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Degenerate Richards Flow with Signorini Seepage: Existence and L^1 Contraction

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Abstract

We establish existence and L^1 contraction for a closed, dry-state constrained formulation of the degenerate Richards equation with mixed Dirichlet, Neumann, and Signorini seepage conditions. In the Kirchhoff variable, the seepage boundary becomes a closed convex constraint. An implicit elliptic resolvent contracts the water content in L^1; its semigroup limit yields an energy solution with strongly continuous storage. The storage, Kirchhoff state, and Darcy flux are unique within the constructed solution class. Initial water content may be prescribed as an admissible L^1 function without compatibility with the Dirichlet trace at time zero. The argument allows conductivity to vanish at the dry endpoint and uses a quantitative modulus for the transformed gravity flux. This modulus is verified for rational Haverkamp-Vachaud laws and Van Genuchten-Mualem laws with $n > 3/2$.

Keywords

Richards Equation, Degenerate Parabolic Equations, Signorini Seepage, Kirchhoff Transformation, L^1 Contraction

1. Introduction

The Richards equation is a nonlinear partial differential equation describing water flow in variably saturated porous media under the combined effects of capillarity and gravity [1]. It is obtained by coupling mass conservation with Darcy's law and by introducing constitutive relationships between the pressure head, the water content, and the hydraulic conductivity. Owing to its ability to describe infiltration, drainage, capillary rise, and groundwater recharge processes, the Richards equation plays a central role in hydrology, soil physics, agriculture, and environmental engineering.

A major difficulty in the analysis and simulation of Richards-type models comes from the strong nonlinearity of the hydraulic functions. In dry regions, the hydraulic conductivity may become very small, or even vanish in idealized models, while the water capacity may also degenerate. As a consequence, the equation may lose its uniformly parabolic character and behave as a degenerate nonlinear parabolic problem. This degeneracy is particularly challenging in the presence of sharp wetting fronts and transitions between saturated and unsaturated zones.

The mathematical analysis of the Richards equation has been extensively studied. Foundational results on nonlinear elliptic-parabolic problems and porous-media type flows were established by Alt and Luckhaus [2] [3], while uniqueness and contraction principles for saturated-unsaturated flows were developed by Otto and Carrillo [4] [5]. Further contributions have addressed degenerate pressure-capillarity relations [6], global existence for subsurface flow models [7] [8], traveling fronts [9], singular-degenerate free boundary problems [10], and related degenerate parabolic equations [11] [12]. These works provide the theoretical background for weak and renormalized formulations of Richards-type equations.

The degenerate Kirchhoff transformation is natural for studying this problem: it controls the capillary flux even when the pressure gradient cannot be bounded in the dry regime. The gravity flux and the unilateral seepage boundary nevertheless require a comparison argument compatible with the closed constitutive graph. Our analysis constructs a contractive elliptic resolvent and then passes to continuous time.

Seepage boundary conditions are especially important in variably saturated flows. They arise when part of the boundary is in contact with the atmosphere and may switch dynamically between an impermeable state and an outflow state. The location of the seepage face is not known in advance and must be determined as part of the solution. This free-boundary character makes the problem more complex than standard Dirichlet or Neumann boundary-value problems, in the analysis. In particular, the active part of the seepage boundary is determined by the solution rather than prescribed in advance.

Earlier contraction and uniqueness results provide the starting point for the present analysis. Otto [4] treats saturated-unsaturated flow with mixed boundary conditions describing impermeable boundaries, reservoirs and contact with the atmosphere. His variational framework allows time-dependent boundary constraints and establishes contraction for finite-energy solutions without assuming an integrable time derivative of the saturation. Carrillo and Wittbold [5] establish comparison and uniqueness for renormalized solutions of degenerate elliptic-parabolic problems with homogeneous Dirichlet boundary conditions, under their structural assumptions on the flux.

In the present work, we develop a direct implicit-resolvent construction for degenerate Richards flow with a Signorini seepage boundary. The analysis uses the Kirchhoff variable $u = \int_0^{\psi} k(s)\mathrm{d}s$, whose gradient remains controlled as the conductivity approaches zero. In this variable, the atmospheric condition takes

the form of the unilateral trace constraint $u \le 0$ on the seepage boundary, while a closed constitutive graph accommodates the saturated plateau. The additional constraint $u \ge \ell$, where ℓ is the dry endpoint of the Kirchhoff range, completes the admissible states at residual storage. We consider a fixed boundary partition, with a Dirichlet portion of positive surface measure and stationary Dirichlet and Neumann data. The initial storage may be any element of the closed admissible set $\mathcal{C}$, without compatibility with the Dirichlet trace at time zero.

Within this setting, we first establish L^1 contraction for the elliptic resolvent and then obtain a strongly continuous contraction semigroup by implicit time discretization. The construction yields existence and uniqueness of the storage, Kirchhoff state and Darcy flux in the associated Rothe energy class. This specifies the solution class covered by our result and its relation to the finite-energy and renormalized frameworks cited above. We also verify the quantitative continuity condition on the transformed gravity flux for rational Haverkamp-Vachaud laws and Van Genuchten-Mualem laws with $n > 3/2$, thereby identifying constitutive models to which the construction applies.

The paper is organized as follows. Section 0 introduces the Richards model and hydraulic laws, and Section 1 describes the boundary conditions. Section 1 presents the Kirchhoff-Signorini formulation and the main theorem, whose proof by implicit resolvents is given in Section. Finally, Section verifies the constitutive continuity condition and discusses its physical scope.

2. Mathematical Model

Water flow in unsaturated porous media is commonly described by the Richards equation. This model is obtained by coupling the mass conservation equation

$$\partial_t \theta(\psi) + \nabla \cdot \boldsymbol{q} = 0, \tag{1}$$

where θ [-] denotes the volumetric water content and $\boldsymbol{q}$ the macroscopic water flux, with the generalized Darcy law

$$\boldsymbol{q} = -\boldsymbol{K}(\psi)\nabla h, \tag{2}$$

where $\boldsymbol{K}$ [LT^{-1}] is the hydraulic conductivity tensor and h [L] the total hydraulic head, defined by

$$h = \psi + z. \tag{3}$$

Here, ψ [L] denotes the water pressure head relative to atmospheric pressure, while z [L] is the vertical coordinate. The water table is the atmospheric-pressure surface $\psi = 0$. Positive pressure heads occur in saturated regions, but saturation does not require a positive pressure head: a capillary fringe above the water table may remain saturated under negative water pressure. The water content θ and hydraulic conductivity $\boldsymbol{K}$ are related to ψ through the chosen hydraulic constitutive laws.

Substituting (2) into (1) leads to the classical formulation of the Richards equation:

$$\partial_t \theta(\psi) - \nabla \cdot \left(\boldsymbol{K}(\psi) \nabla h \right) = 0, \tag{4}$$

This equation is supplemented with the initial condition

$$\psi(x,0) = \psi_0(x), \text{ in } \Omega, \tag{5}$$

where Ω is a bounded domain of $\mathbb{R}^N$, with $N \leq 3$. The boundary $\partial\Omega$ is decomposed as

$$\partial\Omega = \Gamma_{DC} \cup \Gamma_N \cup \Gamma_S,$$

on which Dirichlet, Neumann, and seepage boundary conditions are prescribed on Γ_{DC}, Γ_N, and Γ_S, respectively.

The formulation (4) involves coefficients that depend nonlinearly on the matric potential. It is therefore necessary to specify the hydraulic relationships linking the water content and the hydraulic conductivity to the pressure head.

Hydraulic properties constitute a central element in the modeling of the Richards equation, as they describe how an unsaturated soil stores and transmits water as a function of the matric potential. The soil-water retention relationship expresses the volumetric water content $\theta(\psi)$, which is generally monotone increasing and bounded between the residual water content θ_r and the saturated water content θ_s [3] [4]. It is often reformulated in terms of the effective saturation

$$S_e(\psi) = \frac{\theta(\psi) - \theta_r}{\theta_s - \theta_r}, \tag{6}$$

which normalizes the water content between 0 and 1.

The derivative $C(\psi) = \partial_\psi \theta(\psi)$, called the specific moisture capacity, measures the sensitivity of the water content to pressure variations. It appears in the time-derivative term of the Richards equation and may become very small in dry regions, leading to a parabolic degeneracy of the model.

The air-entry pressure head $\psi_e \leq 0$, when the retention model has a distinct entry threshold, marks the onset of desaturation as the pressure head decreases. It must be distinguished from the water table, which is defined by $\psi = 0$. For $\psi_e < 0$, the interval $\psi_e \leq \psi < 0$ may describe capillary-saturated states above the water table; desaturation occurs below the air-entry threshold. In a hydrostatic profile, $\psi = z_w - z$, where z_w is the water-table elevation, so the idealized saturated capillary fringe extends from z_w to $z_w - \psi_e$. Some empirical laws approach saturation continuously without a separate negative air-entry plateau. In particular, the standard examples used below are extended by saturation at $\psi = 0$; this is a constitutive idealization and does not identify atmospheric pressure with a universal physical air-entry threshold. A piecewise law with a specified threshold can be written as follows:

$$S_e(\psi) = \begin{cases} 1, & \psi \geq \psi_e, \\ S_e^{\text{nsat}}(\psi), & \psi < \psi_e, \end{cases} \quad K_r(\psi) = \begin{cases} 1, & \psi \geq \psi_e, \\ K_r^{\text{nsat}}(\psi), & \psi < \psi_e, \end{cases} \tag{7}$$

where S_e^{nsat} and K_r^{nsat} denote the constitutive laws adopted in the unsaturated

zone.

The hydraulic conductivity, which also depends on the matric potential, is classically written as

$$\boldsymbol{K}(\psi) = K_s K_r(\psi), \tag{8}$$

where K_s denotes the saturated hydraulic conductivity. In the unsaturated zone, $K_r(\psi)$ decreases sharply and may reach extremely small values, thereby enhancing both the nonlinearity and the degeneracy of the equation.

The functions $\theta(\psi)$, $S_e(\psi)$, and $K_r(\psi)$ do not arise from an analytical model but from empirical laws fitted to experimental data. The most commonly used ones are the Gardner-Irmay, Vachaud, and, in particular, the Van Genuchten-Mualem relationships [13]-[16], which have become a standard for representing soil hydraulic properties. These laws are continuous and monotone in their physical ranges; the additional modulus required for uniqueness is checked in Section, rather than being assumed from continuity alone. They play a crucial role in the mathematical analysis of the Richards equation. The main formulations used in this work are summarized in Table 1.

Table 1. Hydraulic constitutive laws used in the unsaturated zone.

Model	Expressions	Main parameters
Gardner-Irmay (1958) [13] [14]	$S_e(\psi) = \mathrm{e}^{\frac{\alpha\psi}{m}}$, $K_r(\psi) = \mathrm{e}^{\alpha\psi}$	α : pore-size distribution; m : tortuosity.
Vachaud (1971) [17]	$S_e(\psi) = \frac{C}{C + \lvert\psi\rvert^D}$, $K_r(\psi) = \frac{A}{A + \lvert\psi\rvert^B}$	A, B : empirical parameters; C, D : empirical parameters.
Van Genuchten-Mualem (1980) [15] [16]	$S_e(\psi) = \left(1 + (\alpha\lvert\psi\rvert)^n\right)^{-m}$, $K_r(\psi) = S_e^i\left[1 - \left(1 - S_e^{1/m}\right)^m\right]^2$	$i = 0.5$: pore connectivity; α : inverse characteristic pressure-head scale; it is not, in general, the inverse of a sharply defined air-entry threshold; $n > 1$: pore-size distribution index; $m = 1 - \frac{1}{n}$: pore-size distribution index.

Beyond the definition of the hydraulic properties of the medium, solving the Richards equation requires the specification of appropriate boundary conditions capable of representing the different flow regimes.

3. Boundary Conditions and *Seepage*

The solution of the Richards equation relies crucially on the choice of appropriate boundary conditions, as they govern the interaction between the porous medium

and its environment and influence the transition between saturated regimes (elliptic in nature) and unsaturated regimes (degenerate parabolic in nature). The three main families of boundary conditions are Dirichlet, Neumann, and the so-called *seepage* condition, which is specific to free-surface flows in unsaturated porous media.

3.1. Dirichlet Condition

On a portion of the boundary Γ_{DC}, the pressure head is prescribed directly as

$$\psi = \psi_{DC} \quad \text{on } \Gamma_{DC},$$

where ψ_{DC} may depend on space and time. This condition is typically used to model an imposed water level, a groundwater table, or a reservoir.

3.2. Neumann Condition

On another part Γ_N of the boundary, the normal water flux is imposed. Denoting by $h = \psi + z$ the total hydraulic head and by

$$\boldsymbol{q} = -\boldsymbol{K}(\psi)\nabla h$$

the Darcy flux, the Neumann condition reads

$$\boldsymbol{q}\cdot\boldsymbol{n} = q_N \quad \text{on } \Gamma_N,$$

where $\boldsymbol{n}$ is the outward unit normal vector. Thus $q_N > 0$ denotes exfiltration, $q_N < 0$ denotes infiltration, and $q_N = 0$ corresponds to an impermeable boundary.

3.3. Limitations of Classical Boundary Conditions

On an exposed soil boundary, the water pressure may evolve between negative values, for which the idealized seepage condition prevents outflow, and atmospheric pressure, at which outflow is allowed. Prescribing either a pressure or a flux in advance does not by itself describe this switching behavior. This phenomenon leads naturally to the *seepage* condition, which is essential for modeling processes such as surface exfiltration, natural drainage of embankments, water outflow along riverbanks, or localized seepage in hydraulic structures [18] [19].

3.4. Physical and Mathematical Formulation of the Seepage Condition

For an exposed boundary in contact with the atmosphere, the idealized seepage condition permits outflow only when the water pressure reaches atmospheric pressure. If $\psi < 0$, the condition imposes zero outward flux; this branch is characterized by negative pressure, rather than by an assertion that the adjacent soil must be unsaturated. If $\psi = 0$, nonnegative outward flux is permitted. These alternatives are expressed by the following complementarity system:

$$\psi \le 0,\ \boldsymbol{q}\cdot\boldsymbol{n} \ge 0,\ \psi(\boldsymbol{q}\cdot\boldsymbol{n}) = 0 \quad \text{on } \Gamma_S.$$

The active outflow subset of the prescribed seepage boundary is unknown in

advance. Its boundary separates active and inactive seepage portions; it should not be identified in general with the entire interface between saturated and unsaturated soil.

3.5. Signorini Interpretation

At each boundary point, the idealized seepage condition has two branches: $\psi < 0$ with zero outward flux, and $\psi = 0$ with nonnegative outward flux. The analytical convex set $\mathcal{K}$ defined in Section 1 imposes the corresponding unilateral trace constraint in the Kirchhoff variable, together with a lower dry-state bound. Recovery of the classical flux complementarity from the variational inequality requires identifiable conormal traces and the absence of an active dry reaction at the boundary, as explained in Section 1.

In this work, the boundary may simultaneously include Dirichlet, Neumann, and seepage portions. The outflow region is part of the solution rather than part of the prescribed boundary partition. **Figure 1** illustrates the atmospheric-pressure water table and the active seepage portion on an inclined wall; a capillary-saturated fringe is not resolved in this schematic.

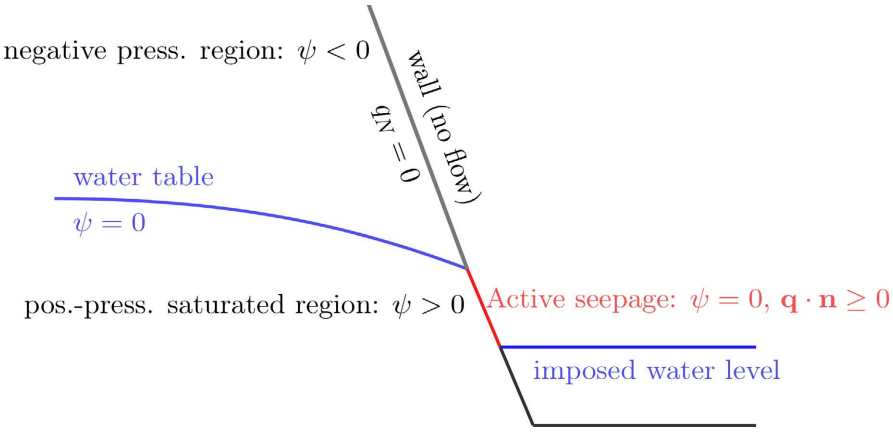

Figure 1. Schematic atmospheric-pressure water table and active seepage boundary on an inclined wall. The surface $\psi = 0$ defines the water table. Negative pressure above it does not exclude a saturated capillary fringe, which is not resolved in this schematic. The active seepage segment is determined by the Signorini condition and is not a prescribed Dirichlet boundary segment.

The previous modeling elements now allow us to rigorously formulate the variational problem associated with the Richards equation and to state the main result of this work.

4. Kirchhoff-Signorini Formulation and Main Result

For the analysis we factor the conductivity tensor as $\boldsymbol{K}(x,\psi) = k(\psi)\boldsymbol{A}(x)$, where $\boldsymbol{A} \in L^{\infty}\left(\Omega; \mathbb{R}^{N\times N}_{sym}\right)$ satisfies

$$a_0|\xi|^2 \leq \boldsymbol{A}(x)\xi \cdot \xi \leq a_1|\xi|^2 \quad \text{for a.e. } x \in \Omega,\ \xi \in \mathbb{R}^N. \tag{9}$$

Let $k \in C(\mathbb{R})$ be positive at every finite pressure, bounded, and constant for $\psi \geq 0$; let $\theta \in C(\mathbb{R})$ be bounded and non-decreasing, and constant for $\psi \geq 0$. We assume that k is integrable at the dry end and write

$$\begin{aligned} &\Phi(s) = \int_0^s k(r)\mathrm{d}r, \ \ell = \lim_{s \to -\infty} \Phi(s) > -\infty, \ u = \Phi(\psi), \\ &\beta(u) = \theta\left(\Phi^{-1}(u)\right), \ \kappa(u) = k\left(\Phi^{-1}(u)\right). \end{aligned} \tag{10}$$

The functions β, κ are extended continuously to $[\ell, \infty)$, with β and κ constant on $[0, \infty)$. Denote the endpoints of the storage range by $b_- = \beta(\ell)$ and $b_+ = \beta(0)$. Set

$$\mathcal{C} = \left\{ b \in L^1(\Omega) : b_- \leq b \leq b_+ \text{ almost everywhere in } \Omega \right\}.$$

The dry endpoint represents the limiting residual-water state, rather than a finite pressure head. The transformed Darcy flux is

$$\boldsymbol{q} = -\boldsymbol{A}\nabla u - \kappa(u)\boldsymbol{A}\nabla z. \tag{11}$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain whose boundary is partitioned into Γ_{DC}, Γ_N, and Γ_S, with $|\Gamma_{DC}|_{N-1} > 0$. Throughout the analytical theorem, the Dirichlet lifting $d \in H^1(\Omega)$ and the outward Neumann flux $g_N \in H^{-1/2}(\Gamma_N)$ are independent of time, and $z \in W^{1,\infty}(\Omega)$.

We use the space of test functions

$$V := \left\{ v \in H^1(\Omega) : \gamma v = 0 \text{ on } \Gamma_{DC} \right\}$$

and the admissible set

$$\mathcal{K} := \left\{ v \in H^1(\Omega) : \gamma v = \gamma d \text{ on } \Gamma_{DC}, \gamma v \leq 0 \text{ on } \Gamma_S, v \geq \ell \text{ a.e. in } \Omega \right\}. \tag{12}$$

We assume $\mathcal{K} \neq \varnothing$. The lower bound $u \geq \ell$ completes the pressure range at its dry endpoint, where the pressure head has no finite value. It is an obstacle constraint, and the theorem concerns the resulting closed-graph problem, including its normal-cone reaction. Because the constraint is imposed on $H^1(\Omega)$ states, it also restricts their boundary traces. Consequently, at dry contact the variational formulation must not automatically be identified with the source-free Richards equation together with every prescribed boundary flux realized as an equality. Where the solution is sufficiently regular and separated from ℓ, two-sided admissible variations recover the corresponding unconstrained balance and boundary conditions.

Since $\Phi(0) = 0$, the atmospheric condition is exactly $u \leq 0$ on Γ_S. For regular states separated from the dry endpoint near the boundary, the variational inequality has the form

$$\begin{aligned} &\partial_t \beta(u), v - u + \int_\Omega \left[\boldsymbol{A}\nabla u + \kappa(u)\boldsymbol{A}\nabla z \right] \cdot \nabla(v - u)\mathrm{d}x \\ &\geq -\left\langle g_N, \gamma(v - u) \right\rangle_{\Gamma_N}, \ v \in \mathcal{K}. \end{aligned} \tag{13}$$

It yields $u \leq 0$, $\boldsymbol{q} \cdot \boldsymbol{n} \geq 0$, and $u\boldsymbol{q} \cdot \boldsymbol{n} = 0$ on the seepage boundary where the conormal trace exists and the dry constraint is inactive.

Dry Contact and Water Balance

For sufficiently regular states admitting a multiplier representation, the lower obstacle can be described in the interior by

$$\partial_t b + \nabla \cdot \boldsymbol{q} = r_\Omega,\ r_\Omega \geq 0,\ r_\Omega \left(u - \ell\right) = 0.$$

On a Neumann boundary portion, its formal boundary counterpart is

$$\boldsymbol{q} \cdot \boldsymbol{n} = g_N - r_N,\ r_N \geq 0,\ r_N \left(\operatorname{tr} u - \ell\right) = 0.$$

Thus, an outward extraction demand need not remain fully realized once the residual-water constraint is active. The term r_N represents the unrealized part of that boundary demand in this multiplier description. If a bulk reaction is present, it contributes to the balance; it cannot be omitted while claiming a source-free equation. A dry reaction can also affect the recovery of the classical seepage trace conditions at a dry boundary point.

When the indicated quantities and normal traces admit integration, the balance associated with these relations reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega b \mathrm{d}x + \int_{\Gamma_{DC}} \boldsymbol{q} \cdot \boldsymbol{n} \mathrm{d}S + \int_{\Gamma_N} g_N \mathrm{d}S + \int_{\Gamma_S} \boldsymbol{q} \cdot \boldsymbol{n} \mathrm{d}S = \int_\Omega r_\Omega \mathrm{d}x + \int_{\Gamma_N} r_N \mathrm{d}S.$$

In particular, the usual balance with the prescribed Neumann flux and no volume source requires the absence of these dry-reaction terms. These formulas explain the constraint at a regular or formal level. The general theorem does not assert that its reactions have integrable densities or that all the displayed normal traces exist; its unconditional statement remains the integrated variational formulation.

A simple example illustrates the distinction between requested and realized extraction. Take a nonnegative, nonzero Neumann datum $g_N \in L^2\left(\Gamma_N\right)$, the constant lifting $d = \ell$, and initial storage $b_0 = b_-$. The state $u \equiv \ell$, $b \equiv b_-$ satisfies every elliptic step: the storage increment and the interior flux vanish, while

$$\int_{\Gamma_N} g_N \left(\operatorname{tr} v - \ell\right) \mathrm{d}S \geq 0 \left(v \in \mathcal{K}\right).$$

Here $\kappa\left(\ell\right) = 0$, as follows from dry-end integrability and the continuous extension of κ. The actual Darcy flux is therefore zero, although the requested outward flux is positive. This example belongs to the closed dry-endpoint formulation; the datum $d = \ell$ represents a limiting dry pressure, not a finite imposed pressure head.

The constrained formulation represents the original source-free Richards problem with its stated boundary conditions when the dry normal-cone contribution vanishes and the balance and boundary fluxes can be identified in the relevant weak or classical sense. A sufficient setting for the usual formal identification is a sufficiently regular solution uniformly separated from ℓ in the region and up to the boundary portions under consideration. Strict separation is a sufficient condition, not a necessary one: contact alone does not imply a nonzero reaction. No unconditional absence of dry reaction is claimed for arbitrary admissible initial storage and extraction data.

The two structural assumptions used below are

$$\beta(r)=\beta(s)\Rightarrow\kappa(r)=\kappa(s),\ r,s\in[\ell,\infty), \tag{14}$$

$$\left|\kappa(r)-\kappa(s)\right|\le\omega_\kappa\left(|r-s|\right),\ \lim_{\delta\downarrow0}\frac{\omega_\kappa(\delta)^2}{\delta}=0. \tag{15}$$

Here ω_κ is non-decreasing and tends to zero at zero. Compatibility means $\kappa=\hat{\kappa}\circ\beta$ for a continuous function $\hat{\kappa}:[b_-,b_+]\to\mathbb{R}$. The saturated plateau of the Haverkamp and Van Genuchten-Mualem laws satisfies this condition.

For a precise weak formulation, when $b_-<b_+$ choose a non-decreasing generalized inverse $\gamma:(b_-,b_+)\to[\ell,0]$ of the storage law and set

$$W(b)=\int_{b_-}^{b}\gamma(s)\mathrm{d}s,\ b\in[b_-,b_+].$$

When the storage range is a singleton, set $W=0$. The function W is convex and Lipschitz. With the subdifferential taken relative to $[b_-,b_+]$,

$$u\in\partial W\left(\beta(u)\right)\ \left(u\in[\ell,\infty)\right). \tag{16}$$

At an endpoint, this subdifferential also includes the outward normal cone of the storage interval; equality with the inverse storage graph is therefore not asserted. In particular, (16) includes the saturated plateau $u\ge0$ and the dry endpoint $u=\ell$. Write

$$a(u,v)=\int_\Omega\left[\boldsymbol{A}\nabla u+\kappa(u)\,\boldsymbol{A}\nabla z\right]\cdot\nabla v\mathrm{d}x+\left\langle g_N,\gamma v\right\rangle_{\Gamma_N}.$$

Definition 1 (Rothe energy solution). A Rothe energy solution with initial storage $b_0\in\mathcal{C}$ is a limit, as the uniform step $\tau\downarrow0$, of the fully implicit scheme (22) initialized by $b^0=b_0$. It has

$$b=\beta(u)\in C\left([0,T];L^1(\Omega)\right),\ u\in L^2\left(0,T;H^1(\Omega)\right),$$

$$u(t)\in\mathcal{K}\ \text{for a.e. } t,\quad b(0)=b_0\ \text{in } L^1(\Omega).$$

No trace compatibility between b_0 and the stationary Dirichlet datum is required; the Dirichlet trace is imposed on $u(t)$ for almost every $t>0$. The integrated variational inequality is, for every $v\in\mathcal{K}$ and $0\le s<t\le T$,

$$\int_\Omega\left[W\left(b(t)\right)-W\left(b(s)\right)\right]\mathrm{d}x\le\int_\Omega\left[b(t)-b(s)\right]v\mathrm{d}x+\int_s^t a\left(u(r),v-u(r)\right)\mathrm{d}r. \tag{17}$$

The pairing on the right is well defined for $v\in H^1(\Omega)$, since the storage is bounded. This formulation records the Signorini normal cone without postulating a separate $L^2(0,T;V')$ bound on $\partial_t b$.

Theorem 1 (Existence and L^1 contraction). Under the geometric, data, and constitutive assumptions above, for every $b_0\in\mathcal{C}$ the degenerate Kirchhoff-Signorini problem has a Rothe energy solution with initial storage $b(0)=b_0$. For two such solutions with initial storages $b_{0,1},b_{0,2}\in\mathcal{C}$ and the same boundary data, their storages satisfy

$$\left\|b_1(t)-b_2(t)\right\|_{L^1(\Omega)}\le\left\|b_{0,1}-b_{0,2}\right\|_{L^1(\Omega)},\ 0\le t\le T. \tag{18}$$

The Kirchhoff state and the flux (11) are uniquely determined by the initial storage within this Rothe energy class. The construction uses the closed degenerate constitutive graph itself; no positive lower bound on k or uniform estimate of $\nabla\psi$ is required.

5. Implicit Construction and Proof of the Main Result

5.1. The Elliptic Resolvent

Let $f\in\mathcal{C}$, let $\lambda>0$, and seek $u\in\mathcal{K}$ such that $b=\beta(u)$ and

$$\int_\Omega (b-f)(v-u)\,\mathrm{d}x+\lambda a(u,v-u)\geq 0\ \ (v\in\mathcal{K}). \tag{19}$$

Although the storage range is bounded, u need not be bounded; the integral in (19) is nevertheless defined since $f,b\in L^\infty(\Omega)$ and $u,v\in H^1(\Omega)$.

Proposition 2 (Resolvent and contraction). Problem (19) has a unique u for each $f\in\mathcal{C}$ and $\lambda>0$. The map $R_\lambda f:=\beta(u)$ satisfies

$$\left\|(R_\lambda f_1-R_\lambda f_2)^+\right\|_{L^1(\Omega)}\leq\left\|(f_1-f_2)^+\right\|_{L^1(\Omega)}. \tag{20}$$

Proof. The operator in (19) is pseudomonotone on the closed convex set $\mathcal{K}$. The Nemytskii storage operator is monotone and demicontinuous. If $u_j\rightharpoonup u$ in $H^1(\Omega)$, Rellich compactness gives convergence in measure, and bounded continuity of κ gives $\kappa(u_j)\to\kappa(u)$ strongly in $L^2(\Omega)$. Thus the gravity term is a strongly continuous perturbation from H^1 into V'. Taking a fixed $w\in\mathcal{K}$, ellipticity, Poincaré's inequality, and the trace theorem give

$$\int_\Omega (b-f)(u-w)\,\mathrm{d}x+\lambda a(u,u-w)\geq c_\lambda\|u-w\|_{H^1}^2-C_{\lambda,f,w}\left(1+\|u-w\|_{H^1}\right).$$

The pseudomonotone variational-inequality theorem gives existence.

For contraction put $v=u_1-u_2$ and $T_\delta^+(r)=\min\{1,r^+/\delta\}$. The functions $u_1-\delta T_\delta^+(v)$ and $u_2+\delta T_\delta^+(v)$ are pointwise between the states, so both are admissible. The Neumann terms cancel. Adding the two inequalities and dividing by δ yields

$$\begin{aligned}&\int_\Omega\left[(b_1-b_2)-(f_1-f_2)\right]T_\delta^+(v)\,\mathrm{d}x+\lambda\int_\Omega \boldsymbol{A}\nabla v\cdot\nabla T_\delta^+(v)\,\mathrm{d}x\\&\leq-\lambda\int_\Omega\left[\kappa(u_1)-\kappa(u_2)\right]\boldsymbol{A}\nabla z\cdot\nabla T_\delta^+(v)\,\mathrm{d}x.\end{aligned} \tag{21}$$

On $E_\delta=\{0<v<\delta\}$, ellipticity and Young's inequality bound the right-hand side, after absorbing half of the diffusion term, by

$$C\lambda|\Omega|\frac{\omega_\kappa(\delta)^2}{\delta}.$$

This tends to zero by (15). Monotonicity of β gives $(b_1-b_2)\mathbf{1}_{\{v>0\}}=(b_1-b_2)^+$; the equality also holds across storage plateaus because $b_1=b_2$ there. Letting $\delta\downarrow 0$ *at the elliptic step* gives

$$\int_\Omega(b_1-b_2)^+\,\mathrm{d}x\leq\int_\Omega(f_1-f_2)\mathbf{1}_{\{v>0\}}\,\mathrm{d}x\leq\int_\Omega(f_1-f_2)^+\,\mathrm{d}x.$$

Interchanging indices gives the full L^1 contraction. For $f_1 = f_2$, the storages coincide. Compatibility (14) then makes the two gravity terms equal. Testing the two original inequalities with u_2 and u_1, respectively, proves $\int_\Omega \boldsymbol{A}\nabla v \cdot \nabla v \le 0$. The common Dirichlet trace implies $u_1 = u_2$. □

5.2. Energy Bound and Temporal Construction

Let $b_0 \in \mathcal{C}$, set $b^0 = b_0$, and for $\tau = T/N$ solve (19) recursively with

$$f = b^{n-1},\ \lambda = \tau,\ b^n = \beta\left(u^n\right),\ n = 1,\cdots,N. \tag{22}$$

Proposition 2 proves existence and uniqueness at every step without a step-size restriction. Fix $w \in \mathcal{K}$, independent of time, and define

$$\mathcal{E}_w(b) = \int_\Omega \left\{W(b) - W\left(\beta(w)\right) - w\left[b - \beta(w)\right]\right\}\mathrm{d}x,\ b \in \mathcal{C}. \tag{23}$$

The subgradient relation (16) implies $\mathcal{E}_w(b) \ge 0$. It is finite for every $b \in \mathcal{C}$, because W and b are bounded and $w \in H^1(\Omega) \subset L^1(\Omega)$. Since $u^n \in \partial W\left(b^n\right)$,

$$\mathcal{E}_w\left(b^n\right) - \mathcal{E}_w\left(b^{n-1}\right) \le \int_\Omega \left(b^n - b^{n-1}\right)\left(u^n - w\right)\mathrm{d}x. \tag{24}$$

Testing the n-th resolvent by w gives

$$\mathcal{E}_w\left(b^n\right) - \mathcal{E}_w\left(b^{n-1}\right) + \tau a\left(u^n, u^n - w\right) \le 0. \tag{25}$$

Put $v = u^n - w \in V$. Ellipticity, boundedness of κ, the trace theorem, and Poincaré's inequality give constants $c_w > 0, C_w < \infty$, independent of n, τ, such that

$$a\left(u^n, u^n - w\right) \ge c_w \left\|u^n - w\right\|^2_{H^1(\Omega)} - C_w. \tag{26}$$

Summation of (25) and nonnegativity of the relative energy prove

$$\max_{0 \le m \le N} \mathcal{E}_w\left(b^m\right) + c_w \sum_{n=1}^{N} \tau \left\|u^n - w\right\|^2_{H^1(\Omega)} \le 2\left(\mathcal{E}_w\left(b_0\right) + C_w T\right). \tag{27}$$

Consequently the piecewise constant interpolants $\overline{u}_\tau, \overline{b}_\tau$ are uniformly bounded in $L^2\left(0,T;H^1(\Omega)\right)$ and $L^\infty\left(\Omega \times (0,T)\right)$, respectively. Neither b_0 nor any pressure preimage of it needs an H^1 trace compatible with the Dirichlet datum.

We now obtain temporal compactness from the resolvents. The family $R_\lambda : \mathcal{C} \to \mathcal{C}$ is nonexpansive by (20). Its resolvent identity follows directly by writing $f = b + \lambda\xi$ in (19): for $0 < \mu < \lambda$,

$$R_\lambda f = R_\mu\left(\frac{\mu}{\lambda} f + \left(1 - \frac{\mu}{\lambda}\right) R_\lambda f\right).$$

For $f = \beta(v)$, $v \in \mathcal{K}$, testing the resolvent by v and using coercivity of $a\left(u_\lambda, u_\lambda - v\right)$ gives a bound on $u_\lambda - v$ in $H^1(\Omega)$, independent of $0 < \lambda \le 1$, and

$$0 \le \int_\Omega \left[R_\lambda f - f\right]\left[u_\lambda - v\right]\mathrm{d}x \le C_v \lambda. \tag{28}$$

For each $\eta>0$, uniform continuity of β on $[\ell,\infty)$ (it is constant on $[0,\infty)$) gives $\rho_\eta>0$ such that $|\beta(r)-\beta(s)|\geq\eta$ implies $|r-s|\geq\rho_\eta$. Monotonicity therefore yields $(\beta(r)-\beta(s))(r-s)\geq\eta\rho_\eta$ on this set. The preceding estimate implies

$$\|R_\lambda f-f\|_{L^1(\Omega)}\leq\eta|\Omega|+\frac{(b_+-b_-)C_v\lambda}{\eta\rho_\eta},$$

and hence $R_\lambda f\to f$ in $L^1(\Omega)$. Moreover $\beta(\mathcal{K})$ is dense in $\mathcal{C}$ for the L^1 norm. Indeed, approximate any $f\in\mathcal{C}$ by simple storages, choose their preimages in $[\ell,0]$, approximate the resulting bounded states in the interior by smooth functions, and join them to a fixed $w\in\mathcal{K}$ in a boundary layer of vanishing measure. Convex interpolation preserves the lower bound $u\geq\ell$, while the fixed boundary trace and seepage constraint come from w. Boundedness of β makes the layer error vanish in L^1. Contraction extends $R_\lambda f\to f$ to every $f\in\mathcal{C}$.

Lemma 3 (Accretive realization of the resolvents). Let $X=L^1(\Omega)$ and define the graph $\mathcal{A}\subset X\times X$ by

$$\mathcal{A}=\left\{\left(R_\lambda f,\frac{f-R_\lambda f}{\lambda}\right):f\in\mathcal{C},?\lambda>0\right\}.$$

Then $\mathcal{A}$ is accretive in X, and

$$\overline{D(\mathcal{A})}^X=\mathcal{C},\ \mathcal{C}\subseteq\mathrm{Ran}(I+\mu\mathcal{A})\ \text{for every}\ \mu>0.$$

Moreover, $(I+\mu\mathcal{A})^{-1}f=R_\mu f$ for every $f\in\mathcal{C}$, and these resolvents leave $\mathcal{C}$ invariant.

Proof. Take $(b_i,\xi_i)\in\mathcal{A}$, $i=1,2$, and choose representations $b_i=R_{\lambda_i}f_i$ and $\xi_i=(f_i-b_i)/\lambda_i$, with $f_i\in\mathcal{C}$ and $\lambda_i>0$. If $0<\mu<\min\{\lambda_1,\lambda_2\}$, convexity of $\mathcal{C}$ gives

$$h_i=b_i+\mu\xi_i=\frac{\mu}{\lambda_i}f_i+\left(1-\frac{\mu}{\lambda_i}\right)b_i\in\mathcal{C}.$$

The resolvent identity yields $R_\mu h_i=b_i$, and contraction gives

$$\|b_1-b_2\|_X\leq\|b_1-b_2+\mu(\xi_1-\xi_2)\|_X.$$

To obtain the same inequality for an arbitrary $\eta>0$, choose $0<\mu<\min\{\eta,\lambda_1,\lambda_2\}$. By convexity of the norm,

$$\begin{aligned}\|b_1-b_2\|_X&\leq\|b_1-b_2+\mu(\xi_1-\xi_2)\|_X\\&\leq\left(1-\frac{\mu}{\eta}\right)\|b_1-b_2\|_X+\frac{\mu}{\eta}\|b_1-b_2+\eta(\xi_1-\xi_2)\|_X.\end{aligned}$$

Rearranging proves accretivity for every $\eta>0$.

For $f\in\mathcal{C}$ and $\mu>0$, the defining graph contains $(R_\mu f,(f-R_\mu f)/\mu)$, so $f\in\mathrm{Ran}(I+\mu\mathcal{A})$. Accretivity makes the inverse single-valued on its range; hence its value at f is exactly $R_\mu f$. The invariance follows from $R_\mu\mathcal{C}\subseteq\mathcal{C}$. Finally,

$D(\mathcal{A}) \subseteq \mathcal{C}$, and $\mathcal{C}$ is closed in X. Conversely, for every $f \in \mathcal{C}$, the already established convergence $R_\lambda f \to f$ in X as $\lambda \downarrow 0$, with $R_\lambda f \in D(\mathcal{A})$, gives $f \in \overline{D(\mathcal{A})}^X$. This proves the asserted equality. □

We apply the following form of the Crandall-Liggett generation theorem [20]: if A is an accretive operator on a Banach space X and $\overline{D(A)} \subseteq \mathrm{Ran}(I + \lambda A)$ for all sufficiently small $\lambda > 0$, the exponential formula generates a strongly continuous contraction semigroup on $\overline{D(A)}$. For the graph $\mathcal{A}$ above, the closure of the domain is $\mathcal{C}$, the required range inclusion holds for every $\lambda > 0$, and the resolvents on $\mathcal{C}$ are precisely the maps already constructed by the variational inequality. Thus

$$S(t) f = \lim_{m \to \infty} (R_{t/m})^m f, \quad f \in \mathcal{C}, \ t \geq 0,$$

with $S(0) f = f$, defines the desired semigroup on $\mathcal{C}$. The associated implicit-Euler convergence, uniform on bounded time intervals, yields the following convergence for the Rothe interpolants:

$$\sup_{0 \leq t \leq T} \left\| \overline{b}_\tau(t) - S(t) b_0 \right\|_{L^1(\Omega)} \to 0. \tag{29}$$

The harmless one-step convention at $t = 0$ is fixed by $\overline{b}_\tau(0) = b_0$. In particular, temporal translations of the storage vanish uniformly along the scheme. This is the temporal compactness needed for the degenerate problem; no estimate of a separate Signorini reaction is invoked.

5.3. Identification of the Limit

The energy bound yields, along a subsequence, $\overline{u}_\tau \rightharpoonup u$ in $L^2(0,T; H^1(\Omega))$. Equation (29) gives $\overline{b}_\tau \to b = S(t) b_0$ in every finite $L^p(\Omega \times (0,T))$, since the storages are uniformly bounded. The monotone-graph inequality

$$\left[\beta(\overline{u}_\tau) - \beta(v)\right](\overline{u}_\tau - v) \geq 0$$

for bounded $v \geq \ell$, followed by weak-strong passage and Minty's argument, identifies $b = \beta(u)$. Convex weak closure gives $u(t) \in \mathcal{K}$ almost everywhere. Compatibility gives

$$\kappa(\overline{u}_\tau) = \hat{\kappa}(\overline{b}_\tau) \to \hat{\kappa}(b) = \kappa(u) \ \text{ strongly in every finite } L^p,$$

and therefore the fluxes converge weakly in L^2 to (11).

For completeness, the limiting Signorini inequality can be obtained without assigning an H^{-1} derivative to the storage. Convexity of W implies

$$\left(b^n - b^{n-1}\right) u^n \geq W(b^n) - W(b^{n-1}).$$

Sum (19) between two time levels with a fixed $v \in \mathcal{K}$, apply this inequality, and pass to the limit. The storage terms converge strongly, including at the two endpoints of the interval, by (29). The negative quadratic form $-\int_\Omega A \nabla \overline{u}_\tau \cdot \nabla \overline{u}_\tau \mathrm{d}x$ is weakly upper semicontinuous; all linear flux terms pass weakly, and the gravity coefficient passes strongly. We obtain precisely (17). Piecewise constant admissible competitors yield the corresponding time-dependent integrated formulation.

This proves existence in Theorem 1.

Apply (20) at successive time levels and pass to (29) to obtain (18). In particular, equal initial storages generate the same continuous storage trajectory. The next lemma proves that this trajectory determines the Kirchhoff state and the flux without pairing a distributional time derivative of the storage with a moving competitor.

Lemma 4 (Reconstruction of the Kirchhoff state). Let (u_1,b) and (u_2,b) satisfy (17) with the same storage $b\in C\left([0,T];L^1(\Omega)\right)$, the same set $\mathcal{K}$, and the same Neumann datum. Assume $b=\beta(u_i)$ almost everywhere, $u_i\in L^2\left(0,T;H^1(\Omega)\right)$, and (14). Then $u_1=u_2$ almost everywhere in $\Omega\times(0,T)$, and their Darcy fluxes coincide.

Proof. Put $F(t)=\int_\Omega W\left(b(t)\right)\mathrm{d}x$. Choose $s\in(0,T)$ such that $u_i(s)\in\mathcal{K}$, $u_i(s)\in\partial W\left(b(s)\right)$ almost everywhere, and s is a Lebesgue point of both $u_i\in L^2\left(0,T;H^1(\Omega)\right)$. These properties hold for almost every s. At this fixed s, use the admissible competitor $v=u_2(s)$ in (17) for u_1, and $v=u_1(s)$ for u_2, on $[s,s+h]$. Adding the two inequalities gives

$$\begin{aligned}2\left[F(s+h)-F(s)\right]\le&\int_\Omega\left[b(s+h)-b(s)\right]\left[u_1(s)+u_2(s)\right]\mathrm{d}x\\&+\int_s^{s+h}a\left(u_1(r),u_2(s)-u_1(r)\right)\mathrm{d}r\\&+\int_s^{s+h}a\left(u_2(r),u_1(s)-u_2(r)\right)\mathrm{d}r.\end{aligned}\tag{30}$$

The subgradient inequality at time s supplies

$$F(s+h)-F(s)\ge\int_\Omega\left[b(s+h)-b(s)\right]u_i(s)\mathrm{d}x\quad(i=1,2).$$

Thus, the sum of the two time integrals in (30) is nonnegative. Divide it by h and let $h\downarrow 0$. At the chosen Lebesgue point, the mean of $\left\|u_i(r)-u_i(s)\right\|_{H^1(\Omega)}^2$ over $[s,s+h]$ tends to zero. Bounded continuity of κ then gives convergence of the gravity term, and the fixed boundary datum passes by the trace theorem. Consequently,

$$0\le a\left(u_1(s),u_2(s)-u_1(s)\right)+a\left(u_2(s),u_1(s)-u_2(s)\right).$$

Because $\beta\left(u_1(s)\right)=\beta\left(u_2(s)\right)$, compatibility implies $\kappa\left(u_1(s)\right)=\kappa\left(u_2(s)\right)$. The gravity and Neumann terms cancel, leaving

$$0\le-\int_\Omega\boldsymbol{A}\nabla\left(u_1(s)-u_2(s)\right)\cdot\nabla\left(u_1(s)-u_2(s)\right)\mathrm{d}x.$$

Ellipticity makes the gradient difference zero. The two states have the same Dirichlet trace on Γ_{DC}, so Poincaré's inequality gives $u_1(s)=u_2(s)$. This holds for almost every s; the flux formula (11) then proves the last claim. □

This completes the proof of Theorem 1.

6. Constitutive Laws and the Contraction Hypothesis

The key order estimate is the exact resolvent inequality (20). The limit $\delta\downarrow 0$ is

taken before the time-step limit, so no time-chain assertion for an unspecified weak solution is needed. The outward Neumann flux cancels because the data are the same and stationary; the seepage cone is preserved by the lattice competitors in the proof.

6.1. Rational and Van Genuchten-Mualem Laws

For the Haverkamp conductivity [21]

$$k(\psi)=K_s\frac{A}{A+|\psi|^B}\ (\psi<0),\quad k(\psi)=K_s\ (\psi\geq 0),$$

$B>1$ makes $\ell>-\infty$, and

$$\frac{\mathrm{d}\kappa}{\mathrm{d}u}=\frac{k'(\psi)}{k(\psi)}=\frac{B|\psi|^{B-1}}{A+|\psi|^B},\ \sup\left|\frac{\mathrm{d}\kappa}{\mathrm{d}u}\right|=(B-1)^{(B-1)/B}A^{-1/B}. \tag{31}$$

Hence κ is Lipschitz for any $A>0$ and $B>1$. The Vachaud rational conductivity has the same form and therefore satisfies the same modulus condition when its exponent is greater than one. A corresponding strictly increasing retention law below saturation, with a compatible saturated plateau, fulfills the constitutive assumptions.

For Van Genuchten-Mualem,

$$S_e=\left(1+(\alpha|\psi|)^n\right)^{-m},\ m=1-\frac{1}{n},\ k(\psi)=K_sS_e^i\left[1-\left(1-S_e^{1/m}\right)^m\right]^2.$$

With the standard $i=1/2$ and $n>3/2$, the dry conductivity behaves as $k(\psi)\asymp|\psi|^{-p}$, $p=2n+i(n-1)>1$. The dry Kirchhoff endpoint is finite and $\kappa(u)\asymp(u-\ell)^{p/(p-1)}$. At saturation $K_s-k(\psi)\asymp|\psi|^{n-1}$ and $|u|\asymp|\psi|$. Therefore

$$|\kappa(r)-\kappa(s)|\leq C|r-s|^\gamma,\ \gamma=\min\{1,n-1\}>\frac{1}{2}, \tag{32}$$

so (15) holds. Both laws have constant storage and conductivity above saturation, which verifies (14). For $1<n\leq 3/2$, the stated modulus hypothesis does not hold; Theorem 1 is asserted for the specified $n>3/2$ range.

6.2. Physical Scope of the Van Genuchten-Mualem Restriction

The condition $n>3/2$ is a sufficient condition for the transformed gravity-flux modulus used in the present contraction proof. Its origin is the behavior near saturation. For the standard choice $i=1/2$,

$$K_s-k(\psi)\asymp|\psi|^{n-1},\ |\Phi(\psi)|\asymp|\psi|\ \text{as}\ \psi\uparrow 0.$$

The resulting Hölder exponent is $\gamma=\min\{1,n-1\}$, so the stated hypothesis $\omega_\kappa(\delta)^2/\delta\to 0$ holds when $\gamma>1/2$. In contrast, the dry-end conductivity satisfies $k(\psi)\asymp|\psi|^{-p}$ with $p=2n+(n-1)/2>1$ for every $n>1$. Thus the restriction $n>3/2$ is not caused by dry-end integrability. For $1<n\leq 3/2$, the present modulus hypothesis fails; this does not prove nonexistence or nonuniqueness

for the original constitutive law.

The restriction excludes physically relevant parameterizations of fine-textured soils. Table 2 lists representative mean values from Carsel and Parrish [22]. Several clay-rich classes have values close to one, and the exclusion also affects some silt-rich classes. These catalog values illustrate the scope of the theorem; they are not universal constants for a texture class. Applicability to a particular soil must be assessed using its fitted hydraulic parameters. Moreover, satisfying this exponent condition does not dispense with the other constitutive, geometric and boundary assumptions of the theorem.

Table 2. Illustrative mean Van Genuchten exponents from Carsel and Parrish [22] and satisfaction of the exponent condition in the present theorem.

Soil textural class	Representative mean n	$n > 3/2$
Sand	2.68	Yes
Sandy loam	1.89	Yes
Loam	1.56	Yes
Silt loam	1.41	No
Clay loam	1.31	No
Clay	1.09	No

One possible extension is to regularize the conductivity near saturation. For a fixed $\varepsilon > 0$, one may retain the original law for $\psi \le -\varepsilon$, replace it on $[-\varepsilon, 0]$ by a positive, monotone Lipschitz connection to K_s, and keep the constant saturated extension. The Kirchhoff transformation must then be recomputed for the modified law. Since the conductivity is bounded away from zero on this transition interval, a bounded derivative there gives a bounded ratio $k_\varepsilon'/k_\varepsilon$ and hence a Lipschitz transformed conductivity on that interval. Together with the unchanged regular dry-end behavior and the compatible storage plateau, this provides a route to a model satisfying the present assumptions for each fixed regularization. Regularizing only the dry tail would not remove the near-saturation obstruction. The relevant constants can deteriorate as $\varepsilon \downarrow 0$, so contraction and uniqueness for the unregularized law cannot be inferred without an additional limiting argument.

An alternative is to fit a constitutive pair with a compatible near-saturation behavior and a transformed conductivity satisfying the required modulus. The rational Haverkamp-Vachaud conductivity with exponent $B > 1$, already examined above, is one such mathematical possibility. Replacing a constitutive law changes the physical model and must therefore be assessed against the hydraulic data of the soil under consideration. Extending the contraction argument to the unmodified Van Genuchten-Mualem law throughout $1 < n \le 3/2$ remains outside the present theorem.

7. Conclusion and Perspectives

We have constructed a Rothe energy solution for the closed dry-state constrained Kirchhoff formulation of degenerate Richards flow with stationary Dirichlet and Neumann data and a Signorini seepage boundary. The initial storage can be any admissible L^1 state; its pressure need not satisfy the Dirichlet trace at time zero. The elliptic resolvent is contractive in the storage, and its semigroup limit is strongly continuous in time. The resulting storage, Kirchhoff state, and Darcy flux are unique in the constructed energy class under the stated gravity modulus. Rational Haverkamp-Vachaud conductivities and Van Genuchten-Mualem laws in the specified exponent range satisfy this modulus.

A modeling limitation is the spatially uniform retention law $\theta = \theta(\psi)$ and the separable conductivity $\boldsymbol{K}(x,\psi) = k(\psi)\boldsymbol{A}(x)$. The measurable uniformly elliptic tensor $\boldsymbol{A}(x)$ permits spatial variations and jumps in conductivity magnitude and anisotropy, but the pressure dependence of the scalar conductivity and the retention curve are common to the medium. Consequently, the theorem does not directly cover general layered soils whose layers possess independently varying retention and relative-conductivity curves, including different residual and saturated water contents or different Van Genuchten parameters. Such extensions require a treatment of spatially dependent constitutive graphs and appropriate interface transmission conditions; they do not follow by simply replacing $\boldsymbol{A}$ in the present proof. At dry contact, the source-free interpretation is subject to the qualifications stated in Section 1.

Extension to time-dependent boundary data requires a corresponding treatment of the moving admissible set.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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