

Recognition by Spectrum of Some Finite Simple Linear Groups over Fields of Characteristic 5

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Abstract

Let L be finite simple linear groups $A_p(5)$ with prime p , $\omega(G)$ denote the spectrum of a finite group G , i.e., the set of orders of its elements. We show that if $p > 3$ is prime, then any finite group G with $\omega(G) = \omega(L)$ is isomorphic to L , where $L = A_p(5)$ has disconnected Gruenberg-Kegel graph. As an immediate consequence of the known results, the recognition problem for all simple groups $A_p(5)$ by spectrum is completely resolved.

Keywords

Finite Simple Groups, Recognition, Spectrum, Gruenberg-Kegel Graph

1. Introduction

Within group theory, quantity is an interesting subject and also occupies an important position. Examples include the order of a group, the order of an element, the element of maximum order in a group, and the set of orders of its elements, and so on. Early results show that we can determine the type of a group using quantities alone. A celebrated result is the Feit-Thompson theorem: every finite group of odd order is solvable. Since the classification of finite simple groups was completed, the quantitative properties within simple groups have become highly interesting. In a recent paper [1], finite simple groups without elements of order 10 are obtained. And then, finite non-solvable groups without elements of order 10 and finite groups without elements of order 10 are obtained successively ([2] and [3]).

All groups in this paper are finite, and all simple groups are nonabelian. For a finite group G , $\pi(G)$ stands for the set of prime divisors of the order of G , and $\omega(G)$ denotes its spectrum, namely the set of orders of elements of G .

Since $\omega(G)$ is closed under divisibility, it is uniquely determined by the subset $\mu(G)$ of maximal elements of $\omega(G)$ under divisibility.

The spectrum $\omega(G)$ of a finite group G defines its *Gruenberg-Kegel* graph $GK(G)$. The vertex set of $GK(G)$ is $\pi(G)$, and two primes $p, q \in \pi(G)$ are adjacent whenever G contains an element of order pq . Clearly, $\omega(G)$ uniquely determines $GK(G)$. Hence, $\omega(G) = \omega(L)$ implies $GK(G) = GK(L)$.

A finite group G is called *quasi-recognizable* by spectrum if every finite group H satisfying $\omega(H) = \omega(G)$ possesses a unique nonabelian composition factor isomorphic to G . A finite group G is *recognizable* by spectrum provided that any finite group H with $\omega(H) = \omega(G)$ is isomorphic to G . Equivalently, G is recognizable if $h(G) = 1$, where $h(G)$ denotes the number of pairwise non-isomorphic finite groups H with $\omega(H) = \omega(G)$. By a well-known observation of W. J. Shi, a finite group admitting a nontrivial normal soluble subgroup cannot be recognizable. Therefore, the recognizability problem is of interest mainly for groups with trivial soluble radical, especially for simple and almost simple groups.

Let S be a finite simple group of Lie type of characteristic r , and let A be an abelian r -group equipped with an S -action. An element $s \in S$ is said to be *unisingular* on A if s has a nonzero fixed point on A . The group S is called *unisingular* if every element $s \in S$ acts unisingularly on every finite abelian r -group A admitting an S -action.

In [4], finite simple linear groups over fields of characteristic 2 are recognized by spectrum. Applying the adjacency criterions of finite simple groups ([5] and [6]), we solve the recognizability problem for simple groups of type $A_p(5)$, which is the goal of this paper. This work enriches the existing body of research on the recognition of finite simple groups by spectrum.

For prime p , if the *Gruenberg-Kegel graph* of simple group $A_p(5)$ is disconnected ([7] Table 2a-2c), then $4 \mid p+1$. After treating with them separately, we draw the following conclusions:

Theorem 1. *Let $L = A_p(5)$ be a finite simple group with disconnected Gruenberg-Kegel graph. Then L is recognizable by spectrum whenever $p > 3$.*

The following conclusion is an immediate consequence of the known results.

Theorem 2. *Let $L = A_p(5)$ be a finite simple group with disconnected Gruenberg-Kegel graph. Then L is recognizable by spectrum for $p > 3$, quasi-recognizable by spectrum for $p = 3$, and satisfies $h(G) = 2$ for $p = 2$.*

2. Preliminary Results

This section contains several auxiliary results required in the sequel.

Gruenberg and Kegel obtained the following description of finite groups with disconnected graph $GK(G)$, which was summarized in ([8] Lemma 1).

Lemma 2.1 (Gruenberg-Kegel Theorem). *If G is a finite group with disconnected graph $GK(G)$, then one of the following holds:*

- (a) $s(G) = 2$, G is a Frobenius group;
- (b) $s(G) = 2$, G is a 2-Frobenius group;

(c) Let K denote a nilpotent normal subgroup of G . Then there exists a non-abelian simple group S satisfying $S \leq \bar{G} = G/K \leq \text{Aut}(S)$; moreover, K and $\bar{G}/S$ are $\pi_1(G)$ -groups, the graph $\text{GK}(S)$ is disconnected, $s(S) \geq s(G)$, and for every i , $2 \leq i \leq s(G)$, there exists j satisfying $2 \leq j \leq s(S)$ such that $\omega_i(G) = \omega_j(S)$.

A more constructive result, which generalizes the Gruenberg-Kegel theorem in a certain sense, was obtained in [5].

Lemma 2.2 ([5] Corollary 7.2). Let L be a finite non-abelian simple group other than $A_2(3)$, ${}^2A_3(3)$, $C_2(3)$, Alt_{10} and Alt_n with n satisfying $\{r \mid n-3 \leq r \leq n, r \text{ is prime}\} = \emptyset$. Let G be a finite group with $\omega(G) = \omega(L)$, and K denote the maximal normal soluble subgroup of G . Then there exists a finite nonabelian simple group S satisfying $S \leq \bar{G} = G/K \leq \text{Aut}(S)$. Moreover, $t(S) \geq t(G) - 1$, and one of the following statements holds:

- (1) $S \cong \text{Alt}_7$ or $A_1(q)$ for some odd q , and $t(S) = t(2, S) = 3$.
- (2) Each prime $p \in \pi(G)$ non-adjacent to 2 in $\text{GK}(G)$ satisfies that the Sylow p -subgroup of G is isomorphic to the Sylow p -subgroup of S . In particular, $t(2, S) \geq t(2, G)$.

For a finite group, we write $\rho(G) (\rho(r, G))$ for some independent set in $\text{GK}(G)$ (containing r) with a maximal number of vertices, and put $t(G) = |\rho(G)|$ and $t(r, G) = |\rho(r, G)|$. Let $s = s(G)$ denote the number of connected components of $\text{GK}(G)$, and let $\pi_i = \pi_i(G)$ be its i th connected component for $i = 1, \dots, s$. For any even order group G , we adopt the convention that $2 \in \pi_1$. For each i , let $\mu_i = \mu_i(G)$ (resp. $\omega_i = \omega_i(G)$) be the collection of all $n \in \mu(G)$ (resp. all $n \in \omega(G)$) such that every prime divisor of n lies in π_i . The symbol $[x]$ stands for the integral part of x .

Lemma 2.3 ([7], Lemma 1.1). Let G be a finite simple group whose graph $\text{GK}(G)$ is disconnected. Then $\mu_i(G)$ is a singleton for $2 \leq i \leq s(G)$. Write n_i for the unique element in $\mu_i(G)$ whenever $i > 1$. The group G , the value $t(G)$, and $n_i(G)$ with $2 \leq i \leq s(G)$ are listed in Tables 1-3, where p stands for an odd prime.

Let q be a natural number and r an odd prime satisfying $(r, q) = 1$. We write $e(r, q)$ for the minimal positive integer number n such that $q^n \equiv 1 \pmod{r}$. If q is odd, we set $e(2, q) = 1$ when $q \equiv 1 \pmod{4}$ and $e(2, q) = 2$ when $q \equiv -1 \pmod{4}$. An odd prime r with $e(r, q) = n$ is called a primitive prime divisor of $q^n - 1$, denoted by r_n . Zsigmondy proved the existence of such a prime in [9].

Lemma 2.4 ([10], Lemma 1). Suppose G is a finite group and $N \triangleleft G$. Assume G/N is a Frobenius group with kernel F and a cyclic complement C . If $(|F|, |N| = 1)$ and $F \not\subseteq N C_G(N)/N$, then there exists a prime divisor p of $|N|$ such that $p|C| \in \omega(G)$.

3. Proof of Theorem 1

In this section, we investigate the spectrum recognizability of the simple group

$A_p(5)$ whose Gruenberg-Kegel graph is disconnected, which is stated in Theorem 1.

At first we assume the following hypothesis:

Let $L = A_p(5)$ where $p > 3$, and suppose G is a finite group satisfying $\omega(G) = \omega(L)$. Then $GK(G) = GK(L)$, $t(L) = \left\lfloor \frac{p+2}{2} \right\rfloor$, $\rho(2, L) = \{2, r_p\}$,

$$\rho(5, L) = \{5, r_p, r_{p+1}\}, \quad s(L) = 2 \quad \text{and} \quad n_2(G) = n_2(L) = \frac{5^p - 1}{4} > 1093.$$

By Lemma 2.1 and Lemma 2.2, if K denotes the maximal normal soluble subgroup of G , then there exists a finite non-abelian simple group S such that $S \leq \bar{G} = G/K \leq \text{Aut}(S)$. Furthermore, $s(S) \geq s(G) = 2$, and

$$n_i(S) = n_2(G) = \frac{5^p - 1}{4} \quad \text{for some } i \text{ with } 2 \leq i \leq s(S).$$

By Lemma 2.3, the group G , the value $t(G)$ and $n_i(G)$ with $2 \leq i \leq s(G)$ are listed in the following Table 1.

Table 1. Finite simple groups G satisfying $s(G) \geq 2$, $t(G) \geq 4$ and $n_i \geq 1093$.

G	Restrictions on G	$t(G)$	n_2	n_3	n_4	n_5
Alt_n	$6 < n = p, p-2$ are primes		$p-2$	p		
Alt_n	$6 < n = p, p+1, p+2$ (not both n and $n-2$ are primes)		p			
$A_{p-1}(q)$	$(p, q) \neq (3, 2), (3, 4)$	$\left\lfloor \frac{p+1}{2} \right\rfloor$ or 4	$\frac{q^p-1}{(q-1)(p, q-1)}$			
$A_p(q)$	$(q-1) \mid (p+1)$	$\left\lfloor \frac{p+1}{2} \right\rfloor$	$\frac{q^p-1}{q-1}$			
${}^2A_{p-1}(q)$	$(q+1) \mid (p+1);$	$\left\lfloor \frac{p+1}{2} \right\rfloor$ or 4	$\frac{q^p+1}{(q+1)(p, q+1)}$			
${}^2A_p(q)$	$(p, q) \neq (3, 3), (5, 2)$	$\left\lfloor \frac{p+1}{2} \right\rfloor$	$\frac{q^p+1}{q+1}$			
$B_n(q)$	$n = 2^m \geq 4, q$ odd	$\left\lfloor \frac{3n+5}{4} \right\rfloor$	$\frac{q^n+1}{2}$			
$B_p(3)$		$\left\lfloor \frac{3p+5}{4} \right\rfloor$	$\frac{3^p-1}{2}$			
$C_n(q)$	$n = 2^m \geq 2$	$\left\lfloor \frac{3n+5}{4} \right\rfloor$	$\frac{q^n+1}{(2, q-1)}$			
$C_p(q)$	$q = 2, 3$	$\left\lfloor \frac{3p+5}{4} \right\rfloor$ or 4	$\frac{q^p-1}{(2, q-1)}$			
$D_p(q)$	$p \geq 5, q = 2, 3, 5$	$\left\lfloor \frac{3p+1}{4} \right\rfloor$	$\frac{q^p-1}{q-1}$			
$D_{p+1}(q)$	$q = 2, 3$	$\left\lfloor \frac{3p+1}{4} \right\rfloor$	$\frac{q^p-1}{(2, q-1)}$			
${}^2D_n(q)$	$n = 2^m \geq 4$	$\left\lfloor \frac{3n+4}{4} \right\rfloor$	$\frac{q^n+1}{(2, q+1)}$			
${}^2D_n(2)$	$n = 2^m + 1 \geq 5$	$\left\lfloor \frac{3n+4}{4} \right\rfloor$	$2^{n-1} + 1$			
${}^2D_n(3)$	$9 \leq n = 2^m + 1 \neq p$	$\left\lfloor \frac{3n+5}{4} \right\rfloor$	$\frac{3^{n-1}+1}{2}$			
${}^2D_p(3)$	$5 \leq p \neq 2^m + 1$	$\left\lfloor \frac{3p+5}{4} \right\rfloor$	$\frac{3^p+1}{4}$			
${}^2D_p(3)$	$p = 2^m + 1$	$\left\lfloor \frac{3p+5}{4} \right\rfloor$	$\frac{3^{p-1}+1}{2}$	$\frac{3^p+1}{4}$		
$E_6(q)$		5	$\frac{q^6+q^3+1}{(3, q-1)}$			
${}^2E_6(q)$	$q > 2$	5	$\frac{q^6-q^3+1}{(3, q+1)}$	1093		
$E_7(3)$		7	757			
$F_4(q)$	q even	4 or 5	$q^4 + 1$	$q^4 - q^2 + 1$		
	q odd	5	$q^4 - q^2 + 1$			
${}^2F_4(q)$	$q = 2^{2m+1} > 2$	5	$q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1$	$q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$		
${}^2G_2(q)$	$q = 3^{2m+1} > 3$	5	$q - \sqrt{3q} + 1$	$q + \sqrt{3q} + 1$		
${}^2B_2(q)$	$q = 2^{2m+1} > 2$	4	$q - 1$	$q - \sqrt{2q} + 1$	$q + \sqrt{2q} + 1$	
$E_8(q)$	$q \equiv 2, 3 \pmod{5}$	12	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	$q^8 - q^4 + 1$	
	$q \equiv 0, 1, 4 \pmod{5}$	12	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	$q^8 - q^4 + 1$	$\frac{q^{10}+1}{q^2+1}$

All $n_i(G)$ ($2 \leq i \leq s(G)$) are quoted from the three tables above and will not be reproduced unless required.

Suppose $K \neq 1$. For every prime $r \in \pi(K)$, we have $O^r(K) \neq K$ ([6] Lemma 2.2). Let $\tilde{G} = G/O^r(K)$ and $\tilde{K} = K/O^r(K)$ denote these factor groups. Then $\tilde{K}$ is a nontrivial r -group. Write $\phi(\tilde{K})$ for the Frattini subgroup of $\tilde{K}$, and define the factor groups $\hat{G} = \tilde{G}/\phi(\tilde{K})$ and $\hat{K} = \tilde{K}/\phi(\tilde{K})$. As $G/K \cong \hat{G}/\hat{K}$, the relation $\omega(\hat{G}) \subsetneq \omega(L)$ implies $\omega(G) \subsetneq \omega(L)$. It follows that we are allowed to replace G and K by $\hat{G}$ and $\hat{K}$, respectively. After this reduction, we may assume that K is a nontrivial elementary abelian r -group ([4], Proposition 1).

Lemma 3.1. *S is not isomorphic to any sporadic groups.*

Proof. When S is a sporadic group, we have $n_i(S) \leq 71 < \frac{5^p - 1}{4}$, then S is not isomorphic to any sporadic groups by Table 1. $\square$

Since $t(L_2(q)) = 3 < t(L) - 1$, we have $S \not\cong A_1(q)$. Thus the second statement of Lemma 2.2 holds, then $r_p \notin \pi(K)$ and $\rho(2, G) = \{2, r_p\} \subseteq \rho(2, S)$. We now examine all possibilities for the non-abelian simple group S listed in Table 1.

Lemma 3.2. *$S \not\cong \text{Alt}_n$ ($n \geq 5$).*

Proof. Let $S \cong \text{Alt}_n$ ($n \geq 5$), then at least one prime among $n, n-1, n-2, n-3$ equals r_p . Since $n_i(S) = n_2(L)$, we have $n \geq 11$ and

$$r_p = \frac{5^p - 1}{4}. \text{ On the other hand, } r_{2p} \mid 5^p + 1 \text{ and } 5^p + 1 \equiv 0 \pmod{4}, \text{ then}$$

$$r_{2p} \leq \frac{5^p + 1}{4} < \frac{5^p - 1}{4} - 6 \leq r_p - 6. \text{ So } r_{2p} \in \pi(S) \text{ and } r_{2p} < n - 3, \text{ then } r_{2p} \text{ is}$$

adjacent to 5 in $\text{GK}(S)$ ([5], Proposition 1.1). However they are non-adjacent in $\text{GK}(L)$, we obtain a contradiction. $\square$

Lemma 3.3. *S is not isomorphic to any simple exceptional groups of Lie type.*

Proof. Let $S \cong {}^3D_4(q)$, then $t(L) \leq t(S) + 1 \leq 4$, therefore $p \leq 5$. However in this case where $n_2(L) = n_2(S) < 1093$, a contradiction.

Let $S \cong E_8(q)$, then $t(L) \leq t(S) + 1 = 13$, therefore $p \leq 17$. And since $31 \in \pi(S)$, then $p = 17$. But in this case, $\text{GK}(L)$ is not disconnected since $q - 1$ does not divide $p + 1$, a contradiction by Table 1.

Let $S \cong E_7(q)$, then $t(L) \leq t(S) + 1 = 8$, therefore $p \leq 7$. If $q \geq 4$, then $s(S) = 1 < s(G)$, a contradiction. If $q < 4$, then $n_i(S) < 1093$, we obtain a contradiction by $n_i(S) = n_2(L)$.

Let $S \cong G_2(q)$, then $t(L) \leq t(S) + 1 = 4$, therefore $p = 3$ or 5. By $n_2(L) = n_i(S)$, $p = 3$ and $q = 4$ or 3, then $21 \notin \omega(L)$ and $30 \notin \omega(L)$. If $q = 4$, then $21 \in \omega(S)$, a contradiction. Thus $q = 3$, then $5 \in \pi(K)$ for $G/K \leq \text{Aut}(S)$. And since $G_2(3)$ includes a Frobenius subgroup with kernel of order 13 and cyclic complement of order 6, then $30 \in \omega(G)$, a contradiction.

Let $S \cong F_4(q)$ or $E_6^c(q)$, then $t(L) \leq t(S) + 1 \leq 6$, therefore $p \leq 7$. If $p = 3$, then $21 \notin \omega(L)$, however $21 \in \omega(S)$, a contradiction. If $p = 5$, then $33 \notin \omega(L)$. Since $11 = r_p \in \pi(S)$, we have $33 \in \omega(S)$, a contradiction. Thus $p = 7$, and $n_2(L) = 1093$. However the equation $n_i(S) = n_2(L)$ admits no solutions ([11] Theorem 3), a contradiction.

Let S be isomorphic to a finite simple Suzuki or Ree group, then $t(L) \leq t(S) + 1 \leq 6$, therefore $p \leq 7$. However in this case the equation $n_i(S) = n_2(L)$ admits no solutions, a contradiction ([11], Theorem 7). $\square$

Lemma 3.4. S is not isomorphic any group of $B_n(q)$, $C_n(q)$ or $D_n^e(q)$.

Proof. If $S \cong B_n(q)$ or $C_n(q)$, then $S \cong C_m(2)$ by the condition

$t(2, S) \geq 3$, with m a prime. Then $n_2(S) = 2^m - 1 = \frac{5^p - 1}{4}$, and this equation

admits no solutions whenever $p > 3$ ([11], Theorem 7).

If $S \cong {}^2D_n(q)$, then $n_i(S) \in \left\{ q^n + 1, 2^{n-1} + 1, \frac{3^{n-1} + 1}{2}, \frac{3^n + 1}{4} \right\}$ by Table 1 and

$t(2, S)$, where q is even. Obviously, when $n_i(S) \in \left\{ q^n + 1, 2^{n-1} + 1, \frac{3^{n-1} + 1}{2} \right\}$, the

equation $n_i(S) = n_2(G) = \frac{5^p - 1}{4}$ has no solutions. And for $n_i(S) = \frac{3^n + 1}{4}$, the equation has also no solutions ([11] Theorem 1).

If $S \cong D_n(q)$, then $S \cong D_n(5)$ or $D_n(2)$ by Table 1 and $t(2, S)$. Obviously, since $n_i(S) = n_2(G)$, we have $S \not\cong D_n(2)$; otherwise,

$2^n - 1 = n_2(D_n(2)) = \frac{5^p - 1}{4}$, which has no solutions whenever $p > 3$ ([11] Theorem

7). When $S \cong D_n(5)$, by $n_i(S) = n_2(G)$, we infer that $n = p$, i.e., $S \cong D_p(5)$.

Noticing that $S < G/K \leq \text{Aut}(S)$ implies $s(G/K) = 1$ ([10] Theorem 5), we

have $S = G/K$. Suppose $K \neq 1$ and $l \in \pi(K)$, then K can taken as an element abelian l -group. Suppose that $C = C_G(K) \neq K$. Since C is normal in G and

G/K is simple, we have $C/K = G/K = S$. Therefore $r \cdot r_p \in \omega(G) = \omega(L)$.

However r_p is non-adjacent to any other primes in $GK(L)$, a contradiction.

Thus $C = K$ and S acts faithfully on K . If $l = 5$, then S acts unisularly on

K ([12] Theorem 1.3). Denote the natural semidirect product $K \rtimes S$ by M ,

then $\omega(M) \subseteq \omega(G)$. Denote by $o(x)$ the order of an element x . Let $x \in K$

and $y \in S$ such that $o(x) = 5$ and $o(y) = r_p$, then $(x, y) \in M$ and

$o(x, y) = 5 \cdot r_p$, therefore $5r_p \in \omega(G)$, a contradiction. If $l \neq 5$, then S includes a

Frobenius subgroup with kernel of order q^{2p} and cyclic complement of order

r_p . Therefore $l \cdot r_p \in \omega(G)$, a contradiction. Thus $K = 1$, i.e., $S = G$. However

$t(S) = \left\lfloor \frac{3p+1}{4} \right\rfloor$ and $t(G) = \frac{p+1}{2}$, we obtain a contradiction. Then $S \not\cong D_p(5)$.

$\square$

Lemma 3.5. $S \cong A_p(5)$.

Proof. Suppose false. Let $S \cong A_n^e(q)$ where $q = r^k$ and $q \neq 5$, then the group S is one of the groups in the following list by Table 1:

$A_1(q)$, where $q > 2$ and q is even; $A_{n-1}(q)$, where n is an odd prime, $(n, q) \neq (3, 2), (3, 4)$; $A_n(q)$, where n is an odd prime, $(q-1)|(n+1)$; ${}^2A_{n-1}(q)$, where n is an odd prime, ${}^2A_n(q)$, where n is an odd prime, $(q+1)|(n+1)$, $(n, q) \neq (3, 3), (5, 2)$.

In this case $n-2 \geq p$ for $p \geq 11$. We have $n_2(S) \geq \frac{q^{n-2}}{2}$, therefore $q < 5$

by $n_2(S) = n_2(G)$. When $q = 2$ and $p \geq 11$, we have $n_2(S) = 2^n - 1$ or $\frac{2^n + 1}{3}$. If $n_2(S) = 2^n - 1$, then $2^{n+2} - 5^p = 3$, which is impossible. If

$n_2(S) = \frac{2^n + 1}{3}$, we have $2^{n+2} = 3 \cdot 5^p - 7$, which is also impossible. Similarly, we infer that $q \neq 3$ or 4 for $p \geq 11$. Thus we have $p < 11$. Let $p = 5$, then $S \cong A_4(5)$ or $A_5(5)$ by $n_2(S) = n_2(G)$. However in this case, $t(2, S) = 2 < t(2, G)$, a contradiction. Similarly we can exclude $p = 7$. Similarly S isn't isomorphic to any other case.

When $S \cong A_n(5)$, by $n_i(S) = n_i(G)$, we infer that $n = p$, i.e., $S \cong A_p(5)$.

Thus the Lemma is proved. $\square$

Lemma 3.6. $S = G$

Proof. By Lemma 2.2, we obtain $S \leq \bar{G} = G/K \leq \text{Aut}(S)$.

Suppose $K \neq 1$, then K may be assumed to be an elementary abelian r -group with $r \in \pi(K)$. Let $r = 5$, then the group $A_p(5)$ acts unisingular on K , so $5 \cdot r_p \in \omega(G)$, which is a contradiction. Hence $r \neq 5$, and $r \cdot r_p \in \omega(G)$. However r_p is non-adjacent to any other prime in $GK(L)$, which gives another contradiction. Therefore $K = 1$, and $S \leq G \leq \text{Aut}(S)$.

If $S < G \leq \text{Aut}(S)$, then there exists an order-2 graph automorphism β such that $G/S \cong \langle \beta \rangle$ ([13] Theorem 5). Consider $C_S(\beta)$, the centralizer of β in S . Since $\pi(C_S(\beta)) = \pi(B_{p-1}(5))$ ([5] Table 4.5.2), we deduce that $5 \cdot r_{2p-2} \in \omega(G)$, which yields a contradiction. Consequently, $G = S \cong A_p(5)$. $\square$

By virtue of Lemmas 3.1-3.6 above, the proof of Theorem 1 is now complete. Grechkoseeva and Zvezdina investigated the spectrum recognition problem for simple classical groups $L_4(q)$ and $U_4(q)$ over finite fields of odd characteristic (see [14]). They proved that every finite group isospectral to $L_4(q)$ is almost simple with socle $L_4(q)$, so $L_4(q)$ is quasi-recognizable. In particular, $L_4(5) \cong A_3(5)$ is only known to be quasi-recognizable, and the pure spectrum-recognition remains open for this group. Mazurov showed in [10] that $h(A_2(5)) = 2$. Hence, the spectrum-recognition problem for all simple groups $A_p(5)$ is fully settled, and **Theorem 2** is proved.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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