2 \tag{5.1}$$

subject to the initial condition $u(1)=-1$, and the solution of this simple ODE is $u(t)=-\frac{1}{t}$. we evolve the equation to time $t=2$. The absolute errors at time $t=2$ are shown in Table 1 and the order of accuracy of four-stage fourth-order SSPRK method is verified to be of 4.

Table 1. The absolute errors and order of four-stage fourth-order SSPRK method.

Δt	absolute errors	order
1.00e−01	9.1525e−07	–
5.00e−02	5.4180e−08	4.078
2.50e−02	3.2919e−09	4.041
1.25e−02	2.0281e−10	4.021

6. Conclusion

In this paper, we rearrange the standard RK methods into a kind of compact forms. This compact RK form is also convenient to be used to analyse the strong ability of method itself as the classical SSPRK method. But, the undetermined coefficients of this compact SSPRK method is less than those of classical SSPRK method. Especially, the objective functions of optimization problem is heavily decreased. With the compact SSPRK form, we re-prove the optimality of two-stage second-order and three-stage third-order SSPRK methods and the procedure of proof is more straightforward. More importantly, we find out the four-stage fourth-order SSPRK method for autonomous scalar ODE. In the end, we test numerically the order of accuracy of four-stage fourth-order SSPRK method by a nonlinear autonomous scalar ODE.

Supported

This work was supported by the Natural Science Foundation of Guangdong Province under Grant No. 2018A030313879, the Technology Foundation of Huizhou under Grant No. 2017C0403019, Huizhou University No. HZUX1201620 and 2018ZX046.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Harten, A. (1983) High Resolution Schemes for Hyperbolic Conservation Laws. *Journal of Computational Physics*, **49**, 357-393. https://doi.org/10.1016/0021-9991(83)90136-5

[2] Harten, A. and Osher, S. (1987) Uniformly High-Order Accurate Nonoscillatory Schemes. I. *SIAM Journal on Numerical Analysis*, **24**, 279-309. https://doi.org/10.1137/0724022

[3] Harten, A., Engquist, B., Osher, S. and Chakravarthy, S.R. (1987) Uniformly High Order Accurate Essentially Non-Oscillatory Schemes, III. *Journal of Computational Physics*, **71**, 231-303. https://doi.org/10.1016/0021-9991(87)90031-3

[4] Liu, X., Osher, S. and Chan, T. (1994) Weighted Essentially Non-Oscillatory Schemes. *Journal of Computational Physics*, **115**, 200-212. https://doi.org/10.1006/jcph.1994.1187

[5] Jiang, G.S. and Shu, C.W. (1996) Efficient Implementation of Weighted ENO

Schemes. *Journal of Computational Physics*, **126**, 202-228. https://doi.org/10.1006/jcph.1996.0130

[6] Gottlieb, S. and Shu, C. (1998) Total Variation Diminishing Runge-Kutta Schemes. *Mathematics of Computation*, **67**, 73-85. https://doi.org/10.1090/s0025-5718-98-00913-2

[7] Shu, C. and Osher, S. (1988) Efficient Implementation of Essentially Non-Oscillatory Shock-Capturing Schemes. *Journal of Computational Physics*, **77**, 439-471. https://doi.org/10.1016/0021-9991(88)90177-5

[8] Shu, C. (1988) Total-Variation-Diminishing Time Discretizations. *SIAM Journal on Scientific and Statistical Computing*, **9**, 1073-1084. https://doi.org/10.1137/0909073

[9] Spiteri, R.J. and Ruuth, S.J. (2002) A New Class of Optimal High-Order Strong-Stability-Preserving Time Discretization Methods. *SIAM Journal on Numerical Analysis*, **40**, 469-491. https://doi.org/10.1137/s0036142901389025

[10] Ruuth, S. (2005) Global Optimization of Explicit Strong-Stability-Preserving Runge-Kutta Methods. *Mathematics of Computation*, **75**, 183-207. https://doi.org/10.1090/s0025-5718-05-01772-2

[11] Gottlieb, S., Ketcheson, D. and Shu, C. (2011) Strong Stability Preserving Runge-Kutta and Multistep Time Discretizations. World Scientific. https://doi.org/10.1142/7498

[12] Ketcheson, D.I. (2019) Relaxation Runge-Kutta Methods: Conservation and Stability for Inner-Product Norms. *SIAM Journal on Numerical Analysis*, **57**, 2850-2870. https://doi.org/10.1137/19m1263662