Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2026, 14(11), 4759-4780
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

A Structural, Plastic, History-Dependent Picture of Emergent Spacetime

Fabrizio Iandiorio

Independent Researcher, Florence, Italy
Email: iandiorio.fabrizio@gmail.com

How to cite this paper: Iandiorio, F. (2026) A Structural, Plastic, History-Dependent Picture of Emergent Spacetime. *Journal of Applied Mathematics and Physics*, **14**, 4759-4780.
https://doi.org/10.4236/jamp.2026.1411232

Received: October 20, 2026
Accepted: November 21, 2026
Published: November 24, 2026

Copyright © 2026 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/

Open Access

Abstract

This work introduces the Sponge Framework, a history-dependent approach to emergent spacetime based on a discrete substrate of interacting microregions connected by permeability channels. Rather than taking geometry as fundamental, the framework assigns primary significance to dynamical interaction pathways whose strengths evolve through memory-dependent, hysteretic processes. The central hypothesis is that effective geometry emerges from accessibility within the interaction network. Distances are defined in terms of traversal costs, while continuum geometric descriptions arise through coarse-graining of the underlying discrete structure. Because channel dynamics depend on both present connectivity and prior evolution, the resulting effective geometry may retain a dependence on the system's history. Under suitable active-corridor conditions, the framework is proposed to admit a continuum description featuring hyperbolic propagation and a local Lorentzian effective structure. Within the regime in which coarse-grained geometric variables remain well defined, General Relativity is treated as a leading-order effective-field-theory description rather than as a theory derived directly from the microscopic substrate. The framework connects conceptually with several existing research directions in emergent spacetime, including network-based models, analogue gravity, causal and pre-geometric approaches, while introducing a distinct emphasis on memory-bearing channels, traversal-cost dynamics, and accessibility-based geometry. Its distinctive emphasis lies in memory-bearing permeability channels, accessibility-based distance, and the conditional emergence of spacetime geometry. These elements provide a conceptual framework for investigating the relationship between discrete substrate dynamics, emergent geometry, causal propagation, and effective gravitational behavior.

Keywords

Emergent Spacetime, Pre-Geometric Substrate, Hysteresis,

Accessibility-Based Geometry, Traversal Cost, Coarse-Graining, Effective Field Theory

1. Introduction

The status of spacetime as a fundamental or emergent entity remains one of the central open questions at the interface between General Relativity and quantum theory. While General Relativity treats spacetime as a dynamical geometric structure, a growing class of approaches—including entanglement-based approaches [1] [2], network-based models [3], causal set theory [4]-[6], causal structure approaches [7]-[9], discrete geometric approaches [10]-[12], and effective-field-theory descriptions of gravity [13] [14]—suggests that spacetime may arise as an effective, large-scale description of deeper, non-spatiotemporal degrees of freedom.

A key challenge for any emergent-spacetime proposal is to provide a minimal and conceptually consistent framework in which geometric notions—such as distance, metric structure, and causal relations—are not presupposed but arise operationally from more primitive elements. At the same time, such frameworks must clarify the precise domain of validity of the emergent description, with respect to Lorentz symmetry and the applicability of General Relativity.

The present work introduces the Sponge Framework, a structural and dynamically explicit proposal for emergent spacetime based on a pre-geometric substrate. The fundamental ontology is defined in terms of discrete micro-regions connected by interaction channels endowed with dynamical properties, including permeability and memory (hysteresis). At this level, no manifold, metric, or causal structure is assumed.

Geometric structure is introduced only at an effective level: distance is defined operationally as minimal traversal cost across the interaction network, leading to an emergent notion of geometry based on accessibility rather than extension. In suitable coarse-grained regimes, this construction admits an effective metric description and supports a hyperbolic sector associated with causal propagation.

A central aspect of the framework is the explicit inclusion of memory and history-dependent dynamics at the level of interaction channels. Consequently, the emergent geometry is not determined solely by instantaneous connectivity, but also by the dynamical history of the substrate. This leads naturally to a picture in which spacetime itself is a history-dependent construction, with possible implications for strongly dynamical or inhomogeneous regimes.

Within this setting, an effective local Lorentzian sector is hypothesized to emerge as an infrared property of the coarse-grained dynamics, expected to hold with high accuracy in suitably structured regions of the substrate (denoted as active corridors), while deviations are associated with boundary or sub-threshold regimes. General Relativity is adopted as the leading-order effective field-theory closure governing the emergent metric whenever the conditions required for met-

ric coarse-graining are satisfied, with corrections arising from the underlying substrate dynamics.

A crucial conceptual point concerns the nature of emergence. In the present framework, spacetime is not recovered as a reformulation of a hidden geometrical micro-theory, nor as an implicit structure encoded in more fundamental variables. Rather than being a fundamental entity, spacetime appears in this framework as an emergent coarse-grained descriptor of accessibility relations within a dynamically evolving substrate. In this picture, geometric notions such as distance and metric structure do not characterize an underlying manifold but instead encode how interactions propagate through the system.

This emergent description is valid only in regimes where a suitable coarse-graining procedure can be defined, allowing for the introduction of effective fields and smooth structures. Outside these regimes, the conditions required for geometric interpretation are no longer satisfied, and spatiotemporal language is not assumed to apply.

The aim of this work is not to provide a complete microscopic theory of quantum gravity, but to establish a minimal, coherent structure that: 1) defines a clear pre-geometric ontology, 2) provides an operational construction of effective geometry, 3) incorporates history-dependent dynamics, and 4) clarifies the domain of validity of emergent spacetime and relativistic descriptions.

The paper is organized as follows. Section 2 introduces the pre-geometric ontology and the basic elements of the substrate. Section 3 derives the static dynamics and its coarse-grained limit. Section 4 develops the notion of emergent geometry from traversal cost. Section 5 introduces memory and hysteresis effects. Section 6 presents the hyperbolic sector and causal structure. Section 7 discusses Lorentz invariance and domain of validity. Section 8 formulates the effective field theory description and the emergence of General Relativity. Section 9 outlines phenomenological implications. Section 10 discusses structural voids and extreme regimes. Section 11 addresses conceptual implications and robustness, followed by conclusions in Section 12.

2. Pre-Geometric Ontology

This section introduces the primitive ontology of the framework. The aim is to define the minimal set of entities and properties required at the fundamental level, without presupposing any geometric or spatiotemporal structure. All subsequent constructions—dynamics, geometry, and effective field-theory behavior—will be derived from these pre-geometric elements. In this framework, all fundamental entities are defined independently of any background geometric structure.

At the most basic level, no spacetime manifold, metric, distance, or curvature is assumed. The primitive ontology is a set of micro-regions (cells) connected by interaction channels. Each channel carries a positive permeability parameter and an internal state encoding memory/hysteresis.

The substrate is not located within spacetime; instead, spacetime is treated as a

macroscopic description that becomes meaningful only when the substrate is sufficiently connected, dynamically stable, and coarse-grainable.

The evolution parameter t appearing in the substrate equations is interpreted as a microscopic ordering parameter describing the sequence of substrate updates. It is not identified with emergent physical time, which becomes meaningful only after coarse-graining within the active-corridor regime. Memory accumulation and relaxation are therefore defined at the substrate level prior to the introduction of any effective spacetime description.

Sponge-Substrate Components (Operational Definitions)

Cells (micro-regions): discrete units of the substrate, represented by vertices $i \in V$.

Channels: interaction conduits between cells, represented by edges $(i,j) \in E$, each carrying a permeability $\kappa_{ij}(t) > 0$ and a memory variable $m_{ij}(t)$.

Active channels: channels that effectively support propagation in the regime of interest (e.g., κ_{ij} above a functional threshold and/or carrying significant gradients $\nabla_{ij}\Phi$).

Active corridors: connected sub-networks of active channels that are sufficiently stable and well-populated to admit coarse-graining; these are the domains where an effective spacetime description (and the hyperbolic sector) is operationally meaningful.

Structural voids: sub-threshold regions in which connectivity and/or permeability is too sparse, too unstable, or too close to $\kappa \to 0$ to support a robust coarse-grained geometry. Spatiotemporal language there is not guaranteed, rather than dynamically 'violated'.

Interfaces (corridor-void boundaries): transition zones where coefficients vary rapidly, memory terms can be comparatively enhanced, and effective-medium scattering/dispersion effects are expected to be most pronounced.

Operational criterion for corridor activation

While active corridors have been introduced as structurally well-connected and dynamically stable regions, an operational definition is required to delimit the domain of validity of the framework and render it testable.

We define an active corridor as a connected region C of the substrate satisfying the condition

$$\Pi(C) = \lambda_{eff} L_C \geq 1$$

where λ_{eff} is an effective transmissivity (or coarse-grained permeability) along the corridor, and L_C is a characteristic path length of the connected region.

The quantity $\Pi(C)$ measures the ability of the substrate to sustain coherent propagation.

When $\Pi(C) < 1$, propagation is effectively damped and the region cannot support a consistent coarse-grained geometry.

When $\Pi(C) \geq 1$, propagation becomes self-sustaining, and the region admits

a stable effective geometric description.

This condition provides a minimal operational threshold linking connectivity, permeability, and dynamical stability. It is not assumed to be unique, but serves as the simplest criterion defining the onset of the active-corridor regime.

Alternative activation indicators may be formulated in terms of connectivity, spectral stability, or propagation persistence. Future developments may relate corridor activation to spectral properties of the propagation operator, including spectral-gap criteria, percolation thresholds, or connectivity eigenmodes. The present condition $\Pi(C) \geq 1$ is adopted only as the simplest operational criterion and is not intended to represent a unique activation threshold.

At the present stage of the framework, no proof is provided that the various activation criteria proposed above are strictly equivalent or individually sufficient for the emergence of coherent propagation and a smooth geometric description. They should therefore be regarded as provisional operational diagnostics whose mutual relationships remain to be established.

The present criterion is adopted because it provides the simplest dimensionless operational characterization of corridor activation.

Equivalent operational formulations may be given. For example, one may express the activation condition as a connectivity threshold $z_{eff} \geq z_c$, *where* z_{eff} *is the effective coordination number, or as a dynamical stability condition* $\Lambda_{max} \geq 0$, *indicating the onset of non-decaying propagation modes. All such formulations are intended as equivalent characterizations of the same effective transition.*

3. Static Substrate Dynamics and Coarse-Grained Limit

We model the fundamental substrate as a weighted, dynamical graph $G = (V, E)$, where vertices $i \in V$ represent micro-regions (cells) and edges $(i, j) \in E$ represent interaction channels. Each channel carries a positive permeability $\kappa_{ij}(t) > 0$ and a memory variable $m_{ij}(t)$.

Propagation of an interaction potential (or intensity) Φ is governed by a weighted graph Laplacian. At the present stage of the framework, Φ should be interpreted as a generic propagation variable rather than a specific physical field. Its precise physical identification depends on the phenomenological realization under consideration.

$$\left(L_\kappa \Phi\right)_i = \sum_j \kappa_{ij} \left(\Phi_i - \Phi_j\right) = \rho_i \tag{1}$$

where ρ_i srepresents localized sources of interaction (e.g., mass/energy density or excitation).

Mathematical consistency of the continuum limit.

While the transition from the discrete graph Laplacian to its continuum counterpart is introduced at a conceptual level, this passage can be formalized within the standard framework of discrete-to-continuum convergence for weighted graphs.

Under appropriate conditions of graph density, statistical homogeneity, and

bounded variation of the coefficients, the discrete operator L_κ can be shown to converge, in a weak sense, to a divergence-form elliptic operator.

More precisely, considering a sequence of graphs with characteristic scale $\ell_0 \to 0$ and suitably rescaled weights κ_{ij}, results from discrete-to-continuum graph convergence [15] suggest that the discrete energy functional (2) may converge, under appropriate regularity assumptions, to the continuum functional $\int \kappa(x)\left|\nabla\Phi(x)\right|^2 \mathrm{d}x$.

The corresponding Euler-Lagrange equation then yields the continuum equation

$$\nabla\cdot\left(\kappa(x)\nabla\Phi(x)\right)=\rho(x).$$

This provides a heuristic continuum justification of Equation (3), consistent with standard discrete-to-continuum convergence results under the assumptions of scale separation and local averaging.

This can be derived from the quadratic functional:

$$E[\Phi]=\frac{1}{2}\sum_{(i,j)}\kappa_{ij}\left(\Phi_i-\Phi_j\right)^2-\sum_i\rho_i\Phi_i \tag{2}$$

whose stationarity yields Equation (1), ensuring the internal consistency of the discrete formulation.

To connect the discrete formulation with an effective continuum description, we now outline the coarse-graining procedure.

Coarse-graining procedure (conceptual outline).

The passage from the discrete graph description to the continuum limit is understood as a coarse-graining procedure over many micro-regions and interaction channels. Consider a region containing many nodes and edges, sufficiently dense and statistically homogeneous on scales larger than the microscopic length ℓ_0.

In this regime, discrete variables can be replaced by smooth effective fields: The node potential Φ_i becomes a continuous field $\Phi(x)$, and the edge-based permeability κ_{ij} is mapped to an effective permeability field $\kappa(x)$, or more generally to a tensor $K(x)$ in anisotropic cases.

The discrete graph Laplacian (1) then approximates a divergence-form operator in the continuum, yielding

$$\nabla\cdot\left(\kappa(x)\nabla\Phi(x)\right)=\rho(x)$$

where $\rho(x)$ represents the coarse-grained source density.

This limit requires that: 1) the network is sufficiently connected, 2) effective coefficients vary slowly over the coarse-graining scale, and 3) fluctuations at the microscopic level average out over the chosen scale. Under these conditions, the discrete dynamics define an effective elliptic operator governing the static sector.

Upon coarse-graining over many cells and channels, the network is approximated by an effective continuous medium characterized by a permeability field $\kappa(x)$, giving:

$$\nabla\cdot\left(\kappa(x)\nabla\Phi(x)\right)=\rho(x) \tag{3}$$

Formal structure: boundary conditions, toy model, and control parameters.

In order to clarify the analytical structure of the framework and delimit its domain of validity, we briefly specify boundary conditions, a minimal toy model, and control parameters governing the discrete-to-continuum approximation.

Boundary conditions.

At the discrete level, the graph Laplacian Equation (1) is solved under standard Dirichlet or Neumann-type conditions imposed on a subset of boundary nodes ∂V. In the continuum limit, these correspond to:

$$\Phi(x)\big|_{\partial\Omega} = \Phi_0$$

(Dirichlet)

or

$$n \cdot \left(\kappa(x)\nabla\Phi(x)\right)\big|_{\partial\Omega} = J_0$$

(Neumann)

where Ω denotes the coarse-grained domain and n its outward normal. These conditions ensure well-posedness of the elliptic problem (3).

Minimal toy model.

The regular lattice construction is introduced solely as a pedagogical example used to illustrate the continuum approximation. The framework itself does not assume a regular embedding, fixed continuum dimensionality, translational symmetry, or nearest-neighbor locality at the fundamental level. These structures are employed only within the toy model and should not be interpreted as part of the primitive ontology.

To make the construction explicit, consider a regular lattice graph with spacing ℓ_0 and nearest-neighbor couplings $\kappa_{ij} = \kappa_0 + \delta\kappa_{ij}$, where $\delta\kappa_{ij}$ represents small fluctuations. In the limit $\ell_0 \to 0$ and under weak heterogeneity, the discrete Laplacian (1) reduces to:

$$\nabla \cdot \left(\kappa_0 \nabla\Phi\right) = \rho$$

at leading order, with higher-order corrections encoding anisotropy and spatial variability. This provides a minimal explicit realization of the continuum approximation used in Equation (3).

Control parameter and error estimate.

The validity of the continuum description is controlled by the scale separation parameter:

$$\varepsilon = \frac{l_0}{L} \ll 1$$

where ℓ_0 is the microscopic spacing and L the characteristic variation scale of Φ and $\kappa(x)$.

In this regime, standard homogenization arguments imply that deviations between discrete and continuum solutions are bounded by $O(\varepsilon)$ in appropriate norms, up to coefficient regularity assumptions.

Accordingly, Equation (3) should be understood as the leading-order term in a

controlled expansion, with higher-order corrections encoding residual discreteness and memory-induced heterogeneity.

In the anisotropic coarse-grained regime, replace $\kappa(x)$ with a positive-definite permeability tensor $K(x)$:

$$\nabla\cdot\left(K(x)\nabla\Phi(x)\right)=\rho(x). \tag{4}$$

4. Emergent Geometry from Accessibility: Traversal Cost and Effective Metric

From dynamics to geometry.

Having established a continuum description of the substrate dynamics in the previous section, we now introduce geometric structure as an effective, derived notion. In the present framework, geometry is not a fundamental ingredient, but an operational descriptor of how interactions propagate through the substrate.

The key idea is to identify geometric relations with accessibility properties: the effective distance between two regions is defined in terms of the minimal cost required for propagation across the underlying network. In this sense, geometry encodes dynamic accessibility rather than pre-existing spatial extension.

This construction makes explicit that geometric quantities—such as distance and metric structure—do not exist at the fundamental level but arise only after coarse-graining as descriptors of propagation efficiency in a structured medium.

Geometric notions are introduced operationally. For each channel define a traversal cost inversely related to permeability:

$$c_{ij}=\frac{l_0}{\kappa_{ij}} \tag{5}$$

where ℓ_0 is a microscopic scale. D

Define effective distance between regions A and B as a shortest-path problem on the weighted network:

$$D_{eff}(A,B)=\min_{\gamma}\sum_{(i,j)\in\gamma}\frac{\ell_0}{\kappa_{ij}} \tag{6}$$

where γ denotes a path through the interaction network.

This makes it explicit that distance is not primitive: it is an emergent quantity reflecting minimal traversal cost through a structured medium. Connectivity does not imply proximity, and geometry codifies accessibility rather than extension.

In the present framework, accessibility rather than geometric extension is taken as the primary operational notion. The effective distance emerges as the minimal propagation cost compatible with the underlying substrate dynamics.

The present formulation realizes accessibility through traversal cost. More general descriptions based directly on relational measures may also be conceivable [16] [17], although they lie beyond the scope of the present work.

In a mature continuum regime, one may write a cost functional:

$$D_{(eff)}(A,B)=\inf_{(\gamma)}\int_{(\gamma)}c(x)\,\mathrm{d}s \tag{7}$$

In the anisotropic case, a convenient equivalent expression is:

$$D_{eff}\left(A,B\right)=\inf_{\gamma}\int_{0}^{1}\left(\dot{x}^{a}g_{ab}^{eff}\left(x\right)\dot{x}^{b}\right)^{1/2}\mathrm{d}\lambda \tag{8}$$

where the effective metric g_{ab}^{eff} is introduced only as a modeling descriptor of accessibility.

A minimal identification is:

$$g_{(ab)}^{(eff)}\propto\left(K^{(-1)}\right)_{(ab)} \tag{9}$$

(in the tensor case).

This identification should be regarded as a minimal modeling assumption rather than as a derived result. No claim is made that Equation (9) provides a unique metric reconstruction from the microscopic substrate. Its role is to provide the simplest coarse-grained representation of accessibility relations emerging from the substrate dynamics.

It is important to emphasize that the effective metric g_{ab}^{eff} is introduced purely as a descriptive tool at the coarse-grained level. It does not represent a fundamental degree of freedom, but a compact encoding of accessibility relations derived from the underlying substrate dynamics.

Remark on metric interpretation.

The identification of an effective metric tensor g_{ab}^{eff} as proportional to the inverse permeability (or inverse permeability tensor) should be understood strictly as a coarse-grained descriptor. Its validity requires that the underlying network admits sufficient local homogeneity and isotropy over scales larger than l_0, and that the effective coefficients vary slowly on the coarse-graining scale $L\left(\varepsilon=l_0/L\ll1\right)$. Only in this regime can g_{ab}^{eff} be treated as smooth and differentiable, thereby supporting a consistent effective field-theory description, and particularly the GR closure discussed below. Outside such regimes, g_{ab}^{eff} should be regarded as a formal representation of accessibility rather than a fundamental geometric degree of freedom.

5. Memory and History-Dependent Substrate Dynamics

Role of memory in the substrate dynamics.

A defining element of the framework is that the substrate does not evolve purely through instantaneous configurations but retains a dynamical memory of past interactions. This introduces a history-dependent layer at the fundamental level, which directly affects propagation properties and, consequently, the emergent geometric structure.

In contrast with models in which geometry is determined solely by static connectivity or instantaneous correlations, the present approach incorporates explicit hysteresis in the interaction channels. As a result, two regions with identical connectivity structures but different internal channel states may exhibit different effective geometries depending on their prior dynamical evolution.

This feature implies that the emergent spacetime is inherently non-Markovian

when described only in terms of connectivity. The effective geometry may depend on internal memory variables that are not fully specified by connectivity alone. Consequently, descriptions based solely on instantaneous connectivity generally do not contain sufficient information to reconstruct the full dynamical state of the substrate. Instead, the effective geometry reflects an accumulated dynamical history encoded in the channel variables.

A defining feature is that propagation modifies the substrate. Channel memory evolves through a minimal accumulation-relaxation law:

$$\frac{\mathrm{d}m_{ij}}{\mathrm{d}t} = \alpha \frac{\Phi_i - \Phi_j}{\ell_0} - \frac{m_{ij}}{\tau} \tag{10}$$

with $\alpha \geq 0$ controlling accumulation and $\tau > 0$ a relaxation timescale.

The memory variable m_{ij} is not assumed to be strictly positive. Positive values correspond to reinforcement of channel transmissivity relative to the reference state, while negative values correspond to effective suppression. The sign of m_{ij} therefore carries physical information about the accumulated propagation history of the channel.

The form of Equation (10) is intentionally minimal and is not assumed to be unique. More elaborate accumulation-relaxation laws may be considered in future developments of the framework.

Permeability is a positive function of memory:

$$\kappa_{(ij)} = \kappa_{(0)} f\left(m_{(ij)}\right), f\left(m\right) > 0 \tag{11}$$

Minimal conditions on the permeability law.

In order for the memory-dependent dynamics to remain stable under sustained propagation, the permeability law $f\left(m\right)$ must satisfy a minimal set of structural conditions.

1) Monotonicity: $\frac{\mathrm{d}f}{\mathrm{d}m} \geq 0$

An increase in the memory variable must not reduce permeability. This ensures that reinforcement through repeated propagation does not lead to counterintuitive suppression effects.

2) Saturation (bounded response): $\lim_{m\to\infty} f\left(m\right) = f_{max} < \infty$

Permeability must remain finite even under prolonged accumulation. This prevents unbounded growth of κ_{ij} and guarantees that effective traversal costs remain well-defined.

3) Controlled growth: $\mathrm{d}f/\mathrm{d}m \leq K$, with K finite.

The rate at which permeability increases with memory must be bounded. This condition avoids runaway amplification of transport channels and ensures that the system remains dynamically stable under sustained gradients.

These conditions should be regarded as necessary structural requirements on the permeability law. A rigorous proof of nonlinear stability for the fully coupled memory-propagation system lies beyond the scope of the present work and is left for future analysis.

Under these conditions, the coupled system defined by Equations (1)-(7) avoids obvious pathological regimes and is expected to admit a well-behaved coarse-grained limit in which propagation remains finite. A rigorous proof of nonlinear stability remains beyond the scope of the present work.

A useful purely graph-local form uses the edge gradient $\nabla_{ij}\Phi := \Phi_i - \Phi_j$:

$$\frac{\mathrm{d}m_{ij}}{\mathrm{d}t} = \alpha \frac{\nabla_{ij}\Phi}{\ell_0} - \frac{m_{ij}}{\tau} \tag{12}$$

with permeability law:

$$\kappa_{(ij)} = \kappa_{(0)} f\left(m_{(ij)}\right), f\left(m\right) > 0 \tag{13}$$

Equations (1)-(13) therefore define a closed history-dependent dynamical system. Two regions with identical instantaneous connectivity may exhibit different effective geometries due solely to different interaction histories. This explicitly breaks the equivalence between geometry and instantaneous structure, and motivates the interpretation of geometry as a dynamic, history-dependent descriptor.

6. Hyperbolic Substrate Dynamics and Causal Structure

From static to dynamical propagation.

The elliptic sector introduced in the previous sections provides a description of static or quasi-static accessibility but does not encode causal propagation. To account for dynamic processes and signal transmission, it is necessary to extend the substrate dynamics to a hyperbolic regime.

This is achieved by introducing inertial properties at the node level, allowing for second-order time evolution. In this way, the framework supports wave-like propagation on the underlying network, and a notion of causal structure emerges as an effective property of the dynamical substrate.

Importantly, causality is not imposed as a primitive structure but arises from the interplay between inertia and channel permeability. This ensures consistency with the general philosophy of the framework, in which spacetime features are derived rather than assumed.

The static/elliptic sector above captures accessibility and distance but does not encode causal structure. Introduce a minimal hyperbolic completion at the graph level by adding node inertance $M_i > 0$:

$$M_i \frac{\mathrm{d}^2\Phi_i}{\mathrm{d}t^2} + \sum_j \kappa_{(ij)}(t)\left(\Phi_i - \Phi_j\right) = \rho_i(t) \tag{14}$$

In the coarse-grained regime, introduce an inertance density $\mu(x)$ and write:

$$\frac{\partial}{\partial t}\left(\mu(x)\frac{\partial\Phi(x,t)}{\partial t}\right) - \nabla\cdot\left(K(x)\nabla\Phi(x,t)\right) = \rho(x,t) \tag{15}$$

The static equations are recovered in the quasi-static limit $\frac{\partial\Phi}{\partial t} \to 0$.

The introduction of the inertial term promotes the dynamics from an elliptic to a hyperbolic type, enabling the existence of finite-speed propagation and signal

cones at the effective level.

Once a hyperbolic sector is established, it becomes possible to introduce an effective Lorentzian line element as a modeling descriptor:

$$\mathrm{d}s^2_{(eff)} = -c^2_{(eff)}(x)\mathrm{d}t^2 + \ell_0^2 \mathrm{d}x^a \left(K^{-1}\right)_{ab}(x)\mathrm{d}x^b \tag{16}$$

with $c^2_{eff}(x) \propto \mu^{-1}(x)$.

The spatial block is determined by accessibility $\left(K^{-1}\right)_{ab}$; Lorentzian signature emerges only once the hyperbolic sector is available.

This Lorentzian structure should be interpreted as an emergent descriptor of propagation constraints, rather than as a fundamental geometric background.

Locally homogeneous limit and wave-cone structure.

To make the emergence of causal propagation and approximate Lorentz symmetry explicit, consider a locally homogeneous region of the substrate within an active corridor, in which effective coefficients vary slowly over the coarse-graining scale. In this regime, one may approximate

$$\mu(x) \approx \mu_0, K(x) \approx K_0$$

with μ_0 constant and K_0 a positive-definite constant tensor.

Under this approximation, Equation (20) reduces to

$$\mu_0 \frac{\partial^2 \Phi}{\partial t^2} - \nabla \cdot \left(K_0 \nabla \Phi\right) = 0$$

In the isotropic case $K_0 = \kappa_0 I$, this becomes

$$\mu_0 \frac{\partial^2 \Phi}{\partial t^2} - \kappa_0 \nabla^2 \Phi = 0$$

Plane-wave solutions of the form $\Phi \sim \exp\left(i\left(k \cdot x - \omega t\right)\right)$ yield the dispersion relation

$$\mu_0 \omega^2 = \kappa_0 k^2$$

or equivalently

$$\omega^2 = c^2_{eff}(k \cdot k)$$

with $c^2_{eff} = \dfrac{\kappa_0}{\mu_0}$.

This relation defines a finite propagation speed c_{eff} and implies the existence of characteristic wave cones in the effective description. Signals are confined within regions satisfying the condition that the spatial distance from the source does not exceed $c_{eff}t$, establishing an emergent causal structure.

The corresponding effective line element (16) reduces locally to a Lorentzian form with constant coefficients, and the propagation equation becomes invariant under approximate Lorentz transformations at leading order [7]-[9].

Deviations from exact Lorentz symmetry arise from spatial and temporal variations of $\mu(x)$ and $K(x)$, as well as from memory-induced corrections, and are therefore naturally interpreted as effective-medium effects within the domain

of validity of the coarse-grained description.

The present construction establishes the emergence of a local Lorentzian propagation sector at leading order. The analysis applies only to the propagation sector explicitly considered in this work. A demonstration of physical local Lorentz symmetry for the complete effective theory would additionally require common causal cones across all relevant fields, suppression of preferred-frame effects, and a separate analysis of higher-order operators, anisotropies, memory corrections, and possible additional propagating modes.

This local analysis provides a direct link between the hyperbolic sector and the effective field-theory description, where the emergent Lorentzian structure provides the geometric setting employed in the effective-field-theory description discussed in Section 8.

7. Domain of Validity and Emergent Lorentz Invariance

Domain of validity of the spacetime description.

At the level of the fundamental substrate, no spacetime structure is assumed. Consequently, geometric notions—including metric structure, causal cones, and Lorentz invariance—are not universally defined, but arise only within specific regimes in which the substrate admits a stable coarse-grained description.

The applicability of spacetime language is therefore conditional on structural properties of the substrate, including sufficient connectivity, dynamical stability, and the possibility of defining slowly varying effective fields. These conditions define the active-corridor regime, in which an effective geometric description becomes meaningful. Conversely, regions failing to satisfy these conditions belong to the structural-void regime, in which geometric notions cease to possess a reliable operational interpretation.

Spacetime and General Relativity are not expected to apply at the level of a single cell, or a small set of weakly connected cells, where the conditions required for coarse-graining are not satisfied. The appropriate statement is not that General Relativity is violated below a threshold, but that it is not yet applicable or defined.

The relevant threshold is not a universal number of cells, but a structural condition combining multiplicity, connectivity, dynamical stability, and coarse-grainability. Only above such a threshold does it become meaningful to introduce effective notions of distance, dimensionality, and macroscopic spacetime structure.

Emergent Lorentz invariance.

Within the active-corridor regime, the hyperbolic dynamics introduced in the previous section support an effective causal structure. Under suitable conditions, namely approximate homogeneity, isotropy, and slow variation of effective coefficients, the resulting dynamics can exhibit approximate Lorentz invariance at large scales.

In this framework, Lorentz symmetry is interpreted as an infrared (IR) emergent property of the coarse-grained dynamics, rather than as a fundamental sym-

metry imposed at the microscopic level.

This perspective implies a clear distinction between genuine symmetry violation and limits of applicability. Apparent deviations from Lorentz invariance are not necessarily indicative of a breakdown of a fundamental symmetry but may instead reflect effective-medium corrections in inhomogeneous regions, or the transition to regimes in which a spacetime description is not well-defined.

This same logic applies to Lorentz symmetry: within active corridors where a local spacetime description is well-defined, local Lorentz invariance is expected to hold to high accuracy, while outside this domain the applicability of spatiotemporal concepts is not guaranteed.

In emergent-spacetime settings, Lorentz invariance arises as an infrared effective symmetry [18]: at large scales and low energies, the coarse-grained dynamics become approximately Lorentz invariant, with deviations suppressed by the ratio of scales ($\varepsilon = \ell_0 / L \ll 1$).

Accordingly, apparent propagation anomalies should be interpreted as effective-medium effects of a non-uniform, history-dependent substrate, rather than as a fundamental breakdown of relativity within its domain of validity.

8. Effective Field Theory and Emergence of General Relativity

Effective field-theory description.

The present framework does not claim a microscopic derivation of the Einstein-Hilbert action from the underlying substrate dynamics [13] [14]. The purpose of this section is more limited. Once an effective Lorentzian metric description becomes available, General Relativity is adopted as the leading-order effective closure compatible with diffeomorphism invariance and standard effective field-theory principles.

Once an effective metric description becomes available in the coarse-grained active-corridor regime, it is natural to formulate the large-scale dynamics in the language of effective field theory. In this setting, the metric g_{eff} is not fundamental, but serves as a macroscopic descriptor encoding the accessibility structure of the substrate.

The effective action is constructed as the most general functional of g_{eff} and additional fields, consistent with the symmetries that emerge at large scales. In particular, diffeomorphism invariance is understood as an emergent property of the coarse-grained description, rather than a fundamental symmetry of the underlying substrate.

Emergence of General Relativity.

General Relativity is adopted as the leading-order effective description governing the dynamics of the emergent metric. The Einstein-Hilbert action is therefore interpreted not as a fundamental postulate, but as the dominant term in a low-energy expansion.

Accordingly, the Einstein-Hilbert term should be interpreted as an effective in-

frared description rather than as a quantity derived directly from the microscopic substrate dynamics.

Within the effective-field-theory perspective, the Einstein-Hilbert term is adopted because it represents the lowest-order generally covariant scalar constructed from the effective metric. Higher-curvature operators may also be present, but are expected to be suppressed in the infrared regime where the coarse-grained description remains valid.

The effective action can be written as:

$$S_{eff}\left[g_{eff},\psi,m\right]=\left(16\pi G\right)^{-1}\int \mathrm{d}^4x\sqrt{-g_{eff}}\,R\left(g_{eff}\right)+S_{matter}\left[g_{eff},\psi\right]+S_{memory}\left[g_{eff},m\right] \tag{17}$$

The stress-energy tensors are defined variationally as:

$$T_{\mu\nu}^{memory}\equiv-\frac{2}{\sqrt{-g_{eff}}}\frac{\delta S_{memory}}{\delta g_{eff}^{\mu\nu}} \tag{18}$$

Variation yields the effective Einstein equations:

$$G_{\mu\nu}\left(g_{eff}\right)=8\pi G\left(T_{\mu\nu}^{matter}+T_{\mu\nu}^{memory}\right) \tag{19}$$

If S_{matter} and S_{memory} are diffeomorphism-invariant, covariant consistency follows:

$$\nabla_{\mu}\left(T_{matter}^{\mu\nu}+T_{memory}^{\mu\nu}\right)=0 \tag{20}$$

The conservation law follows from the contracted Bianchi identities together with the assumed diffeomorphism invariance of both the matter and memory sectors.

A control criterion for the validity of the effective-theory regime is:

$$\varepsilon=\frac{T_{\mu\nu}^{memory}}{T_{\mu\nu}^{matter}},\ \varepsilon\ll 1 \tag{21}$$

Here ε represents a dimensionless measure of the relative contribution of the memory sector to the effective dynamics, where $\|\cdot\|$ denotes a suitable measure of the magnitude of the corresponding stress-energy tensor.

In regimes where $\varepsilon\ll 1$, General Relativity is recovered as the leading-order effective description, while memory and hysteresis provide controlled subleading corrections.

This hierarchy clarifies the conceptual status of General Relativity within the framework: it is neither assumed at the microscopic level nor expected to be universally valid, but arises as a robust effective description within a well-defined domain. Outside this domain, deviations should be interpreted in terms of the underlying substrate dynamics rather than as inconsistencies of the theory itself.

9. Phenomenology and Physical Implications

Although the present framework is primarily structural and conceptual, it suggests several qualitative physical implications that may guide future phenomeno-

logical investigations.

The first implication follows from the identification of distance with traversal cost. Since effective geometry reflects the dynamical state of channel permeability, spatial relations become state-dependent. In regions where the substrate is strongly inhomogeneous, one may therefore expect effective-metric variations that do not admit a simple geometric interpretation in standard terms.

A second implication concerns the role of memory and hysteresis. Because channel properties depend on past interactions, the resulting effective geometry is history-dependent. This suggests that spacetime configurations may retain imprints of past dynamical processes, potentially leading to observable effects in highly dynamical or non-equilibrium regimes.

A third aspect involves the emergence of Lorentz invariance. Within well-formed active corridors, Lorentz symmetry is expected to hold to high accuracy as an infrared property. However, near structural boundaries or in rapidly varying regions, effective-medium corrections may lead to small, controlled deviations, which should be interpreted as signatures of the underlying substrate rather than as fundamental symmetry violations.

More generally, the framework predicts the existence of regimes in which a spacetime description is not applicable. In such cases, apparent anomalies should not be interpreted as breakdowns of physical laws, but as limitations of the effective geometric description. This perspective provides a unified interpretation of both standard relativistic behavior and its possible deviations within a single structural framework.

These considerations do not yet constitute precise experimental predictions but establish a clear set of phenomenological directions. The framework identifies the qualitative classes of physical effects that may arise, while their quantitative magnitude and observational detectability remain to be determined.

The phenomenological considerations presented below should be interpreted as illustrative parametrizations intended to identify potentially observable consequences of the framework. They are not claimed to represent quantitative predictions derived directly from the microscopic substrate dynamics.

One possible observational signature is memory-induced dispersion.

The presence of memory and hysteresis at the level of interaction channels suggests that propagation may depend not only on instantaneous structure but also on the history of the substrate. A direct consequence is the possibility of memory-induced dispersion effects in the hyperbolic sector.

In a coarse-grained description, the effective permeability $K(x)$ and inertance $\mu(x)$ may acquire a weak, history-dependent time dependence through the memory variables m_{ij}. Linearizing around a slowly varying background and introducing small perturbations, one expects corrections to the local dispersion relation of the form

$$\omega^2 = c_{eff}^2 k^2 + \Delta_m(k,\omega)$$

where $\Delta_m(k,\omega)$ encodes memory-dependent contributions arising from the evolution of m.

To leading order, these corrections generically produce a weak frequency dependence of the propagation speed,

$$c_{eff} \to c_{eff}(\omega)$$

with $\varepsilon(\omega) \ll 1$ in regimes where the effective-field-theory description remains valid (cf. Section 8).

In the context of gravitational-wave propagation over large distances, this implies that signals traveling through regions with non-negligible memory accumulation may exhibit small dispersive distortions, including frequency-dependent arrival times or phase shifts. The magnitude of such effects is controlled by the ratio between the memory-induced contribution and the leading dynamical terms,

$$\varepsilon \sim \frac{T_{memory}}{T_{matter}}$$

and is therefore expected to be strongly suppressed in well-formed active corridors but potentially enhanced near structural boundaries or in highly dynamical regions.

A complementary signature concerns transient relaxation effects: following a strong perturbation (e.g., a localized excitation or passage through an inhomogeneous region), the memory variables relax on a timescale τ, leading to delayed adjustments in effective permeability. This may produce observable time-dependent propagation effects or hysteresis-like lag in signal response.

These signatures follow directly from the memory-dependent structure of the substrate and provide a concrete pathway toward testing the framework once quantitative estimates are specified.

Illustrative phenomenological extensions.

Building on the qualitative considerations outlined above, one may consider more explicit phenomenological projections based on the memory-dependent dynamics of the substrate. These projections are not intended as precise predictions, but as indicative extensions illustrating possible observational implications of the framework.

A first example concerns modified dispersion effects for gravitational waves. In regimes where the substrate response acquires a weak frequency dependence due to memory (hysteresis), one may parametrize the effective propagation speed as:

$$v_{eff}(\omega) \approx c\left[1-\alpha\left(\frac{E}{E_{QG}}\right)^2-\beta\frac{\omega}{\omega_c}\right]$$

where ω_c represents a characteristic frequency associated with the density of active corridors, and β is a dimensionless parameter encoding memory coupling effects, and E_{QG} denotes the effective quantum-gravity energy scale used in the phenomenological parametrization [19]. In this picture, deviations from General Relativity would appear as small, frequency-dependent phase shifts in gravita-

tional-wave signals.

A second example concerns primordial fluctuations. The plasticity and history-dependence of the substrate suggest that early-universe density fluctuations may acquire weak non-Gaussian components induced by memory effects. At an effective level, one may write:

$$f_{NL}^{sponge} \propto \left(\frac{\delta\rho_{mem}}{\rho_{tot}} \right)$$

where ρ_{mem} represents the contribution arising from substrate hysteresis and ρ_{tot} denotes the total effective energy-density contribution used for normalization. Detection of such signatures, if within observational sensitivity, could in principle provide an indirect probe of the underlying pre-geometric structure.

More generally, these examples illustrate how the framework may be connected to observational programs, including gravitational-wave interferometry and cosmological probes. A systematic quantitative analysis, however, lies beyond the scope of the present work.

10. Structural Voids and Limits of Geometric Description

Limits of the emergent geometric description.

The framework developed in the previous sections defines spacetime as an effective description valid only within specific structural regimes of the underlying substrate. It is therefore natural to consider configurations in which these conditions are not satisfied.

Structural voids represent such regimes: domains in which connectivity, permeability, or dynamical stability fall below the threshold required for defining consistent coarse-grained geometry. In these regions, geometric notions such as distance or causal structure are not expected to be well-defined.

The framework adopts a structural distinction between 1) active corridors (well-connected regimes supporting coarse-grained spacetime) and 2) structural voids (sub-threshold regimes lacking sufficient connectivity or stability for a geometric description).

Clarifying note (*black-hole-like regimes*).

Within this framework, a "black-hole-like" object is modeled as a persistent structural void together with a surrounding boundary region in which accessibility becomes sharply suppressed. Operationally, the horizon corresponds to an emergent interface across which the minimal traversal cost becomes effectively divergent for paths attempting to penetrate deeper into the void-dominated domain.

At the substrate level, this behavior is associated with near-vanishing permeability channels ($\kappa_{ij} \to 0 \Rightarrow c_{ij} = \frac{\ell_0}{\kappa_{ij}} \to \infty$), *which eliminate finite-cost paths without presupposing a metric singularity. In the active-corridor regime outside the boundary, the usual spacetime description may remain valid, and General Relativity can serve as a leading-order effective closure. The "interior" is instead*

treated as outside the domain of applicability of coarse-grained geometric language.

This interpretation is fully consistent with the domain-of-validity perspective introduced earlier. Rather than signaling a breakdown of physical laws, these regimes identify the limits of applicability of the geometric description itself.

Horizon-like behavior can thus be interpreted as an accessibility divergence. From the cost definition:

$$\kappa_{ij} \to 0 \Rightarrow c_{ij} = \frac{\ell_0}{\kappa_{ij}} \to \infty \tag{22}$$

Hence, no finite-cost optimal path can cross regions dominated by vanishing-permeability channels: accessibility fails without requiring a fundamental metric singularity.

This description maintains full consistency with the domain-of-validity logic: in regions where spacetime is well-defined, General Relativity and Lorentz invariance apply as effective descriptions; in sub-threshold regimes, the geometric language loses operational meaning rather than being dynamically violated.

11. Conceptual Synthesis and Interpretative Framework

The framework developed in this work provides a unified interpretation of spacetime, geometry, and gravitational dynamics as emergent phenomena arising from propagation constraints in a structured, memory-bearing substrate.

At a fundamental level, geometry is not a primitive input but a derived description: distance, metric structure, and causal relations encode accessibility, *i.e.*, the cost and organization of possible paths. In this sense, spacetime is a language that becomes meaningful only under specific structural conditions.

This perspective allows for a natural comparison with familiar physical systems. For example, different phases of matter such as ice, liquid water, and vapor exhibit distinct structural organizations (density, connectivity, and response to perturbations) despite sharing the same microscopic constituents. The analogy refers to differences in effective organization and propagation behavior, not necessarily to a thermodynamic phase transition of the substrate itself. In a similar way, the substrate described here can support qualitatively different regimes of propagation depending on its structural state. Related emergence scenarios have been explored in causal dynamical triangulations, analogue-gravity models, and other approaches to emergent spacetime [11] [12] [20] [21].

In highly connected, stable, and sufficiently homogeneous regions, propagation becomes coherent and supports the emergence of an effective continuum description. These regions correspond to active corridors, where well-defined notions of distance, causal structure, and approximate Lorentz invariance arise.

By contrast, in regimes where connectivity or permeability is strongly suppressed—analogous to disordered or rarefied phases—propagation becomes fragmented or inhibited. Such configurations correspond to structural voids, in which geometric notions lose their operational meaning. In this sense, the transition be-

tween "geometric" and "non-geometric" regimes is not tied to a fundamental breakdown of laws, but to a change in the organization of the underlying medium.

The role of memory is essential in this picture: the substrate is dynamically shaped by past propagation, leading to the formation, reinforcement, or dissolution of transport channels. This introduces a history-dependent structure that can stabilize or destabilize emergent geometric behavior, in close analogy with systems exhibiting hysteresis or path-dependent responses.

From this viewpoint, spacetime, Lorentz invariance, and gravitational dynamics do not represent fundamental ingredients, but rather effective descriptors associated with specific structural regimes of the substrate. Their emergence, stability, and eventual breakdown are governed by the same underlying principle: the organization of accessible paths and their costs.

This interpretative framework emphasizes a clear separation between fundamental structure and effective description, providing a coherent conceptual basis for connecting microscopic dynamics with macroscopic spacetime behavior.

12. Conclusions

The Sponge Framework provides a minimal and structurally explicit approach to emergent spacetime, in which geometric notions are not assumed at the fundamental level but arise as effective descriptors of a dynamically evolving, memory-bearing substrate.

Starting from a pre-geometric ontology defined by micro-regions and interaction channels, the framework establishes a consistent progression from discrete dynamics to a coarse-grained continuum description, in which geometry is identified with accessibility and distance corresponds to minimal traversal cost.

A key feature is the presence of memory and hysteresis at the microscopic level, leading to a non-Markovian structure in which geometric properties are not solely determined by instantaneous configurations, but retain dependence on the history of propagation.

The introduction of hyperbolic dynamics enables causal propagation, allowing for the emergence of an effective Lorentzian structure under appropriate structural conditions. In this setting, Lorentz invariance is interpreted as an infrared property of the coarse-grained dynamics.

At larger scales, the framework admits an effective field-theory formulation in which General Relativity is adopted as the leading-order effective closure of the emergent metric description, with controlled corrections associated with the underlying memory-dependent substrate.

A central outcome is the identification of a precise domain of validity: spacetime is not universal, but an emergent regime defined by connectivity, stability, and coarse-grainability. Outside this domain, apparent deviations are naturally understood as limits of applicability of the geometric description, rather than violations of fundamental laws.

Overall, the framework provides a coherent and operational basis for interpret-

ing spacetime, causality, and gravitational dynamics as emergent phenomena governed by propagation constraints in a structured substrate, and suggests clear directions for future theoretical and phenomenological developments.

The phenomenological implications discussed in this work, including gravitational-wave propagation effects, cosmological signatures, and horizon-like interpretations, should be regarded as illustrative consequences of the effective framework rather than as results established directly from the microscopic substrate dynamics.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Verlinde, E. (2011) On the Origin of Gravity and the Laws of Newton. *Journal of High Energy Physics*, **2011**, Article No. 29. https://doi.org/10.1007/jhep04(2011)029

[2] Van Raamsdonk, M. (2010) Building up Spacetime with Quantum Entanglement. *General Relativity and Gravitation*, **42**, 2323-2329. https://doi.org/10.1007/s10714-010-1034-0

[3] Konopka, T., Markopoulou, F. and Severini, S. (2008) Quantum Graphity: A Model of Emergent Locality. *Physical Review D*, **77**, Article ID: 104029. https://doi.org/10.1103/physrevd.77.104029

[4] Bombelli, L., Lee, J., Meyer, D. and Sorkin, R.D. (1987) Space-Time as a Causal Set. *Physical Review Letters*, **59**, 521-524. https://doi.org/10.1103/physrevlett.59.521

[5] Sorkin, R.D. (2005) Causal Sets: Discrete Gravity. In: Gomberoff, A. and Marolf, D., Eds., *Lectures on Quantum Gravity*, Springer, 305-327.

[6] Surya, S. (2019) The Causal Set Approach to Quantum Gravity. *Living Reviews in Relativity*, **22**, Article No. 5. https://doi.org/10.1007/s41114-019-0023-1

[7] Zeeman, E.C. (1964) Causality Implies the Lorentz Group. *Journal of Mathematical Physics*, **5**, 490-493. https://doi.org/10.1063/1.1704140

[8] Hawking, S.W., King, A.R. and McCarthy, P.J. (1976) A New Topology for Curved Space-Time Which Incorporates the Causal, Differential, and Conformal Structures. *Journal of Mathematical Physics*, **17**, 174-181. https://doi.org/10.1063/1.522874

[9] Malament, D.B. (1977) The Class of Continuous Timelike Curves Determines the Topology of Spacetime. *Journal of Mathematical Physics*, **18**, 1399-1404. https://doi.org/10.1063/1.523436

[10] Regge, T. (1961) General Relativity without Coordinates. *Il Nuovo Cimento*, **19**, 558-571. https://doi.org/10.1007/bf02733251

[11] Ambjørn, J., Jurkiewicz, J. and Loll, R. (2004) Emergence of a 4D World from Causal Quantum Gravity. *Physical Review Letters*, **93**, Article ID: 131301. https://doi.org/10.1103/physrevlett.93.131301

[12] Ambjørn, J., Jurkiewicz, J. and Loll, R. (2010) Causal Dynamical Triangulations and the Quest for Quantum Gravity. https://arxiv.org/abs/1004.0352

[13] Donoghue, J.F. (1994) General Relativity as an Effective Field Theory: The Leading Quantum Corrections. *Physical Review D*, **50**, 3874-3888. https://doi.org/10.1103/physrevd.50.3874

[14] Donoghue, J.F. (1995) Introduction to the Effective Field Theory Description of Gravity. https://arxiv.org/abs/gr-qc/9512024

[15] Chung, F. (1997) Spectral Graph Theory. CBMS Regional Conference Series in Mathematics 92, American Mathematical Society. https://doi.org/10.1090/cbms/092

[16] Benamou, J. and Brenier, Y. (2000) A Computational Fluid Mechanics Solution to the Monge-Kantorovich Mass Transfer Problem. *Numerische Mathematik*, **84**, 375-393. https://doi.org/10.1007/s002110050002

[17] Villani, C. (2009) Optimal Transport: Old and New. Springer.

[18] Collins, J., Perez, A., Sudarsky, D., Urrutia, L. and Vucetich, H. (2004) Lorentz Invariance and Quantum Gravity: An Additional Fine-Tuning Problem? *Physical Review Letters*, **93**, Article ID: 191301. https://doi.org/10.1103/physrevlett.93.191301

[19] Amelino-Camelia, G. (2013) Quantum-Spacetime Phenomenology. *Living Reviews in Relativity*, **16**, Article No. 5. https://doi.org/10.12942/lrr-2013-5

[20] Barceló, C., Liberati, S. and Visser, M. (2005) Analogue Gravity. *Living Reviews in Relativity*, **8**, Article No. 12. https://doi.org/10.12942/lrr-2005-12

[21] Hořava, P. (2009) Quantum Gravity at a Lifshitz Point. *Physical Review D*, **79**, Article ID: 084008. https://doi.org/10.1103/physrevd.79.084008