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Two-Route Corrections: A Master Equation for Motion, Constraints and Conserved Quantities in Analytical Mechanics

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Abstract

Many errors in analytical mechanics are errors of omission: a Coriolis force that never appears, a centrifugal term set to zero, or a term that vanishes because a rolling constraint was imposed one step too early. This paper shows that such structural terms (inertial, gyroscopic, transport, anholonomy and constraint terms) share a single origin, the symmetry of mixed partial derivatives, and that they can be found by comparing two routes to the same quantity rather than by recalling formulas. We call the difference between the routes a *two-route correction*, and list the corrections produced by the standard operations of a derivation. Non-potential forces such as friction and actuation are not of this kind; they enter as independent physical input. The central result is a master equation of motion, valid in any moving frame, in which every such term has a definite place: inertial and gyroscopic terms come from the Lagrangian written in the chosen velocity variables, and transport and anholonomy terms from the commutators of the frame's vector fields. Lagrange's, Euler's, Hamel's and Kane's equations all follow from it directly. The same idea reaches beyond the equations of motion: in Hamilton-Jacobi theory, the gap between partial and total time derivatives is what carries the dynamics, and Routh reduction, Noether's theorem and Poisson brackets fit the same pattern. The approach also gives the exact term lost when a nonholonomic constraint is substituted too early, with a simple test for when this shortcut is safe. It also turns the search for conserved quantities into a linear problem needing no guessed symmetry or gauge choice. Since this problem also admits driven and damped systems with a time-independent kinetic metric, it yields Lyapunov functions too. Worked examples range from the Chaplygin sleigh, solved in closed form, to the hidden integral of Euler's two-centre problem. A decision chart, reference tables and a catalogue of standard cases make the paper a working reference.

Keywords

Analytical Mechanics, Lagrange Equations, Hamilton-Jacobi Equations, Hamel Equations, Nonholonomic Systems, Kane's Method, Routh Reduction, First Integrals, Killing Tensors, Lyapunov Functions, Noether's Theorem, Poisson Brackets

1. Introduction

Lagrange's equations follow from Newton's second law by projecting $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$ onto the vectors $\partial \boldsymbol{r}/\partial q^i$, using two textbook identities [1]-[3]:

$$\frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{q}^i} = \frac{\partial \boldsymbol{r}}{\partial q^i}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{r}}{\partial q^i} = \frac{\partial \dot{\boldsymbol{r}}}{\partial q^i}. \tag{1}$$

These are usually presented as computational tricks. Both are instances of the equality of mixed partial derivatives (Clairaut-Schwarz [4]). This paper shows that this one fact, used systematically, is a complete working method. It produces the equations of motion of difficult systems together with the terms most often lost. It also produces equations whose solutions are the conserved quantities of a system.

Errors in analytical mechanics almost always come from a step used outside its range of validity. Typical cases are treating $\partial T/\partial q^i$ as zero in curvilinear coordinates, confusing a partial with a total time derivative, identifying canonical with kinetic momentum in a magnetic field, taking the conserved energy in a rotating frame to be $T+U$, or substituting a nonholonomic constraint into the Lagrangian before differentiating. The historical record includes such errors by first-rate analysts; Lindelöf's 1895 treatment of a rolling body, corrected by Chaplygin, is the classical example [5] [6]. In the method developed here each step carries its assumption, and when the assumption fails, the missing piece is a computable term with a physical name.

1.1. A First Example: The Bead on a Rotating Hoop

A bead of mass m slides without friction on a hoop of radius R that a motor turns at constant rate Ω about its vertical diameter. Let θ be the angle of the bead from the lowest point. In the inertial frame,

$$T = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\Omega^2\sin^2\theta, \quad U = -mgR\cos\theta. \tag{2}$$

Three plausible shortcuts give three different wrong answers.

a) Writing the problem "in the rotating frame" as a pendulum, with $\partial T/\partial\theta$ set to zero, gives $R\ddot{\theta} = -g\sin\theta$. This misses the bifurcation of the equilibrium. The term lost is the *inertial correction* $\partial T_0/\partial\theta$ of Section 4.3(a).

b) Taking $E = T+U$ as conserved is wrong: the motor does work. The quantity that is conserved is $h = E - 2T_0$ (Result 13).

c) Adding the Coriolis force by hand to the tangential equation adds a term that is not there: the Coriolis force is normal to the hoop and is absorbed by the constraint. With the coordinate velocity $\dot{\theta}$ the Coriolis force is the gyroscopic correction of $T_1 = b_\theta \dot{\theta}$ (Section 4.3(b)), and its coefficient $b_\theta = m\partial_\theta \boldsymbol{r} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r})$ vanishes, because $\boldsymbol{\Omega} \times \boldsymbol{r}$ is normal to the hoop.

The method gives the correct equation in one line,

$$R\ddot{\theta} = \left(R\Omega^2 \cos\theta - g\right)\sin\theta, \tag{3}$$

with the off-bottom equilibria $\cos\theta_* = g/\left(R\Omega^2\right)$ for $\Omega^2 > g/R$. The rest of the paper shows that corrections of this kind can be enumerated in advance, read off from one equation, and used to construct conserved quantities as well as equations of motion.

1.2. Contributions

The classical equations recovered below are known. What the paper adds is a single mechanism that produces all of them: a master equation and three tools that we have not found stated in this form.

1) **A master equation in time-dependent moving frames** (Result 3). Hamel's equations for affine, time-dependent (rheonomic) quasi-velocities [7] [8] are written on the extended configuration space. Inertial and gyroscopic terms then come from the Lagrangian L^* in the chosen velocity variables, and all transport and anholonomy terms from the commutators $\left[e_\beta, e_\mu\right]$ of the frame vector fields, so that rotating frames, rigid bodies, vehicles and charged particles are covered by one formula.

2) **A commutator table** (Table 1). Each standard step of a derivation is a linear operation, and each correction is the commutator of two such operations. The table makes the audit of a derivation built from these operations a check against a fixed list.

3) **The constraint-substitution correction** (Result 7). The terms lost by early substitution of nonholonomic constraints are known in principle from the work of Voronets, Chaplygin and Hamel [6] and, geometrically, as the curvature of the nonholonomic connection [9] [10]. We give them as one explicit product, *constraint momentum* times *anholonomy*, with a one-line Frobenius test for when the shortcut is safe.

4) **A gauge-free determining system for first integrals and Lyapunov functions** (Result 15). Determining equations for quadratic integrals in magnetic fields are known [11]-[13]. We give a form that contains only field strengths, applies unchanged to Coriolis and Routh-reduced systems, and, for systems with a time-independent kinetic metric, accepts explicit time dependence and arbitrary forces linear in the velocity, and therefore also constructs Lyapunov functions for damped systems.

Routh reduction, Noether's theorem, Hamilton-Jacobi theory, the Helmholtz conditions and Poisson brackets are shown to fit the same pattern, so that they

can be combined with these tools rather than used side by side. The mathematics required is multivariable calculus, and every statement is written in components. A dictionary to the language of differential geometry is given in **Appendix B** for readers who want it, and can be ignored by those who do not.

[How to use this paper]

Design engineers can start at the decision chart of Section 12 and the problem guide of Section 13, which lead from a type of problem to the variables to use, the corrections that must appear and the integrals to look for. Section 4.3 (moving frames, robot arms), Section 5 (vehicles, Kane's method) and Example 12 (Lyapunov functions) are the most directly applicable parts. **Physicists** will find the new structural results in Sections 4, 5 and 10. **Everyone** can use Section 13 and **Appendix A** as a quick reference. Every formula there is derived in the main text, and every worked example has been checked symbolically.

2. The Basic Identity and the Commutator Table

2.1. Commutation Rules

Let $q=\left(q^1,\cdots,q^n\right)$ be generalized coordinates; repeated indices are summed. We treat time t, positions q^i, velocities v^i and accelerations a^i as independent variables and set $v^i=\dot{q}^i$, $a^i=\ddot{q}^i$ only along an actual motion $q(t)$. This is what is done implicitly whenever one writes $\partial L/\partial\dot{q}$. The total time derivative is

$$D_t=\frac{\partial}{\partial t}+v^i\frac{\partial}{\partial q^i}+a^i\frac{\partial}{\partial v^i}, \tag{4}$$

which reproduces $\mathrm{d}/\mathrm{d}t$ along every motion. The part $v^i\partial q^i+a^i\partial v^i$, the *convective term*, is what separates D_t from ∂t.

Result 1 (Commutation rules). *For any twice continuously differentiable* $f(t,q,v)$,

$$\begin{aligned}&\frac{\partial}{\partial v^i}\left(D_tf\right)-D_t\left(\frac{\partial f}{\partial v^i}\right)=\frac{\partial f}{\partial q^i},\\&\frac{\partial}{\partial q^i}\left(D_tf\right)=D_t\left(\frac{\partial f}{\partial q^i}\right),\quad\frac{\partial}{\partial t}\left(D_tf\right)=D_t\left(\frac{\partial f}{\partial t}\right).\end{aligned} \tag{5}$$

Proof. Expand D_tf using (4) and differentiate. All terms cancel in pairs once mixed partials are equated, except in the first rule, where $\partial/\partial v^i$ also acts on the coefficient v^i of D_t and leaves $\partial f/\partial q^i$. □

For the position $\boldsymbol{r}(t,q)$, which does not depend on v, the first rule gives the first identity in (1) and the second rule the second. Conversely, if (5) holds for a function, its mixed partials involving a velocity are symmetric: the rules are Clairaut's theorem in the variables of mechanics. Together with the chain and product rules and the Poincaré lemma ("a field with symmetric derivatives is locally a gradient" [14]), they are the only mathematical input of this paper.

2.2. The Two-Route Principle

Every standard step of a derivation is a linear operation on functions: differentia-

tion with respect to t, q^i or v^i; application of D_t; a change of velocity variables; restriction to a constraint or to a level set of a momentum. If two chains of such operations lead from the same quantity to the same target, their difference is a sum of *commutators* of the operations involved. Under some assumption the commutator vanishes and the two routes agree; when the assumption fails, the difference is a definite, computable term, the *two-route correction* (Figure 1).

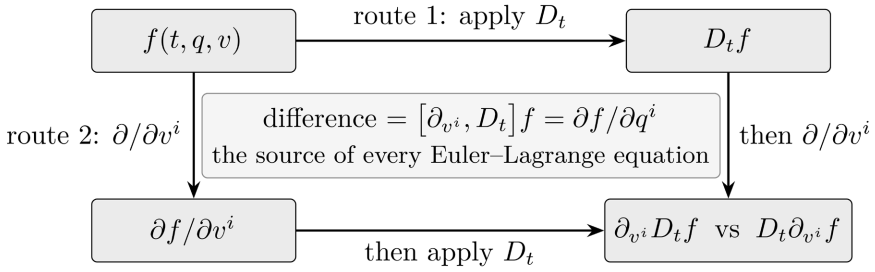


Figure 1. The two-route principle. Two orders of the same two operations differ by their commutator. Here the commutator is the first rule of (5); all structural corrections in this paper arise in the same way from the operations listed in Table 1, while constitutive forces (Section 4.3(e)) are independent physical input.

Table 1 lists the commutators that occur in the rest of the paper, the physical term each produces, and where it is derived. It is the organizing table of the method: a derivation that applies two of these operations in an order different from the one its assumptions justify loses exactly the entry in the corresponding row.

Table 1. The commutator table. Each row compares two orders of operations and names the correction that their commutator produces. Symbols are defined in the sections cited and collected in the notation table of Section 13.

Operations compared	Commutator	Correction it produces	Where
∂_{v^i} and D_t	$\left[\partial_{v^i}, D_t\right] = \partial_{q^i}$	Euler-Lagrange structure; inertial terms (Christoffel, centrifugal)	Res. 1, 2
∂_{q^i}, ∂_t and D_t	0	none: identities (1), gauge invariance	Res. 1, 9
D_t and ∂_t	$D_t - \partial_t = v^i \partial_{q^i} + a^i \partial_{v^i}$	convective term; Hamilton's equations from Hamilton-Jacobi	Res. 11
Velocity-linear L: $\mathcal{E}_i$ and gradient	$\mathcal{F}_{ij} = \partial_i \lambda_j - \partial_j \lambda_i$, $\partial_t \lambda_i - \partial_i \lambda_0$	gyroscopic terms: Lorentz, Coriolis, Euler forces	Res. 5
Frame derivatives e_β, e_δ	$\left[e_\beta, e_\delta\right] = \gamma^\alpha_{\beta\delta} e_\alpha$	anholonomy: Euler's rigid-body terms, nonholonomic terms	Res. 3
Frame derivative e_β and frame flow e_0	$\left[e_\beta, e_0\right] = \gamma^\alpha_{\beta 0} e_\alpha$	transport: rotating and accelerating frames	Res. 3
Constraint $\omega^c = 0$ and differentiation	$\gamma^c_{ad} P_c \omega^d$	constraint-substitution correction	Res. 7
Level set $p_\chi = c$ and $\mathcal{E}_i$	transfer to $(c-\sigma)a_i$, V_{eff}	reduction: gyroscopic term, effective potential	Res. 8
Hamiltonian flows X_g, X_k	$\left[X_g, X_k\right] = -X_{\{g,k\}}$	Poisson bracket; central charges	Sec. 11

3. Bookkeeping: Quantities and Assumptions

3.1. Quantities Kept Apart

Using one symbol for two different things is a frequent source of error, so we keep the following distinct:

$$\begin{aligned} &T(t,q,v),\quad U(t,q),\quad E:=T+U,\quad L:=T-U,\\ &h:=v^i\frac{\partial L}{\partial v^i}-L,\quad H(t,q,p),\quad S(t,q), \end{aligned} \tag{6}$$

with the canonical momentum $p_i:=\partial L/\partial v^i$ and the kinetic momentum $\pi_i:=\partial T/\partial v^i$. For particles with positions $\boldsymbol{r}(t,q)$, the kinetic energy splits into parts quadratic, linear and independent of the velocities,

$$T=T_2+T_1+T_0,\quad T_2=\frac{1}{2}g_{ij}(t,q)v^iv^j,\quad T_1=b_i(t,q)v^i,\quad T_0(t,q), \tag{7}$$

with $g_{ij}=m\partial_i\boldsymbol{r}\cdot\partial_j\boldsymbol{r}$, $b_i=m\partial_i\boldsymbol{r}\cdot\partial_t\boldsymbol{r}$, $T_0=\frac{1}{2}m\left|\partial_t\boldsymbol{r}\right|^2$ (summed over particles) and $\partial_i:=\partial/\partial q^i$. T_1 and T_0 are nonzero only for moving coordinates.

3.2. Assumptions

All functions are twice continuously differentiable (three times where the Helmholtz conditions are used). Each step is labelled by the assumptions it uses. They are separated so that a failed check points to exactly one cause.

(H1) *Cartesian coordinates*: $g_{ij}=m\delta_{ij}$ is constant.

(H2) *Fixed coordinates in an inertial frame*: $b_i=0$, $T_0=0$, $\partial_t g_{ij}=0$.

(H3) *Constant mass, closed system*: no mass enters or leaves.

(H4) *Potential forces*: the generalized force $Q_i:=\boldsymbol{F}\cdot\partial\boldsymbol{r}/\partial q^i$ equals $-\partial U/\partial q^i$ with $U=U(t,q)$.

(H5) *Regular Legendre map*: $\det\left(\partial^2L/\partial v^i\partial v^j\right)\neq 0$.

(H6) *On shell*: the relation holds along actual motions, not identically in (t,q,v,a).

(H7) *Natural system*: $T=T_2$ and U is velocity independent.

(H8) *Holonomic velocities*: the velocity variables used are time derivatives of coordinates (Section 4).

3.3. Derivative Operations

Table 2 lists how energy, momentum, force and velocity are obtained from each generating quantity. Every entry is a derivative operation, so entries can be chained; a chain is valid when all assumptions along it hold. Relations such as $\boldsymbol{\pi}=m\boldsymbol{v}$ or $W=\Delta T$ are statements about values in special circumstances, not operations, and are never chained.

4. The Master Equation

[In brief]

What you get: one equation of motion valid in any coordinates, any moving frame

and any choice of velocity variables, with every inertial, gyroscopic, transport and anholonomy term identified. **When you need it:** rotating or accelerating frames, rigid bodies, vehicles, robots, and any problem where body-fixed velocity components are the natural variables. **What you compute:** the Lagrangian L^* in the chosen velocity variables and the commutators $\left[e_\beta, e_\mu\right]$ of the frame vector fields.

Table 2. Derivative operations relating mechanical quantities. Labels are the assumptions of Section 3.2; unlabelled entries hold for all smooth functions. The Force entry of the action column is the partial time derivative of a momentum field and equals $\dot{p}_i$ only after the convective correction (Section 9).

	Action S	Lagrangian L	Hamiltonian H	Kinetic T, potential U
Energy	$-\dfrac{\partial S}{\partial t}$	$h = v^i \dfrac{\partial L}{\partial v^i} - L$	H (H5)	$E = T + U$
Momentum	$\dfrac{\partial S}{\partial q^i}$	$p_i = \dfrac{\partial L}{\partial v^i}$	—	$\pi_i = \dfrac{\partial T}{\partial v^i}$
Force	$\dfrac{\partial^2 S}{\partial t\, \partial q^i}$	$\dfrac{\partial L}{\partial q^i}$	$-\dfrac{\partial H}{\partial q^i}$ (H5, H6)	$\dfrac{\partial T}{\partial q^i}$; $-\dfrac{\partial U}{\partial q^i}$ (H4)
Velocity	—	—	$\dfrac{\partial H}{\partial p_i}$ (H5)	—

4.1. From Newton's Second Law

Let $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$ hold in an inertial frame, with $\boldsymbol{r} = \boldsymbol{r}(t,q)$ in arbitrary, possibly time-dependent coordinates; the extension to systems of particles with ideal holonomic constraints is standard [1] [2]. Define the Euler-Lagrange expression of any $f(t,q,v)$,

$$\mathcal{E}_i(f) := D_t \frac{\partial f}{\partial v^i} - \frac{\partial f}{\partial q^i}, \tag{8}$$

which is linear in f, so pieces of a Lagrangian can be treated separately.

Result 2 (Lagrange's equations). *Under* (*H*3), $\mathcal{E}_i(T) = m\ddot{\boldsymbol{r}} \cdot \partial_i \boldsymbol{r}$ *identically. Hence Newton's law is equivalent to* $\mathcal{E}_i(T) = Q_i$, *and under* (*H*4) *to*

$$\mathcal{E}_i(L) = D_t \frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial q^i} = 0, \quad L = T - U. \tag{9}$$

Proof. By the first rule, $\partial \dot{\boldsymbol{r}} / \partial v^i = \partial_i \boldsymbol{r}$, so $\partial T / \partial v^i = m\dot{\boldsymbol{r}} \cdot \partial_i \boldsymbol{r}$. By the product rule and the second rule, $D_t\left(\partial T / \partial v^i\right) = m\ddot{\boldsymbol{r}} \cdot \partial_i \boldsymbol{r} + m\dot{\boldsymbol{r}} \cdot \partial_i \dot{\boldsymbol{r}}$, and the last term is $\partial T / \partial q^i$. Under (H4), $Q_i = -\partial_i U$ and $\partial U / \partial v^i = 0$. □

No variational principle is used; that (9) also makes $\int L \mathrm{d}t$ stationary is a separate fact [15] [16]. In the language of Table 2, (9) states that two routes from L to the Force row agree: the direct one, $\partial L / \partial q^i$, and the one through the momentum, $L \to p_i \to D_t p_i$. If nonpotential forces are present, $\mathcal{E}_i(L) = Q_i^{\mathrm{nc}}$.

4.2. Moving Frames and the Master Equation

Many problems are naturally written not in the velocities v^i but in velocity com-

ponents resolved along a moving body or frame. We allow the most general linear choice, including explicit time dependence and an offset,

$$\begin{aligned} &\omega^{\alpha} = A_i^{\alpha}(t,q)v^i + A_0^{\alpha}(t,q), \quad v^i = B_{\alpha}^i\omega^{\alpha} + B_0^i, \\ &B = A^{-1}, \quad B_0^i := -B_{\alpha}^i A_0^{\alpha}. \end{aligned} \tag{10}$$

Examples are body-frame angular velocities, velocities resolved along a vehicle, and inertial velocities resolved in a rotating frame (for which $A_0^{\alpha} \neq 0$). With these variables come $n+1$ *frame vector fields* on (t,q)-space,

$$e_{\alpha} := B_{\alpha}^i\partial_i, \quad e_0 := \partial_t + B_0^i\partial_i, \tag{11}$$

chosen so that along any motion the time derivative of a function of (t,q) splits as $\mathrm{d}/\mathrm{d}t = e_0 + \omega^{\alpha}e_{\alpha}$. Their commutators close on the spatial fields,

$$\left[e_{\beta}, e_{\mu}\right] = \gamma^{\alpha}{}_{\beta\mu}e_{\alpha} \quad (\mu = 0,1,\cdots,n), \quad \gamma^{\alpha}{}_{\beta\mu} = -\gamma^{\alpha}{}_{\mu\beta}, \tag{12}$$

because e_{β} and e_0 have constant t-components. Explicitly, for spatial indices,

$$\begin{aligned} \gamma^{\alpha}{}_{\beta\delta} &= \left(\partial_j A_i^{\alpha} - \partial_i A_j^{\alpha}\right)B_{\beta}^i B_{\delta}^j, \\ \gamma^{\alpha}{}_{\beta 0} &= \left(\partial_t A_i^{\alpha} - \partial_i A_0^{\alpha}\right)B_{\beta}^i + \left(\partial_j A_i^{\alpha} - \partial_i A_j^{\alpha}\right)B_{\beta}^i B_0^j. \end{aligned} \tag{13}$$

We call $\gamma^{\alpha}{}_{\beta\delta}$ the *anholonomy* and $\gamma^{\alpha}{}_{\beta 0}$ the *transport* coefficients; in Hamel's notation the former are the transitivity (Boltzmann-Hamel) coefficients [6]-[8]. All of them vanish if and only if each ω^{α} is the total time derivative of a function $\varphi^{\alpha}(t,q)$, which is assumption (H8).

Result 3 (Master equation). *Let* $L^*(t,q,\omega) := L\left(t,q,v(t,q,\omega)\right)$, $P_{\alpha} := \partial L^*/\partial\omega^{\alpha}$ *and* $\omega^0 := 1$. *Then, identically,*

$$B_{\beta}^i\mathcal{E}_i(L) = \dot{P}_{\beta} - e_{\beta}L^* + \gamma^{\alpha}{}_{\beta\mu}P_{\alpha}\omega^{\mu} \quad (\textit{sum over } \mu = 0,1,\cdots,n), \tag{14}$$

and Newton's law takes the form

$$\dot{P}_{\beta} - e_{\beta}L^* + \gamma^{\alpha}{}_{\beta\delta}P_{\alpha}\omega^{\delta} + \gamma^{\alpha}{}_{\beta 0}P_{\alpha} = Q_{\beta}^*, \quad Q_{\beta}^* := B_{\beta}^i Q_i^{\mathrm{nc}}. \tag{15}$$

Proof. By the chain rule, $\partial L/\partial v^i = A_i^{\alpha}P_{\alpha}$ and $\partial_i L\big|_{t,v} = \partial_i L^*\big|_{\omega} + P_{\alpha}\left(\partial_i A_j^{\alpha}v^j + \partial_i A_0^{\alpha}\right)$. The product rule gives $D_t\left(A_i^{\alpha}P_{\alpha}\right) = A_i^{\alpha}\dot{P}_{\alpha} + P_{\alpha}\left(\partial_t A_i^{\alpha} + v^j\partial_j A_i^{\alpha}\right)$. Subtracting,

$$\mathcal{E}_i(L) = A_i^{\alpha}\dot{P}_{\alpha} - \partial_i L^* + P_{\alpha}\left[\left(\partial_j A_i^{\alpha} - \partial_i A_j^{\alpha}\right)v^j + \partial_t A_i^{\alpha} - \partial_i A_0^{\alpha}\right].$$

Contract with B_{β}^i and insert $v^j = B_{\delta}^j\omega^{\delta} + B_0^j$; the bracket becomes the right-hand side of (13). That these coefficients are the commutator coefficients (12) follows from $\omega^{\alpha}(e_{\mu}) = \text{const}$ and the formula $\mathrm{d}\theta(X,Y) = X\theta(Y) - Y\theta(X) - \theta\left([X,Y]\right)$ for the one-forms $\theta^{\alpha} = A_i^{\alpha}\mathrm{d}q^i + A_0^{\alpha}\mathrm{d}t$, which in components is again the symmetry of mixed partials. □

Equation (15) is the Hamel (Boltzmann-Hamel) equation [6] [7] [17] for time-dependent, affine velocity variables, whose rheonomic terms are treated in [8]; the form used here collects them into the transport coefficients $\gamma^{\alpha}{}_{\beta 0}$. The only new input it needs is the table of commutators $\left[e_{\beta}, e_{\mu}\right]$, which is computed once for

a given frame. A potential is included in L^*; Q_β^* collects the rest. Table 3 shows where each named term of classical mechanics appears in (15) and when it vanishes.

Table 3. Anatomy of the master equation (15): where each named term of classical mechanics lives.

Term	Physical content	Vanishes when
$\dot{P}_\beta$	rate of change of momentum, including the Christoffel terms of curvilinear coordinates	—
$-e_\beta L^*$	potential forces; centrifugal term ∂T_0; gyroscopic forces from velocity-linear terms in L^*	L^* independent of q
$\gamma^\alpha{}_{\beta\delta} P_\alpha \omega^\delta$	anholonomy: Euler's rigid-body terms, nonholonomic vehicle terms	(H8): ω^α are coordinate velocities
$\gamma^\alpha{}_{\beta 0} P_\alpha$	transport: frame rotation and acceleration (Coriolis, Euler, centrifugal in compact form)	frame fixed in an inertial frame
Q_β^*	nonpotential forces: friction, actuation, constraint forces	(H4)

4.3. Special Cases

(a) Coordinate velocities: inertial corrections. With $\omega^\alpha = v^i$ all γ vanish and (15) is (9). Writing $\mathcal{E}_i(T) = Q_i$ out with (7) gives the general equation of motion in arbitrary coordinates,

$$g_{ij}a^j + \Gamma_{i,jk} v^j v^k = Q_i - \partial_t g_{ij} v^j + \beta_{ij} v^j - \partial_i b_i + \partial_i T_0, \tag{16}$$

with the Christoffel symbols $\Gamma_{i,jk} = \frac{1}{2}\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}\right)$ and $\beta_{ij} := \partial_i b_j - \partial_j b_i$. (Directly: $\mathcal{E}_i(T_2) = g_{ij}a^j + \partial_t g_{ij} v^j + \Gamma_{i,jk} v^j v^k$, $\mathcal{E}_i(T_1) = \partial_t b_i - \beta_{ij} v^j$, $\mathcal{E}_i(T_0) = -\partial_i T_0$.) The terms from T_2 and T_0 are *inertial corrections*: they exist because the kinetic coefficients depend on position and time, so that $\partial T/\partial q^i \neq 0$ and $D_t\left(g_{ij}v^j\right) \neq g_{ij}a^j$.

Example 1 (Plane polar coordinates). For $T = \frac{1}{2}m\left(v_r^2 + r^2 v_\theta^2\right)$, (16) gives $m\ddot{r} - mr\dot{\theta}^2 = Q_r$ and $mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = Q_\theta$. The centrifugal term is the correction $\partial T/\partial r$; the term $2mr\dot{r}\dot{\theta}$ comes from D_t acting on $\pi_\theta = mr^2 v_\theta$. Assuming $\partial T/\partial q^i = 0$ loses the first; also taking $\pi_i = mv^i$ loses the second.

Result 4 (Matrix form and passivity). *For* $T = T_2$, (16) *is the manipulator equation*

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla U = \tau, \quad M = \left(g_{ij}\right), \quad C_{ij} := \Gamma_{i,jk} v^k, \tag{17}$$

with τ *the actuator forces, and* $\dot{M} - 2C$ *is antisymmetric.*

Proof. $\left(\dot{M} - 2C\right)_{ij} = \left(\partial_i g_{jk} - \partial_j g_{ik}\right)v^k$. □

The antisymmetry, widely used in passivity-based and adaptive control of robots [18], is the matrix statement that inertial corrections do no work, $\dot{q}^{\mathrm{T}}\left(\frac{1}{2}\dot{M} - C\right)\dot{q} = 0$. Defining C from the Christoffel symbols, as here, is what guarantees it; other factorizations of the same velocity-quadratic terms do not

have this property. The two-link arm is worked out in **Appendix A.4**.

(b) Velocity-linear terms: gyroscopic corrections. Let L contain a part linear in the velocities,

$$L_\Lambda = \lambda_i(t,q)v^i + \lambda_0(t,q), \tag{18}$$

such as the vector potential ($\lambda_i = eA_i$, $\lambda_0 = -e\phi$) or the term $T_1 = b_i v^i$ of a moving frame.

Result 5 (Gyroscopic correction). *With the field strength* $\mathcal{F}_{ij} := \partial_i\lambda_j - \partial_j\lambda_i$,

$$\mathcal{E}_i(L_\Lambda) = (\partial_t\lambda_i - \partial_i\lambda_0) - \mathcal{F}_{ij}v^j. \tag{19}$$

It vanishes for all motions if and only if (λ_0, λ_i) *is locally the gradient of a function of* (t,q).

Proof. $D_t\lambda_i = \partial_t\lambda_i + v^j\partial_j\lambda_i$ and $\partial L_\Lambda/\partial q^i = v^j\partial_i\lambda_j + \partial_i\lambda_0$; subtract. The vanishing conditions are Clairaut's integrability conditions. □

Because the velocity-dependent part of (19) is an antisymmetric array contracted with v^j, its contraction with v^i vanishes: neither magnetic nor Coriolis forces do work.

Result 6 (Lorentz and inertial forces). (*a*) *For a charge in potentials* $(\phi, \boldsymbol{A})$, $L = T - e\phi + e\boldsymbol{A}\cdot\boldsymbol{v}$, (19) *gives* $-\mathcal{E}_i(L_\Lambda) = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})_i$ *with* $\boldsymbol{E} = -\nabla\phi - \partial_t\boldsymbol{A}$, $\boldsymbol{B} = \nabla\times\boldsymbol{A}$; *the canonical and kinetic momenta differ by* $p_i - \pi_i = eA_i$.

(b) *In a frame rotating with angular velocity* $\boldsymbol{\Omega}(t)$ *about a fixed origin, with coordinate velocities,* $g_{ij} = m\delta_{ij}$, $\boldsymbol{b} = m\boldsymbol{\Omega}\times\boldsymbol{r}$, $T_0 = \frac{1}{2}m|\boldsymbol{\Omega}\times\boldsymbol{r}|^2$, *and* (16) *becomes*

$$m\boldsymbol{a} = \boldsymbol{F} - 2m\boldsymbol{\Omega}\times\boldsymbol{v} - m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) - m\dot{\boldsymbol{\Omega}}\times\boldsymbol{r}. \tag{20}$$

The Coriolis force is the gyroscopic correction of $\boldsymbol{b}$ ($\nabla\times\boldsymbol{b} = 2m\boldsymbol{\Omega}$), *the Euler force its time component, and the centrifugal force the inertial correction* ∇T_0.

Proof. (a) Insert $\lambda_i = eA_i$, $\lambda_0 = -e\phi$ and use $\mathcal{F}_{ij}v^j = e(\boldsymbol{v}\times\boldsymbol{B})_i$. (b) $\beta_{ij}v^j = -2m(\boldsymbol{\Omega}\times\boldsymbol{v})_i$, $\partial_t\boldsymbol{b} = m\dot{\boldsymbol{\Omega}}\times\boldsymbol{r}$ and $\nabla T_0 = -m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$; the Christoffel terms vanish. □

Example 2 (Larmor's theorem as a cancellation). A charge in a uniform field $\boldsymbol{B}$ ($\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}$) observed in a frame rotating with $\boldsymbol{\Omega}\parallel\boldsymbol{B}$ has the Lagrangian $L = \frac{1}{2}m|\boldsymbol{v} + \boldsymbol{\Omega}\times\boldsymbol{r}|^2 + e\boldsymbol{A}\cdot(\boldsymbol{v} + \boldsymbol{\Omega}\times\boldsymbol{r}) - U$. Its velocity-linear part has $\boldsymbol{\lambda} = \left(\frac{e}{2}\boldsymbol{B} + m\boldsymbol{\Omega}\right)\times\boldsymbol{r}$ and total field strength $\nabla\times\boldsymbol{\lambda} = e\boldsymbol{B} + 2m\boldsymbol{\Omega}$. Choosing $\boldsymbol{\Omega} = -e\boldsymbol{B}/2m$ removes the gyroscopic correction entirely. Two velocity-independent pieces remain: the centrifugal term $T_0 = \frac{1}{2}m|\boldsymbol{\Omega}\times\boldsymbol{r}|^2$ and the cross term $e\boldsymbol{A}\cdot(\boldsymbol{\Omega}\times\boldsymbol{r})$. Both are of second order in B and together give the effective potential

$$V_{\text{eff}} = U + \frac{e^2}{8m}\left|\boldsymbol{B}\times\boldsymbol{r}\right|^2, \tag{21}$$

a confining harmonic term, not a centrifugal one: the cross term outweighs T_0 by a factor of two. This is Larmor's theorem [1] [19], obtained as the cancellation of two gyroscopic terms of the same class, with the diamagnetic residual computed rather than guessed.

(c) Moving frames: the transport term. Let $\boldsymbol{\rho}$ be Cartesian coordinates in a frame with angular velocity $\boldsymbol{\Omega}(t)$ whose origin moves with acceleration $\boldsymbol{a}_0$ (frame components), and let ω^α be the components of the *inertial* velocity resolved in the frame, $\boldsymbol{\omega} = \dot{\boldsymbol{\rho}} + \boldsymbol{\Omega}\times\boldsymbol{\rho} + \boldsymbol{w}(t)$, with $\boldsymbol{w}$ the frame components of the origin's velocity. Then $A = I$, $A_0^\alpha = (\boldsymbol{\Omega}\times\boldsymbol{\rho} + \boldsymbol{w})^\alpha$, $L^* = \frac{1}{2}m\left|\boldsymbol{\omega}\right|^2 - U$ and (13) gives $\gamma^\alpha{}_{\beta\delta} = 0$, $\gamma^\alpha{}_{\beta 0} = \epsilon_{\alpha\beta k}\Omega_k$. The master Equation (15) reads

$$m\left(\dot{\boldsymbol{\omega}} + \boldsymbol{\Omega}\times\boldsymbol{\omega}\right) = \boldsymbol{F}, \tag{22}$$

the familiar "rate of change in a rotating frame". A single transport term replaces the Coriolis, centrifugal, Euler and translational terms; expanding $\boldsymbol{\omega}$ gives back (55) of **Appendix A.2**. Equations (20) and (22) describe the same physics with different velocity variables: in the first the frame motion appears as gyroscopic and inertial corrections, in the second as transport. The master equation makes their agreement an identity rather than a coincidence. For the bead on the hoop of Section 1.1, the Coriolis force is normal to the hoop whichever form is used; in the coordinate-velocity form its gyroscopic coefficient b_θ vanishes, which is why it does not enter the equation for θ.

(d) Body velocities: anholonomy. With A independent of t and $A_0^\alpha = 0$, only $\gamma^\alpha{}_{\beta\delta}$ survives and (15) is the classical Hamel equation.

Example 3 (Euler's equations of the rigid body). For body-frame angular velocities written in Euler angles, (13) gives the constants $\gamma^\alpha{}_{\beta\delta} = \epsilon_{\alpha\beta\delta}$ (the Levi-Civita symbol). With $T^* = \frac{1}{2}\sum_\alpha I_\alpha\left(\omega^\alpha\right)^2$ independent of q, (15) gives $I_1\dot{\omega}^1 + \left(I_3 - I_2\right)\omega^2\omega^3 = M_1$ and cyclic permutations. All gyroscopic coupling of the free rigid body is anholonomy; no Euler-angle Christoffel symbols are needed.

(e) Constitutive corrections. If (H4) fails, write $Q_i = -\partial_i U + Q_i^{\text{nc}}$; then $\mathcal{E}_i(L) = Q_i^{\text{nc}}$. This correction comes not from the structure of the operations but from a force law that is not of potential form. Forces linear in velocity with symmetric coefficients can be written with a Rayleigh function [1] [20]. Whether they can be absorbed into a modified Lagrangian is the inverse problem of Section 8; how they enter conserved quantities and Lyapunov functions is treated in Section 10.3. If (H3) fails (rockets, belts, cables), the momentum flux of entering or leaving mass is a further constitutive term on the right-hand side of (15).

5. Constraints: Vehicles, Kane's Method and the Substitution Correction

[In brief]

What you get: the correct equations of motion of rolling, skating and towed systems, the constraint forces, and the exact term lost if the constraints are substituted into the Lagrangian too early. **When you need it:** wheels, blades, skates, trailers, rolling bodies. **What you compute:** the anholonomy γ^{c}_{ad} and the constraint momentum P_c.

5.1. The Constraint-Substitution Correction

For linear nonholonomic constraints it is convenient to choose the velocity variables so that the constraints read $\omega^{c}=0$, where indices c label constrained and a,b,d free variables. Ideal constraint forces have $Q^{*}_{a}=0$. A common shortcut is to substitute $\omega^{c}=0$ into L^{*} and then write Lagrange-type equations for the remaining variables. We take the frame time-independent here ($\gamma^{\alpha}_{\beta 0}=0$); in a time-dependent frame the transport term $\gamma^{\alpha}_{a0}P_{\alpha}$ splits in the same way, its free part $\gamma^{b}_{a0}P^{\circ}_{b}$ joining the left-hand side and its constrained part $\gamma^{c}_{a0}P_{c}$ joining the correction.

Result 7 (Constraint-substitution correction). *Let* $L^{\circ}\left(t,q,\omega^{a}\right):=L^{*}\big|_{\omega^{c}=0}$ *and* $P^{\circ}_{a}:=\partial L^{\circ}/\partial\omega^{a}$. *The correct equations for the free variables are*

$$\dot{P}^{\circ}_{a}-e_{a}L^{\circ}+\gamma^{b}_{ad}P^{\circ}_{b}\omega^{d}=Q^{*}_{a}-\underbrace{\gamma^{c}_{ad}\left(P_{c}\right)_{\omega^{c}=0}\omega^{d}}_{\text{lost by early substitution}}, \tag{23}$$

where $\left(P_{c}\right)_{\omega^{c}=0}=\left(\partial L^{*}/\partial\omega^{c}\right)_{\omega^{c}=0}$ *is the constraint momentum. The constraint forces follow from the constrained components of* (15). *Applying the Hamel equations to* L° *alone is correct if and only if the underbraced term vanishes along the motion; treating the* ω^{a} *moreover as coordinate velocities also drops* $\gamma^{b}_{ad}P^{\circ}_{b}\omega^{d}$. *The correction vanishes in particular when* $\gamma^{c}_{ad}=0$ *for all free* a,d, *which is the Frobenius condition for the constraints to be integrable, that is, holonomic.*

Proof. In (15) set $\omega^{c}=0$ after differentiation. Since e_{a} differentiates only in q, $\left(e_{a}L^{*}\right)_{\omega^{c}=0}=e_{a}L^{\circ}$ and $\left(P_{a}\right)_{\omega^{c}=0}=P^{\circ}_{a}$. In the sum over α, split α into free b and constrained c; the sum over δ runs only over free d. □

The criterion is again a mixed-partial condition. The constraints can be integrated to relations between the coordinates exactly when the antisymmetrized derivatives of A^{c}_{i}, evaluated on pairs of allowed directions, vanish. The correction can be nonzero only when the kinetic energy couples allowed and forbidden directions ($P_{c}\neq 0$ although $\omega^{c}=0$). Lindelöf's error for a rolling body of revolution [6] is of exactly this type: rolling constraints are nonintegrable and the kinetic energy couples rolling to slipping directions.

Example 4 (Chaplygin sleigh: equations and closed-form solution). A rigid body of mass m slides on a plane on a knife edge at point P; its mass centre lies a distance ℓ ahead of P on the blade axis, with central moment of inertia I_{c}. Use $q=(x,y,\theta)$ for P and the heading, and $u=\dot{x}\cos\theta+\dot{y}\sin\theta$ (along the blade), $w=-\dot{x}\sin\theta+\dot{y}\cos\theta$ (sideways) and $\Omega=\dot{\theta}$. The knife edge im-

poses $w=0$. Then

$$T^{*}=\frac{1}{2}m[u^{2}+\left(w+\ell\Omega\right)^{2}]+\frac{1}{2}I_{c}\Omega^{2}, \tag{24}$$

and (13) gives the only nonzero coefficients $\gamma^{u}_{w\Omega}=-\gamma^{w}_{u\Omega}=1$ (with their antisymmetric partners). With $J:=I_{c}+m\ell^{2}$, $T^{\circ}=\frac{1}{2}mu^{2}+\frac{1}{2}J\Omega^{2}$, and the naive equations would be $\dot{u}=0$, $\dot{\Omega}=0$: motion in a straight line or a fixed circle, which is wrong. The constraint momentum is $P_{w}=m\ell\Omega\neq 0$, and (23) restores the missing terms:

$$m\dot{u}=m\ell\Omega^{2},\quad J\dot{\Omega}=-m\ell u\Omega, \tag{25}$$

with lateral blade force $\Lambda=m\left(\ell\dot{\Omega}+u\Omega\right)$ from the w-component of (15). The energy $\frac{1}{2}mu^{2}+\frac{1}{2}J\Omega^{2}=\frac{1}{2}mu_{\infty}^{2}$, with $u_{\infty}>0$ the terminal sliding speed, is conserved, since the correction does no work for the same antisymmetry reason as in Section 4.3(b), and (25) integrates in closed form:

$$\begin{aligned}u\left(t\right)&=u_{\infty}\tanh\left(\frac{m\ell u_{\infty}}{J}\left(t-t_{0}\right)\right),\\ \Omega\left(t\right)&=\pm u_{\infty}\sqrt{\frac{m}{J}}\operatorname{sech}\left(\frac{m\ell u_{\infty}}{J}\left(t-t_{0}\right)\right).\end{aligned} \tag{26}$$

Every motion with $\Omega\not\equiv 0$ ends in straight-line sliding with the mass centre leading, the characteristic asymmetric behaviour of the sleigh [6] [10] [21] (**Figure 2**). Integrating Ω over a complete motion gives the total change of heading in closed form,

$$\Delta\theta=\int_{-\infty}^{\infty}\Omega\mathrm{d}t=\pi\sqrt{\frac{J}{m\ell^{2}}}=\pi\sqrt{1+\frac{I_{c}}{m\ell^{2}}}>\pi, \tag{27}$$

independent of the speed u_{∞}: the sleigh always turns through more than a half circle, and the blade point P, which stops and reverses at $t=t_{0}$, traces a cusp. The two restored terms in (25) are, respectively, a centripetal and a gyroscopic anholonomy correction.

5.2. Kane's Equations and the Gibbs-Appell Form

Engineers usually treat constrained multibody systems with Kane's method [22]. In Kane's language the free variables ω^{a} are *generalized speeds*, the vectors $B_{a}^{i}\partial_{i}\boldsymbol{r}$ are the *partial velocities*, and the generalized inertia force is $F_{a}^{*}=-m\ddot{\boldsymbol{r}}\cdot B_{a}^{i}\partial_{i}\boldsymbol{r}$. By Result 2, $F_{a}^{*}=-B_{a}^{i}\mathcal{E}_{i}\left(T\right)$, and by Result 3,

$$-F_{a}^{*}=\left[\dot{P}_{a}-e_{a}T^{*}+\gamma^{\alpha}_{a\mu}P_{\alpha}\omega^{\mu}\right]_{\omega^{c}=0}. \tag{28}$$

Kane's equations $F_{a}+F_{a}^{*}=0$ are therefore the free components of the master equation, evaluated on the constraint *after* differentiation. This explains why Kane's method never makes the substitution error: it projects Newton's law instead of differentiating a constrained energy. Equation (28) adds a closed form for F_{a}^{*} that requires no accelerations of material points, only the rates $\dot{\omega}^{a}$ through

$\dot{P}_a$, and Result 7 says which terms of it come from the constraint. The Gibbs-Appell equations [8] are the same projection written through the acceleration energy.

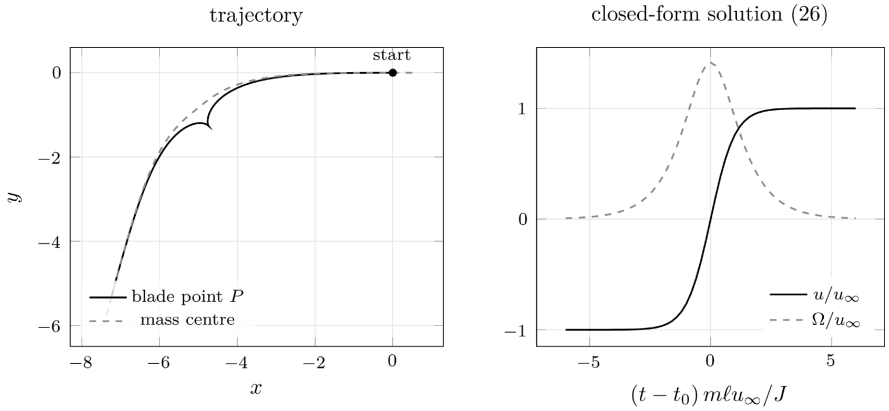


Figure 2. Chaplygin sleigh with $m=1$, $\ell=0.5$, $I_c=0.25$, $u_\infty=1$. Starting by sliding backwards (mass centre trailing), the sleigh turns through $\Delta\theta=\pi\sqrt{2}\approx 255^\circ$ and ends sliding forwards with the mass centre leading; the blade point traces a cusp where it stops and reverses. The naive equations obtained by early substitution of $w=0$ would predict an almost straight line (a circle of very large radius, since Ω is initially small but nonzero). The numerically integrated motion agrees with (26) to 10^{-10}.

Another route is sometimes confused with the one used here. Imposing nonholonomic constraints with multipliers inside Hamilton's principle (vakonomic dynamics) gives different equations in general; for mechanical systems with ideal constraints the correct equations are those of d'Alembert's principle used above [10]. Constraint forces, when wanted, are the constrained components $\beta=c$ of (15).

6. Reduction: Cyclic Coordinates Become Gyroscopic Fields

Eliminating a cyclic coordinate is a two-route operation ("solve, then substitute" against "substitute, then solve"), and the mechanism of Section 4.3(b) predicts what it produces.

Result 8 (Routh reduction as transfer of field strength). *Let* χ *be cyclic in*

$$L=\frac{1}{2}g_{ij}v^iv^j+\frac{1}{2}I_\chi\left(v^\chi+a_iv^i\right)^2+\sigma v^\chi+\lambda_iv^i-V, \tag{29}$$

where $g_{ij},I_\chi\neq 0,a_i,\sigma,\lambda_i,V$ *depend on the remaining coordinates* q *only. On the level set* $p_\chi=c$, *the motion of* q *obeys* $\mathcal{E}_i(R_c)=0$ *with*

$$R_c=\frac{1}{2}g_{ij}v^iv^j+\left[\lambda_i+(c-\sigma)a_i\right]v^i-V_{\text{eff}},\quad V_{\text{eff}}=V+\frac{(c-\sigma)^2}{2I_\chi}. \tag{30}$$

The cyclic coordinate thus becomes a gyroscopic field with "charge" c, *with reduced field strength* $\mathcal{F}_{ij}+\partial_i\left[(c-\sigma)a_j\right]-\partial_j\left[(c-\sigma)a_i\right]$, *plus an inertial-type correction in the effective potential.*

Proof. $p_\chi = I_\chi\left(v^\chi + a_i v^i\right) + \sigma = c$ fixes v^χ. The Routhian $R = L - cv^\chi$ satisfies $\partial R/\partial v^\chi = 0$ there, so its derivatives with respect to q^i and v^i at fixed c equal those of L (the dependence through v^χ drops out: two routes agree). Hence $\mathcal{E}_i(R) = \mathcal{E}_i(L)$ on the level set. Substituting $v^\chi = \left(c - \sigma\right)/I_\chi - a_i v^i$ gives (30). □

This is the classical Routh procedure [23]-[25]; the point here is that its output is automatically of the form (39) below, so all tools for magnetic-type systems (brackets, integrals) apply to reduced systems without change.

Example 5 (Heavy symmetric top in two reductions). With Euler angles (θ, ϕ, ψ), $L = \frac{1}{2} I_1\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2} I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgl\cos\theta$. Reducing ψ ($I_\chi = I_3$, $a_\phi = \cos\theta$, $\sigma = 0$) on $p_\psi = c$ gives the velocity-linear term $c\cos\theta\dot{\phi}$, whose field strength $\mathcal{F}_{\theta\phi} = -c\sin\theta$ is the gyroscopic force:

$$I_1\ddot{\theta} - I_1\dot{\phi}^2 \sin\theta\cos\theta + c\dot{\phi}\sin\theta - Mgl\sin\theta = 0. \tag{31}$$

Reducing ϕ next ($I_\chi = I_1\sin^2\theta$, $a = 0$, $\sigma = c\cos\theta$) on $p_\phi = b$ gives, up to a constant, $V_{\text{eff}} = Mgl\cos\theta + \left(b - c\cos\theta\right)^2/\left(2I_1\sin^2\theta\right)$, the classical one-dimensional problem of the top [1]. The gyroscopic term of the first stage and the σ-term of the second are the same object seen at two stages of reduction. The same two steps apply to a spacecraft with a momentum wheel, where the wheel angle is the cyclic coordinate and the wheel momentum supplies the gyroscopic stiffness.

7. Gauge Freedom, Null Lagrangians and Generating Functions

Result 9 (Null Lagrangians). *On a star-shaped region of* (t, q)*-space,* $\mathcal{E}_i(L) \equiv 0$ *identically if and only if* $L = D_t F = \partial_t F + v^j \partial_j F$ *for some* $F(t, q)$.

Proof. If $L = D_t F$, the second rule gives $\mathcal{E}_i(L) = D_t\partial_i F - \partial_i D_t F = 0$. Conversely, the coefficient of a^j in $\mathcal{E}_i(L)$ is $\partial^2 L/\partial v^i \partial v^j$, so L is affine in v; Result 5 gives the integrability conditions, and the Poincaré lemma gives F. □

This is the first-order case of the classical characterization [26] [27]. Consequences: 1) *gauge invariance*— L and $L + D_t F$ give the same equations, and only field strengths are physical; 2) on a region with a hole, a velocity-linear term may satisfy the integrability conditions without being a gradient, and is then detected only by loop integrals, as in the Aharonov-Bohm configuration [28]; 3) *generating functions*—a time-independent change of variables $(q, p) \to (Q, P)$ preserves the form of Hamilton's equations when $p_i \mathrm{d}q^i - P_i \mathrm{d}Q^i$ has symmetric mixed partials, and the Poincaré lemma then supplies the generating function $F_1(q, Q)$ with $p_i = \partial F_1/\partial q^i$, $P_i = -\partial F_1/\partial Q^i$ [1]. Canonical transformations are, in this sense, gauge transformations of the phase-space Lagrangian $p_i\dot{q}^i - H$.

8. The Inverse Problem: Does an Equation Have a Lagrangian?

Given second-order expressions $\varepsilon_i(t, q, v, a)$, is there an L with $\varepsilon_i = \mathcal{E}_i(L)$

[27] [29]-[31]? The necessary conditions are once more the commutation rules.

Result 10 (Helmholtz conditions). *If* $\varepsilon_i = \mathcal{E}_i(L)$ *for some* L *with three continuous derivatives, then*

$$\frac{\partial \varepsilon_i}{\partial a^j} = \frac{\partial \varepsilon_j}{\partial a^i}, \tag{32}$$

$$\frac{\partial \varepsilon_i}{\partial v^j} + \frac{\partial \varepsilon_j}{\partial v^i} = 2D_t \frac{\partial \varepsilon_i}{\partial a^j}, \tag{33}$$

$$\frac{\partial \varepsilon_i}{\partial q^j} - \frac{\partial \varepsilon_j}{\partial q^i} = \frac{1}{2} D_t \left(\frac{\partial \varepsilon_i}{\partial v^j} - \frac{\partial \varepsilon_j}{\partial v^i} \right). \tag{34}$$

Proof sketch. Only D_t introduces accelerations, so $\partial \varepsilon_i / \partial a^j = \partial^2 L / \partial v^i \partial v^j$ is symmetric. Differentiating ε_i in v^j with the first rule, the symmetric part is $D_t \left(\partial^2 L / \partial v^i \partial v^j \right)$ and the antisymmetric part is $L_{\overline{ij}} - L_{\overline{ji}}$, with $L_{\overline{ij}} := \partial^2 L / \partial v^i \partial q^j$. Differentiating in q^j with the second rule, the antisymmetric part is $D_t \left(L_{\overline{ij}} - L_{\overline{ji}} \right)$. □

These conditions are also locally sufficient [26] [27] [31]. For equations $\ddot{q}^i = f^i(t, q, \dot{q})$ one looks for a multiplier g_{ij} such that $g_{ij}\left(a^j - f^j\right)$ passes the test (Douglas [29], Sarlet [32]).

Example 6 (Damped oscillator). For $\varepsilon = ma + \zeta m v + kq$ with $\zeta \neq 0$, (33) reads $2\zeta m = 0$ and fails. The multiplied expression $\mathrm{e}^{\zeta t}\varepsilon$ passes, giving the Caldirola-Kanai Lagrangian $L = \mathrm{e}^{\zeta t}\left(\frac{1}{2} m v^2 - \frac{1}{2} k q^2 \right)$ [33] [34], with canonical momentum $p = m\mathrm{e}^{\zeta t} v$ and an energy function that is not the physical energy: a multiplier changes which entries of **Table 2** carry physical meaning.

9. Hamilton-Jacobi Theory

Under (H5), $H(t, q, p) := p_i v^i - L$ (otherwise Dirac's constraint analysis is needed [35] [36]). Any $S(t, q)$ defines a momentum field $p_i(t, q) := \partial_i S$ and an energy field $-\partial_t S$, with

$$\partial_i p_j = \partial_j p_i, \quad \partial_t p_i = \partial_i (\partial_t S) \tag{35}$$

for every S. The Hamilton-Jacobi equation requires the energy field to equal the Hamiltonian on the momentum field,

$$\partial_t S + H\left(t, q, \partial_q S\right) = 0. \tag{36}$$

Result 11 (Characteristics). *Let* S *solve* (36) *and let* $q(t)$ *satisfy* $\dot{q}^j = \partial H / \partial p_j$ *at* $p = \partial_q S$. *Then* $p_i(t) := \partial_i S\left(t, q(t)\right)$ *satisfies* $\dot{p}_i = -\partial H / \partial q^i$, *and* $\mathrm{d}\left(-\partial_t S\right)/\mathrm{d}t = \partial H / \partial t$.

Proof. Along the curve, $\dot{p}_i = \partial_t p_i + \dot{q}^j \partial_j p_i$: partial derivative plus convective correction. Differentiating (36) in q^i and using (35), $\partial_t p_i = -\partial H / \partial q^i - \left(\partial H / \partial p_j\right) \partial_j p_i$; the last term cancels the convective correction exactly. □

The convective correction is thus the carrier of the dynamics: the method of

characteristics [37] [38]. The same argument gives Jacobi's theorem: for a complete integral $S(t,q,\alpha)$, differentiating (36) in α_k and exchanging mixed partials shows that $\partial S/\partial\alpha_k$ is constant along every characteristic [1] [39].

Example 7 (Free particle). For $H = p^2/2m$, $S = mq^2/(2t)$ has momentum field $p = mq/t$ with $\partial_t p = -mq/t^2$, cancelled by the convective term $\dot{q}\partial_q p = mq/t^2$ along $q = ct$. Mistaking $\partial_t p$ for the force would assign a spurious force to a free particle.

10. Conserved Quantities and Lyapunov Functions

[In brief]

What you get: conserved quantities without guessing symmetries, integrals beyond the point-symmetry class considered here, adiabatic-type invariants of driven systems, and Lyapunov functions for damped systems, all from one linear system of equations (for the class of systems stated in Section 10.3). **When you need it:** integrability, stability, error control of simulations, design of controllers. **What you compute:** Killing tensors of the kinetic metric (once per geometry), then a linear compatibility test on the forces and a quadrature.

10.1. Energy Function versus Total Energy

Result 12 (Energy balance). *For any Lagrangian, identically,* $D_t h = v^i \mathcal{E}_i(L) - \partial L/\partial t$. *Along solutions of* $\mathcal{E}_i(L) = Q_i^{\mathrm{nc}}$, $D_t h = v^i Q_i^{\mathrm{nc}} - \partial_t L$.

Proof. $D_t(v^i p_i) = a^i p_i + v^i D_t p_i$ and $D_t L = \partial_t L + v^i \partial_i L + a^i p_i$; subtract. □

Result 13 (Energy correction). *Under* (*H*4), *by Euler's theorem on homogeneous functions,*

$$h = T_2 - T_0 + U, \quad h - E = -(T_1 + 2T_0), \tag{37}$$

so $h = E$ *only for natural systems* (*H*7).

In a uniformly rotating frame this gives the Jacobi integral $h = \frac{1}{2}m|\boldsymbol{v}|^2 - \frac{1}{2}m|\boldsymbol{\Omega}\times\boldsymbol{r}|^2 + U$, which is conserved although E is not; for a charge, $h = T + e\phi$; for the bead on the hoop, $h = \frac{1}{2}mR^2(\dot{\theta}^2 - \Omega^2\sin^2\theta) - mgR\cos\theta$.

10.2. Noether's Theorem

Let $q^i \to q^i + \epsilon\xi^i(t,q)$, so that v^i changes by $\epsilon D_t\xi^i$ and L by $\epsilon\delta_\xi L$, $\delta_\xi L := \xi^i\,\partial L/\partial q^i + (D_t\xi^i)\partial L/\partial v^i$ [26].

Result 14 (Noether's theorem for point symmetries). *If* $\delta_\xi L = D_t F$ *for some* $F(t,q)$, *then* $I := \xi^i p_i - F$ *satisfies* $D_t I = \xi^i \mathcal{E}_i(L)$ *identically.*

Proof. $D_t(\xi^i p_i) = (D_t\xi^i)p_i + \xi^i D_t p_i = \delta_\xi L + \xi^i \mathcal{E}_i(L)$. □

The term F is needed whenever L is symmetric only up to a null Lagrangian [26] [40]. For a uniform field, $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}$, a translation by $\boldsymbol{c}$ has

$F=\frac{e}{2}(\boldsymbol{B}\times\boldsymbol{c})\cdot\boldsymbol{r}$, and the conserved pseudo-momentum is $\boldsymbol{c}\cdot(m\boldsymbol{v}+e\boldsymbol{B}\times\boldsymbol{r})$: neither $\boldsymbol{p}$ nor $m\boldsymbol{v}$ is conserved.

Noether's theorem requires a symmetry to be found first, and in the point-symmetry form of Result 14 it depends on the gauge through F. Quadratic integrals such as those of Examples 8 and 10 lie outside this point-symmetry class, but they can correspond to generalized (velocity-dependent) symmetries in Noether's general sense [26], which must likewise be found first. Both drawbacks are removed by turning the problem around: instead of guessing a symmetry, we impose a condition on $D_t I$ for an ansatz I and solve for its coefficients.

10.3. Determining Equations for Polynomial Integrals

Scope. This subsection assumes a time-independent kinetic metric $g_{ij}(q)$, forces of the form $N_{ij}(t,q)v^j-\partial_i V(t,q)$ (velocity-independent forces of gradient form plus forces linear in the velocity), and coordinate velocities. Its applicability therefore does not automatically extend to every time-dependent moving-frame system considered earlier: time-dependent metrics, frames rotating at a variable rate (whose Euler force $-m\dot{\boldsymbol{\Omega}}\times\boldsymbol{r}$ is not a gradient), time-varying magnetic fields (whose induced electric field is not a gradient), and quasi-velocity or nonholonomic formulations (Sections 4 and 5) require separate treatment.

Consider second-order systems of the form

$$g_{ij}(q)a^j=-\Gamma_{i,jk}v^jv^k+N_{ij}(t,q)v^j-\partial_i V(t,q), \tag{38}$$

where N_{ij} is an arbitrary array. Its antisymmetric part $\mathcal{F}_{ij}$ contains all gyroscopic forces; its symmetric part $-R_{ij}$ contains linear damping (R positive semi-definite) or anti-damping. The Lagrangian systems

$$L=\frac{1}{2}g_{ij}(q)v^iv^j+\lambda_i(q)v^i-V(t,q),\quad N_{ij}=\mathcal{F}_{ij}:=\partial_i\lambda_j-\partial_j\lambda_i, \tag{39}$$

include particles in curvilinear coordinates, charges in magnetic fields, uniformly rotating frames and every reduced system of Section 6; by (16) and (19) only the field strength enters their equations of motion. We seek functions quadratic in the velocities,

$$I=\frac{1}{2}K_{ij}(t,q)v^iv^j+K_i(t,q)v^i+k(t,q), \tag{40}$$

which include linear integrals ($K_{ij}=0$). Indices are raised with g^{ij}, ∇ is the covariant derivative of g_{ij} (in Cartesian coordinates simply ∂), and round brackets denote symmetrization over the enclosed indices.

Result 15 (Gauge-free determining system). *Along solutions of* (38), $D_tI=C_3+C_2+C_1+C_0$, *with*

$$C_3=\frac{1}{2}\nabla_{(l}K_{ij)}v^lv^iv^j, \tag{41}$$

$$C_2=\left(\frac{1}{2}\partial_tK_{ij}+\nabla_iK_j+K_i{}^mN_{mj}\right)v^iv^j, \tag{42}$$

$$C_1 = \left(\partial_t K_i + \partial_i k - K_i{}^m \partial_m V + K^m N_{mi}\right) v^i, \tag{43}$$

$$C_0 = \partial_t k - K^m \partial_m V. \tag{44}$$

Hence I *is a first integral if and only if each coefficient vanishes*:

$$\nabla_{(l} K_{ij)} = 0, \quad \frac{1}{2}\partial_t K_{ij} + \nabla_{(i} K_{j)} + K_{(i}{}^m N_{|m|j)} = 0,$$

$$\partial_i k = K_i{}^m \partial_m V - K^m N_{mi} - \partial_t K_i, \quad \partial_t k = K^m \partial_m V, \tag{45}$$

and, if K_{ij}, K_i *and* k *do not depend on* t, I *is a Lyapunov function for an equilibrium on a region about it where* I *is positive definite relative to the equilibrium and* $C_3 + C_2 + C_1 + C_0 \le 0$; *if they depend on* t, *the bounds uniform in time of item* (*d*) *below are required as well.*

Proof. Compute $D_t I$, insert (38) for a^j, and collect powers of v. The cubic part combines $\frac{1}{2}\partial_l K_{ij}$ with the Christoffel terms into $\frac{1}{2}\nabla_{(l} K_{ij)}$ (inertial correction); the quadratic part combines $\partial_t K_{ij}$, $\partial_i K_j$, its Christoffel terms and the contraction of K_{ij} with N (gyroscopic and dissipative correction); the linear and constant parts give (43) and (44). For a first integral each order must vanish separately, since the initial velocity is arbitrary. □

For time-independent coefficients, (45) says that K_{ij} is a Killing tensor and, for $K_{ij} = 0$, that K_i is a Killing vector [41] [42]; these depend only on the metric. For $N = \mathcal{F}$ the static system is equivalent to the determining equations used in [11] [12], here written without reference to a vector potential. The features that make it a practical tool are the following.

a) *Gauge freedom never appears*. Only N_{ij} enters, so the system can be solved for fields without a global vector potential (monopoles) and for Coriolis and reduced systems without returning to potentials. For linear integrals, comparison with Noether's $I = \xi^i p_i - F$ gives $K_i = \xi_i$ and $k = \xi^i \lambda_i - F$: the gauge-dependent pieces of Noether's theorem recombine into the gauge-independent k.

b) *The potential enters through an integrability condition*. The third equation of (45) asks that the right-hand side be a gradient. On a simply connected region this holds if and only if

$$\partial_j \left(K_i{}^m \partial_m V - K^m N_{mi} - \partial_t K_i\right) = \partial_i \left(K_j{}^m \partial_m V - K^m N_{mj} - \partial_t K_j\right), \tag{46}$$

a linear partial differential equation for V (and N) for each admissible K. For static V, $N = 0$ in the plane this is the classical Bertrand-Darboux equation [2] [42]; (46) extends it to magnetic, Coriolis, reduced, time-dependent and damped systems within the class (38). Once (46) holds, k follows by quadrature.

c) *The search is finite-dimensional at each step*. Killing vectors and tensors of a given metric form finite-dimensional spaces (in Euclidean space they are spanned by products of translation and rotation generators), so the top equation is solved once for the geometry and (46) is then a linear test on the forces.

d) *Inequalities are allowed*. For damped and controlled systems one does not require $D_t I = 0$ but $D_t I \le 0$: the same coefficients $C_0, \cdots, C_3$ are then re-

quired to form a nonpositive polynomial in v, so that Lyapunov functions are built from the same coefficient equations together with a definiteness condition (for quadratic I, a matrix inequality). These conditions yield the stability conclusions of the autonomous Example 12 only when the coefficients of I do not depend on t. If K_{ij}, K_i or k depend on t, pointwise positivity of I and $D_t I \le 0$ are not sufficient: with $x=(q-q_*,v)$ and an equilibrium q_* that is an equilibrium for all t, uniform stability requires bounds $W_1(x) \le I(t,x) \le W_2(x)$ by time-independent functions positive definite about $x=0$, and uniform asymptotic or exponential stability requires in addition a bound $D_t I \le -W_3(x)$ with W_3 time independent and positive definite [43].

Example 8 (Runge-Lenz vector by quadrature). For $V=-\kappa/r$ in Cartesian coordinates ($g_{ij}=m\delta_{ij}$, $N=0$) take the Killing tensor $K_{ij}=m^2\left[2(\boldsymbol{c}\cdot\boldsymbol{r})\delta_{ij}-c_i x_j-c_j x_i\right]$ for a constant vector $\boldsymbol{c}$, and $K_i=0$. Then $K_i{}^m\partial_m V=m\kappa\left[(\boldsymbol{c}\cdot\boldsymbol{r})x_i-c_i r^2\right]/r^3$, which is the gradient of $k=-m\kappa\,\boldsymbol{c}\cdot\boldsymbol{r}/r$. The integral is $I=\boldsymbol{c}\cdot\left[m\boldsymbol{v}\times(\boldsymbol{r}\times m\boldsymbol{v})-m\kappa\,\boldsymbol{r}/r\right]$, the Runge-Lenz vector [1] [44], found here without any guessing beyond the choice of Killing tensor.

Example 9 (Poincaré vector of a magnetic monopole, without a vector potential). For a charge e in the field $\boldsymbol{B}=g\,\boldsymbol{r}/r^3$, $N_{ij}=\mathcal{F}_{ij}=e\epsilon_{ijk}B_k$, and no vector potential exists on all of space with the origin removed. Take the rotational Killing vector $K_i=m(\boldsymbol{c}\times\boldsymbol{r})_i$ and $V=0$. The determining system gives $\partial_i k=-(\boldsymbol{c}\times\boldsymbol{r})_m\mathcal{F}_{mi}$, which integrates to $k=-eg\,\boldsymbol{c}\cdot\boldsymbol{r}/r$, and the conserved quantity is

$$\boldsymbol{J}=\boldsymbol{r}\times m\boldsymbol{v}-eg\frac{\boldsymbol{r}}{r}, \tag{47}$$

the Poincaré vector [45] [46]. Noether's theorem would need a vector potential with a string singularity and a compensating gauge term; the determining system needs neither.

Example 10 (Euler's two-centre problem: a hidden integral). A unit mass moves in the plane under $V=-\kappa_1/r_1-\kappa_2/r_2$, with fixed centres at $(\pm d,0)$, $r_{1,2}=\sqrt{(x\mp d)^2+y^2}$. Energy is the only obvious integral; there is no continuous point symmetry. Take the Killing tensor of the plane defined by $\frac{1}{2}K_{ij}v^i v^j=(xv_y-yv_x)^2+d^2v_x^2$ (rotation squared plus d^2 times translation squared), $K_i=0$. For this K, condition (46) becomes

$$xy(V_{yy}-V_{xx})+(x^2-y^2-d^2)V_{xy}+3xV_y-3yV_x=0, \tag{48}$$

which the two-centre potential satisfies identically; the quadrature then gives $k=2dx(\kappa_2/r_2-\kappa_1/r_1)$. The hidden integral

$$I=(xv_y-yv_x)^2+d^2v_x^2+2dx\left(\frac{\kappa_2}{r_2}-\frac{\kappa_1}{r_1}\right) \tag{49}$$

makes the problem integrable (Euler; separable in elliptic coordinates [2] [47]).

Equation (48) is also a classification tool: it is linear in V, so any superposition of its solutions retains a quadratic integral of the same type. For instance, an isotropic harmonic term $\frac{1}{2}\varpi^2\left(x^2+y^2\right)$ satisfies (48) and may be added to the two-centre potential without destroying integrability.

Example 11 (Ermakov-Lewis invariant of a driven oscillator). For $\ddot{q}=-\varpi(t)^2 q$ (unit mass, $V=\frac{1}{2}\varpi(t)^2 q^2$, $N=0$), energy is not conserved. In one dimension every K_{11} independent of q is Killing; take $K_{11}=2\rho(t)^2$. The quadratic condition gives $K_1=-2\rho\dot{\rho}q$; the linear condition gives $k=\left(\rho^2\varpi^2+\dot{\rho}^2+\rho\ddot{\rho}\right)q^2$; the constant condition holds if and only if $\rho^3\left(\ddot{\rho}+\varpi^2\rho\right)$ is constant. Normalizing this constant to 1 (possible by rescaling ρ whenever it is positive) gives the Ermakov-Pinney equation $\ddot{\rho}+\varpi^2\rho=\rho^{-3}$, and the result is the invariant

$$I=(\rho\dot{q}-\dot{\rho}q)^2+q^2/\rho^2 \tag{50}$$

of Ermakov [48] and of Lewis and Riesenfeld [49], the standard tool for parametrically excited systems. It is obtained here by the same three steps as the Runge-Lenz vector.

Example 12 (A strict Lyapunov function by determining equations). For the damped oscillator $\ddot{q}=-\zeta\dot{q}-\omega_0^2 q$ ($\zeta,\omega_0>0$), the energy gives only $D_t E=-\zeta v^2$, which is not strictly negative and so does not by itself give an exponential decay rate. Take the ansatz $K_{11}=1$, $K_1=\varepsilon q$ and choose k from the linear condition to remove the mixed term qv: $k=\frac{1}{2}\left(\omega_0^2+\varepsilon\zeta\right)q^2$. Then (42)-(44) give

$$I=\frac{1}{2}\dot{q}^2+\varepsilon q\dot{q}+\frac{1}{2}\left(\omega_0^2+\varepsilon\zeta\right)q^2,\quad D_t I=-(\zeta-\varepsilon)\dot{q}^2-\varepsilon\omega_0^2 q^2, \tag{51}$$

which is negative definite for $0<\varepsilon<\zeta$, while I is positive definite for $\varepsilon^2<\omega_0^2+\varepsilon\zeta$. This is the classical strict Lyapunov function used in control design [43], obtained here from the determining equations: the only choice is the trial term $K_1=\varepsilon q$, and k and $D_t I$ then follow. The same procedure applies unchanged to multi-degree-of-freedom systems with gyroscopic and damping matrices.

Remark. Higher-degree ansätze give the same structure: the top-degree condition is a Killing-tensor equation (pure inertial correction), each lower degree couples the previous coefficients to N and to ∂V, and the lowest degree is a gradient condition settled by the Poincaré lemma. Explicit time dependence also covers integrals such as Galilean boost charges.

11. Phase Space: Corrections as Poisson Brackets

Under (H5) we use $\left(q^i,p_i\right)$ and the Poisson bracket [50]

$$\{f,g\} := \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}, \quad D_t f = \frac{\partial f}{\partial t} + \{f,H\} \quad \text{(H6)}. \tag{52}$$

The convective correction on phase space is the bracket with H: $\{q^i,H\}$ and $\{p_i,H\}$ are the Velocity and Force entries of Table 2, and $D_t H = \partial_t H$ because $\{H,H\}=0$. The divergence of the Hamiltonian flow, $\partial_{q^i}\partial_{p_i}H - \partial_{p_i}\partial_{q^i}H$, vanishes by symmetry of mixed partials: Liouville's theorem.

11.1. What the Bracket Measures

Each g generates a family of canonical transformations with rate $X_g f := \{f,g\}$. The Jacobi identity $\{f,\{g,k\}\} + \text{cyclic} = 0$ is again symmetry of mixed partials (all remaining terms are differences of mixed second derivatives), and is equivalent to $X_g X_k - X_k X_g = -X_{\{g,k\}}$: *the bracket of two generators generates the two-route correction between their transformations*. A quantity I is conserved iff $\partial_t I + \{I,H\} = 0$, the phase-space form of Result 15, and the bracket of two integrals is an integral (Poisson's theorem). When n independent integrals are mutually in involution ($\{I_a,I_b\}=0$: all two-route corrections between their flows vanish), the flows commute and the system is integrable by quadratures in the sense of Liouville and Arnold [39]; for Example 10, H and I are such a pair.

11.2. Field Strength and Anholonomy in the Bracket

For (39), introduce the mechanical momentum $\kappa_i := g_{ij}v^j = p_i - \lambda_i$.

Result 16 (Magnetic bracket). *In the variables* (q,κ), $\{q^i,q^j\}=0$, $\{q^i,\kappa_j\}=\delta^i_j$, $\{\kappa_i,\kappa_j\}=\mathcal{F}_{ij}$, *and the Hamiltonian* $H_\kappa = \frac{1}{2}g^{ij}\kappa_i\kappa_j + V$ *does not contain* λ.

Proof. $\{\kappa_i,\kappa_j\} = -\{p_i,\lambda_j\} - \{\lambda_i,p_j\} = \partial_i\lambda_j - \partial_j\lambda_i$. □

Inertial corrections sit in H_κ, gyroscopic ones in the bracket. The Jacobi identity of this bracket reduces to $\partial_i\mathcal{F}_{jk} + \partial_j\mathcal{F}_{ki} + \partial_k\mathcal{F}_{ij} = 0$ ($\nabla\cdot\boldsymbol{B}=0$ in three dimensions): *Jacobi is Bianchi*, and if $\mathcal{F}$ comes from a λ it is symmetry of mixed partials one level up. For a monopole it fails at the origin, and the bracket is consistent only with the origin removed [51].

Anholonomy has the same phase-space form. For a time-independent frame and quasi-momenta $P_\alpha := B^i_\alpha p_i$, a direct calculation gives

$$\{P_\beta,P_\delta\} = -\gamma^\alpha{}_{\beta\delta}P_\alpha, \tag{53}$$

so for the rigid body ($\gamma=\epsilon$) one obtains the body angular momentum bracket $\{P_1,P_2\}=-P_3$ and cyclic: the Lie-Poisson structure underlying Euler's equations [25] [52]. Field strength, anholonomy and reduction all appear as noncommuting momenta. Table 4 classifies all the corrections met so far and shows where each of them lives on phase space.

Table 4. Classification of corrections and where each lives on phase space. All classes except the constitutive one are two-route corrections; constitutive terms are independent physical input (Section 4.3(e)).

Class	Origin	Failed	Examples	Phase-space form
Inertial	g_{ij}, T_0 depend on (t,q)	(H1), (H2)	Christoffel terms, centrifugal force	in H_κ, via g^{ij} and V
Gyroscopic	velocity-linear part of L not a gradient, Equation (19)	(H2) or (H4)	Coriolis, Euler, Lorentz forces	$\{\kappa_i,\kappa_j\}=\mathcal{F}_{ij}$; Jacobi $\Leftrightarrow$ Bianchi
Transport	frame moves, $\gamma^{\alpha}{}_{\beta 0}\neq 0$, Equation (13)	(H2)	rotating-frame derivative (22)	as gyroscopic
Anholonomy	velocity variables not time derivatives, Equation (12)	(H8)	Euler's rigid-body equations; nonholonomic terms, Equation (23)	$\{P_\beta,P_\delta\}=-\gamma^{\alpha}{}_{\beta\delta}P_\alpha$
Reduction	cyclic coordinate eliminated, Equation (30)	—	gyroscopic terms, effective potentials	as gyroscopic, with charge c
Constitutive	force not from a potential; mass flux	(H3), (H4)	friction, damping, actuation, rockets	not in the canonical bracket of the physical variables (Liouville); see Section 11.3
Convective	$D_t-\partial_t$	(always)	Hamilton's equations from Hamilton-Jacobi	bracket with H
Energy	$h-E=-(T_1+2T_0)$	(H7)	Jacobi integral	—

11.3. Central Charges, and Why Friction Does Not Fit the Canonical Bracket

For point symmetries ξ, η with gauge terms F_ξ, F_η, the charges $I_\xi=\xi^i p_i-F_\xi$ satisfy

$$\{I_\xi,I_\eta\}=-[\xi,\eta]^i\,p_i+\xi^i\partial_i F_\eta-\eta^i\partial_i F_\xi, \tag{54}$$

with $[\xi,\eta]^i=\xi^j\partial_j\eta^i-\eta^j\partial_j\xi^i$. A constant extra term is a *central charge*. For magnetic translations, $\{I_1,I_2\}=-e\boldsymbol{B}\cdot(\boldsymbol{c}_1\times\boldsymbol{c}_2)$ (minus e times the flux through the parallelogram of the two translations); for a Galilean boost and a translation of a free particle, $\{I_\xi,I_\eta\}=-m\boldsymbol{c}\cdot\boldsymbol{c}'$: the mass is the central charge of the Galilei algebra [53].

In the physical variables, constitutive corrections cannot be placed in the canonical Poisson bracket: the damped oscillator has phase-space flow of divergence $-\zeta$ in the variables (q,mv), whereas every flow that is Hamiltonian with respect to the canonical bracket preserves the canonical phase-space volume and is therefore divergence free in these variables. The argument is restricted to these variables, this bracket and this volume. Alternative variables, structures or enlarged descriptions require separate consideration: the Caldirola-Kanai description restores a Hamiltonian flow by changing the bracket to the time-dependent

$\{q, mv\} = e^{-\zeta t}$; Bateman's dual system doubles the degrees of freedom [54]; and metriplectic [55] [56] and port-Hamiltonian [57] structures supplement a Poisson bracket with a symmetric dissipative bracket or a resistive structure. The determining system of Result 15, which works directly with the equations of motion, does not need a bracket at all.

12. The Method as a Procedure

The results combine into the following procedure for a given system.

1) *Ingredients*. Write T, U and any nonconservative Q^{nc}; split $T = T_2 + T_1 + T_0$; collect all velocity-linear terms into (λ_0, λ_i). Choose velocity variables (10): coordinate velocities, body velocities, or inertial velocities resolved in a moving frame. For nonholonomic constraints, choose them so that the constraints read $\omega^c = 0$.

2) *Assumptions*. Record (H1)-(H8) used at each step.

3) *Corrections*. Compute the frame commutators (13) and write the master equation (15); read off inertial, gyroscopic, transport and anholonomy terms with Table 3. Impose constraints after differentiation; the constraint correction is (23).

4) *Reduce*. Eliminate cyclic coordinates with (30); the result is again of the form (39).

5) *Integrals and stability*. Use the energy balance and (37); then solve the determining system (45) degree by degree, testing the forces with the compatibility condition (46) and obtaining k by quadrature. For damped or controlled systems require $D_t I \le 0$ instead, with bounds uniform in time if I depends on t (Section 10.3(d)). Check involution with Poisson brackets.

6) *Lagrangian test*, if only equations of motion are given: (32)-(34), with a multiplier if necessary.

Every step is partial differentiation, an application of D_t, or a linear system of partial differential equations, so the procedure is directly implementable in computer algebra; all results and examples in this paper were checked symbolically in this way. It doubles as an audit of an existing derivation: an equation used beyond its assumptions produces a nonzero correction, and Table 1 and Table 4 name the missing term. Figure 3 condenses the procedure into a decision chart that shows which corrections a given problem needs.

13. Quick Reference

This section collects, without derivation, what is needed to apply the method. Table 5 fixes notation, Table 6 lists the master formulas with their assumptions, and Table 7 maps common types of problem to the steps of Section 12.

[Common errors and the terms they lose]

- $\partial T/\partial q^i$ set to zero outside (H1)-(H2): lost inertial correction (centrifugal, Christoffel terms), Equation (16), Example 1, Section 1.1.
- D_t replaced by ∂_t: lost convective correction; in Hamilton-Jacobi theory a spurious force (Example 7).

- p_i identified with π_i or mv^i: lost gyroscopic term; $p_i - \pi_i = \lambda_i$ for velocity-linear terms not contained in T (Result 6).
- h identified with E outside (H7): wrong energy integral; $h - E = -(T_1 + 2T_0)$ (Result 13).
- Nonholonomic constraints substituted before differentiating: lost $\gamma^c_{ad} P_c \omega^d$ (Result 7).
- Body or frame velocities treated as coordinate velocities: lost anholonomy and transport terms $\gamma^\alpha_{\beta\mu} P_\alpha \omega^\mu$ (Result 3).
- Coriolis matrix C not built from Christoffel symbols: loss of the skew-symmetry of $\dot{M} - 2C$ used in control (Result 4).
- Gauge term F dropped in Noether's theorem: wrong conserved quantity (pseudo-momentum, boost charges); use instead the gauge-free system (45).

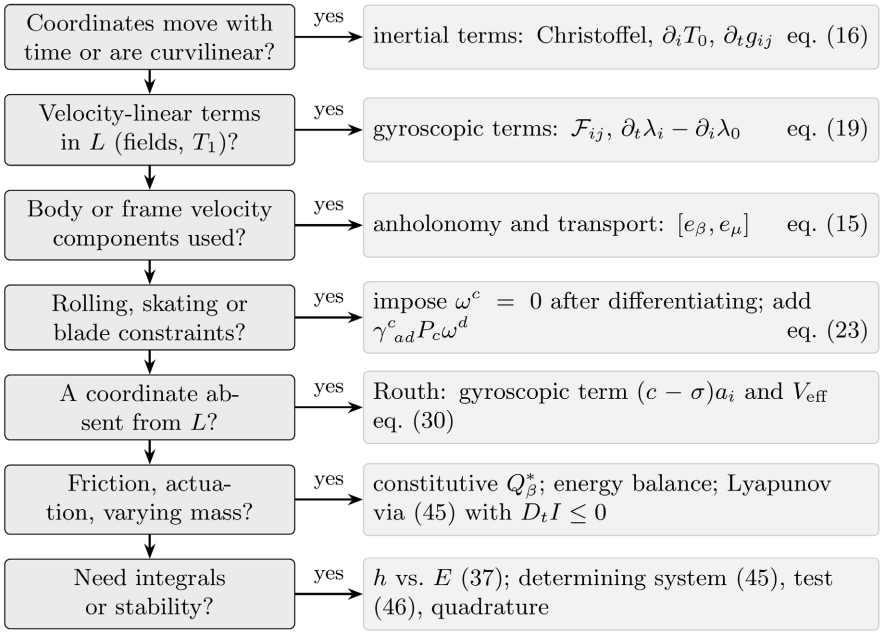


Figure 3. Decision chart: which corrections a problem needs. Work down the left column; every "yes" adds the term on the right. All answers are read off from the master equation (15).

Table 5. Notation.

Symbol	Meaning
q^i, v^i, a^i	coordinates, velocities, accelerations (independent until evaluated on a motion)
D_t; ∂_i	total time derivative (4); $\partial/\partial q^i$
$\mathcal{E}_i(f)$	Euler-Lagrange expression $D_t(\partial f/\partial v^i) - \partial f/\partial q^i$
g_{ij}, b_i, T_0	kinetic coefficients, $T = \frac{1}{2} g_{ij} v^i v^j + b_i v^i + T_0$
$\Gamma_{i,jk}$; ∇	Christoffel symbols of g_{ij}; covariant derivative

Continued

λ_i, λ_0; $\mathcal{F}_{ij}$	velocity-linear part of L; field strength $\partial_i\lambda_j - \partial_j\lambda_i$
N_{ij}	coefficient of all velocity-linear forces in (38) (gyroscopic and dissipative)
p_i; π_i; κ_i	canonical $\partial L/\partial v^i$; kinetic $\partial T/\partial v^i$; mechanical $g_{ij}v^j$ momentum
h; E	energy function $v^i p_i - L$; total energy $T+U$
$\omega^\alpha = A_i^\alpha v^i + A_0^\alpha$	velocity variables (quasi-velocities, generalized speeds)
e_α, e_0	frame vector fields (11); $B = A^{-1}$
$\gamma^\alpha_{\beta\delta}$; $\gamma^\alpha_{\beta 0}$	anholonomy; transport coefficients (12)
L^*; P_α	Lagrangian in velocity variables; momenta $\partial L^*/\partial\omega^\alpha$
ω^c, ω^a	constrained ($\omega^c = 0$) and free velocity variables
K_{ij}, K_i, k	coefficients of a trial integral (40)
$\{f,g\}$	Poisson bracket (52)

Table 6. Master formulas. "Assumes" lists the hypotheses of Section 3.2 that the formula needs beyond smoothness.

Purpose	Formula	Assumes	Where
Master equation	$\dot{P}_\beta - e_\beta L^* + \gamma^\alpha_{\beta\delta}P_\alpha\omega^\delta + \gamma^\alpha_{\beta 0}P_\alpha = Q^*_\beta$, $\left[e_\beta, e_\mu\right] = \gamma^\alpha_{\beta\mu}e_\alpha$	(H3)	(15)
Any coordinates	$g_{ij}a^j + \Gamma_{i,jk}v^j v^k = Q_i - \partial_t g_{ij}v^j + \beta_{ij}v^j - \partial_t b_i + \partial_i T_0$	(H3)	(16)
Robot form	$M\ddot{q} + C\dot{q} + \nabla U = \tau$, $C_{ij} = \Gamma_{i,jk}v^k$, $\dot{M} - 2C$ antisymmetric	(H2), (H3)	(17)
Velocity-linear terms	$\mathcal{E}_i\left(\lambda_j v^j + \lambda_0\right) = \partial_t\lambda_i - \partial_i\lambda_0 - \mathcal{F}_{ij}v^j$	—	(19)
Moving frame	$m\left(\dot{\boldsymbol{\omega}} + \boldsymbol{\Omega}\times\boldsymbol{\omega}\right) = \boldsymbol{F}$; equivalently (55)	(H3)	(22)
Nonholonomic shortcut	missing term $-\gamma^c_{ad}P_c\omega^d$; zero if constraints integrable	ideal constraints	(23)
Kane's method	$-F^*_a$ = free components of the master equation at $\omega^c = 0$	ideal constraints	(28)
Cyclic coordinate	$R_c = \frac{1}{2}g_{ij}v^i v^j + \left[\lambda_i + (c-\sigma)a_i\right]v^i - V - (c-\sigma)^2/2I_\chi$	form (29)	(30)
Gauge freedom	$L \sim L + D_t F(t,q)$	star-shaped region	Res. 9
Lagrangian exists?	Helmholtz conditions	—	(32)-(34)
Energy	$D_t h = v^i Q_i^{\mathrm{nc}} - \partial_t L$; $h - E = -(T_1 + 2T_0)$	(H4) for $h-E$	Res. 12, (37)
Point symmetry	$\delta_\xi L = D_t F \Rightarrow I = \xi^i p_i - F$	—	Res. 14
Integrals, Lyapunov	$\nabla_{(l}K_{ij)} = 0$, $\frac{1}{2}\partial_t K_{ij} + \nabla_{(i}K_{j)} + K_{(i}{}^m N_{|m|j)} = 0$, $\partial_i k = K_i{}^m\partial_m V - K^m N_{mi} - \partial_t K_i$, $\partial_t k = K^m\partial_m V$	form (38)	(45), (46)
Phase space	$D_t f = \partial_t f + \{f,H\}$; $\{\kappa_i,\kappa_j\} = \mathcal{F}_{ij}$; $\{P_\beta, P_\delta\} = -\gamma^\alpha_{\beta\delta}P_\alpha$	(H5)	Sec. 11

Table 7. Guide from problem type to method. "Corrections" are the terms that must appear in the equations of motion; "Integrals" are the conserved quantities (or Lyapunov functions) to look for first.

Problem	Variables	Corrections	Integrals	See
Particle in curvilinear coordinates	q^i, v^i	inertial (Christoffel, centrifugal)	$h = E$; Killing vectors of g	Ex. 1, App. A.1

Continued

Robot arm, linkage	joint angles	inertial; C from Christoffel symbols	energy; passivity	Res. 4, App. A.4
Rotating or accelerating frame	frame Cartesian, or frame-resolved inertial velocity	gyroscopic and inertial, or one transport term	Jacobi integral $h \neq E$	Res. 6, Sec. 4.3(c)
Charge in electromagnetic field	Cartesian	gyroscopic (Lorentz), only via $\boldsymbol{E}$, $\boldsymbol{B}$	$h = T + e\phi$; determining system with $\mathcal{F}$	Ex. 2, 9
Rigid body	body angular velocities	anholonomy ($\gamma = \epsilon$)	energy, $\lvert P \rvert^2$ (free body)	Ex. 3, App. A.3
Rolling, skating, vehicles	generalized speeds with $\omega^c = 0$	anholonomy; constraint-substitution term	energy (static constraints)	Res. 7, Ex. 4
Tops, gyrostats, central forces	coordinates with cyclic ones	reduction: gyroscopic term, V_{eff}	cyclic momenta; reduced system (39)	Res. 8, Ex. 5
Is the system integrable?	reduced coordinates	—	Killing tensors, test (46), involution	Ex. 8, 10, App. A.5
Driven, parametric	q, v, t	time-dependent V	time-dependent determining system	Ex. 11
Damping, control	any	constitutive Q^{nc}	energy balance; Lyapunov via $D_t I \leq 0$	Ex. 12
Equations given, L unknown	q, v, a	—	Helmholtz test with multiplier	Res. 10, Ex. 6
Separation of variables	(q, p)	convective	complete integral, Jacobi's theorem	Sec. 9

14. Discussion and Conclusions

The symmetry of mixed partial derivatives, written as three commutation rules and applied to operations that carry their assumptions, is, together with Newton's second law and the force law, enough to derive Lagrange's equations in moving coordinates, the Hamel and Kane equations, reduced equations of Routh type, the Helmholtz conditions, Hamilton's equations, Jacobi's theorem and Noether's theorem. Hamilton's principle explains why physical motions satisfy (9) but is not needed to derive them.

The organizing results are the commutator table (Table 1) and the master equation (Result 3). The table turns "what can go wrong" in a derivation built from the operations of Section 2.2 into a fixed list; the master equation turns "which terms belong in this equation" into writing L^* in the chosen velocity variables and computing the frame commutators $\left[e_\beta, e_\mu\right]$. Besides the commutator table, two further tools follow. First, the constraint-substitution correction (23) gives, for nonholonomic systems, the exact term lost by a common shortcut, together with

a Frobenius criterion for when the shortcut is safe; it also explains why Kane's method is immune to the error. Second, the determining system (45) finds first integrals without guessing symmetries and without choosing a gauge, accepts, within the class (38) with a time-independent kinetic metric, explicit time dependence and linear dissipative forces, and yields Lyapunov functions from the same coefficient equations; its compatibility condition (46) generalizes the Bertrand-Darboux equation. In addition, Routh reduction appears as a transfer of a cyclic coordinate into a gyroscopic field, so reduced systems inherit every tool for magnetic-type systems.

Relation to existing work. Each ingredient has antecedents, and the comparison below states what the present formulation adds to each.

Moving frames. Hamel's equations for rheonomic, affine quasi-velocities $\omega^{\alpha} = A_i^{\alpha} v^i + A_0^{\alpha}$, including the terms produced by the explicit time dependence and the offset A_0^{α}, are treated comprehensively by Papastavridis [8]; the theory of quasi-velocities in arbitrary configuration-dependent frames, with and without nonholonomic constraints, is developed by Bloch, Marsden and Zenkov [58], and the geometry of nonholonomic corrections by Bloch and co-workers [9] [10]. Result 3 does not add new terms to these equations. What it adds is a presentation in which the rheonomic terms are the commutator coefficients $\gamma^{\alpha}_{\beta 0}$ of the frame fields on extended configuration space, so that anholonomy and transport follow from the single formula (13), and the use of this form to place rotating-frame, rigid-body and vehicle equations in one commutator table. The substitution correction (23) likewise restates the classical Voronets-Chaplygin-Hamel terms [6] as one explicit product of constraint momentum and anholonomy, with the Frobenius test attached.

Integrals and Lyapunov functions. Determining equations for quadratic integrals with magnetic fields go back to [11] and its successors [12] [13], and for static systems (45) is equivalent to them. Time-dependent integrals have been sought by direct ansätze since Lewis and Leach [59], in a gauge-invariant formulation of time-dependent symmetry mappings by Katzin and Levine [60], and, for potentials of the form $\omega(t)V(q)$ on a general kinetic metric, with the Killing-tensor condition as the top-degree equation, by Mitsopoulos and Tsamparlis [61]; polynomial Lyapunov functions are constructed algorithmically by sum-of-squares programming [62]. Killing tensors and the Bertrand-Darboux equation are classical [41] [42]. What (45) adds is a single linear system, written only in terms of the force coefficients N_{ij} and the potential V, in which the gyroscopic, Coriolis, reduced, time-dependent and linearly damped cases of the class (38) are treated together, so that first integrals and Lyapunov candidates come from the same coefficient equations and the compatibility condition (46) extends the Bertrand-Darboux equation to all of them. Its scope is that stated at the start of Section 10.3.

Overall, what is new is the reduction of these ingredients to one mechanism and one equation, and their organization as a checkable procedure.

Limitations. The method finds integrals within a chosen polynomial degree; the absence of an integral of degree two does not exclude integrals of higher degree or non-polynomial ones, for which singularity (Painlevé-type) analysis and related methods are needed [42]. For nonholonomic systems the equations are exact, but conservation of quantities other than energy is not guaranteed by symmetry alone [10]. The determining system assumes a time-independent kinetic metric and forces of the form (38), that is, gradient forces plus forces linear in the velocity (Section 10.3), and time-dependent Lyapunov candidates need bounds uniform in time. Dissipative systems are outside the canonical Poisson framework in their physical variables (Section 11.3), although not outside the determining system; alternative variables, structures or enlarged descriptions require separate consideration.

Extensions. Natural next steps are determining systems for nonholonomic systems, where (23) supplies the extra terms; field theory, where the total derivatives D_μ obey the analogous commutation rules and yield the field-theoretic Euler operator and null Lagrangians by the same arguments [26] [27]; and numerical integration, where the leading error of a splitting method is the commutator of the vector fields being split [63], a two-route correction in discrete time. In thermodynamics, the Maxwell relations are the same symmetry of mixed partials applied to thermodynamic potentials [64].

For practical use, the method is condensed into **Figure 3**, Section 13 and **Appendix A**: a reader can go from the type of problem (**Table 7**) to the master formulas (**Table 6**) and take the kinetic data from **Appendix A**.

[The paper at a glance]

- **One identity** (Result 1), the symmetry of mixed partials, generates every structural correction in the paper; constitutive forces enter as independent physical input (Section 4.3(e)).
- **One master equation** (Result 3), valid in any moving frame, with or without constraints, contains every standard equation of motion as a special case (**Table 3**).
- **One commutator table** (**Table 1**) lists the structural corrections produced by the operations of Section 2.2 and names the physical term each produces.
- **One determining system** (Result 15) finds conserved quantities by linear algebra and quadrature, and Lyapunov functions from the same coefficients, without guessing symmetries or choosing a gauge, for systems with a time-independent kinetic metric.
- **Ready to use**: a six-step procedure (Section 12), a decision chart (**Figure 3**), master-formula tables (Section 13) and a catalogue of standard cases (**Appendix A**).

Data Availability

No new data were generated or analyzed in this study. All results can be reproduced from the equations provided in the paper.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A. Catalogue of Standard Cases

The entries below are ready to insert into step 1 of the procedure. All were checked symbolically.

A.1. Common Coordinate Systems

For a particle of mass m ($b_i = T_0 = 0$), the general Equation (16) gives:

Plane polar (r,θ), $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$:

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = Q_r, \quad m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = Q_\theta.$$

Cylindrical (R,ϕ,z), $T = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2\right)$:

$$m\left(\ddot{R} - R\dot{\phi}^2\right) = Q_R, \quad m\left(R^2\ddot{\phi} + 2R\dot{R}\dot{\phi}\right) = Q_\phi, \quad m\ddot{z} = Q_z.$$

Spherical (r,θ,ϕ), $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)$:

$$m\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2\right) = Q_r, \quad m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2\right) = Q_\theta,$$

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\sin^2\theta\dot{\phi}\right) = Q_\phi.$$

Here $Q_i = \boldsymbol{F}\cdot\partial\boldsymbol{r}/\partial q^i$ (for example $Q_\theta = rF_\theta$ in polar coordinates, with F_θ the physical component). All velocity-quadratic terms are inertial corrections; their removal is the error of Example 1.

A.2. General Moving Frame

Let the inertial position be $\boldsymbol{R}_0(t) + \boldsymbol{R}(t)\boldsymbol{\rho}$, where $\boldsymbol{R}(t)$ is a rotation, $\boldsymbol{\rho}$ are Cartesian coordinates in the moving frame, $\boldsymbol{\Omega}$ is the frame's angular velocity and $\boldsymbol{a}_0 := \boldsymbol{R}^{\mathrm{T}}\ddot{\boldsymbol{R}}_0$ the acceleration of its origin, both in frame components.

Coordinate-velocity route. $g_{ij} = m\delta_{ij}$, $\boldsymbol{b} = m\left(\boldsymbol{\Omega}\times\boldsymbol{\rho} + \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}}_0\right)$, $T_0 = \frac{1}{2}m\left|\boldsymbol{\Omega}\times\boldsymbol{\rho} + \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}}_0\right|^2$, and (16) gives [47]

$$m\ddot{\boldsymbol{\rho}} = \boldsymbol{F} - m\boldsymbol{a}_0 - 2m\boldsymbol{\Omega}\times\dot{\boldsymbol{\rho}} - m\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\boldsymbol{\rho}\right) - m\dot{\boldsymbol{\Omega}}\times\boldsymbol{\rho}. \tag{55}$$

The translational part of $\boldsymbol{b}$ is independent of $\boldsymbol{\rho}$ and has no curl, so it produces no Coriolis-type term; through $\partial_t\boldsymbol{b}$ and ∇T_0 it produces exactly the term $-m\boldsymbol{a}_0$.

Transport route. With $\boldsymbol{\omega} = \dot{\boldsymbol{\rho}} + \boldsymbol{\Omega}\times\boldsymbol{\rho} + \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}}_0$, $\gamma^{\alpha}_{\beta 0} = \epsilon_{\alpha\beta k}\Omega_k$ and the master equation is (22). Expanding $\dot{\boldsymbol{\omega}}$, with $\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{R}^{\mathrm{T}}\dot{\boldsymbol{R}}_0\right) = \boldsymbol{a}_0 - \boldsymbol{\Omega}\times\boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}}_0$, reproduces (55) term by term.

For $\dot{\boldsymbol{R}}_0 = 0$ and constant $\boldsymbol{\Omega}$, the Jacobi integral is $h = \frac{1}{2}m\left|\dot{\boldsymbol{\rho}}\right|^2 - \frac{1}{2}m\left|\boldsymbol{\Omega}\times\boldsymbol{\rho}\right|^2 + U$.

A.3. Quasi-Velocity Frames

Planar rigid body. With $q = (x,y,\theta)$ and velocities resolved in the body,

$u=\dot{x}\cos\theta+\dot{y}\sin\theta$, $w=-\dot{x}\sin\theta+\dot{y}\cos\theta$, $\Omega=\dot{\theta}$, the nonzero anholonomy coefficients are $\gamma^{u}_{w\Omega}=-\gamma^{u}_{\Omega w}=1$ and $\gamma^{w}_{u\Omega}=-\gamma^{w}_{\Omega u}=-1$. For any $T^{*}(u,w,\Omega)$ independent of q, (15) reads

$$\dot{P}_u-\Omega P_w=Q^{*}_u,\quad \dot{P}_w+\Omega P_u=Q^{*}_w,\quad \dot{P}_\Omega+uP_w-wP_u=Q^{*}_\Omega, \tag{56}$$

which covers sleighs, skates, wheeled and towed vehicles; a knife edge or non-slipping wheel at the reference point imposes $w=0$.

Rigid body in space. With Euler angles (ϕ,θ,ψ) in the z-x-z convention, the body angular velocities are

$$\omega^1=\dot{\phi}\sin\theta\sin\psi+\dot{\theta}\cos\psi,\quad \omega^2=\dot{\phi}\sin\theta\cos\psi-\dot{\theta}\sin\psi,\quad \omega^3=\dot{\phi}\cos\theta+\dot{\psi},$$

the anholonomy coefficients are $\gamma^{\alpha}_{\beta\delta}=\epsilon_{\alpha\beta\delta}$, and (15) is $\dot{\boldsymbol{P}}+\boldsymbol{\omega}\times\boldsymbol{P}=\boldsymbol{M}$ in body components, with $\boldsymbol{M}$ the torque (including potential torques $-e_\beta U$). For principal axes, $\boldsymbol{P}=(I_1\omega^1,I_2\omega^2,I_3\omega^3)$, giving Euler's equations.

A.4. Two-Link Planar Arm

For links of masses m_1,m_2, length l_1 of the first link, distances l_{c1},l_{c2} from the joints to the link mass centres, central moments of inertia I_1,I_2, and relative joint angles (θ_1,θ_2):

$$M_{11}=m_1l_{c1}^2+m_2\left(l_1^2+l_{c2}^2+2l_1l_{c2}\cos\theta_2\right)+I_1+I_2,$$

$$M_{12}=M_{21}=m_2\left(l_{c2}^2+l_1l_{c2}\cos\theta_2\right)+I_2,\quad M_{22}=m_2l_{c2}^2+I_2,$$

and, with $\eta:=-m_2l_1l_{c2}\sin\theta_2$, the Christoffel-based Coriolis matrix of (17) is

$$C=\begin{pmatrix}\eta\dot{\theta}_2 & \eta\left(\dot{\theta}_1+\dot{\theta}_2\right)\\ -\eta\dot{\theta}_1 & 0\end{pmatrix},$$

for which $\dot{M}-2C$ is antisymmetric [18].

A.5. Killing Tensors of the Plane and the Bertrand-Darboux Equation

For a unit mass in the Euclidean plane, every quadratic Killing tensor has the form

$$\begin{aligned}&\frac{1}{2}K_{ij}v^iv^j=sL_z^2+\mu_1L_zv_x+\mu_2L_zv_y+\nu_{11}v_x^2+\nu_{12}v_xv_y+\nu_{22}v_y^2,\\&L_z:=xv_y-yv_x,\end{aligned} \tag{57}$$

with six constants. For $N=0$ and static V, the compatibility condition (46) becomes

$$\begin{aligned}&\left(2sxy-\mu_1x+\mu_2y-\nu_{12}\right)\left(V_{xx}-V_{yy}\right)\\&-2\left[s\left(x^2-y^2\right)+\mu_1y+\mu_2x-\nu_{11}+\nu_{22}\right]V_{xy}\\&+3\left(2sy-\mu_1\right)V_x-3\left(2sx+\mu_2\right)V_y=0.\end{aligned} \tag{58}$$

Equation (48) is the case $s=1$, $\nu_{11}=d^2$, all other constants zero. Apart from the trivial case $K_{ij}\propto g_{ij}$, which reproduces the energy, the Killing tensor can be

brought by a Euclidean motion to one of four normal forms, and the potentials satisfying (58) are exactly those separable in Cartesian, polar, parabolic or elliptic coordinates [2] [42]. For a magnetic or Coriolis field in the plane, $N_{12}=-N_{21}=B(x,y)$, the term $-K^{m}N_{mi}$ must be added before testing compatibility, and the quadratic condition imposes the additional coupling between K_i, K_{ij} and B.

Appendix B. Dictionary to Differential Geometry

The method is written in components so that it can be used with multivariable calculus alone. For readers who prefer intrinsic language, Table B1 lists the corresponding geometric objects.

Table B1. Geometric meaning of the objects used in the paper.

In this paper	Geometric object
(t,q,v,a) independent; D_t	jet coordinates; total derivative (contact vector field) on the second jet bundle
Rules (5)	commutators of the total derivative with vertical and horizontal coordinate derivatives
g_{ij}; inertial corrections	kinetic metric; Levi-Civita connection (Christoffel symbols)
$\mathcal{F}_{ij}$; gyroscopic corrections	closed two-form $\mathrm{d}\lambda$ (magnetic form); Lorentz force
e_α, e_0; $\gamma^{\alpha}{}_{\beta\mu}$	moving frame on extended configuration space; its structure functions (anholonomy object)
$\gamma^{c}{}_{ad}$ on allowed directions	curvature of the nonholonomic (Ehresmann) connection [9]
Reduced field strength in (30)	curvature of the mechanical connection [24]
K_{ij}, K_i	Killing tensor, Killing vector of g
Commutation of Hamiltonian flows	Lie algebra of Hamiltonian vector fields; Jacobi identity