Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2026, 14(11), 4819-4841
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Bilinear Fractional Integral Operators Associated with Generalized Kernels and Their Commutators on Generalized Morrey Spaces

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How to cite this paper: Gou, J. and Wang, J.Q. (2026) Bilinear Fractional Integral Operators Associated with Generalized Kernels and Their Commutators on Generalized Morrey Spaces. *Journal of Applied Mathematics and Physics*, **14**, 4819-4841.
https://doi.org/10.4236/jamp.2026.1411235

Received: October 16, 2026
Accepted: November 23, 2026
Published: November 26, 2026

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Abstract

The aim of this paper is to investigate the boundedness of a bilinear fractional integral operator T_α with generalized kernels and its commutator T_{α,b_1,b_2} formed by $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, under assumption that the Lebesgue measurable function φ_1, φ_2 and φ belong to the class $\mathbb{W}_\tau$ and satisfies $\varphi_1\varphi_2 = \varphi$, we prove that the T_α on product of generalized Morrey spaces $\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)$ and product of Morrey spaces $\mathcal{Q}^{p_1,\lambda}(\mathbb{R}^n) \times \mathcal{Q}^{p_2,\lambda}(\mathbb{R}^n)$; furthermore, the authors show that the T_{α,b_1,b_2} is bounded from product of spaces $\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)$, and it is also bounded from product of spaces $\mathcal{Q}^{p_1,\lambda}(\mathbb{R}^n) \times \mathcal{Q}^{p_2,\lambda}(\mathbb{R}^n)$ into spaces $\mathcal{Q}^{p,\lambda}(\mathbb{R}^n)$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$ for $1 < p_1, p_2 < \frac{2n}{\alpha}$ and $0 < \alpha < 2n$.

Keywords

Bilinear Fractional Integral Operator with Generalized Kernels, Commutator, Space $\mathrm{BMO}(\mathbb{R}^n)$, Generalized Morrey Spaces

1. Introduction

In the study of local regularity for solutions of second-order quasilinear elliptic partial differential equations, C.B. Morrey [1] first introduced Morrey spaces, since then, the integral operators on Morrey space have begun to receive widespread (see, for example [2]-[4]). T. Mizuhara and E. Nakai [5] [6] respectively proposed generalized Morrey spaces, and investigated the boundedness of the

Hardy-Littlewood maximal operator M and the Calderón-Zygmund integral operators T on these spaces. Since then, many scholars have conducted research on the various generalized Morrey spaces, for example, in 2024, Lu and Tao [7] proved that the T_σ is bounded from product of generalized fractional weighted Morrey spaces $L^{p_1,\eta_1,\varphi}\left(w_1\right)\times L^{p_2,\eta_2,\varphi}\left(w_2\right)$ into spaces $L^{p,\eta,\varphi}\left(v_{\vec{w}}\right)$, and proved that the commutator $\left[b_1,b_2,T_\sigma\right]$ generated by b_1, $b_2\in \mathrm{BMO}\left(\mathbb{R}^n\right)$ and the T_σ is bounded from the product of generalized fractional weighted Morrey spaces $L^{p_1,\eta_1,\varphi}\left(\mathbb{R}^n\right)\times L^{p_2,\eta_2,\varphi}\left(\mathbb{R}^n\right)$ into spaces $L^{p,\eta,\varphi}\left(\mathbb{R}^n\right)$; in 2025, Yu and Sun [8] introduced the generalized mixed Morrey spaces $M_{\vec{q}}^{\vec{p}}\left(\mathbb{R}^n\right)$, and proved that the Hardy-Littlewood maximal operator M and the generalized fractional maximal operator M_ρ are bounded on spaces $M_{\vec{q}}^{\varphi}\left(\mathbb{R}^n\right)$; recently, M. A. Dakoury and J. Feuto [9] proved that a Calderón-Zygmund operator T is bounded on Hardy-Morrey spaces $\mathcal{H}^{p,q,\alpha}\left(\mathbb{R}^d\right)$. Further research about the bounded properties of integral operators in harmonic analysis on various generalized Morrey spaces can be seen in [10]-[15].

The boundedness of generalized fractional integral operators has also been widely concerned. Krantz [16] established the boundedness of the Riesz potential operator I^α on Hardy spaces $H^p\left(\mathbb{R}^n\right)$, this result was later extended to the full range $0<p<\dfrac{n}{\alpha}$, and the mapping properties of integral operators with more general kernels on local Hardy spaces were also investigated. L. Grafakos [17] first proposed the generalized multilinear fractional integral operator I_α and established its basic boundedness on the $L_p\left(\mathbb{R}^n\right)$. In an RD-space, Lu *et al.* [18] proved that a bilinear generalized fractional integral operator $\tilde{T}_\alpha$ is bounded from the product of generalized Morrey spaces $\mathcal{L}^{\varphi_1,p_1}\left(X\right)\times\mathcal{L}^{\varphi_2,p_2}\left(X\right)$ into spaces $\mathcal{L}^{\varphi,q}\left(X\right)$, it is also bounded from the product of spaces $\mathcal{L}^{\varphi_1,p_1}\left(X\right)\times\mathcal{L}^{\varphi_2,p_2}\left(X\right)$ into generalized weak Morrey spaces $W\mathcal{L}^{\varphi,q}\left(X\right)$. Lu and Tao [19] proved that the bilinear θ-type generalized fractional integral operator $\tilde{T}_{\alpha,\theta}$ is bounded from the product of spaces $L^{p_1}\left(\mu\right)\times L^{p_2}\left(\mu\right)$ into space $L^{q}\left(\mu\right)$, and also from the product of spaces $M_{p_1}^{q_1}\left(\mu\right)\times M_{p_2}^{q_2}\left(\mu\right)$ into space $M_{p}^{q}\left(\mu\right)$. More research about generalized fractional integral operators can be found in references [20]-[25].

In 2025, Lin *et al.* [26] defined a class of multilinear fractional integral operators with generalized kernels, established the boundedness of these operators on the weighted Lebesgue spaces $L^{q,\infty}\left(v_w^q\right)$ and weighted variable exponent Lebesgue spaces $L^{q,\infty}\left(v_{\vec{w}}^{q/p_0'}\right)$, and investigated the boundedness of multilinear commutators associated with spaces $\mathrm{BMO}\left(\mathbb{R}^n\right)$. Lu *et al.* [27] established the boundedness of a bilinear integral operator $\tilde{T}$ with generalized kernels on product of generalized fractional Morrey spaces $L^{p_1,\eta_1,\varphi}\left(\mathbb{R}^n\right)\times L^{p_2,\eta_2,\varphi}\left(\mathbb{R}^n\right)$ and product of generalized fractional weighted Morrey spaces $L_{w_1}^{p_1,\eta_1,\varphi}\left(\mathbb{R}^n\right)\times L_{w_2}^{p_2,\eta_2,\varphi}\left(\mathbb{R}^n\right)$, and product of grand generalized fractional weighted Morrey spaces $L_{w_1}^{p_1,\eta_1,\varphi,\theta,\sigma}\left(\mathbb{R}^n\right)\times L_{w_2}^{p_2,\eta_2,\varphi,\theta,\sigma}\left(\mathbb{R}^n\right)$. Recently, Wang and Lu [28] establish the boundedness of a bilinear integral operator T with generalized kernels and its com-

mutator T_{b_1,b_2} on product of generalized variable Morrey spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}\left(\mathbb{R}^n\right)\times\mathcal{M}^{p_2(\cdot),\varphi_2}\left(\mathbb{R}^n\right)$.

Motivated by the above results, in this paper, we prove that the bilinear fractional integral operator T_α with generalized kernels and its commutator T_{α,b_1,b_2} formed by $b_1,b_2\in\mathrm{BMO}\left(\mathbb{R}^n\right)$ and the T_α are bounded from product of generalized Morrey spaces $\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)\times\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)$ into spaces $\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)$, and they are also bounded from the product of Morrey spaces $\mathcal{Q}^{p_1,\lambda}\left(\mathbb{R}^n\right)\times\mathcal{Q}^{p_2,\lambda}\left(\mathbb{R}^n\right)$ into spaces $\mathcal{Q}^{p,\lambda}\left(\mathbb{R}^n\right)$.

Before stating the organization of this article, we need to recall some necessary notion. The following definition of spaces $\mathrm{BMO}\left(\mathbb{R}^n\right)$ is from [29].

Definition 1.1. A real-valued function $b\in L^1_{loc}\left(\mathbb{R}^n\right)$ is said to be in the space $BMO\left(\mathbb{R}^n\right)$ if

$$\|b\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}=\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B\left|b(x)-b_B\right|\mathrm{d}y,$$

where b_B represents the mean value of the function b over the ball B, *i.e.*,

$$b_B=\frac{1}{|B|}\int_B b(y)\mathrm{d}y.$$

The following definition of the bilinear fractional integral operator with generalized kernels is from [26].

Definition 1.2. Let $0<\alpha<2n$ and $K_\alpha(x,y_1,y_2)$ be a locally integrable function defined away from the diagonal $x=y_1=y_2$ in $\left(\mathbb{R}^n\right)^3$, which is called the bilinear generalized fractional integral kernel, if there exists a constant $A>0$ such that,

1) for all $x,y_1,y_2\in\mathbb{R}^n$ with $x\neq y_i$, $i=1,2$,

$$\left|K_\alpha(x,y_1,y_2)\right|\le\frac{A}{\left(|x-y_1|+|x-y_2|\right)^{2n-\alpha}}. \tag{1.1}$$

2) for all $x,x',y_1,y_2\in\mathbb{R}^n$ with $x\neq x'$, for $k\in\mathbb{Z}^+$,

$$\left(\int_{\left(Q\left(x,2^{k+2}\sqrt{2n}|x-x'|\right)\right)^2\setminus\left(Q\left(x,2^{k+1}\sqrt{2n}|x-x'|\right)\right)^2}\left|K_\alpha(x,y_1,y_2)-K_\alpha(x',y_1,y_2)\right|^{p_0}\mathrm{d}y_1\mathrm{d}y_2\right)^{\frac{1}{p_0}} \tag{1.2}$$
$$\le CC_k2^{k\left(\alpha-\frac{2n}{p_0'}\right)}\left|x-x'\right|^{\alpha-\frac{2n}{p_0'}},$$

where $Q\left(x,2^{k+2}\sqrt{2n}|x-x'|\right)$ denotes the cube centered at x with the side length $2^{k+2}\sqrt{2n}|x-x'|$, (p_0,p_0') is a pair of positive numbers satisfying $\frac{1}{p_0}+\frac{1}{p_0'}=1$, $1<p_0<\infty$ and C_k is a positive constant depending on k.

3) for all $x,x',y_1,y_2\in\mathbb{R}^n$ with $y_1\neq y_1'$, for $k\in\mathbb{Z}^+$,

$$\left(\int_{\left(Q\left(y_1,2^{k+2}\sqrt{2n}|y_1-y_1'|\right)\right)^2\setminus\left(Q\left(y_1,2^{k+1}\sqrt{2n}|y_1-y_1'|\right)\right)^2}\left|K_\alpha(x,y_1,y_2)-K_\alpha(x,y_1',y_2)\right|^{p_0}\mathrm{d}x\mathrm{d}y_2\right)^{\frac{1}{p_0}} \tag{1.3}$$
$$\le CC_k2^{k\left(\alpha-\frac{2n}{p_0'}\right)}\left|y_1-y_1'\right|^{\alpha-\frac{2n}{p_0'}},$$

where $Q\left(y_1, 2^{k+2}\sqrt{2n}\left|y_1-y_1'\right|\right)$ denotes the cube centered at y_1 with the side length $2^{k+2}\sqrt{2n}\left|y_1-y_1'\right|$, $\left(p_0, p_0'\right)$ is a pair of positive numbers satisfying $\frac{1}{p_0}+\frac{1}{p_0'}=1$, $1<p_0<\infty$ and C_k is a positive constant depending on k.

4) for all $x, x', y_1, y_2 \in \mathbb{R}^n$ with $y_2 \neq y_2'$, for $k \in \mathbb{Z}^+$,

$$\begin{aligned}&\left(\int_{\left(Q\left(y_2, 2^{k+2}\sqrt{2n}\left|y_2-y_2'\right|\right)\right)^2 \setminus \left(Q\left(y_2, 2^{k+1}\sqrt{2n}\left|y_2-y_2'\right|\right)\right)^2}\left|K_\alpha\left(x, y_1, y_2\right)-K_\alpha\left(x, y_1, y_2'\right)\right|^{p_0} \mathrm{d}x\mathrm{d}y_1\right)^{\frac{1}{p_0}}\\ &\le CC_k 2^{k\left(\alpha-\frac{2n}{p_0'}\right)}\left|y_2-y_2'\right|^{\alpha-\frac{2n}{p_0'}},\end{aligned} \tag{1.4}$$

where $Q\left(y_2, 2^{k+2}\sqrt{2n}\left|y_2-y_2'\right|\right)$ denotes the cube centered at y_2 with the side length $2^{k+2}\sqrt{2n}\left|y_2-y_2'\right|$, $\left(p_0, p_0'\right)$ is a pair of positive numbers satisfying $\frac{1}{p_0}+\frac{1}{p_0'}=1$, $1<p_0<\infty$ and C_k is a positive constant depending on k.

The following definition of the bilinear fractional integral operator T_α with generalized kernel is from [26].

Definition 1.3. A bilinear operator $T_\alpha: \mathscr{S}\left(\mathbb{R}^n\right)\times\mathscr{S}\left(\mathbb{R}^n\right)\to\mathscr{S}'\left(\mathbb{R}^n\right)$ is called a bilinear fractional integral operator with generalized kernels K satisfying (1.1)-(1.2)-(1.3)-(1.4) if, for any $x \notin \left(\operatorname{supp}\left(f_1\right)\cap\operatorname{supp}\left(f_2\right)\right)$ and any $f_1, f_2 \in \mathscr{S}\left(\mathbb{R}^n\right)$,

$$T_\alpha\left(f_1, f_2\right)(x)=\int_{\left(\mathbb{R}^n\right)^2} K_\alpha\left(x, y_1, y_2\right) f_1\left(y_1\right) f_2\left(y_2\right)\mathrm{d}y_1\mathrm{d}y_2, \tag{1.5}$$

and the T_α can extend to product of spaces $L^{s_1}\left(\mathbb{R}^n\right)\times L^{s_2}\left(\mathbb{R}^n\right)$ into $L^{\frac{sn}{n-\alpha s},\infty}\left(\mathbb{R}^n\right)$, where $\frac{1}{s}=\frac{1}{s_1}+\frac{1}{s_2}$ for fixed $1\le s_1, s_2 \le p_0'$.

Given $b_1, b_2 \in \mathrm{BMO}\left(\mathbb{R}^n\right)$, the commutator T_{α, b_1, b_2} formed by b_1, b_2 and the T_α is defined by

$$\begin{aligned}&T_{\alpha, b_1, b_2}\left(f_1, f_2\right)(x)\\ &=\int_{\left(\mathbb{R}^n\right)^2} K\left(x, y_1, y_2\right)\left(b_1(x)-b_1\left(y_1\right)\right)\left(b_2(x)-b_2\left(y_2\right)\right) f_1\left(y_1\right) f_2\left(y_2\right)\mathrm{d}y_1\mathrm{d}y_2\\ &=b_1(x) b_2(x) T_\alpha\left(f_1, f_2\right)(x)-b_1(x) T_\alpha\left(f_1, b_2(\cdot) f_2\right)(x)\\ &\quad -b_2(x) T_\alpha\left(b_1(\cdot) f_1, f_2\right)(x)+T_\alpha\left(b_1(\cdot) f_1, b_2(\cdot) f_2\right)(x).\end{aligned} \tag{1.6}$$

We now recall the following notion of a generalized Morrey space $\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)$ introduced in [6].

Definition 1.4. Let $1\le p<\infty$ and a Lebesgue measurable function $\varphi: \mathbb{R}^n\times(0,\infty)\to(0,\infty)$, the generalized Morrey space $\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)$ consists of all locally p-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}=\sup_{x\in\mathbb{R}^n, r>0}\frac{1}{\varphi(x, r)}\left(\int_{B(x,r)}\left|f(y)\right|^p \mathrm{d}y\right)^{\frac{1}{p}}<\infty, \tag{1.7}$$

where $B(x, r)$ denotes the open ball centered at x with radius r.

Remark 1.5. 1) If we take $\varphi(r)=r^{\lambda}\ (0\le\lambda<n)$, then the space $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ reduces to the classical Morrey space $\mathcal{Q}^{p,\lambda}(\mathbb{R}^n)$ introduced in [1], *i.e.*, let $1\le p<\infty$ and $0\le\lambda<n$, then the Morrey space norm $\|\cdot\|_{\mathcal{Q}^{p,\lambda}(\mathbb{R}^n)}$ is defined by

$$\|f\|_{\mathcal{Q}^{p,\lambda}(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0}r^{-\frac{\lambda}{p}}\left(\int_{B(x,r)}|f(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}<\infty.$$

2) If we take $\varphi(x,r)=1$, then the space $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ is just the Lebesgue space $L^p(\mathbb{R}^n)$.

We now define a class of functions as follows in [19] [30].

Definition 1.6. Let $0<\tau<2$. A Lebesgue measurable function $\varphi(\cdot,\cdot)$ defined on $\mathbb{R}^n\times(0,\infty)$ is said to belong to the class $\mathbb{W}_{\tau}$ if there exists some positive constant C_1 such that, for all balls $B(x,r)\subset\mathbb{R}^n$

$$\sum_{k=0}^{\infty}\frac{|2B|^{\tau}}{|2^{k+1}B|^{\tau}}\varphi(x,2^{k+1}r)\le C_1\varphi(x,r),\tag{1.8}$$

and there exists a positive constant C_2, such that

$$\varphi(x,r)\ge C_2,\ \text{for all } r\ge 1.$$

The main structure of this article is as follows. In Section 2, we prove that the T_{α} is bounded from the product of generalized Morrey spaces $\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)\times\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)$ into spaces $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$, and it is also bounded from the product of Morrey spaces $\mathcal{Q}^{p_1,\lambda}(\mathbb{R}^n)\times\mathcal{Q}^{p_2,\lambda}(\mathbb{R}^n)$ into spaces $\mathcal{Q}^{p,\lambda}(\mathbb{R}^n)$, where Lebesgue measurable functions $\varphi_i\,(i=1,2)$ belong to the class $\mathbb{W}_{\tau}$ and satisfy $\varphi_1\varphi_2=\varphi$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$ for $0<\alpha<2n$. In Section 3, we show that the commutator T_{α,b_1,b_2} associated with spaces $\mathrm{BMO}(\mathbb{R}^n)$ is bounded from product of spaces $\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)\times\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)$ into spaces $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$, and it is also bounded from the product of spaces $\mathcal{Q}^{p_1,\lambda}(\mathbb{R}^n)\times\mathcal{Q}^{p_2,\lambda}(\mathbb{R}^n)$ into spaces $\mathcal{Q}^{p,\lambda}(\mathbb{R}^n)$.

Finally, we make some conventions on notation. Throughout this paper, we use the letter C to represent a positive constant which is independent of the main parameters, but it varies from line to line. Furthermore, for any given $p\in(1,\infty)$, if $\frac{1}{p}+\frac{1}{p'}=1$, then we use p' to represent the conjugate index of p. For any subset $E\subset\mathbb{R}^n$, χ_E represents its characteristic function.

2. Estimate for T_{α} on Product of Spaces $\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)\times\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)$

In this section, we show that the bilinear fractional integral operator T_{α} is bounded from the product of generalized Morrey spaces

$\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)\times\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)$ into the space $\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)$, and it is also bounded from the product of Morrey spaces $\mathcal{Q}^{p_1,\lambda}\left(\mathbb{R}^n\right)\times\mathcal{Q}^{p_2,\lambda}\left(\mathbb{R}^n\right)$ into the space $\mathcal{Q}^{p,\lambda}\left(\mathbb{R}^n\right)$.

It is now position to state the main theorems of this section as follows.

Theorem 2.1. *Let* $0<\alpha<2n$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$ *with* $1<p_1,p_2<\frac{2n}{\alpha}$, *and* φ_1,φ_2 *and* φ *are Lebesgue measurable functions defined as in Definition* 1.6 *and meet* $\varphi_1\varphi_2=\varphi$. *Suppose that* T_α *is a bilinear fractional integral operators with generalized kernel defined as in* (1.5). *Then there exists a positive constant* C *such that, for all* $f_i\in\mathcal{M}^{p_i,\varphi_i}\left(\mathbb{R}^n\right)$, $i=1,2$,

$$T_\alpha\left(f_1,f_2\right)_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}.$$

Theorem 2.2 *Let* $0<\alpha<2n$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$ *with* $1<p_1,p_2<\frac{2n}{\alpha}$ *and* $0\le\lambda<n$. *Suppose that* T_α *is a bilinear fractional integral operators with generalized kernel defined as in* (1.5). *Then there exists some positive constant* C *such that, for all* $f_i\in\mathcal{Q}^{p_i,\lambda}\left(\mathbb{R}^n\right)$, $i=1,2$,

$$T_\alpha\left(f_1,f_2\right)_{\mathcal{Q}^{p,\lambda}\left(\mathbb{R}^n\right)}\le C\left\|f_1\right\|_{\mathcal{Q}^{p_1,\lambda}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{Q}^{p_2,\lambda}\left(\mathbb{R}^n\right)}.$$

To prove the above theorems, we should recall the following lemmas [26] [28], respectively.

Lemma 2.3. *Let* $\tau\in(0,1)$ *and* $\varphi\in\mathbb{W}_\tau$. *Then there exists a positive constant* C *such that, for any* $x\in\mathbb{R}^n$ *and* $r>0$, *the following inequality*

$$\varphi(x,2r)\le C\varphi(x,r)$$

holds. In other words, the $\varphi(x,r)$ is a doubling function on the second variable r.

Lemma 2.4. *Let* $0<\alpha<2n$, $1<p_1,p_2<\frac{2n}{\alpha}$ *and* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$. *Suppose that* T_α *defined as in* (1.5) *is bounded from product of spaces* $L^{s_1}\left(\mathbb{R}^n\right)\times L^{s_2}\left(\mathbb{R}^n\right)$ *into the space* $L^{\frac{sn}{n-s\alpha},\infty}\left(\mathbb{R}^n\right)$ *for* $1\le s_1,s_2\le q'$. *Then there exists a positive constant* C *such that, for all* $f_i\in L^{p_i}\left(\mathbb{R}^n\right)$, $i=1,2$,

$$\left\|T_\alpha\left(f_1,f_2\right)\right\|_{L^p\left(\mathbb{R}^n\right)}\le C\left\|f_1\right\|_{L^{p_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{L^{p_2}\left(\mathbb{R}^n\right)}.$$

Proof of Theorem 2.1. Let $B=B(x,r)$ be the ball centered at $x\in\mathbb{R}^n$ with radius $r>0$. And decompose functions f_i as

$$f_i=f_i^0+f_i^\infty=f_i\chi_{(2B)}+f_i\chi_{\mathbb{R}^n\setminus(2B)},\ i=1,2.$$

Then, by the Minkowski's inequality, write

$$\begin{aligned}\left\|T_\alpha\left(f_1,f_2\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}&\le\left\|T_\alpha\left(f_1^0,f_2^0\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}+\left\|T_\alpha\left(f_1^0,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\&\quad+\left\|T_\alpha\left(f_1^\infty,f_2^0\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}+\left\|T_\alpha\left(f_1^\infty,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\&=\mathrm{D}_1+\mathrm{D}_2+\mathrm{D}_3+\mathrm{D}_4.\end{aligned}$$

Applying (1.5), $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$, $\varphi_1\varphi_2=\varphi$, Lemmas 2.3 and 2.4, we obtain

$$\begin{aligned}
\mathrm{D}_1 &= \sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left(\int_{B(x,r)}\left|T_\alpha\left(f_1^0,f_2^0\right)(y)\right|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\\
&\le C\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left(\int_{\mathbb{R}^n}\left|T_\alpha\left(f_1^0,f_2^0\right)(y)\right|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\\
&\le C\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}\frac{1}{\varphi_1(x,2r)}\left(\int_{B(x,2r)}\left|f_1(y_1)\right|^{p_1}\mathrm{d}y_1\right)^{\frac{1}{p_1}}\\
&\quad\times\frac{1}{\varphi_2(x,2r)}\left(\int_{B(x,2r)}\left|f_2(y_2)\right|^{p_2}\mathrm{d}y_2\right)^{\frac{1}{p_2}}\\
&\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi(x,2r)}{\varphi(x,r)}\\
&\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}.
\end{aligned}$$

To estimate D_2, we first consider the $\left|T_\alpha\left(f_1^0,f_2^\infty\right)(y)\right|$. For any $y\in B(x,r)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$ by applying Hölder's inequality, (1.1), (1.5) and (1.7), we have

$$\begin{aligned}
&\left|T_\alpha\left(f_1^0,f_2^\infty\right)(y)\right|\\
&\le C\int_{\left(\mathbb{R}^n\right)^2}\frac{\left|f_1^0(y_1)\right|\left|f_2^\infty(y_2)\right|}{\left(|x-y_1|+|x-y_2|\right)^{2n-\alpha}}\mathrm{d}y_1\mathrm{d}y_2\\
&\le C\int_{2B}\left|f_1(y_1)\right|\mathrm{d}y_1\int_{\mathbb{R}^n\setminus(2B)}\frac{\left|f_2(y_2)\right|}{\left|y-y_2\right|^{2n-\alpha}}\mathrm{d}y_2\\
&\le C\left(\int_{2B}\left|f_1(y_1)\right|^{p_1}\mathrm{d}y_1\right)^{\frac{1}{p_1}}\left|2B\right|^{1-\frac{1}{p_1}}\left(\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus\left(2^kB\right)}\frac{\left|f_2(y_2)\right|}{\left|y-y_2\right|^{2n-\alpha}}\mathrm{d}y_2\right)\\
&\le C\varphi_1(x,2r)\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left|2B\right|^{1-\frac{1}{p_1}}\left(\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\int_{2^{k+1}B}\left|f_2(y_2)\right|\mathrm{d}y_2\right)\\
&\le C\varphi_1(x,2r)\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left|2B\right|^{1-\frac{1}{p_1}}\\
&\quad\times\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\left(\int_{2^{k+1}B}\left|f_2(y_2)\right|^{p_2}\mathrm{d}y_2\right)^{\frac{1}{p_2}}\left|2^{k+1}B\right|^{1-\frac{1}{p_2}}\right]\\
&\le C\varphi_1(x,2r)\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\left|2B\right|^{1-\frac{1}{p_1}}\\
&\quad\times\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\left|2^{k+1}B\right|^{1-\frac{1}{p_2}}\right]
\end{aligned}$$

$$\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\varphi_1\left(x,2r\right)\left|2B\right|^{1-\frac{1}{p_1}}$$

$$\times\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\right],$$

by this (1.7), (1.8), $\varphi_1\varphi_2=\varphi$ and Lemma 2.3, we deduce

$$\mathrm{D}_2\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi\left(x,r\right)}\left|2B\right|^{\frac{1}{p}}\varphi_1\left(x,2r\right)\left|2B\right|^{1-\frac{1}{p_1}}$$

$$\times\left[\sum_{k=1}^{\infty}\frac{1}{\left|B\left(x,2^{k+1}r\right)\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\right]$$

$$\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1\left(x,2r\right)}{\varphi\left(x,r\right)}$$

$$\times\left[\sum_{k=1}^{\infty}\frac{\left|2B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\right]$$

$$\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1\left(x,2r\right)\varphi_2\left(x,2r\right)}{\varphi\left(x,r\right)}$$

$$\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}.$$

With an argument similar to that used in D_2, it is easy to obtain

$$\mathrm{D}_3\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}.$$

To estimate D_4, we first consider the $\left|T_\alpha\left(f_1^\infty,f_2^\infty\right)\left(y\right)\right|$. For any $y\in B\left(x,r\right)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$, by applying Hölder's inequality, (1.1), (1.5) and (1.7), we have

$$\left|T_\alpha\left(f_1^\infty,f_2^\infty\right)\left(y\right)\right|$$

$$\le C\int_{\left(\mathbb{R}^n\right)^2}\frac{\left|f_1^\infty\left(y_1\right)\right|\left|f_2^\infty\left(y_2\right)\right|}{\left(\left|x-y_1\right|+\left|x-y_2\right|\right)^{2n-\alpha}}\mathrm{d}y_1\mathrm{d}y_2$$

$$\le\int_{\mathbb{R}^n\setminus\left(2B\right)}\frac{\left|f_1\left(y_1\right)\right|}{\left|y-y_1\right|^{n-\frac{\alpha}{2}}}\mathrm{d}y_1\int_{\mathbb{R}^n\setminus\left(2B\right)}\frac{\left|f_2\left(y_2\right)\right|}{\left|y-y_2\right|^{n-\frac{\alpha}{2}}}\mathrm{d}y_2$$

$$\le\left[\sum_{k=1}^{\infty}\frac{1}{\left(2^kr\right)^{n-\frac{\alpha}{2}}}\int_{2^{k+1}B\setminus\left(2^kB\right)}\left|f_1\left(y_1\right)\right|\mathrm{d}y_1\right]\left[\sum_{l=1}^{\infty}\frac{1}{\left(2^lr\right)^{n-\frac{\alpha}{2}}}\int_{2^{l+1}B\setminus\left(2^lB\right)}\left|f_2\left(y_2\right)\right|\mathrm{d}y_2\right]$$

$$\le\left[\sum_{k=1}^{\infty}\frac{1}{\left(2^kr\right)^{n-\frac{\alpha}{2}}}\left(\int_{2^{k+1}B}\left|f_1\left(y_1\right)\right|^{p_1}\mathrm{d}y_1\right)^{\frac{1}{p_1}}\left|2^{k+1}B\right|^{1-\frac{1}{p_1}}\right]$$

$$\times\left[\sum_{l=1}^{\infty}\frac{1}{\left(2^lr\right)^{n-\frac{\alpha}{2}}}\left(\int_{2^{l+1}B}\left|f_2\left(y_2\right)\right|^{p_2}\mathrm{d}y_2\right)^{\frac{1}{p_2}}\left|2^{l+1}B\right|^{1-\frac{1}{p_2}}\right]$$

$$\le C\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2,\varphi_2}(\mathbb{R}^n)}\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{1-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\left|2^{k+1}B\right|^{1-\frac{1}{p_1}}\right]$$
$$\times\left[\sum_{l=1}^{\infty}\frac{1}{\left|2^lB\right|^{1-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\left|2^{l+1}B\right|^{1-\frac{1}{p_2}}\right]$$
$$\le C\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]$$
$$\times\left[\sum_{l=1}^{\infty}\frac{1}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right],$$

from this, Lemma 2.3, (1.7) and (1.8), it follows that

$$\mathrm{D}_4\le C\|f_1\|_{M^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}|2B|^{\frac{1}{p}}$$
$$\times\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]\times\left[\sum_{l=1}^{\infty}\frac{1}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right]$$
$$\le C\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{M^{p_2,\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}\frac{|2B|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]$$
$$\times\left[\sum_{l=1}^{\infty}\frac{|2B|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right]$$
$$\le C\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}$$
$$\le C\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)},$$

which, combining all estimates for D_1, D_2, D_3 and D_4, we complete the proof of Theorem 2.1. □

Proof of Theorem 2.2. By applyng Remark 1.5 (1) and an argument similar to that used in the proof of Theorem 2.1, it is not difficult to show that Theorem 2.2 holds. Hence, to avoid repetition, here we omit the proof. □

3. Estimate for T_{α,b_1,b_2} on Product of Spaces $\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)\times\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)$

In this section, we show that the commutator T_{α,b_1,b_2} associated with spaces $\mathrm{BMO}(\mathbb{R}^n)$ is bounded from product of spaces $\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)\times\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)$, and it is also bounded from the product of Morrey spaces $\mathcal{Q}^{p_1,\lambda}(\mathbb{R}^n)\times\mathcal{Q}^{p_2,\lambda}(\mathbb{R}^n)$

into the space $\mathcal{Q}^{p,\lambda}\left(\mathbb{R}^n\right)$.

It is now position to state the main theorems of this section as follows.

Theorem 3.1. *Let* $b_1, b_2 \in \mathrm{BMO}\left(\mathbb{R}^n\right)$, $0<\alpha<2n$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$ *with* $1<p_1,p_2<\frac{2n}{\alpha}$, *Lebesgue measurable functions* $\varphi_i\left(i=1,2\right)$ *meet*

$$\sum_{k=0}^{\infty}(k+1)\frac{|2B|^{\tau}}{\left|2^{k+1}B\right|^{\tau}}\varphi_i\left(x,2^{k+1}r\right)\le C\varphi_i\left(x,r\right), \text{ for } 0<\tau<2, \tag{3.1}$$

and $\varphi_1\varphi_2=\varphi$. Suppose that T_{α} is a bilinear fractional operators with generalized kernels defined as in (1.5). Then there exists a positive constant C such that, for all $f_i\in\mathcal{M}^{p_i,\varphi_i}\left(\mathbb{R}^n\right)$, $i=1,2$,

$$\left\|T_{\alpha,b_1,b_2}\left(f_1,f_2\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\le C\|b_1\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\|b_2\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}.$$

Theorem 3.2. *Let* $b_1, b_2 \in \mathrm{BMO}\left(\mathbb{R}^n\right)$, $0<\alpha<2n$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$ *with* $1<p_1,p_2<\frac{2n}{\alpha}$, $0\le\lambda<n$. *Suppose that* T_{α,b_1,b_2} *is a bilinear commutator operators with generalized kernels defined as in* (1.5). *Then there exists a positive constant* C *such that, for all* $f_i\in\mathcal{Q}^{p_i,\lambda}\left(\mathbb{R}^n\right)$, $i=1,2$,

$$\left\|T_{\alpha,b_1,b_2}\left(f_1,f_2\right)\right\|_{\mathcal{Q}^{p,\lambda}\left(\mathbb{R}^n\right)}\le C\|b_1\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\|b_2\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\|f_1\|_{\mathcal{Q}^{p_1,\lambda}\left(\mathbb{R}^n\right)}\|f_2\|_{\mathcal{Q}^{p_2,\lambda}\left(\mathbb{R}^n\right)}.$$

To prove the above theorems, we recall the following characterizations of spaces $\mathrm{BMO}\left(\mathbb{R}^n\right)$ introduced in [29].

Lemma 3.3. *Let* $b\in\mathrm{BMO}\left(\mathbb{R}^n\right)$ *and* $s\in[1,\infty)$. *Then*

1) there exists some positive constant C such that, for any ball $B\subset\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B\left|b(y)-b_B\right|^s\mathrm{d}y\right)^{\frac{1}{s}}\le C\|b\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)};$$

2) there exists a positive constant C such that, for any $k\in\mathbb{N}$,

$$\left|b_B-b_{2^{k+1}B}\right|\le C(k+1)\|b\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}.$$

The following result about the T_{α,b_1,b_2} is from [26].

Lemma 3.4. *Let* $b_1, b_2 \in \mathrm{BMO}\left(\mathbb{R}^n\right)$ *and* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$, *for* $1<p_1,p_2<\frac{2n}{\alpha}$, *Suppose that* T_{α,b_1,b_2} *defined as in* (1.6) *is bounded from the product of spaces* $L^{s_1}\left(\mathbb{R}^n\right)\times L^{s_2}\left(\mathbb{R}^n\right)$ *into the space* $L^{\frac{sn}{n-s\alpha},\infty}\left(\mathbb{R}^n\right)$ *for* $1\le s_1,s_2\le q'$. *Then there exists some positive constant* C *such that, for all* $f_i\in L^{p_i}\left(\mathbb{R}^n\right)$, $i=1,2$,

$$\left\|T_{\alpha,b_1,b_2}\left(f_1,f_2\right)\right\|_{L^{p}\left(\mathbb{R}^n\right)}\le C\|b_1\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\|b_2\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\|f_1\|_{L^{p_1}\left(\mathbb{R}^n\right)}\|f_2\|_{L^{p_2}\left(\mathbb{R}^n\right)}.$$

Proof of Theorem 3.1. Let $B=B(x,r)$ be the open ball centered at $x\in\mathbb{R}^n$ with radius $r>0$. Decompose the functions f_i ($i=1,2$) using the form in Theorem 2.1, that is,

$$f_i=f_i^0+f_i^\infty,\ i=1,2.$$

where $f_i^0=f_i\chi_{(2B)}$ and $f_i^\infty=f_i\chi_{\mathbb{R}^n\setminus(2B)}$. Then, by Minkowski's inequality, write

$$\begin{aligned}\left\|T_{\alpha,b_1,b_2}\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}&\le\left\|T_{\alpha,b_1,b_2}\left(f_1^0,f_2^0\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}+\left\|T_{\alpha,b_1,b_2}\left(f_1^0,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}\\&\quad+\left\|T_{\alpha,b_1,b_2}\left(f_1^\infty,f_2^0\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}+\left\|T_{\alpha,b_1,b_2}\left(f_1^\infty,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}\\&=\mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3+\mathrm{E}_4.\end{aligned}$$

From (1.5), (1.7), $\varphi_1\varphi_2=\varphi$, the Hölder's inequality, Lemma 2.3 and Lemma 3.4, it follows that

$$\begin{aligned}\mathrm{E}_1&\le\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left(\int_{(\mathbb{R}^n)^2}\left|T_{\alpha,b_1,b_2}\left(f_1^0,f_2^0\right)(y)\right|^p\mathrm{d}y\right)^{\frac{1}{p}}\\&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\&\quad\times\left(\int_{2B}\left|f_1(y_1)\right|^{p_1}\mathrm{d}y\right)^{\frac{1}{p_1}}\left(\int_{2B}\left|f_2(y_2)\right|^{p_2}\mathrm{d}y\right)^{\frac{1}{p_2}}\\&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\&\quad\times\frac{\varphi_1(x,2r)}{\varphi_1(x,2r)}\left(\int_{2B}\left|f_1(y_1)\right|^{p_1}\mathrm{d}y\right)^{\frac{1}{p_1}}\frac{\varphi_2(x,2r)}{\varphi_2(x,2r)}\left(\int_{2B}\left|f_2(y_2)\right|^{p_2}\mathrm{d}y\right)^{\frac{1}{p_2}}\\&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\\&\quad\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}\\&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}.\end{aligned}$$

For $y\in B(x,r)$, since

$$\begin{aligned}T_{\alpha,b_1,b_2}\left(f_1^0,f_2^\infty\right)(y)&=\left(b_1(y)-(b_1)_{2B}\right)\left(b_2(y)-(b_2)_{2B}\right)T_\alpha\left(f_1^0,f_2^0\right)(y)\\&\quad+\left(b_1(y)-(b_1)_{2B}\right)T_\alpha\left(\left(b_2(y)-(b_2)_{2B}\right)f_1^0,f_2^\infty\right)(y)\\&\quad+\left(b_2(y)-(b_2)_{2B}\right)T_\alpha\left(f_1^0,\left(b_1(y)-(b_1)_{2B}\right)f_2^\infty\right)(y)\\&\quad+T_\alpha\left(\left(b_1(y)-(b_1)_{2B}\right)f_1^0,\left(b_2(y)-(b_2)_{2B}\right)f_2^\infty\right)(y),\end{aligned}$$

then, by the Minkowski's inequality, write

$$\begin{aligned}&\left\|T_{\alpha,b_1,b_2}\left(f_1^0,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}\\&\le\left\|\left(b_1(\cdot)-(b_1)_{2B}\right)\left(b_2(\cdot)-(b_2)_{2B}\right)T_\alpha\left(f_1^0,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}\\&\quad+\left\|\left(b_1(\cdot)-(b_1)_{2B}\right)T_\alpha\left(\left(b_2(\cdot)-(b_2)_{2B}\right)f_1^0,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}\end{aligned}$$

$$+\left\|\left(b_2(\cdot)-(b_2)_{2B}\right)T_\alpha\left(f_1^0,\left(b_1(\cdot)-(b_1)_{2B}\right)f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}$$

$$+\left\|T_\alpha\left(\left(b_1(\cdot)-(b_1)_{2B}\right)f_1^0,\left(b_2(\cdot)-(b_2)_{2B}\right)f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}$$

$$=\mathrm{E}_{21}+\mathrm{E}_{22}+\mathrm{E}_{23}+\mathrm{E}_{24}.$$

By applying the estimate of $\left|T_\alpha\left(f_1^0,f_2^\infty\right)\right|$ in D_2, the Hölder's inequality, $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{p}$ and $\frac{1}{q_i}=\frac{1}{p_i}-\frac{\alpha}{n}$ $(i=1,2)$, $\varphi_1\varphi_2=\varphi$, (1.5), (1.7), (3.1), Lemma 2.3 and Lemma 3.3, we have

$$\begin{aligned}
\mathrm{E}_{21}&=\left\|\left(b_1(\cdot)-(b_1)_{2B}\right)\left(b_2(\cdot)-(b_2)_{2B}\right)T_\alpha\left(f_1^0,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\
&\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}\frac{|2B|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\right]\\
&\quad\times\left[\int_{B(x,r)}\left|\left(b_1(y)-(b_1)_{2B}\right)\left(b_2(y)-(b_2)_{2B}\right)\right|^p\mathrm{d}y\right]^{\frac{1}{p}}\\
&\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}\frac{|2B|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\right]\\
&\quad\times\left(\frac{1}{|2B|}\int_{2B}\left|b_1(y)-(b_1)_{2B}\right|^{q_1}\mathrm{d}y\right)^{\frac{1}{q_1}}\left(\frac{1}{|2B|}\int_{2B}\left|b_2(y)-(b_2)_{2B}\right|^{q_2}\mathrm{d}y\right)^{\frac{1}{q_2}}\\
&\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}\\
&\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}.
\end{aligned}$$

To estimate E_{22}, we first consider the $\left|T_\alpha\left(\left(b_2(y_2)-(b_2)_{2B}\right)f_1^0,f_2^\infty\right)(y)\right|$ with $y\in B(x,r)$. By (1.1), (1.5), (1.7), the Hölder's inequality, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$ and Lemma 3.3, we obtain

$$\begin{aligned}
&\left|T_\alpha\left(\left(b_2(\cdot)-(b_2)_{2B}\right)f_1^0,f_2^\infty\right)(y)\right|\\
&\le C\int_{\left(\mathbb{R}^n\right)^2}\frac{\left|b_2(y_2)-(b_2)_{2B}\right|\left|f_1^0(y_1)\right|\left|f_2^\infty(y_2)\right|}{\left(|y-y_1|+|y-y_2|\right)^{2n-\alpha}}\mathrm{d}y_1\mathrm{d}y_2\\
&\le C\int_{2B}\left|f_1(y_1)\right|\mathrm{d}y_1\int_{\mathbb{R}^n\setminus(2B)}\frac{\left|b_2(y_2)-(b_2)_{2B}\right|\left|f_2(y_2)\right|}{|y-y_2|^{2n-\alpha}}\mathrm{d}y_2\\
&\le C\left(\int_{2B}\left|f_1(y_1)\right|^{p_1}\mathrm{d}y_1\right)^{\frac{1}{p_1}}|2B|^{1-\frac{1}{p_1}}\frac{\varphi_1(x,2r)}{\varphi_1(x,2r)}\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\int_{2^{k+1}B}\left|b_2(y_2)-(b_2)_{2B}\right|\left|f_2(y_2)\right|\mathrm{d}y_2\right)
\end{aligned}$$

$$\le C\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\varphi_1(x,2r)|2B|^{1-\frac{1}{p_1}}$$
$$\times\left[\sum_{k=1}^{\infty}\frac{1}{|2^kB|^{2-\frac{\alpha}{n}}}\left(\left|(b_2)_{2B}-(b_2)_{2^{k+1}B}\right|\int_{2^{k+1}B}|f_2(y_2)|\,\mathrm{d}y_2\right.\right.$$
$$\left.\left.+\int_{2^{k+1}B}\left|b_2(y_2)-(b_2)_{2^{k+1}B}\right||f_2(y_2)|\,\mathrm{d}y_2\right)\right]$$
$$\le C\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\varphi_1(x,2r)|2B|^{1-\frac{1}{p_1}}$$
$$\times\left\{\sum_{k=1}^{\infty}\frac{1}{|2^kB|^{2-\frac{\alpha}{n}}}\left[\left|(b_2)_{2B}-(b_2)_{2^{k+1}B}\right|\left(\int_{2^{k+1}B}|f_2(y_2)|^{p_2}\,\mathrm{d}y_2\right)^{\frac{1}{p_2}}|2^{k+1}B|^{1-\frac{1}{p_2}}\right.\right.$$
$$+\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}\left|b_2(y_2)-(b_2)_{2^{k+1}B}\right|^{p_2'}\mathrm{d}y_2\right)^{\frac{1}{p_2'}}$$
$$\left.\left.\times\left(\int_{2^{k+1}B}|f_2(y_2)|^{p_2}\,\mathrm{d}y_2\right)^{\frac{1}{p_2}}|2^{k+1}B|^{1-\frac{1}{p_2}}\right]\right\}$$
$$\le C\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\varphi_1(x,2r)|2B|^{1-\frac{1}{p_1}}$$
$$\times\left\{\sum_{k=1}^{\infty}\frac{1}{|2^kB|^{2-\frac{\alpha}{n}}}\left[(k+1)\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\left(\int_{2^{k+1}B}|f_2(y_2)|^{p_2}\,\mathrm{d}y_2\right)^{\frac{1}{p_2}}|2^{k+1}B|^{1-\frac{1}{p_2}}\right.\right.$$
$$\left.\left.+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\left(\int_{2^{k+1}B}|f_2(y_2)|^{p_2}\,\mathrm{d}y_2\right)^{\frac{1}{p_2}}|2^{k+1}B|^{1-\frac{1}{p_2}}\right]\right\}$$
$$\le C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)$$
$$\times\left[\sum_{k=1}^{\infty}(k+1)\frac{|2B|^{1-\frac{1}{p_1}}}{|2^{k+1}B|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2(x,2^{k+1}r)\right],$$

by this, $\varphi_1\varphi_2=\varphi$, (1.7), (3.1), Lemma 2.3, we deduce

$$\mathrm{E}_{22}=\left\|\left(b_1(\cdot)-(b_1)_{2B}\right)T_\alpha\left(\left(b_2(\cdot)-(b_2)_{2B}\right)f_1^0,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}$$
$$\le C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}$$
$$\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}(k+1)\frac{|2B|^{\frac{1}{p}}|2B|^{1-\frac{1}{p_1}}}{|2^{k+1}B|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\right.$$
$$\left.\times\varphi_2(x,2^{k+1}r)\left(\frac{1}{|2B|}\int_{2B}|b_1(y)-b_{2B}|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\right]$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}$$
$$\times \sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}(k+1)\frac{|2B|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}{|2^{k+1}B|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2(x,2^{k+1}r)\right]$$
$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}$$
$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}.$$

To estimate E_{23}, we first consider $\left|T_\alpha\left(f_1^0,\left(b_2(y_2)-(b_2)_{2B}\right)f_2^\infty\right)(y)\right|$ with $y\in B(x,r)$, by (1.1), (1.5), (1.7), $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$, the Hölder's inequality and Lemma 3.3, we obtain

$$\left|T_\alpha\left(f_1^0,\left(b_2(\cdot)-(b_2)_{2B}\right)f_2^\infty\right)(y)\right|$$
$$\le C\int_{(\mathbb{R}^n)^2}\frac{\left|b_1(y_1)-(b_1)_{(2B)}\right|\left|f_1^0(y_1)\right|\left|f_2^\infty(y_2)\right|}{\left(|y-y_1|+|y-y_2|\right)^{2n-\alpha}}\mathrm{d}y_1\mathrm{d}y_2$$
$$\le C\int_{2B}\left|b_1(y_1)-(b_1)_{2B}\right|\left|f_1(y_1)\right|\mathrm{d}y_1\int_{\mathbb{R}^n\setminus(2B)}\frac{\left|f_2(y_2)\right|}{|y-y_2|^{2n-\alpha}}\mathrm{d}y_2$$
$$\le C\left(\frac{1}{|2B|}\int_{2B}\left|b_1(y_1)-(b_1)_{2B}\right|^{p_1'}\mathrm{d}y_1\right)^{\frac{1}{p_1'}}\left(\int_{2B}\left|f_1(y_1)\right|^{p_1}\mathrm{d}y_1\right)^{\frac{1}{p_1}}|2B|^{1-\frac{1}{p_1}}$$
$$\times\left(\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\int_{2^{k+1}B}\left|f_2(y_2)\right|\mathrm{d}y_2\right)$$
$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\left(\int_{2B}\left|f_1(y_1)\right|^{p_1}\mathrm{d}y_1\right)^{\frac{1}{p_1}}|2B|^{1-\frac{1}{p_1}}$$
$$\times\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\left(\int_{2^{k+1}B}\left|f_2(y_2)\right|^{p_2}\mathrm{d}y_2\right)^{\frac{1}{p_2}}\left|2^{k+1}B\right|^{1-\frac{1}{p_2}}\right]$$
$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)$$
$$\times\left[\sum_{k=1}^{\infty}\frac{|2B|^{1-\frac{1}{p_1}}}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2(x,2^{k+1}r)\right],$$

by this, $\varphi_1\varphi_2=\varphi$, (1.7), (3.1) and Lemma 2.3, we deduce

$$\mathrm{E}_{23}=\left\|\left(b_2(\cdot)-(b_2)_{2B}\right)T_\alpha\left(f_1^0,\left(b_2(\cdot)-(b_2)_{2B}\right)f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}$$
$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}$$
$$\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}\frac{|2B|^{\frac{1}{p}}|2B|^{1-\frac{1}{p_1}}}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2(x,2^{k+1}r)\right.$$

$$\times\left(\frac{1}{|2B|}\int_{2B}\left|b_2(y)-(b_2)_{2B}\right|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\Bigg]$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}$$

$$\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}(k+1)\frac{|2B|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\right]$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}.$$

To estimate E_{24}, we first consider $\left|T_\alpha\left(\left(b_1(y_1)-(b_1)_{2B}\right)f_1^0,\left(b_2(y_2)-(b_2)_{2B}\right)f_2^\infty\right)(y)\right|$ with $y\in B(x,r)$, by (1.1), (1.5), (1.7), $\varphi_1\varphi_2=\varphi$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$, the Hölder's inequality and Lemma 3.3, we obtain

$$\left|T_\alpha\left(\left(b_1(\cdot)-(b_1)_{2B}\right)f_1^0,\left(b_2(\cdot)-(b_2)_{2B}\right)f_2^\infty\right)(y)\right|$$

$$\le C\int_{(\mathbb{R}^n)^2}\frac{\left|b_1(y_1)-(b_1)_{2B}\right|\left|b_2(y_2)-(b_2)_{2B}\right|\left|f_1^0(y_1)\right|\left|f_2^\infty(y_2)\right|}{\left(|y-y_1|+|y-y_2|\right)^{2n-\alpha}}\mathrm{d}y_1\mathrm{d}y_2$$

$$\le C\int_{2B}\left|b_1(y_1)-(b_1)_{2B}\right|\left|f_1(y_1)\right|\mathrm{d}y_1\int_{\mathbb{R}^n\setminus(2B)}\frac{\left|b_2(y_2)-(b_2)_{2B}\right|\left|f_2(y_2)\right|}{|y-y_2|^{2n-\alpha}}\mathrm{d}y_2$$

$$\le C\left(\frac{1}{2B}\int_{2B}\left|b_1(y_1)-(b_1)_{2B}\right|^{p_1'}\mathrm{d}y_1\right)^{\frac{1}{p_1'}}\left(\int_{2B}\left|f_1(y_1)\right|^{p_1}\mathrm{d}y_1\right)^{\frac{1}{p_1}}|2B|^{1-\frac{1}{p_1}}$$

$$\times\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\int_{2^{k+1}B}\left|b_2(y_2)-(b_2)_{2B}\right|\left|f_2(y_2)\right|\mathrm{d}y_2\right]$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\varphi_1(x,2r)|2B|^{1-\frac{1}{p_1}}$$

$$\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\Bigg[\left|(b_2)_{2B}-(b_2)_{2^{k+1}B}\right|\int_{2^{k+1}B}\left|f_2(y_2)\right|\mathrm{d}y_2$$

$$+\int_{2^{k+1}B}\left|b_2(y_2)-(b_2)_{2^{k+1}B}\right|\left|f_2(y_2)\right|\mathrm{d}y_2\Bigg]\Bigg\}$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\varphi_1(x,2r)|2B|^{1-\frac{1}{p_1}}$$

$$\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\Bigg[\left|(b_2)_{2B}-(b_2)_{2^{k+1}B}\right|\int_{2^{k+1}B}\left|f_2(y_2)\right|\mathrm{d}y_2$$

$$+\left(\frac{1}{\left|2^{k+1}B\right|}\int_{2^{k+1}B}\left|b_2(y_2)-(b_2)_{2^{k+1}B}\right|^{p_2'}\mathrm{d}y_2\right)^{\frac{1}{p_2'}}$$

$$\times\left(\int_{2^{k+1}B}\left|f_2(y_2)\right|^{p_2}\mathrm{d}y_2\right)^{\frac{1}{p_2}}\left|2^{k+1}B\right|^{1-\frac{1}{p_2}}\Bigg]\Bigg\}$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\varphi_1(x,2r)|2B|^{1-\frac{1}{p_1}}$$

$$\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{2-\frac{\alpha}{n}}}\Bigg[(k+1)\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\left|2^{k+1}B\right|^{1-\frac{1}{p_2}}\varphi_2\left(x,2^{k+1}r\right)$$

$$+\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\left|2^{k+1}B\right|^{1-\frac{1}{p_2}}\varphi_2\left(x,2^{k+1}r\right)\Bigg]\Bigg\}$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)$$

$$\times\Bigg\{\sum_{k=1}^{\infty}(k+1)\frac{|2B|^{1-\frac{1}{p_1}}}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\Bigg\},$$

by this, (1.7), (3.1), $\varphi_1\varphi_2=\varphi$ and Lemma 2.3, we deduce

$$\mathrm{E}_{24}=\left\|T_\alpha\left(\left(b_1(\cdot)-(b_1)_{2B}\right)f_1^0,\left(b_2(\cdot)-(b_2)_{2B}\right)f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}(\mathbb{R}^n)}$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}|2B|^{\frac{1}{p}}$$

$$\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}(k+1)\frac{|2B|^{1-\frac{1}{p}}}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\right]$$

$$\le C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}$$

$$\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}(k+1)\frac{|2B|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}{\left|2^{k+1}B\right|^{1+\frac{1}{p_2}-\frac{\alpha}{n}}}\varphi_2\left(x,2^{k+1}r\right)\right]$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}$$

$$\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)},$$

which, together with the estimates for $\mathrm{E}_{21},\mathrm{E}_{22}$ and E_{23}, we deduce

$$\mathrm{E}_2\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}.$$

With an argument similar to that used in the estimate for E_2, it is easy to get

$$\mathrm{E}_3\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}.$$

For $y \in B(x,r)$, since

$$\begin{aligned}T_{\alpha,b_1,b_2}\left(f_1^{\infty},f_2^{\infty}\right)(y)&=\left(b_1(y)-(b_1)_{2B}\right)\left(b_2(y)-(b_2)_{2B}\right)T_{\alpha}\left(f_1^{\infty},f_2^{\infty}\right)(y)\\&+\left(b_1(y)-(b_1)_{2B}\right)T_{\alpha}\left(\left(b_2(y)-(b_2)_{2B}\right)f_1^{\infty},f_2^{\infty}\right)(y)\\&+\left(b_2(y)-(b_2)_{2B}\right)T_{\alpha}\left(f_1^{\infty},\left(b_1(y)-(b_1)_{2B}\right)f_2^{\infty}\right)(y)\\&+T_{\alpha}\left(\left(b_1(y)-(b_1)_{2B}\right)f_1^{\infty},\left(b_2(y)-(b_2)_{2B}\right)f_2^{\infty}\right)(y),\end{aligned}$$

then, by the Minkowski's inequality, write

$$\begin{aligned}&\left\|T_{\alpha,b_1,b_2}\left(f_1^{\infty},f_2^{\infty}\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\&\le\left\|\left(b_1(\cdot)-(b_1)_{2B}\right)\left(b_2(\cdot)-(b_2)_{2B}\right)T_{\alpha}\left(f_1^{\infty},f_2^{\infty}\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\&\quad+\left\|\left(b_1(\cdot)-(b_1)_{2B}\right)T_{\alpha}\left(\left(b_1(\cdot)-(b_1)_{2B}\right)f_1^{\infty},f_2^{\infty}\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\&\quad+\left\|\left(b_2(\cdot)-(b_2)_{2B}\right)T_{\alpha}\left(f_1^{\infty},\left(b_1(\cdot)-(b_1)_{2B}\right)f_2^{\infty}\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\&\quad+\left\|T_{\alpha}\left(\left(b_1(\cdot)-(b_1)_{2B}\right)f_1^{\infty},\left(b_2(\cdot)-(b_2)_{2B}\right)f_2^{\infty}\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\&=\mathrm{E}_{41}+\mathrm{E}_{42}+\mathrm{E}_{43}+\mathrm{E}_{44}.\end{aligned}$$

To estimate E_{41}, by applying the estimate of $\left|T_{\alpha}\left(f_1^{\infty},f_2^{\infty}\right)\right|$ in D_4, (1.1), (1.5), (1.7), $\varphi_1\varphi_2=\varphi$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$, $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{p}$ and $\frac{1}{q_i}=\frac{1}{p_i}-\frac{\alpha}{n}$ $(i=1,2)$, the Hölder's inequality, Lemma 2.3 and Lemma 3.3, we obtain

$$\begin{aligned}\mathrm{E}_{41}&=\left\|\left(b_1(\cdot)-(b_1)_{2B}\right)\left(b_2(\cdot)-(b_2)_{2B}\right)T_{\alpha}\left(f_1^{\infty},f_2^{\infty}\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\&\le C\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\\&\quad\times\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]\left[\sum_{l=1}^{\infty}\frac{1}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right]\\&\quad\times\left[\left(\frac{1}{|2B|}\int_{2B}\left|b_1(y)-(b_1)_{2B}\right|^{q_1}\mathrm{d}y\right)^{\frac{1}{q_1}}\left(\frac{1}{|2B|}\int_{2B}\left|b_2(y)-(b_2)_{2B}\right|^{q_2}\mathrm{d}y\right)^{\frac{1}{q_2}}\right]|2B|^{\frac{1}{p}}\\&\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\\&\quad\times\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}\frac{|2B|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]\left[\sum_{l=1}^{\infty}\frac{|2B|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right]\\&\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi(x,r)}{\varphi_1(x,r)\varphi_2(x,r)}\\&\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}.\end{aligned}$$

To estimate E_{42}, we first consider $\left|T_{\alpha}\left(\left(b_2(y)-(b_2)_{2B}\right)f_1^{\infty},f_2^{\infty}\right)(y)\right|$ with

$y\in B(x,r)$, by (1.1), (1.5), (1.7), $\varphi_1\varphi_2=\varphi$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$, the Hölder's inequality and Lemma 3.3, we obtain

$$\begin{aligned}
&\left|T_\alpha\left(\left(b_2(\cdot)-(b_2)_{2B}\right)f_1^\infty,f_2^\infty\right)(y)\right|\\
&\le C\int_{(\mathbb{R}^n)^2}\frac{\left|b_2(y_2)-(b_2)_{2B}\right|\left|f_1^\infty(y_1)\right|\left|f_2^\infty(y_2)\right|}{\left(|y-y_1|+|y-y_2|\right)^{2n-\alpha}}\mathrm{d}y_1\mathrm{d}y_2\\
&\le C\int_{\mathbb{R}^n\setminus(2B)}\frac{|f_1(y_1)|}{|y-y_1|^{n-\frac{\alpha}{2}}}\mathrm{d}y_1\int_{\mathbb{R}^n\setminus(2B)}\frac{\left|b_2(y_2)-(b_2)_{2B}\right||f_2(y_2)|}{|y-y_2|^{n-\frac{\alpha}{2}}}\mathrm{d}y_2\\
&\le C\left[\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus\left(2^kB\right)}\frac{|f_1(y_1)|}{|y-y_1|^{n-\frac{\alpha}{2}}}\mathrm{d}y_1\right]\left[\sum_{l=1}^{\infty}\int_{2^{l+1}B\setminus\left(2^lB\right)}\frac{\left|b_2(y_2)-(b_2)_{2B}\right||f_2(y_2)|}{|y-y_2|^{n-\frac{\alpha}{2}}}\mathrm{d}y_2\right]\\
&\le C\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{1-\frac{\alpha}{2n}}}\left(\int_{2^{k+1}B}|f_1(y_1)|^{p_1}\mathrm{d}y_1\right)^{\frac{1}{p_1}}\left|2^{k+1}B\right|^{1-\frac{1}{p_1}}\right]\\
&\quad\times\left\{\sum_{l=1}^{\infty}\frac{1}{\left|2^lB\right|^{1-\frac{\alpha}{2n}}}\left(\left|(b_2)_{2B}-(b_2)_{2^{l+1}B}\right|\int_{2^{l+1}B}|f_2(y_2)|\mathrm{d}y_2\right.\right.\\
&\quad\left.\left.+\int_{2^{l+1}B}\left|b_2(y_2)-(b_2)_{2^{l+1}B}\right||f_2(y_2)|\mathrm{d}y_2\right)\right\}\\
&\le C\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left[\sum_{k=1}^{\infty}\frac{\left|2^{k+1}B\right|^{1-\frac{1}{p_1}}}{\left|2^kB\right|^{1-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]\\
&\quad\times\left\{\sum_{l=1}^{\infty}\frac{1}{\left|2^lB\right|^{1-\frac{\alpha}{2n}}}\left[(l+1)\|b_2\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left(\int_{2^{l+1}B}|f_2(y_2)|^{p_2}\mathrm{d}y_2\right)^{\frac{1}{p_2}}\left|2^{l+1}B\right|^{1-\frac{1}{p_2}}\right]\right\}\\
&\le C\|b_2\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\\
&\quad\times\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]\left[\sum_{l=1}^{\infty}(l+1)\frac{1}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right],
\end{aligned}$$

by this, (1.7), (3.1), $\varphi_1\varphi_2=\varphi$ and Lemma 2.3, we deduce

$$\begin{aligned}
\mathrm{E}_{42}&=\left\|\left(b_1(\cdot)-(b_1)_{2B}\right)T_\alpha\left(\left(b_1(\cdot)-(b_1)_{2B}\right)f_1^\infty,f_2^\infty\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}\\
&\le C\|b_2\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}|2B|^{\frac{1}{p}}\\
&\quad\times\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]\left[\sum_{l=1}^{\infty}(l+1)\frac{1}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right]
\end{aligned}$$

$$\times\left(\frac{1}{|2B|}\int_{2B}\left|b_1(y)-(b_1)_{2B}\right|^p\mathrm{d}y\right)^{\frac{1}{p}}$$

$$\le C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}$$

$$\times\left[\sum_{k=1}^{\infty}\frac{|2B|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]\left[\sum_{l=1}^{\infty}(l+1)\frac{|2B|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right]$$

$$\times\left(\frac{1}{|2B|}\int_{2B}\left|b_1(y)-(b_1)_{2B}\right|^p\mathrm{d}y\right)^{\frac{1}{p}}$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}$$

$$\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}.$$

With an argument similar to that used in the estimate for E_{42}, it is easy to get

$$\mathrm{E}_{43}\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}.$$

To estimate E_{44}, we first consider the form $\left|T_\alpha\left(\left(b_1(\cdot)-(b_1)_{2B}\right)f_1^\infty,\left(b_2(\cdot)-(b_2)_{2B}\right)f_2^\infty\right)(y)\right|$ with $y\in B(x,r)$. Applying (1.1), (1.5), (1.7), $\varphi_1\varphi_2=\varphi$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$, the Hölder's inequality and Lemma 3.3, we obtain

$$\left|T_\alpha\left(\left(b_1(\cdot)-(b_1)_{2B}\right)f_1^\infty,\left(b_2(\cdot)-(b_2)_{2B}\right)f_2^\infty\right)(y)\right|$$

$$\le C\int_{(\mathbb{R}^n)^2}\frac{\left|b_1(y_1)-(b_1)_{2B}\right|\left|b_2(y_2)-(b_2)_{2B}\right|}{\left(|y-y_1|+|y-y_2|\right)^{2n-\alpha}}\left|f_1^\infty(y_1)\right|\left|f_2^\infty(y_2)\right|\mathrm{d}y_1\mathrm{d}y_2$$

$$\le C\int_{\mathbb{R}^n\setminus(2B)}\frac{\left|b_1(y_1)-(b_1)_{(2B)}\right|}{|y-y_1|^{n-\frac{\alpha}{2}}}\left|f_1(y_1)\right|\mathrm{d}y_1\int_{\mathbb{R}^n\setminus(2B)}\frac{\left|b_2(y_2)-(b_2)_{(2B)}\right|}{|y-y_2|^{n-\frac{\alpha}{2}}}\left|f_2(y_2)\right|\mathrm{d}y_2$$

$$\le C\left(\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{1-\frac{\alpha}{2n}}}\int_{2^{k+1}B\setminus\left(2^kB\right)}\left|b_1(y_1)-(b_1)_{2B}\right|\left|f_1(y_1)\right|\mathrm{d}y_1\right)$$

$$\times\left(\sum_{l=1}^{\infty}\frac{1}{\left|2^lB\right|^{1-\frac{\alpha}{2n}}}\int_{2^{l+1}B\setminus\left(2^lB\right)}\left|b_2(y_2)-(b_2)_{2B}\right|\left|f_2(y_2)\right|\mathrm{d}y_2\right)$$

$$\le C\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^kB\right|^{1-\frac{\alpha}{2n}}}\left(\left|(b_1)_{2B}-(b_1)_{2^{k+1}B}\right|\int_{2^{k+1}B}\left|f_1(y_1)\right|\mathrm{d}y_1\right.\right.$$

$$\left.\left.+\int_{2^{k+1}B}\left|b_1(y_1)-(b_1)_{2^{k+1}B}\right|\left|f_1(y_1)\right|\mathrm{d}y_1\right)\right]$$

$$\times\left[\sum_{l=1}^{\infty}\frac{1}{\left|2^{l}B\right|^{1-\frac{\alpha}{2n}}}\left(\left|\left(b_2\right)_{2B}-\left(b_2\right)_{2^{l+1}B}\right|\int_{2^{l+1}B}\left|f_2\left(y_2\right)\right|\mathrm{d}y_2\right.\right.$$

$$\left.\left.+\int_{2^{l+1}B}\left|b_2\left(y_2\right)-\left(b_2\right)_{2^{l+1}B}\right|\left|f_2\left(y_2\right)\right|\mathrm{d}y_2\right)\right]$$

$$\le C\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^{k}B\right|^{1-\frac{\alpha}{2n}}}(k+1)\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left(\int_{2^{l+1}B}\left|f_1\left(y_1\right)\right|^{p_1}\right)^{\frac{1}{p_1}}\left|2^{k+1}B\right|^{1-\frac{1}{p_1}}\right]$$

$$\times\left[\sum_{l=1}^{\infty}\frac{1}{\left|2^{l}B\right|^{1-\frac{\alpha}{2n}}}(l+1)\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left(\int_{2^{l+1}B}\left|f_2\left(y_2\right)\right|^{p_2}\right)^{\frac{1}{p_2}}\left|2^{l+1}B\right|^{1-\frac{1}{p_2}}\right]$$

$$\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}$$

$$\times\left[\sum_{k=1}^{\infty}(k+1)\frac{1}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]\left[\sum_{l=1}^{\infty}(l+1)\frac{1}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right],$$

by this, (1.7), (3.1), $\varphi_1\varphi_2=\varphi$ and Lemma 2.3, we deduce

$$\mathrm{E}_{44}=\left\|T_\alpha\left(\left(b_1(\cdot)-\left(b_1\right)_{2B}\right)f_1^{\infty},\left(b_2(\cdot)-\left(b_2\right)\right)_{2B}f_2^{\infty}\right)\right\|_{\mathcal{M}^{p,\varphi}\left(\mathbb{R}^n\right)}$$

$$\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\left|2B\right|^{\frac{1}{p}}$$

$$\times\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}\frac{1}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]$$

$$\times\left[\sum_{l=1}^{\infty}\frac{1}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right]$$

$$\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}$$

$$\times\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left[\sum_{k=1}^{\infty}\frac{\left|2B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}{\left|2^{k+1}B\right|^{\frac{1}{p_1}-\frac{\alpha}{2n}}}\varphi_1\left(x,2^{k+1}r\right)\right]$$

$$\times\left[\sum_{l=1}^{\infty}\frac{\left|2B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}{\left|2^{l+1}B\right|^{\frac{1}{p_2}-\frac{\alpha}{2n}}}\varphi_2\left(x,2^{l+1}r\right)\right]$$

$$\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}$$

$$\le C\left\|b_1\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|b_2\right\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}\left\|f_1\right\|_{\mathcal{M}^{p_1,\varphi_1}\left(\mathbb{R}^n\right)}\left\|f_2\right\|_{\mathcal{M}^{p_2,\varphi_2}\left(\mathbb{R}^n\right)},$$

which, combining the estimates for $\mathrm{E}_{41},\mathrm{E}_{42}$ and E_{43}, we deduce

$$\mathrm{E}_4 \le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1,\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2,\varphi_2}(\mathbb{R}^n)}.$$

combining the above estimates for $\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4$, we complete the proof of Theorem 3.1. □

Proof of Theorem 3.2. By applyng Remark 1.5 (1) and an argument similar to that used in the proof of Theorem 3.1, it is not difficult to show that Theorem 3.2 holds. Hence, to avoid repetition, here we do not state the proof of Theorem 3.2. □

Acknowledgments

This work is supported by the Gansu Outstanding Graduate Students "Foundation of China" Innovation Star Fund (2026CXZX-421).

Author Contributions

Jiao Gou: Carried out the main derivation of operator boundedness, completed the proof of the commutator estimates and drafted the manuscript.

Jinqi Wang: Proposed the research problem, supervised the work, checked and revised the proofs and the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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